Mathematical Logic

Short note: Least fixed points versus least closed points

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Abstract

This short note is on the question whether the intersection of all fixed points of a positive arithmetic operator and the intersection of all its closed points can proved to be equivalent in a weak fragment of second order arithmetic.

1 Introduction

Fixed points and least fixed points of positive operators are well-studied objects; cf., e.g., Moschovakis [1] for the general theory and Pohlers [2] for applications in proof theory. Given a monotone operator

$$\Gamma : Pow(\mathbb{N}) \rightarrow Pow(\mathbb{N}),$$

the Knaster–Tarski theorem tells us that Γ has a least fixed point I_Γ and that

$$I_\Gamma = \bigcap \{X \subseteq \mathbb{N} : \Gamma(X) = X\} = \bigcap \{X \subseteq \mathbb{N} : \Gamma(X) \subseteq X\}.$$

Thus the least fixed point of Γ can be characterized as the intersection of all fixed points of Γ as well as the intersection of all Γ-closed points.

However, according to my knowledge, it has never been discussed whether this equivalence can be proved in weak subsystems of second order arithmetic. It is the purpose of this note to shed some light on this issue. I thank Kentaro Sato for bringing this question to my attention.

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2 The general environment

Let $\mathcal{L}_2$ be a standard language of second order arithmetic with countably infinite supplies of *number variables* k, m, n and *set variables* U, V, X, Y, Z. We also have function and relation symbols for all primitive recursive functions and relations. The formulas θ, φ, ψ of $\mathcal{L}_2$ are built up as usual. See, for example, Simpson [3].

A formula of $\mathcal{L}_2$ without bound set variables is called *arithmetical*. We refer to arithmetical formulas $\varphi[X, n]$ in which the set variable X occurs only positively as *X-positive operator forms*. Each such X-positive operator form $\varphi[X, n]$ defines a monotone function

$$\Gamma_\varphi : Pow(\mathbb{N}) \to Pow(\mathbb{N}); \quad \Gamma_\varphi(S) := \{n \in \mathbb{N} : \varphi[S, n]\}.$$

Throughout this paper we work in classical logic with equality for the first sort. Equality for sets in $\mathcal{L}_2$ is defined by saying that two sets are identical iff they contain the same elements.

ACA_0 is the system of second order arithmetic whose non-logical axioms comprise the defining axioms for all primitive recursive functions and relations, the axiom schema of *arithmetical comprehension*

$$\exists X \forall n (n \in X \leftrightarrow \varphi[n])$$

for all arithmetic formulas $\varphi[n]$, and the *induction axiom*

$$\forall X (0 \in X \wedge \forall n (n \in X \to n{+}1 \in X) \to \forall n (n \in X)).$$

It is well known that ACA_0 is a conservative extension of Peano arithmetic PA.

Now we turn to the notions of *fixed point* and *least fixed point* as well as *closed point* and *least closed point* of X-positive operator forms. Let $\varphi[X, n]$ be such an operator form; then we set:

$$\begin{aligned}
FP_\varphi[Z] &:= \forall n (\varphi[Z, n] \leftrightarrow n \in Z), \\
LFP_\varphi[Z] &:= FP_\varphi[Z] \wedge \forall X (FP_\varphi[X] \to Z \subseteq X), \\
CP_\varphi[Z] &:= \forall n (\varphi[Z, n] \to n \in Z), \\
LCP_\varphi[Z] &:= CP_\varphi[Z] \wedge \forall X (CP_\varphi[X] \to Z \subseteq X).
\end{aligned}$$

Before we turn to some systems of second order arithmetic based on these principles, let me point out a significant difference between fixed points and closed points.

Obviously, every fixed point of an X-positive operator form is also a closed point of this form. The converse is clearly not true in general. More interesting is the observation that closed points are closed under intersections whereas this is in general not the case for fixed points.

Remark 1 Clearly, the closed points of any X-positive operator form $\varphi[X, n]$ form are closed under intersections, i.e.

$$\mathsf{ACA}_0 \vdash \forall X, Y(CP_\varphi[X] \wedge CP_\varphi[Y] \rightarrow CP_\varphi[X \cap Y]).$$

On the other hand, consider the X-positive operator form

$$\psi[X, n] := (\exists m \in X)((n = m \wedge m < 2) \vee n = m + 2 \vee n = m + 5).$$

Then it is easy to check that

$$P_\psi := \{2m + 5n : m, n \in \mathbb{N}\} \quad \text{and} \quad Q_\psi := \{1 + 2m + 5n : m, n \in \mathbb{N}\}$$

are fixed points of ψ whereas $P_\psi \cap Q_\psi$ is not a fixed point of ψ.

In the following we write FP_0, LFP_0, and LCP_0 for the extensions of ACA_0 by all formulas

$$\exists Z FP_\varphi[Z], \quad \exists Z LFP_\varphi[Z], \text{ and } \exists Z LCP_\varphi[Z],$$

respectively, where $\varphi[X, n]$ ranges over all X-positive operator forms. Keep in mind that $\varphi[X, n]$ may also contain parameters, i.e., additional free number and set variables.

3 *LFP* versus *LCP*

We first observe (trivial) that least fixed points and least closed points of X-positive operator forms are uniquely determined, provided that they can be shown to exist. It is also easy to see that the least closed point of any X-positive operator form is its least fixed point. And, as the following lemma states, this fact can be proved in ACA_0.

Lemma 2 *Let $\varphi[X, n]$ be an X-positive operator form. Then we have that*

$$\mathsf{ACA}_0 \vdash \forall Z(LCP_\varphi[Z] \rightarrow LFP_\varphi[Z]).$$

Proof Let Z be the least closed point of φ. We form, by means of arithmetic comprehension, the set

$$Y := \{n : \varphi[Z, n]\}.$$

Because of $CP_\varphi[Z]$ we have $Y \subseteq Z$, and thus the monotonicity of φ yields

$$\forall n(\varphi[Y, n] \rightarrow n \in Y).$$

So we have $CP_\varphi[Y]$, hence $Z \subseteq Y$. Therefore, Z is a fixed point of φ. On the other hand, every fixed point of φ is also φ-closed. Hence Z is the least fixed point of φ. $\square$

More interesting is the question of whether the least fixed point of an X-positive operator is its least closed point. In order to obtain this result, we have to make use of a specific fixed point construction and thus work in FP_0.

Theorem 3 *Let $\varphi[X, n]$ be an X-positive operator form. Then we have that*

$$\mathsf{FP}_0 \ \vdash \ \forall Z(LFP_\varphi[Z] \ \rightarrow \ LCP_\varphi[Z]).$$

Proof Working in FP_0 we assume that Z is the least fixed point of φ. We must show that

$$CP_\varphi[Y] \ \rightarrow \ Z \subseteq Y \tag{*}$$

for any set Y. Given a Y with $CP_\varphi[Y]$, consider the X-positive operator form

$$\psi[X, n] \ := \ \varphi[X, n] \wedge n \in Y.$$

Since we work in FP_0, we know that ψ has a fixed point U, i.e.

$$\forall n(n \in U \ \leftrightarrow \ \psi[U, n] \ \leftrightarrow \ \varphi[U, n] \wedge n \in Y),$$

in particular, $U \subseteq Y$. Therefore, for any n,

$$\varphi[U, n] \ \rightarrow \ \varphi[Y, n] \ \rightarrow \ n \in Y.$$

Hence,

$$\varphi[U, n] \ \leftrightarrow \ \varphi[U, n] \wedge n \in Y \ \leftrightarrow \ \psi[U, n] \ \leftrightarrow \ n \in U.$$

This means that U is a fixed point of φ, and we obtain

$$Z \subseteq U \subseteq Y.$$

Thus we have (*), and our theorem is proved. $\qquad\square$

Corollary 4 *The theories LFP_0 and LCP_0 are equivalent, i.e., for every formula θ of $\mathcal{L}_2$,*

$$\mathsf{LFP}_0 \vdash \theta \ \Longleftrightarrow \ \mathsf{LCP}_0 \vdash \theta.$$

Proof For the direction from left to right we only have to show that LCP_0 proves $\exists Z LFP\varphi[Z]$ for any X-positive operator form $\varphi[X, n]$. This is immediate from the previous lemma.

Conversely, for the direction from right to left, all we have to do is to prove that LFP_0 proves $\exists Z LCP_\varphi[Z]$ for any X-positive operator form $\varphi[X, n]$. But since FP_0 is a subsystem of LFP_0, this is a direct consequence of the previous theorem. $\qquad\square$

The equivalence of LFP_0 and LCP_0 can also be shown more directly: $\mathsf{LFP}_0 \subseteq \mathsf{LCP}_0$ is obvious (see above). For the inclusion $\mathsf{LCP}_0 \subseteq \mathsf{LFP}_0$, pick an arbitrary X-positive operator form $\varphi[X, n]$. We have to show in LFP_0 that φ has a least closed point. To achieve this, consider – following a suggestion of Sato – the X-positive operator form

$$\psi[X, n] := \varphi[X, n] \vee n \in X.$$

Then it is easy to check that

$$\mathsf{ACA}_0 \vdash \forall Z(FP_\psi[Z] \leftrightarrow CP_\varphi[Z]).$$

As a consequence of that we know that the least fixed point of ψ is the least closed point of φ. This argument shows $\mathsf{LCP}_0 \subseteq \mathsf{LFP}_0$.

There is an interesting questions in this context, which is still open as far as I know: Can the assertion $\forall Z(LFP_\varphi[Z] \rightarrow LCP_\varphi[Z])$ be proved in ACA_0? And if not, what do we need? Is there a theory weaker than FP_0 that does the job?

References

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