

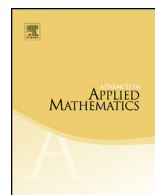


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Convex bodies generated by sublinear expectations of random vectors [☆]

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ABSTRACT

We show that many well-known transforms in convex geometry (in particular, centroid body, convex floating body, and Ulam floating body) are special instances of a general construction, relying on applying sublinear expectations to random vectors in Euclidean space. We identify the dual representation of such convex bodies and describe a construction that serves as a building block for all so defined convex bodies. Sublinear expectations are studied in mathematical finance within the theory of risk measures. In this way, tools from mathematical finance yield a whole variety of new geometric constructions.

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1. Introduction

The concept of sublinear expectation is essential in mathematical finance, where it is used to quantify the operational risk, see [13,17]. The sublinearity property reflects the financial paradigm, saying that the diversification decreases the risk, and so the risk of a diversified portfolio is dominated by the sum of the risks of its components. Sublinear expectations are closely related to solutions of backward stochastic differential equations, see [44].

A sublinear expectation \mathbf{e} is a sublinear (positively homogeneous and convex) map from the space $L^p(\mathbb{R})$ (or another linear space of random variables) to $(-\infty, \infty]$, and so may be regarded as a convex function on an infinite-dimensional space, see [58] for a thorough account of convex analysis tools in the infinite-dimensional setting.

In this paper we use a sublinear expectation \mathbf{e} to associate with each p -integrable random vector ξ in \mathbb{R}^d a convex closed set $\mathcal{E}_{\mathbf{e}}(\xi)$ in \mathbb{R}^d . This is done by letting the support function of $\mathcal{E}_{\mathbf{e}}(\xi)$ be the sublinear expectation \mathbf{e} applied to the scalar product $\langle \xi, u \rangle$. For instance, if \mathbf{e} is the L^p -norm and ξ is symmetric, $\mathcal{E}_{\mathbf{e}}(\xi)$ becomes the centroid body associated to the distribution of ξ as introduced by Petty [45] for $p = 1$ and Lutwak and Zhang [37] for a general p . If \mathbf{e} is the average quantile of $\langle \xi, u \rangle$, one obtains convex closed sets called metronoids and studied by Huang and Slomka [26]. Further examples are given by expected random polytopes, which also form a special case of our construction.

We commence with Section 2, giving the definition of sublinear expectation of random variables, explaining their dual representation and presenting several examples. We mention the particularly important Kusuoka representation which expresses any law-determined sublinear expectation in terms of integrated quantiles and describe a novel construction (called the maximum extension) suitable to produce parametric families of sublinear expectations from each given one.

Section 3 presents our construction of convex closed sets $\mathcal{E}_{\mathbf{e}}(\xi)$ generated by a random vector ξ and a given sublinear expectation \mathbf{e} . Section 4 describes a generalisation based on relaxing some properties of the underlying numerical sublinear expectations, namely, replacing them with gauge functions. This construction yields centroid bodies [37] and half-space depth-trimmed regions [42], the latter are closely related to convex floating bodies introduced in [48] and their weighted variant from [8].

One of the most important sublinear expectations is based on using weighted integrals of the quantile function. The corresponding convex bodies are studied in Section 5, where we show their close connection to metronoids [26] and zonoid-trimmed regions [31]. The Kusuoka representation of numerical sublinear expectations yields Theorem 5.4, which provides a representation of a general convex set $\mathcal{E}_{\mathbf{e}}(\xi)$ (derived from ξ using a sublinear expectation \mathbf{e}) in terms of Aumann integrals of metronoids. We further provide a uniqueness result for the distribution of ξ on the basis of a family of convex bodies generated by it, and also a concentration result for random convex sets constructed from the empirical distribution of ξ .

Section 6 specialises our general construction to the case when ξ is uniformly distributed on a convex body K (that is, a compact convex set in \mathbb{R}^d with nonempty interior), and so $\mathcal{E}_e(\xi)$ yields a transform $K \mapsto \mathcal{E}_e(K) = \mathcal{E}_e(\xi)$. We derive several properties of this transformation for general e , in particular, establish the continuity of such maps in the Hausdorff metric.

In special cases, our construction yields L^p -centroid bodies (see [37] and [47, Sec. 10.8]) and Ulam floating bodies recently introduced in [27]. The latter form a particularly important special setting, which is confirmed by showing that all transformations $K \mapsto \mathcal{E}_e(K)$ can be expressed in terms of Ulam floating bodies. For instance, Corollary 6.8 provides a representation of the centroid body of an origin symmetric K as the convex hull of dilated Ulam floating bodies of K . In this course, results for sublinear expectations yield a new insight into the well-known aforementioned constructions of convex bodies, deliver some new relations between them, and provide a general source of nonlinear transformations of convex bodies. Finally, we formulate several conjectures.

2. Sublinear expectations of random variables

2.1. Definition and dual representation

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a nonatomic probability space, and let $L^p(\mathbb{R}^d)$ denote the family of all p -integrable random vectors in \mathbb{R}^d , with $p \in [1, \infty]$. Endow $L^p(\mathbb{R}^d)$ with the $\sigma(L^p, L^q)$ -topology, which is the weak-star topology based on the pairing of $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ with $p^{-1} + q^{-1} = 1$, see [3, Sec. 5.14]. Denote $\mathbb{R}_+ = [0, \infty)$.

The following definition amends the standard definition of sublinear expectations of random variables (see, e.g., [44]) by including the extra lower semicontinuity property, which is often additionally imposed.

Definition 2.1. A *sublinear expectation* is a function $e : L^p(\mathbb{R}) \rightarrow (-\infty, \infty]$ with $p \in [1, \infty]$, satisfying the following properties for all $\beta, \beta' \in L^p(\mathbb{R})$:

- i) monotonicity: $e(\beta) \leq e(\beta')$ if $\beta \leq \beta'$ a.s.;
- ii) translation equivariance: $e(\beta + a) = e(\beta) + a$ for all $a \in \mathbb{R}$, and $e(0) = 0$;
- iii) positive homogeneity: $e(c\beta) = ce(\beta)$ for all $c > 0$;
- iv) subadditivity: $e(\beta + \beta') \leq e(\beta) + e(\beta')$,
- v) lower semicontinuity in $\sigma(L^p, L^q)$, that is,

$$e(\beta) \leq \liminf_{n \rightarrow \infty} e(\beta_n)$$

for each sequence $\{\beta_n, n \geq 1\}$ converging to β in the weak-star topology $\sigma(L^p, L^q)$.

The sublinear expectation e is often referred to as *numerical* one, in contrast with the set-valued expectation introduced in Section 3. The translation equivariance property

implies that $e(a) = a$ for each deterministic a . The sublinear expectation e is said to be *finite* if it takes finite values on all $\beta \in L^p(\mathbb{R})$.

Example 2.2 (*Relation to coherent risk measures*). For $\beta \in L^p(\mathbb{R})$, define $r(\beta) = e(-\beta)$. The obtained antimonotonic and subadditive function is called a *coherent risk measure* of β , see [13] and [17, Def. 4.5]. The negative of the risk is said to be a utility function, see [13].

A random variable β is said to be acceptable if its risk is at most zero. If β is the financial position at the terminal time, its risk $r(\beta)$ yields the smallest amount a of capital which should be reserved at the initial time to render $\beta + a$ acceptable; this amount may be negative if $r(\beta) < 0$, and then capital can be released or invested. The subadditivity property of the risk (equivalently, of e) is the manifestation of the financial principle, saying that diversification decreases the risk. Many results from the theory of risk measures can be easily reformulated for sublinear expectations. For instance, from the theory of risk measures, it is known that the lower semicontinuity property always holds if $p \in [1, \infty)$ and e takes only finite values, see [28].

While the following result is well known for risk measures [17, Cor. 4.18] and sublinear expectations [44, Th. 1.2.1], we provide its proof for completeness.

Theorem 2.3. *A functional $e : L^p(\mathbb{R}) \rightarrow (-\infty, \infty]$ is a sublinear expectation if and only if*

$$e(\beta) = \sup_{\gamma \in \mathcal{M}_e, \mathbf{E}\gamma=1} \mathbf{E}(\gamma\beta), \quad (2.1)$$

where \mathcal{M}_e is a convex $\sigma(L^q, L^p)$ -closed cone in $L^q(\mathbb{R}_+)$.

Proof. *Sufficiency* is easy to confirm by a direct check of the properties.

Necessity. Let \mathcal{A} be the family of $\beta \in L^p(\mathbb{R})$, such that $e(\beta) \leq 0$. The sublinearity property yields that \mathcal{A} is a convex cone. The lower semicontinuity property implies that this cone is weak-star closed. The polar cone to \mathcal{A} is defined as

$$\mathcal{A}^\circ = \{\gamma \in L^q(\mathbb{R}) : \mathbf{E}(\gamma\beta) \leq 0 \text{ for all } \beta \in \mathcal{A}\}. \quad (2.2)$$

Since $-1_A \in \mathcal{A}$ for the indicator of any event A , all random variables from \mathcal{A}° are a.s. nonnegative. The bipolar theorem from functional analysis (see, e.g., [3, Th. 5.103]) yields that $(\mathcal{A}^\circ)^\circ = \mathcal{A}$. Hence,

$$\begin{aligned} e(\beta) &= \inf\{a \in \mathbb{R} : (\beta - a) \in \mathcal{A}\} \\ &= \inf\{a \in \mathbb{R} : \mathbf{E}((\beta - a)\gamma) \leq 0 \text{ for all } \gamma \in \mathcal{A}^\circ\} \\ &= \inf\{a \in \mathbb{R} : \mathbf{E}(\gamma\beta) \leq a\mathbf{E}(\gamma) \text{ for all } \gamma \in \mathcal{A}^\circ\}. \end{aligned}$$

Thus, (2.1) holds with $\mathcal{M}_e = \mathcal{A}^\circ$. \square

Representation (2.1) is called the *dual* representation of \mathbf{e} . It is easy to see that each γ in (2.1) can be chosen to be a function of β , namely, the conditional expectations $\mathbf{E}(\gamma|\beta)$.

A sublinear expectation is said to be *law-determined* (often named law invariant) if it attains the same value on identically distributed random variables, and this is the case for all examples considered in this paper. In terms of the representation (2.1), this means that, for each $\gamma \in \mathcal{M}_{\mathbf{e}}$, the set $\mathcal{M}_{\mathbf{e}}$ contains all random variables sharing the same distribution with γ .

A sublinear expectation is said to be *continuous from below* if it is continuous on all almost surely convergent increasing sequences of random variables in $L^p(\mathbb{R})$. It follows from [28] that each finite sublinear expectation on $L^p(\mathbb{R})$ with $p \in [1, \infty)$ is continuous from below. Every law-determined continuous from below sublinear expectation on a nonatomic probability space is *dilatation monotonic*, meaning that

$$\mathbf{e}(\mathbf{E}(\beta|\mathfrak{A})) \leq \mathbf{e}(\beta) \tag{2.3}$$

for each sub- σ -algebra \mathfrak{A} of \mathfrak{F} , see [17, Cor. 4.59]. In particular, $\mathbf{E}\beta \leq \mathbf{e}(\beta)$ for all $\beta \in L^p(\mathbb{R})$.

2.2. Average quantiles and the Kusuoka representation

For a fixed value of $\alpha \in (0, 1]$ and $\beta \in L^1(\mathbb{R})$, define

$$\mathbf{e}_{\alpha}(\beta) = \frac{1}{\alpha} \int_{1-\alpha}^1 q_t(\beta) dt, \tag{2.4}$$

where

$$q_t(\beta) = \sup\{s \in \mathbb{R} : \mathbf{P}\{\beta \leq s\} < t\} = \inf\{s \in \mathbb{R} : \mathbf{P}\{\beta \leq s\} \geq t\} \tag{2.5}$$

is the t -quantile of β . Because of integration, the choice of a particular quantile in case of multiplicities is immaterial. This sublinear expectation is subsequently called the *average quantile*. In particular, $\mathbf{e}_1(\beta) = \mathbf{E}\beta$ is the mean. If β has a nonatomic distribution, then $\mathbf{e}_{\alpha}(\beta) = \mathbf{E}(\beta|\beta \geq q_{1-\alpha}(\beta))$.

The value of $\mathbf{r}(\beta) = \mathbf{e}_{\alpha}(-\beta)$ is obtained by averaging the quantiles of β at levels between 0 and α . This risk measure is well studied in finance and widely applied in practice under the name of the average Value-at-Risk or expected shortfall, see, e.g., [2]. By computing the dual cone at (2.2) or rephrasing the representation of the risk measure $\mathbf{e}_{\alpha}(-\beta)$ from [28, Th. 4.1], one can derive the following dual representation

$$\mathbf{e}_{\alpha}(\beta) = \sup_{\gamma \in L^{\infty}([0, \alpha^{-1}]), \mathbf{E}\gamma=1} \mathbf{E}(\gamma\beta). \tag{2.6}$$

This immediately yields that the average quantiles satisfy all properties imposed in Definition 2.1.

Average quantiles form a building block for all other law-determined sublinear expectations. The following result for risk measures is known as the *Kusuoka representation*: it was first obtained by Kusuoka [32] in case $p = \infty$ and can also be found in [17, Cor. 4.58] and [13, Th. 32]; the L^p -variant follows from the Orlicz space version proved in [21]. For its validity, it is essential that the probability space is nonatomic.

Theorem 2.4. *Each law-determined sublinear expectation on $L^p(\mathbb{R})$ with $p \in [1, \infty]$ can be represented as*

$$e(\beta) = \sup_{\nu \in \mathcal{P}_e} \int_{(0,1)} e_\alpha(\beta) \nu(d\alpha), \tag{2.7}$$

where \mathcal{P}_e is the family of probability measures ν on $(0, 1]$ such that $\int_{(0,1)} e_\alpha(\beta) \nu(d\alpha) \leq 0$ whenever $e(\beta) \leq 0$.

It is possible to show that e is finite on $L^p(\mathbb{R})$ if and only if the function $t \mapsto \int_{(t,1]} s^{-1} \nu(ds)$ is q -integrable on $(0, 1]$ with respect to the Lebesgue measure for all $\nu \in \mathcal{P}_e$. If e is finite and $p \in [1, \infty)$, one can provide a constructive representation of \mathcal{P}_e in terms of the extremal points of the set $\mathcal{M}_e^1 = \{\gamma \in \mathcal{M}_e : \mathbf{E}\gamma = 1\}$, where \mathcal{M}_e is defined in (2.1). The case $p = \infty$ requires extra arguments, since a norm bounded set in L^1 is not necessarily weakly compact, hence, the supremum in (2.1) is not necessarily attained. Since $(\Omega, \mathcal{F}, \mathbf{P})$ is nonatomic, we can assume without loss of generality that Ω is the interval $[0, 1]$ equipped with its Borel σ -algebra and the Lebesgue measure \mathbf{P} . Let $\gamma : [0, 1] \rightarrow [0, \infty)$ be a nondecreasing right-continuous function that is extremal in \mathcal{M}_e^1 . Define the probability measure ν_γ on $(0, 1]$ by letting

$$\nu_\gamma((0, \alpha)) = \int_{[1-\alpha, 1)} (\gamma(t) - \gamma(1 - \alpha)) dt,$$

and $\nu(\{1\}) = \gamma(0)$. It is shown in [49] that \mathcal{P}_e can be chosen to be the set of ν_γ for the family of all right-continuous nondecreasing functions γ which are extremal in \mathcal{M}_e^1 .

2.3. Examples of sublinear expectations

A simple example of a sublinear expectation is provided by the essential supremum

$$e(\beta) = \text{ess sup } \beta,$$

which is finite for all $\beta \in L^\infty(\mathbb{R})$. If $\alpha \downarrow 0$, then the average quantile $e_\alpha(\beta)$ increases to the (possibly, infinite) value $e_0(\beta)$, which is equal to the essential supremum of β . Next, we discuss more involved constructions of sublinear expectations.

Example 2.5 (*Spectral sublinear expectation*). Let $\varphi : (0, 1] \rightarrow \mathbb{R}_+$ be a nonincreasing function such that $\int_0^1 \varphi(t)dt = 1$, φ is called a spectral function. Then

$$e_{f_\varphi}(\beta) = \int_{(0,1]} q_{1-t}(\beta)\varphi(t)dt \tag{2.8}$$

is called a spectral sublinear expectation, see [1] for the closely related definition of the spectral risk measure. By Fubini’s theorem, e_{f_φ} admits the following equivalent representation

$$e_{f_\varphi}(\beta) = \int_{(0,1]} e_\alpha(\beta)\nu(d\alpha), \tag{2.9}$$

where e_α is given by (2.4) and ν is the probability measure on $(0, 1]$ with

$$\varphi(t) = \int_{(t,1]} s^{-1}\nu(ds), \quad t \in (0, 1]. \tag{2.10}$$

Conversely, for any probability measure ν on $(0, 1]$, (2.9) yields a spectral sublinear expectation. The set \mathcal{P}_e in the Kusuoka representation of $e_{f_\varphi}(\beta)$ consists of the single probability measure ν , so the right-hand side of (2.7) is the supremum over a family of spectral sublinear expectations.

Example 2.6 (*One-sided moments*). The L^p -norm $\|\beta\|_p$ satisfies all properties of a sublinear expectation but the monotonicity and translation equivariance. It is possible to come up with a norm-based sublinear expectation on $L^p(\mathbb{R})$ with $p \in [1, \infty)$ by letting

$$e_{p,a}(\beta) = \mathbf{E}\beta + a(\mathbf{E}(\beta - \mathbf{E}\beta)_+^p)^{1/p} \tag{2.11}$$

with $a \in [0, 1]$, where $x_+ = \max(x, 0)$ denotes the positive part of $x \in \mathbb{R}$. The corresponding risk measure was introduced in [14]. Note that $e_{p,a}(\beta) = \frac{a}{2}\|\beta\|_p$ if β is symmetric. Translation equivariance and positive homogeneity of $e_{p,a}$ are obvious. The subadditivity of the second term follows from $(t + s)_+ \leq (t)_+ + (s)_+$ and the subadditivity of the L^p -norm. To prove the monotonicity, we first observe that since $e_{p,a}$ is subadditive, we only need to show that $e_{p,a}(\gamma) \leq 0$ for any almost surely negative integrable γ . Indeed, substituting $(\gamma - \mathbf{E}\gamma)_+ \leq -\mathbf{E}\gamma$ in (2.11) implies that $e_{p,a}(\gamma) \leq \mathbf{E}\gamma - a\mathbf{E}\gamma \leq 0$.

The sublinear expectation given by (2.11) admits the dual representation (2.1) with the cone \mathcal{M}_e generated by the family of random variables $\gamma = 1 + a(\zeta - \mathbf{E}\zeta)$ for all $\zeta \in L^q(\mathbb{R}_+)$ with $\|\zeta\|_q \leq 1$, see [13, p. 46]. The family \mathcal{P}_e from (2.7) is explicitly known only for $p = 1$; it consists of probability measures obtained as $(1 - at)\delta_1 + at\delta_t$, which is the weighted sum of the Dirac measures at 1 and t for $t \in [0, 1]$. Then

$$e_{1,a}(\beta) = \sup_{t \in [0,1]} \left[(1 - at)\mathbf{E}\beta + ate_t(\beta) \right] = \mathbf{E}\beta + a \sup_{t \in [0,1]} te_t(\beta - \mathbf{E}\beta). \tag{2.12}$$

Recall in this relation that

$$te_t(\beta) = \int_{1-t}^1 q_s(\beta) ds,$$

so that the supremum on the right-hand side of (2.12) is indeed the expectation of $(\beta - \mathbf{E}\beta)_+$.

Example 2.7 (*Expectile*). Following [7], define the *expectile* $e_{[\tau]}(\beta)$ of a random variable $\beta \in L^1(\mathbb{R})$ at level $\tau \in (0, 1)$ as the (necessarily, unique) solution $x \in \mathbb{R}$ of

$$\tau\mathbf{E}(\beta - x)_+ = (1 - \tau)\mathbf{E}(x - \beta)_+.$$

If $\tau \in [1/2, 1)$, then the expectile is a sublinear expectation, see [7]. For $\tau = 1/2$, we obtain the mean of β . For $\tau \in [1/2, 1)$, the dual representation holds with \mathcal{M}_e being the set of $\gamma \in L^\infty(\mathbb{R}_+)$ such that the ratio between the essential supremum and the essential infimum of γ is at most $\tau/(1 - \tau)$. The Kusuoka representation holds with

$$e_{[\tau]}(\beta) = \sup_{t \in [0, 2-1/\tau]} \left[(1 - t)\mathbf{E}\beta + te_{\frac{(1-\tau)t}{(2\tau-1)(1-t)}}(\beta) \right].$$

2.4. Maximum extension

Let e be a law-determined sublinear expectation on $L^p(\mathbb{R})$ with $p \in [1, \infty]$. The following construction suggests a way of extending e to a monotone parametric family of sublinear expectations. For a fixed $m \geq 1$, define

$$e^{\vee m}(\beta) = e(\max(\beta_1, \dots, \beta_m)), \tag{2.13}$$

where β_1, \dots, β_m are independent copies of $\beta \in L^p(\mathbb{R})$. All properties in Definition 2.1 are straightforward and we refer to this sublinear expectation as the *maximum extension* of e . Let us stress that this extension applies only to law-determined sublinear expectations.

It is possible to obtain a family of such expectations $e^{\vee(\lambda)}$ continuously parametrised by $\lambda \in (0, 1]$. For this, m is replaced by a geometrically distributed random variable N with parameter λ , that is, $\mathbf{P}\{N = k\} = (1 - \lambda)^{k-1}\lambda$, $k \geq 1$. Define

$$e^{\vee(\lambda)} = e(\max(\beta_1, \dots, \beta_N)), \quad \lambda \in (0, 1].$$

This family of sublinear expectations interpolates between $e^{\vee(1)}(\beta) = e(\beta)$ and $e^{\vee(0)}(\beta)$ which is set to be $\text{ess sup } \beta$.

Example 2.8. The maximum extension can be applied to the average quantile risk measure e_α ; the result is denoted by $e_\alpha^{\vee m}$. For $\alpha = 1$, we obtain the *expected maximum*

$$e_1^{\vee m}(\beta) = \mathbf{E} \max(\beta_1, \dots, \beta_m). \tag{2.14}$$

Note that

$$e_1^{\vee m}(\beta) = \int_0^1 q_t(\max\{\beta_1, \dots, \beta_m\})dt = \int_0^1 q_{t^{\frac{1}{m}}}(\beta)dt = m \int_0^1 t^{m-1}q_t(\beta)dt.$$

For $m \geq 2$, $e_1^{\vee m}$ is the spectral sublinear expectation given at (2.8) with $\varphi(t) = m(1 - t)^{m-1}$, equivalently, (2.9) with $\nu(dt) = m(m - 1)t(1 - t)^{m-2}dt$. Similar calculations yield that

$$e_\alpha^{\vee m}(\beta) = e_\alpha(\max(\beta_1, \dots, \beta_m)) = \frac{m(m - 1)}{\alpha} \int_0^{1-(1-\alpha)^{1/m}} t(1 - t)^{m-2}e_t(\beta)dt + \frac{m}{\alpha}(1 - \alpha)^{(m-1)/m}(1 - (1 - \alpha)^{1/m})e_{1-(1-\alpha)^{1/m}}(\beta). \tag{2.15}$$

3. Measure-generated convex sets

Fix a law-determined sublinear expectation e on $L^p(\mathbb{R})$, $p \in [1, \infty]$. For a p -integrable probability measure μ on \mathbb{R}^d , equivalently, for a random vector $\xi \in L^p(\mathbb{R}^d)$ with distribution μ , define

$$h(u) = e(\langle \xi, u \rangle), \quad u \in \mathbb{R}^d, \tag{3.1}$$

where $\langle \xi, u \rangle$ denotes the scalar product in \mathbb{R}^d . The function h is subadditive

$$h(u + u') = e(\langle \xi, u + u' \rangle) \leq e(\langle \xi, u \rangle) + e(\langle \xi, u' \rangle) = h(u) + h(u'),$$

and homogeneous

$$h(cu) = e(\langle \xi, cu \rangle) = ce(\langle \xi, u \rangle) = ch(u), \quad c \geq 0.$$

Furthermore, h is lower semicontinuous, since $\langle \xi, u_n \rangle \rightarrow \langle \xi, u \rangle$ in $\sigma(L^p, L^q)$ if $u_n \rightarrow u$ as $n \rightarrow \infty$ and e is assumed to be lower semicontinuous. These three properties identify support functions of convex closed sets, see [47, Th. 1.7.1]. Therefore, there exists a (possibly, unbounded) convex closed set F such that its support function

$$h(F, u) = \sup\{\langle x, u \rangle : x \in F\}$$

is given by (3.1). This set is denoted by $\mathcal{E}_e(\xi)$ or $\mathcal{E}_e(\mu)$. The construction can be summarised by the equality

$$h(\mathcal{E}_e(\xi), u) = e(\langle \xi, u \rangle), \quad u \in \mathbb{R}^d. \tag{3.2}$$

The following result shows that $\mathcal{E}_e(\xi)$ is a set-valued sublinear function of ξ , called a *set-valued sublinear expectation* generated by e . In other instances, we pass to \mathcal{E}_e the sub- and superscripts of e , e.g., $\mathcal{E}_{[e]}$ is obtained by choosing e to be the expectile $e_{[\tau]}$.

For convex closed sets F, F' , their (closed) Minkowski sum $F + F'$ is the closure of $\{x + x' : x \in F, x' \in F'\}$, and the dilation of F by $c > 0$ is $cF = \{cx : x \in F\}$.

Theorem 3.1. *Fix $p \in [1, \infty]$ and a law-determined sublinear expectation e defined on $L^p(\mathbb{R})$. The corresponding map \mathcal{E}_e (given at (3.2)) from $L^p(\mathbb{R}^d)$ to the family of convex closed sets in \mathbb{R}^d satisfies the following properties:*

- i) monotonicity: if $\xi \in F$ a.s. for a convex closed F , then $\mathcal{E}_e(\xi) \subseteq F$;*
- ii) singleton preserving: $\mathcal{E}_e(a) = \{a\}$ for all deterministic a ;*
- iii) affine equivariance $\mathcal{E}_e(A\xi + a) = A\mathcal{E}_e(\xi) + a$ for all matrices A and $a \in \mathbb{R}^d$;*
- iv) subadditivity: $\mathcal{E}_e(\xi + \eta) \subseteq \mathcal{E}_e(\xi) + \mathcal{E}_e(\eta)$;*
- v) lower semicontinuity of support functions, that is, $h(\mathcal{E}_e(\xi), u) \leq \liminf_{n \rightarrow \infty} h(\mathcal{E}_e(\xi_n), u)$ for all $u \in \mathbb{R}^d$ if $\xi_n \rightarrow \xi$ in $\sigma(L^p, L^q)$;*
- vi) if $e(\beta)$ is finite for all $\beta \in L^p(\mathbb{R})$, then the map $\xi \mapsto \mathcal{E}_e(\xi)$ is continuous in the Hausdorff metric (see [47, Sec. 1.8]) with respect to the norm on L^p ;*
- vii) if e is continuous from below, then $\mathcal{E}_e(\xi)$ contains the expectation $\mathbf{E}\xi$.*

Proof. Property (i) holds since $\langle \xi, u \rangle \leq h(F, u)$ and in view of the monotonicity property of e . Property (ii) directly follows from the construction, and, for the affine equivariance, note that

$$h(\mathcal{E}_e(A\xi + a), u) = e(\langle \xi, A^\top u \rangle) + \langle a, u \rangle = h(\mathcal{E}_e(\xi), A^\top u) + \langle a, u \rangle = h(A\mathcal{E}_e(\xi) + a, u).$$

The subadditivity follows from

$$h(\mathcal{E}_e(\xi + \eta), u) = e(\langle \xi + \eta, u \rangle) \leq e(\langle \xi, u \rangle) + e(\langle \eta, u \rangle) = h(\mathcal{E}_e(\xi), u) + h(\mathcal{E}_e(\eta), u).$$

If $\xi_n \rightarrow \xi$ in $\sigma(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$, then $\langle \xi_n, u \rangle \rightarrow \langle \xi, u \rangle$ in $\sigma(L^p(\mathbb{R}), L^q(\mathbb{R}))$. By the lower semicontinuity of e ,

$$e(\langle \xi, u \rangle) \leq \liminf_{n \rightarrow \infty} e(\langle \xi_n, u \rangle).$$

This implies the lower semicontinuity of the support functions.

Property (vi) follows from the Extended Namioka Theorem, which says that every finite sublinear expectation is continuous with respect to the norm topology, see [9].

Recall that sublinear expectations on L^∞ are also Lipschitz. Hence, $e(\langle \xi_n, u \rangle) \rightarrow e(\langle \xi, u \rangle)$ if $\xi_n \rightarrow \xi$ in L^p . The convergence of support functions implies the convergence of the corresponding sets in the Hausdorff metric, see [47, Th. 1.8.15].

Finally, (vii) is a consequence of the dilatation monotonicity property (2.3). \square

Example 3.2. If e is the essential supremum, then $\mathcal{E}_e(\xi)$ equals the closed convex hull of the support of ξ .

If $p = \infty$, then an easy argument shows that the map $\xi \mapsto \mathcal{E}_e(\xi)$ between $L^\infty(\mathbb{R}^d)$ and the family of convex compact sets in \mathbb{R}^d is 1-Lipschitz, that is, the Hausdorff distance between $\mathcal{E}_e(\xi)$ and $\mathcal{E}_e(\eta)$ is at most $\|\xi - \eta\|_\infty$ for all $\xi, \eta \in L^\infty(\mathbb{R}^d)$. Indeed,

$$\begin{aligned} h(\mathcal{E}_e(\xi), u) - h(\mathcal{E}_e(\eta), u) &= e(\langle \xi, u \rangle) - e(\langle \eta, u \rangle) \\ &\leq e(\langle \eta, u \rangle + \|\xi - \eta\|_\infty) - e(\langle \eta, u \rangle) = \|\xi - \eta\|_\infty \end{aligned}$$

for all unit $u \in \mathbb{R}^d$.

If $\xi, \eta \in L^p(\mathbb{R}^d)$ and $\mathbf{E}(\eta|\xi) = 0$ a.s., then the dilatation monotonicity property (2.3) implies that

$$\mathcal{E}_e(\xi + \eta) \supseteq \mathcal{E}_e(\mathbf{E}(\xi + \eta|\xi)) = \mathcal{E}_e(\xi).$$

Hence, if ξ_1, ξ_2, \dots is a sequence of i.i.d. centred p -integrable random vectors, then $\mathcal{E}_e(\xi_1 + \dots + \xi_n)$, $n \geq 1$, is a growing sequence of nested convex sets in \mathbb{R}^d .

Remark 3.3. If ξ is dominated by η in the *convex order*, meaning that $\mathbf{E}f(\xi) \leq \mathbf{E}f(\eta)$ for all convex functions f , then $\mathcal{E}_e(\xi) \subseteq \mathcal{E}_e(\eta)$, see [17, Cor. 4.59]. In particular, the sequence $\mathcal{E}_e(\xi_n)$, $n \geq 1$, grows if $(\xi_n)_{n \geq 0}$ is a martingale.

Example 3.4. Let $\langle \xi, u \rangle$ be distributed as $\zeta \|u\|_L$, where ζ is a random variable and $\|\cdot\|_L$ is a certain norm on \mathbb{R}^d with L being the unit ball; then ξ is called pseudo-isotropic, see, e.g., [22]. In this case, $\mathcal{E}_e(\xi) = cL^\circ$, where

$$L^\circ = \{u : h(L, u) \leq 1\} \tag{3.3}$$

is the *polar set* to L and $c = e(\zeta) = e(\langle \xi, u \rangle)$ for any given $u \in \partial L$. For instance, this is the case if ξ is symmetric α -stable with $\alpha \in (1, 2]$; then $\mathcal{E}_e(\xi)$ is expressed in terms of the associated convex body of ξ , see [39]. If ξ is Gaussian, then L° is the ellipsoid determined by the covariance matrix of ξ and translated by the mean of ξ .

The dual representation of e given by Theorem 2.3 immediately implies the following result.

Corollary 3.5. *The set-valued sublinear expectation generated by \mathbf{e} can be represented as*

$$\mathcal{E}_{\mathbf{e}}(\xi) = \text{cl}\{\mathbf{E}(\xi\gamma) : \gamma \in \mathcal{M}_{\mathbf{e}}, \mathbf{E}\gamma = 1\}, \tag{3.4}$$

where cl denotes the topological closure in \mathbb{R}^d and $\mathcal{M}_{\mathbf{e}}$ is the family of probability measures from (2.1).

The convexity of $\mathcal{M}_{\mathbf{e}}$ implies that the set on the right-hand side of (3.4) is convex. This set can be written as the intersection of the closure of the cone $\{(\mathbf{E}\gamma, \mathbf{E}(\xi\gamma)) : \gamma \in \mathcal{M}_{\mathbf{e}}\}$ with the set $\{1\} \times \mathbb{R}^d$ and then projected on its last d -components.

Remark 3.6. It is possible to construct a variant of the set $\mathcal{E}_{\mathbf{e}}(\xi)$ by applying the underlying sublinear expectation \mathbf{e} to the positive part $(\langle \xi, u \rangle)_+$ of the scalar product of ξ and u . The obtained function is the support function of a convex closed set, which may be considered a sublinear expectation of the segment $[0, \xi]$, see [41] for a study of sublinear expectations with set-valued arguments.

4. Convex gauges

We sometimes consider a variant of the sublinear expectation which is a positive homogeneous, subadditive and lower semicontinuous function $\mathbf{g} : L^p(\mathbb{R}) \rightarrow (-\infty, \infty]$ and so is not necessarily monotone or translation equivariant. We refer to this function as a *convex gauge*. The most important example is the L^p -norm, so that $\mathbf{g}(\beta) = \|\beta\|_p$, which is convex but not translation equivariant.

For a lower semicontinuous convex gauge \mathbf{g} , we define $\mathcal{G}(\xi)$ as the convex closed set such that

$$h(\mathcal{G}(\xi), u) = \mathbf{g}(\langle \xi, u \rangle), \quad u \in \mathbb{R}^d.$$

It is easily seen that $\mathbf{g}(\langle \xi, u \rangle)$ is indeed a support function.

Example 4.1. Let $\mathbf{g}(\beta) = \|\beta\|_p$. For $\xi \in L^p(\mathbb{R}^d)$, the convex body $\mathcal{G}(\xi)$ is the L^p -centroid of ξ (or of its distribution μ). These convex bodies have been introduced in [45] for $p = 1$ and in [37] for a general p , and further thoroughly studied, see, e.g., [15,24,43].

In some cases, \mathbf{g} fails to be convex. For instance, this is the case for L^p -norm with $p \in (0, 1)$. Another important case arises when $\mathbf{g}(\beta)$ is the quantile function $q_t(\beta)$ given by (2.5) for a fixed $t \in (0, 1)$, which is known to be not necessarily subadditive in β . In the absence of subadditivity, it is natural to consider the largest convex set whose support function is dominated by the quantile function of $\langle x, u \rangle$, namely, let

$$D_{\delta}(\xi) = \bigcap_{u \in \mathbb{R}^d} \{x \in \mathbb{R}^d : \langle x, u \rangle \leq q_{1-\delta}(\langle \xi, u \rangle)\}. \tag{4.1}$$

The set $D_\delta(\xi)$ is called the *depth-trimmed region* of ξ . The support function of $D_\delta(\xi)$ may be strictly less than $q_{1-\delta}(\langle \xi, u \rangle)$, for example, if ξ is uniformly distributed on a triangle on the plane, see [33]. The set $D_\delta(\xi)$ is necessarily empty if ξ is nonatomic and $\delta \in (1/2, 1]$.

The set $D_\delta(\xi)$ is related to the *Tukey (or half-space) depth* (see [52]), which associates to a point x the smallest μ -content of a half-space containing x , where μ is the distribution of ξ . The depth-trimmed region of ξ is the excursion set of the Tukey depth, so that

$$D_\delta(\xi) = \bigcap_{\mu(H) > 1-\delta} H, \tag{4.2}$$

where H runs through the collection of all closed half-spaces. If ξ has *contiguous support* (that is, the support of $\langle \xi, u \rangle$ is connected for every u), then (4.1) holds with q being any other quantile function in case of multiplicities, and the intersection in (4.2) can be taken over half-spaces H with $\mu(H) \geq 1 - \delta$, see [12,30].

Example 4.2. Let ξ be uniformly distributed on a convex body K . Then $D_\delta(\xi)$ is the *convex floating body* of K , see [48] and [56]. A variant of this concept for nonuniform distributions on K has been studied in [8].

Recall that a random vector ξ with distribution μ is said to have *k-concave distribution*, with $k \in [-\infty, \infty]$, if

$$\mu(\theta A + (1 - \theta)B) \geq \begin{cases} \min\{\mu(A), \mu(B)\} & \text{if } k = -\infty, \\ \mu(A)^\theta \mu(B)^{(1-\theta)} & \text{if } k = 0, \\ (\theta\mu(A)^k + (1 - \theta)\mu(B)^k)^{1/k} & \text{otherwise,} \end{cases}$$

for all Borel sets A and B and $\theta \in [0, 1]$. In case of $k = 0$, the measure μ is called *log-concave*. The next theorem establishes some conditions under which $q_\delta(\langle \xi, u \rangle)$ is a support function; it is a direct consequence of [10, Th. 6.1].

Theorem 4.3. *Let ξ be a symmetric k-concave random vector with $k \geq -1$ and such that the support of ξ is full-dimensional. Then*

$$h(D_\delta(\xi), u) = q_{1-\delta}(\langle \xi, u \rangle), \quad u \in \mathbb{R}^d,$$

for all $\delta \in (0, 1/2)$.

5. Convex bodies generated by average quantiles

5.1. Metronoids and zonoid-trimmed regions

For $\xi \in L^1(\mathbb{R}^d)$ and $\alpha \in (0, 1]$, denote by $\mathcal{E}_\alpha(\xi)$ the convex set generated by the average quantile sublinear expectation e_α given by (2.4). Such convex sets are hereafter

called *average quantile sets*. In particular, $\mathcal{E}_1(\xi) = \mathbf{E}\xi$. Since e_α is finite on $L^1(\mathbb{R})$, the set $\mathcal{E}_\alpha(\xi)$ is compact. Noticing that $q_t(-\beta) = -q_{1-t}(\beta)$, it is easy to see that $\mathcal{E}_\alpha(\xi)$ has nonempty interior for all $\alpha \in (0, 1)$, hence, is a convex body. The set $\mathcal{E}_\alpha(\xi)$ increases as α decreases to zero with limit $\mathcal{E}_0(\xi)$, being the convex hull of the support of ξ .

The following result relates average quantile sets and the *zonoid-trimmed regions* introduced in [31] as

$$Z_\alpha(\xi) = \{ \mathbf{E}(\xi f(\xi)) : f : \mathbb{R}^d \rightarrow [0, \alpha^{-1}] \text{ measurable and } \mathbf{E}f(\xi) = 1 \}.$$

Proposition 5.1. *For all $\alpha \in (0, 1]$, $\mathcal{E}_\alpha(\xi) = Z_\alpha(\xi)$.*

Proof. Representation (2.6) yields that

$$h(\mathcal{E}_\alpha(\xi), u) = e_\alpha(\langle \xi, u \rangle) = \sup_{\gamma \in L^\infty([0, \alpha^{-1}]), \mathbf{E}\gamma=1} \langle \mathbf{E}(\gamma \xi), u \rangle.$$

Noticing that any γ in the last expression can be replaced with $\mathbf{E}(\gamma|\xi)$ yields that

$$\sup_{\substack{\gamma \in L^\infty([0, \alpha^{-1}]) \\ \mathbf{E}\gamma=1}} \langle \mathbf{E}(\gamma \xi), u \rangle = \sup_{\substack{f: \mathbb{R}^d \rightarrow [0, \alpha^{-1}] \\ \mathbf{E}f(\xi)=1}} \langle \mathbf{E}(\xi f(\xi)), u \rangle. \quad \square$$

Let μ be a locally finite Borel measure on \mathbb{R}^d . Denote by $L^1_\mu([0, 1])$ the family of functions $f : \mathbb{R}^d \rightarrow [0, 1]$ such that $\int x f(x) \mu(dx)$ exists. The set

$$M(\mu) = \left\{ \int_{\mathbb{R}^d} x f(x) \mu(dx) : f \in L^1_\mu([0, 1]), \int_{\mathbb{R}^d} f d\mu = 1 \right\}$$

has the support function

$$h(M(\mu), u) = \sup_{f \in L^1_\mu([0, 1]), \int f d\mu=1} \int \langle x, u \rangle f(x) \mu(dx), \quad u \in \mathbb{R}^d.$$

The set $M(\mu)$ was introduced in [26] and called the *metronoid* of μ . This definition applies also for possibly infinite measures μ , e.g., if μ is the Lebesgue measure, then $M(\mu) = \mathbb{R}^d$, since each point $x \in \mathbb{R}^d$ can be obtained by letting f be the indicator of the unit ball centred at x normalised by the volume of the unit ball. Furthermore, $M(\mu)$ is empty if the total mass of μ is less than one, and $M(\mu)$ is the singleton $\int x \mu(dx)$ if μ is an integrable probability measure. The following result establishes a relation between metronoids and average quantile sets.

Proposition 5.2. *Let μ be an integrable probability measure on \mathbb{R}^d . Then $M(\alpha^{-1}\mu) = \mathcal{E}_\alpha(\mu)$ for any $\alpha \in (0, 1]$.*

Proof. Consider a random vector ξ with distribution μ . By (2.6), for every $u \in \mathbb{R}^d$ the support function of $M(\alpha^{-1}\mu)$ is

$$\begin{aligned} h(M(\alpha^{-1}\mu), u) &= \sup_{0 \leq f \leq 1, \int f d\mu = \alpha} \int \langle x, u \rangle f(x) \alpha^{-1} \mu(dx) \\ &= \sup_{0 \leq f \leq \alpha^{-1}, \mathbf{E}f(\xi) = 1} \mathbf{E}(\langle \xi, u \rangle f(\xi)) \\ &= \sup_{\gamma \in L^\infty([0, \alpha^{-1}]), \mathbf{E}\gamma = 1} \mathbf{E}(\langle \xi, u \rangle \gamma) \\ &= e_\alpha(\langle \xi, u \rangle) = h(\mathcal{E}_\alpha(\xi), u), \end{aligned}$$

where in the second equality $f\alpha^{-1}$ was replaced by f and later $f(\xi)$ by γ . \square

Example 5.3. Let ξ have a discrete distribution with atoms at x_1, \dots, x_n of probabilities p_1, \dots, p_n . Then $\mathcal{E}_\alpha(\xi)$ is the polytope

$$\mathcal{E}_\alpha(\xi) = \left\{ \sum_{i=1}^n \lambda_i p_i x_i : \lambda_1, \dots, \lambda_n \in [0, \alpha^{-1}], \sum_{i=1}^n \lambda_i p_i = 1 \right\},$$

see [26, Prop. 2.3], where this is proved for metronoids.

5.2. A representation of general $\mathcal{E}_e(\xi)$

Fix $\xi \in L^1(\mathbb{R}^d)$ and consider the average quantile sets $\mathcal{E}_\alpha(\xi)$ as a set-valued function of $\alpha \in (0, 1]$. Let ν be a probability measure on $(0, 1]$, which appears in the spectral sublinear expectation (2.9) from Example 2.5. The closed Aumann integral (see [5]) of the set-valued function $\alpha \mapsto \mathcal{E}_\alpha(\xi)$ is the convex closed set $\mathcal{E}_{f_\varphi}(\xi)$, whose support function at any direction u equals the integral of the support function, so that

$$h(\mathcal{E}_{f_\varphi}(\xi), u) = \int_{(0,1]} h(\mathcal{E}_\alpha(\xi), u) \nu(d\alpha), \quad u \in \mathbb{R}^d. \tag{5.1}$$

Recognising the right-hand side as $e_{f_\varphi}(\langle \xi, u \rangle)$, it is immediately seen that $\mathcal{E}_{f_\varphi}(\xi)$ is the set-valued sublinear expectation generated by the spectral numerical one from Example 2.5. Equivalently, $\mathcal{E}_{f_\varphi}(\xi)$ equals the closure of the set of integrals of all measurable integrable functions $f(\alpha)$, $\alpha \in (0, 1]$, such that $f(\alpha) \in \mathcal{E}_\alpha(\xi)$ for all α , see [5] and [40, Sec. 2.1.2]. This is reflected by writing

$$\mathcal{E}_{f_\varphi}(\xi) = \text{cl} \int_{(0,1]} \mathcal{E}_\alpha(\xi) \nu(d\alpha). \tag{5.2}$$

Since $\mathcal{E}_\alpha(\xi)$ increases to the closed convex hull of the support of ξ as $\alpha \downarrow 0$, the set $\mathcal{E}_{f_\varphi}(\xi)$ is not necessarily bounded.

The following result provides a representation of the set $\mathcal{E}_e(\xi)$ constructed using a general law-determined sublinear expectation e . It confirms that the average quantile sets (equivalently, metronoids) are building blocks for a general $\mathcal{E}_e(\xi)$. Denote by $\text{conv } A$ the closed convex hull of a set A in \mathbb{R}^d .

Theorem 5.4. *For each $\xi \in L^p(\mathbb{R}^d)$ and a set-valued sublinear expectation $\mathcal{E}_e(\xi)$ generated by a law-determined sublinear expectation e , we have*

$$\mathcal{E}_e(\xi) = \text{conv} \bigcup_{\nu \in \mathcal{P}_e(0,1]} \int \mathcal{E}_\alpha(\xi) \nu(d\alpha),$$

where \mathcal{P}_e is the family probability measures ν on $(0, 1]$ from the Kusuoka representation of e , see (2.7).

Proof. By Theorem 2.4,

$$e(\langle \xi, u \rangle) = \sup_{\nu \in \mathcal{P}_e(0,1]} \int e_\alpha(\langle \xi, u \rangle) \nu(d\alpha) = \sup_{\nu \in \mathcal{P}_e(0,1]} \int h(\mathcal{E}_\alpha(\xi), u) \nu(d\alpha).$$

The proof is completed by noticing (5.1), using the notation (5.2) and the fact that the supremum of support functions is the support function of the closed convex hull of the involved sets. \square

5.3. Average quantile sets as integrated depth-trimmed regions

Under the symmetry and log-concavity assumptions on ξ , the average quantile sets $\mathcal{E}_\alpha(\xi)$ can be characterised as set-valued integrals of the depth-trimmed regions (equivalently, weighted floating bodies) $D_\delta(\xi)$ introduced in (4.1). Similarly to (5.1), the closed Aumann integral of the function $t \mapsto D_t(\xi)$ with respect to a measure ν on $[0, 1]$ is defined as the convex set whose support function equals the integral of the support functions of $D_t(\xi)$, that is,

$$h\left(\int_0^1 D_t(\xi) \nu(dt), u\right) = \int_0^1 h(D_t(\xi), u) \nu(dt), \quad u \in \mathbb{R}^d.$$

If the measure ν attaches positive mass to the set of $t \in [0, 1]$ where $D_t(\xi)$ is empty, the integral is set to be the empty set.

The following result establishes relationships between average quantile sets (or metronoids) and depth-trimmed regions. Its second part generalises [27, Th. 1.1], which concerns the case of ξ supported by a convex body.

Theorem 5.5. *Let $\xi \in L^1(\mathbb{R}^d)$. Then*

$$D_\alpha(\xi) \subseteq \frac{1}{\alpha} \int_0^\alpha D_t(\xi) dt \subseteq \mathcal{E}_\alpha(\xi). \tag{5.3}$$

If ξ has a log-concave distribution, then

$$D_{\frac{e-1}{e}\alpha}(\xi) \subseteq \mathcal{E}_\alpha(\xi) \subseteq D_{\frac{\alpha}{e}}(\xi) \tag{5.4}$$

for every $\alpha \in (0, 1]$.

Proof. By definition of the average quantile set,

$$h(\mathcal{E}_\alpha(\xi), u) = \frac{1}{\alpha} \int_{1-\alpha}^1 q_t(\langle \xi, u \rangle) dt \geq \frac{1}{\alpha} \int_{1-\alpha}^1 h(D_{1-t}(\xi), u) dt = \frac{1}{\alpha} \int_0^\alpha h(D_t(\xi), u) dt,$$

where the inequality follows from (4.1). Finally, (5.3) follows from the monotonicity of $D_t(\xi)$.

Fix $u \in \mathbb{R}^d$. Consider $\beta = \langle \xi, u \rangle$ and note that the distribution ν of β is log-concave by the invariance of the log-concavity property under projection. For (5.4), it suffices to show that

$$q_{(1-\frac{e-1}{e}\alpha)}(\beta) \leq e_\alpha(\beta) \leq q_{(1-\frac{1}{e}\alpha)}(\beta). \tag{5.5}$$

Being the projection of a log-concave vector, β is either deterministic or absolutely continuous with connected support. In the first case (5.5) becomes trivial, thus we can assume that β is absolutely continuous with connected support. In particular, q in (5.5) can be equivalently chosen to be the left- or the right-quantile function. Observe that, for measurable sets A and B , convex C and $\theta \in [0, 1]$,

$$\begin{aligned} \nu(A \cap C)^\theta \nu(B \cap C)^{1-\theta} &\leq \nu(\theta(A \cap C) + (1-\theta)(B \cap C)) \\ &= \nu((\theta A \cap \theta C) + ((1-\theta)B \cap (1-\theta)C)) \\ &\leq \nu((\theta A + (1-\theta)B) \cap (\theta C + (1-\theta)C)) \\ &= \nu((\theta A + (1-\theta)B) \cap C). \end{aligned}$$

Therefore, the probability measure obtained by restricting ν to the interval $(q_{1-\alpha}(\beta), \infty)$ and normalising by the factor $\alpha-1$ is log-concave, and we consider a random variable X with such distribution. It follows from the theory of risk measures (see, e.g., [51, Prop. 2.1]), that for the case of absolutely continuous random variables, supremum in the characterisation of $e_\alpha(\beta)$ in (2.6) is attained at $\gamma = \alpha^{-1} \mathbf{1}_{\{\beta > q_{1-\alpha}(\beta)\}}$, which implies

$$\mathbf{E}X = \alpha^{-1} \mathbf{E} (\beta \mathbf{1}_{\{\beta > q_{1-\alpha}(\beta)\}}) = \mathbf{e}_\alpha(\beta). \tag{5.6}$$

It follows from [10, Eq. (5.7)] that for any log-concave random variable X ,

$$e^{-1} \leq \mathbf{P} \{X > \mathbf{E}X\} \leq 1 - e^{-1}. \tag{5.7}$$

Therefore, (5.6) and (5.7) yield that

$$e^{-1} \leq \alpha^{-1} \nu(\mathbf{e}_\alpha(\beta), \infty) \leq 1 - e^{-1}.$$

Hence,

$$e^{-1} \alpha \leq \mathbf{P} \{\beta > \mathbf{e}_\alpha(\beta)\} \leq (1 - e^{-1}) \alpha,$$

which implies (5.5), given that β has connected support. \square

5.4. A uniqueness result for maximum extensions

A single set $\mathcal{E}_e(\xi)$ surely does not characterise the distribution of ξ . However, families of such sets can be sufficient to recover the distribution of ξ .

Example 5.6. Assume that $\xi, \eta \in L^1(\mathbb{R}^d)$ and consider the average quantile sets $\mathcal{E}_\alpha(\xi)$ and $\mathcal{E}_\alpha(\eta)$. If $\mathcal{E}_\alpha(\xi) = \mathcal{E}_\alpha(\eta)$ for all $\alpha \in (0, 1/2]$, then ξ and η have the same distribution. This follows from Proposition 5.1 and [31, Th. 5.6].

Since the definition of $\mathcal{E}_e(\xi)$ is based on the univariate sublinear expectation \mathbf{e} applied to the projections of ξ , the following result is a straightforward application of the Cramér–Wold theorem, see, e.g., [29, Cor. 5.5].

Proposition 5.7. *A family of sets $\mathcal{E}_e(\xi)$, $\mathbf{e} \in E$, generated by sublinear expectations \mathbf{e} from a certain family E uniquely identifies the distribution of $\xi \in L^p(\mathbb{R}^d)$ if and only if the family of the underlying univariate sublinear expectations $\mathbf{e}(\beta)$, $\mathbf{e} \in E$, uniquely identifies the distribution of any $\beta \in L^p(\mathbb{R})$.*

Natural families of sublinear expectations arise by applying the maximum extension to a given sublinear expectation.

Example 5.8. Consider the expected maximum sublinear expectation $\mathbf{e}_1^{\vee m}$ given by (2.14). Then the convex body $\mathcal{E}_1^{\vee m}(\xi)$ is the expectation $\mathbf{E}P_m$ of the random polytope P_m obtained as the convex hull of m independent copies of ξ , see [40, Sec. 2.1]. It is well known that the sequence $\mathbf{e}_1^{\vee m}(\beta)$, $m \geq 1$, uniquely identifies the distribution of $\beta \in L^1(\mathbb{R})$, see [25] and [20]. As a consequence, the nested sequence $\mathbf{E}P_m$, $m \geq 1$, of convex bodies uniquely determines the distribution of ξ , see [54].

Applying the maximum extension (2.13) to the spectral sublinear expectation $e_{f\varphi}(\cdot)$ yields the sublinear expectation $e_{f\varphi}^{\vee m}(\cdot)$ and the corresponding sequence of nested convex bodies $\mathcal{E}_{f\varphi}^{\vee m}(\xi)$, $m \geq 1$.

Theorem 5.9. *Let $\xi, \eta \in L^1(\mathbb{R}^d)$. For any constant $c \geq 0$, consider the spectral function $\varphi(t) = (c + 1)(1 - t)^c$. If*

$$\mathcal{E}_{f\varphi}^{\vee m}(\xi) = \mathcal{E}_{f\varphi}^{\vee m}(\eta), \quad m \geq 1,$$

then ξ and η have the same distribution.

Proof. In view of Proposition 5.7, it suffices to prove this result for two random variables β and γ . For any integer $m \geq 1$, we have

$$\int_0^1 q_{1-t}(\max(\beta_1, \dots, \beta_m)) \varphi(t) dt = \int_0^1 q_{1-t}(\max(\gamma_1, \dots, \gamma_m)) \varphi(t) dt,$$

where β_i, γ_i , $i = 1, \dots, m$, are independent copies of β, γ , respectively. By a change of variables,

$$\begin{aligned} \int_0^1 q_{1-t}(\max(\beta_1, \dots, \beta_m)) \varphi(t) dt &= (c + 1) \int_0^1 q_t(\max(\beta_1, \dots, \beta_m)) t^c dt \\ &= (c + 1) \int_0^1 q_{t^{\frac{1}{m}}}(\beta) t^c dt \\ &= m(c + 1) \int_0^1 q_s(\beta) s^{cm+m-1} ds. \end{aligned}$$

Therefore,

$$\int_0^1 f(s) s^{(c+1)(m-1)} ds = 0, \quad m \geq 1,$$

with

$$f(s) = s^c (q_s(\beta) - q_s(\gamma)) \in L^1([0, 1]).$$

The family

$$\mathcal{A} = \left\{ c_0 + \sum_{i=1}^n c_i x^{(c+1)m_i} : n, m_1, \dots, m_n \in \mathbb{N}, c_0, \dots, c_n \in \mathbb{R} \right\}$$

is an algebra of continuous functions separating the points on $[0, 1]$. By linearity of the Lebesgue integral

$$\int_0^1 f(s)a(s)ds = 0$$

for all $a \in \mathcal{A}$. The Stone–Weierstrass theorem (see, e.g., [16, Th. 4.45]) yields that

$$\int_0^1 f(s)g(s)ds = 0$$

for all continuous functions g on $[0, 1]$. Therefore, f vanishes almost everywhere, so the proof is complete. \square

5.5. Concentration of empirical average quantile sets

Let $\xi \in \mathbb{L}^p(\mathbb{R}^d)$ with distribution μ . Consider the empirical random measure constructed by n independent copies ξ_1, \dots, ξ_n of ξ as

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i}, \quad n \geq 1, \tag{5.8}$$

where δ_x is the one point mass measure at $x \in \mathbb{R}^d$. The average quantile convex body $\mathcal{E}_\alpha(\hat{\mu}_n)$ generated by $\hat{\mu}_n$ is a random convex set, which approximates the body $\mathcal{E}_\alpha(\mu)$ as n grows to infinity. In fact, the sequence $\{\mathcal{E}_\alpha(\hat{\mu}_n), n \geq 1\}$ almost surely converges to $\mathcal{E}_\alpha(\mu)$ in the Hausdorff metric, as directly follows from [31, Th. 5.2] and Proposition 5.1. The following theorem provides probabilistic bounds for this convergence.

Theorem 5.10. *Let μ be a probability measure with bounded support of diameter R , and let r be the largest radius of a centred Euclidean ball contained in the average quantile set $\mathcal{E}_\alpha(\mu)$ for some $\alpha \in (0, 1)$. For all $\varepsilon > 0$ and $n \in \mathbb{N}$,*

$$\mathbf{P}\left\{(1 - \varepsilon)\mathcal{E}_\alpha(\mu) \subseteq \mathcal{E}_\alpha(\hat{\mu}_n) \subseteq (1 + \varepsilon)\mathcal{E}_\alpha(\mu)\right\} \geq 1 - 6^{d+1}(1 + 1/\varepsilon)^d \exp\left\{-\frac{\alpha\varepsilon^2r^2n}{44R^2}\right\}.$$

We use the following auxiliary result.

Lemma 5.11 (see [19, Lemma 5.2]). *Let K be a convex body which contains the origin in its interior. For each $\delta \in (0, 1/2)$, there exists a set $\mathcal{N} \subseteq \partial K$ with cardinality at most $(3/\delta)^d$ such that each $v \in \partial K$ satisfies*

$$v = w_0 + \sum_{i=1}^{\infty} \delta_i w_i$$

for $w_i \in \mathcal{N}$, $i \geq 0$, and $\delta_i \in [0, \delta^i]$, $i \geq 1$.

Proof of Theorem 5.10. On a (possibly, enlarged) probability space $\Omega \times \Omega'$, let ξ be a μ -distributed random vector, and let $\hat{\xi}_n$ take one of the values ξ_1, \dots, ξ_n with equal probabilities. For any fixed $u \in \mathbb{R}^d$,

$$h(\mathcal{E}_\alpha(\xi), u) = \frac{1}{\alpha} \int_{1-\alpha}^1 q_t(\langle \xi, u \rangle) dt$$

and

$$h(\mathcal{E}_\alpha(\hat{\mu}_n), u) = \frac{1}{\alpha} \int_{1-\alpha}^1 q_t(\langle \hat{\xi}_n, u \rangle) dt.$$

Clearly, $\langle \hat{\xi}_n, u \rangle$ is distributed according to the empirical distribution function generated by the sample $\langle \xi_i, u \rangle$, $i = 1, \dots, n$. Thus, the right-hand sides of the two equations are, respectively, the conditional value at risk of $\beta = \langle \xi, u \rangle$ and its sample-based estimator, see [11,55]. Note that the support of β is a subset of an interval of length R . By [55, Th. 3.1], for any $\eta > 0$,

$$\mathbf{P} \{h(\mathcal{E}_\alpha(\hat{\mu}_n), u) \leq h(\mathcal{E}_\alpha(\xi), u) - \eta\} \leq 3 \exp \left\{ -\frac{\alpha \eta^2 n}{5R^2} \right\}$$

and

$$\mathbf{P} \{h(\mathcal{E}_\alpha(\hat{\mu}_n), u) \geq h(\mathcal{E}_\alpha(\xi), u) + \eta\} \leq 3 \exp \left\{ -\frac{\alpha \eta^2 n}{11R^2} \right\}.$$

Noticing that the second bound is larger than the first one and that $h(\mathcal{E}_\alpha(\xi), u) \geq r$ by the imposed condition, we obtain

$$\begin{aligned} \mathbf{P} \left\{ (1 - \varepsilon/2)h(\mathcal{E}_\alpha(\xi), u) \leq h(\mathcal{E}_\alpha(\hat{\mu}_n), u) \leq (1 + \varepsilon/2)h(\mathcal{E}_\alpha(\xi), u) \right\} \\ \geq 1 - 6 \exp \left\{ -\frac{\alpha \varepsilon^2 r^2 n}{44R^2} \right\}. \end{aligned} \tag{5.9}$$

Let $\mathcal{N} \subseteq \partial \mathcal{E}_\alpha(\xi)^\circ$ be a set from Lemma 5.11, with $\delta = \frac{\varepsilon}{2+2\varepsilon}$, where $\mathcal{E}_\alpha(\xi)^\circ$ is the polar set to $\mathcal{E}_\alpha(\xi)$, see (3.3). Since $h(\mathcal{E}_\alpha(\xi), u) = 1$ for all $u \in \partial \mathcal{E}_\alpha(\xi)^\circ$, the union bound applied to (5.9) yields that

$$(1 - \varepsilon/2) \leq h(\mathcal{E}_\alpha(\hat{\mu}_n), w) \leq (1 + \varepsilon/2) \quad \text{for all } w \in \mathcal{N} \tag{5.10}$$

with probability at least

$$1 - 6 \left(\frac{6 + 6\varepsilon}{\varepsilon} \right)^d \exp \left\{ -\frac{\alpha\varepsilon^2 r^2 n}{44R^2} \right\}.$$

For any $v \in \partial\mathcal{E}_\alpha(\xi)^\circ$ and some sequences $w_i \in \mathcal{N}$ and $\delta_i \geq 0, i \geq 1$, the sublinearity of h , Lemma 5.11 and (5.10) imply that

$$\begin{aligned} h(\mathcal{E}_\alpha(\hat{\mu}_n), v) &= h \left(\mathcal{E}_\alpha(\hat{\mu}_n), w_0 + \sum_{i=1}^\infty \delta_i w_i \right) \\ &\leq (1 + \varepsilon/2) \sum_{i=0}^\infty \left(\frac{\varepsilon}{2 + 2\varepsilon} \right)^i \\ &= (1 + \varepsilon/2) \frac{1}{1 - \left(\frac{\varepsilon}{2+2\varepsilon} \right)} = (1 + \varepsilon) h(\mathcal{E}_\alpha(\xi), v) \end{aligned}$$

and

$$\begin{aligned} h(\mathcal{E}_\alpha(\hat{\mu}_n), v) &= h \left(\mathcal{E}_\alpha(\hat{\mu}_n), w_0 + \sum_{i=1}^\infty \delta_i w_i \right) \\ &\geq (1 - \varepsilon/2) - (1 + \varepsilon/2) \sum_{i=1}^\infty \left(\frac{\varepsilon}{2 + 2\varepsilon} \right)^i \\ &= (1 - \varepsilon/2) - (1 + \varepsilon/2) \frac{\left(\frac{\varepsilon}{2+2\varepsilon} \right)}{1 - \left(\frac{\varepsilon}{2+2\varepsilon} \right)} = (1 - \varepsilon) h(\mathcal{E}_\alpha(\xi), v), \end{aligned}$$

which deliver the desired assertion. \square

6. Floating-like bodies

6.1. Sublinear transform

Consider the set-valued sublinear expectation \mathcal{E}_e generated by a law-determined numerical sublinear expectation e . Let ξ be a random vector uniformly distributed on a convex body $K \subset \mathbb{R}^d$. Recall that K is assumed to have a nonempty interior. In the following, we write $\mathcal{E}_e(K)$ instead of $\mathcal{E}_e(\xi)$ and refer to $K \mapsto \mathcal{E}_e(K)$ as a *sublinear transform* of K generated by the numerical sublinear expectation e . We also refer to $\mathcal{E}_e(K)$ as a *floating-like body*.

Denoting by \mathcal{K} the family of convex bodies in \mathbb{R}^d , the sublinear transform is a map $\mathcal{E}_e : \mathcal{K} \rightarrow \mathcal{K}$. It is easy to see that $\mathcal{E}_e(K) \subseteq K$ for all K . If ξ is uniformly distributed on K and A is a nondegenerate matrix, then $A\xi$ is uniformly distributed on AK . Thus,

$$\mathcal{E}_e(AK + a) = A\mathcal{E}_e(K) + a, \quad a \in \mathbb{R}^d.$$

The sublinear transform $\mathcal{E}_e(B)$ of a centred Euclidean ball B is another centred Euclidean ball, which is contained in B . Furthermore, the sublinear transform of an ellipsoid is also an ellipsoid.

The sublinear transform is not necessarily monotone for inclusion, see Example 6.6. In view of Remark 3.3, $\mathcal{E}_e(K) \subseteq \mathcal{E}_e(L)$ for all sublinear transforms \mathcal{E}_e if

$$\frac{1}{V_d(K)} \int_K f(x)dx \leq \frac{1}{V_d(L)} \int_L f(x)dx$$

for all convex functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, where $V_d(\cdot)$ denotes the d -dimensional Lebesgue measure. The latter condition implies that K and L share the same barycentre.

If $K_n \rightarrow K$ in the Hausdorff metric as $n \rightarrow \infty$ and ξ_n, ξ are uniformly distributed on K_n, K , respectively, then $\xi_n \rightarrow \xi$ in $\sigma(L^p, L^q)$ for any $p \in [1, \infty]$ by the dominated convergence theorem. By Theorem 3.1(v), $h(\mathcal{E}_e(K), u) \leq \liminf h(\mathcal{E}_e(K_n), u)$.

The continuity of the sublinear map in the Hausdorff metric follows from the next result, which we find interesting in its own right. Denote by $\text{diam}(K)$ the diameter of K and by $K \Delta L$ the symmetric difference of K and L .

Theorem 6.1. *Assume that $p \in [1, \infty)$. For any two convex bodies K and L , there exist random vectors ξ and η uniformly distributed on K and L , respectively, such that*

$$\|\xi - \eta\|_p \leq \left(\frac{V_d(K \Delta L)}{\max(V_d(L), V_d(K))} \right)^{\frac{1}{p}} \text{diam}(K \cup L). \tag{6.1}$$

Proof. It suffices to prove the statement for $p = 1$. Indeed,

$$\|\xi - \eta\|_p \leq \text{diam}(K \cup L)^{(p-1)/p} \|\xi - \eta\|_1^{1/p}.$$

Consider Monge’s optimal transport problem of finding

$$\mathfrak{C}(\mu, \nu) = \inf_{T_{\#}\mu = \nu} \int_{\mathbb{R}^d} \|x - T(x)\| \mu(dx), \tag{6.2}$$

where μ and ν are the uniform distributions on K and L , respectively, and $T_{\#}\mu$ denotes the push-forward of the measure μ by T . It is known from the theory of optimal mass transportation (see, e.g., [4] or [53]) that the infimum in (6.2) is attained on an optimal transport map T . Moreover, under our assumptions, [46, Th. B] yields the equivalence between Monge’s transport problem and its alternative formulation by Kantorovich. Namely,

$$\mathfrak{C}(\mu, \nu) = \min_{\gamma \in \Pi(\mu, \nu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \gamma(dx, dy),$$

where $\Pi(\mu, \nu)$ denotes the family of probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν . In other words, $\mathfrak{C}(\mu, \nu)$ is the 1-Wasserstein distance between μ and ν . The dual representation of Kantorovich’s problem (e.g. [53, Th. 1.14]) yields that

$$\min_{\gamma \in \Pi(\mu, \nu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \gamma(dx, dy) = \max_{f \in \text{Lip}_1} \left\{ \int_{\mathbb{R}^d} f(x) \mu(dx) - \int_{\mathbb{R}^d} f(x) \nu(dx) \right\}, \quad (6.3)$$

where Lip_1 is the family of 1-Lipschitz functions on \mathbb{R}^d .

By adding a constant to f , one can restrict the maximisation in (6.3) to the set of 1-Lipschitz functions with values in $[0, \text{diam}(K \cup L)]$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) \mu(dx) - \int_{\mathbb{R}^d} f(x) \nu(dx) &= \frac{1}{V_d(K)} \int_K f(x) dx - \frac{1}{V_d(L)} \int_L f(x) dx \\ &= \frac{V_d(L) - V_d(K)}{V_d(K)V_d(L)} \int_{K \cap L} f(x) dx + \frac{1}{V_d(K)} \int_{K \setminus L} f(x) dx - \frac{1}{V_d(L)} \int_{L \setminus K} f(x) dx \\ &\leq \left(\frac{V_d(L \setminus K)}{V_d(K)} \frac{V_d(K \cap L)}{V_d(L)} + \frac{V_d(K \setminus L)}{V_d(K)} \right) \text{diam}(K \cup L) \\ &\leq \left(\frac{V_d(L \setminus K)}{V_d(K)} + \frac{V_d(K \setminus L)}{V_d(K)} \right) \text{diam}(K \cup L) \\ &= \frac{V_d(K \triangle L)}{V_d(K)} \text{diam}(K \cup L). \end{aligned}$$

Changing the order of summands, one obtains a similar bound with $V_d(K)$ replaced by $V_d(L)$, hence the result. \square

Theorem 6.2. *Let \mathfrak{e} be a sublinear expectation defined on $L^p(\mathbb{R})$ for some $p \in [1, \infty)$ and having finite values. Then the map $K \mapsto \mathfrak{E}_{\mathfrak{e}}(K)$ is continuous in the Hausdorff metric.*

Proof. Note that the convergence of convex bodies (with nonempty interiors) in the Hausdorff metric is equivalent to their convergence in the symmetric difference metric, see [50]. If $K_n \rightarrow K$ in the Hausdorff metric, then $\cup_n K_n$ is bounded and $\inf_n V_d(K_n)$ is strictly positive. By Theorem 6.1, it is possible to find a sequence of random vectors $\{\xi_n, n \geq 1\}$ such that ξ_n is uniformly distributed on K_n and ξ_n converges in L^p to a random vector ξ uniformly distributed on K . The result follows from Theorem 3.1(vi). \square

Example 6.3. The construction of the sublinear transform can be amended by replacing the underlying sublinear expectation \mathfrak{e} with a (not necessarily convex) gauge function. For example, if the gauge function is a quantile, one obtains the set $D_\alpha(K)$, which is the *convex floating body* of K , see [6] and [48].

6.2. Ulam floating bodies

Consider the sublinear transform $K \mapsto \mathcal{E}_\alpha(K)$ generated by the average quantile sub-linear expectation \mathbf{e}_α . Note that $\mathcal{E}_1(K) = \{x_K\}$ is the barycentre of K (the expectation of ξ uniformly distributed in K), and $\mathcal{E}_0(K) = K$.

The metronoid $M(\mu)$ of the measure μ with density $\delta^{-1}\mathbf{1}_K$ is called the *Ulam floating body* of K at level δ and is denoted by $M_\delta(K)$, see [27]. This measure μ is the uniform probability distribution on K scaled by $\delta^{-1}V_d(K)$. Proposition 5.2 yields that

$$\mathcal{E}_\alpha(K) = M_{\alpha V_d(K)}(K). \tag{6.4}$$

Affine equivariance of sublinear transforms implies that $M_\delta(cK) = cM_{\delta c^{-d}}(K)$. Since the uniform probability distribution on K is log-concave, (5.4) yields a relationship between convex floating bodies of K (denoted by $D_\alpha(K)$) and Ulam floating bodies, proved in [27, Th. 1.1].

The following result for $\alpha \in (0, 1/2)$ follows from Theorem 4.3, see also [38]. Together with (6.4), it implies that Ulam floating bodies can be obtained as Aumann integrals of convex floating bodies. The case $\alpha = 1/2$ follows by continuity.

Corollary 6.4. *For each origin symmetric convex body K and $\alpha \in (0, 1/2]$, we have*

$$\mathcal{E}_\alpha(K) = \frac{1}{\alpha} \int_0^\alpha D_t(K) dt.$$

Hence, $\alpha\mathcal{E}_\alpha(K)$ grows in α for $\alpha \in (0, 1/2]$, equivalently, the dilated Ulam floating body $tM_t(K)$ grows for $t \in (0, V_d(K)/2]$.

The next result follows from Theorem 5.4; it implies that Ulam floating bodies are building blocks for all sublinear transforms.

Corollary 6.5. *For each law-determined sublinear expectation \mathbf{e} , the corresponding sub-linear transform $\mathcal{E}_\mathbf{e}$ can be represented as*

$$\mathcal{E}_\mathbf{e}(K) = \text{conv} \bigcup_{\nu \in \mathcal{P}_\mathbf{e}(0,1]} \int \mathcal{E}_\alpha(K) \nu(d\alpha), \tag{6.5}$$

where ν runs through a family $\mathcal{P}_\mathbf{e}$ of probability measures on $(0, 1]$ that yields the Kusuoka representation of \mathbf{e} , see (2.7).

It is possible to replace \mathcal{E}_α with $M_{\alpha V_d(K)}$ on the right-hand side of (6.5). While the integration domain in (6.5) excludes 0, it is always possible to approximate $\mathcal{E}_0(K) = K$ by a sequence $\mathcal{E}_{\alpha_n}(K)$ as $\alpha_n \downarrow 0$. Thus, the Kusuoka representation can be equivalently written using probability measures on $[0, 1]$.

Example 6.6. The map $K \mapsto \mathcal{E}_\alpha(K)$ is not necessarily monotone in K . An easy counterexample is provided by two segments $[0, 1]$ and $[0, 2]$ on the line. However, the monotonicity fails even for origin symmetric convex bodies. Consider two convex bodies on the plane: $L = [-a, a] \times [-\varepsilon, \varepsilon]$ with $a + \varepsilon \leq 1$ and the ℓ_1 -ball K . We show that for suitable values of a and α , the support function of $\mathcal{E}_\alpha(L)$ is not smaller than the support function of $\mathcal{E}_\alpha(K)$ in direction $u = (1, 0)$. Let $\beta = \langle \xi, u \rangle$ for ξ uniformly distributed in K . Note that $\gamma = \langle \eta, u \rangle$ is uniformly distributed on $[-a, a]$ if η is uniform on L . The quantile functions are

$$q_t(\beta) = 1 - \sqrt{2(1-t)}, \quad q_t(\gamma) = (2t-1)a, \quad t \in [1/2, 1].$$

For $\alpha \in [0, 1/2]$,

$$e_\alpha(\beta) = 1 - 2\sqrt{2}\alpha^{1/2}/3$$

and

$$e_\alpha(\gamma) = a(1 - \alpha).$$

If $\alpha = 1/2$, then $e_\alpha(\beta) < e_\alpha(\gamma)$ if $\frac{2}{3} < a < 1$, meaning that $\mathcal{E}_\alpha(L)$ is not necessarily a subset of $\mathcal{E}_\alpha(K)$.

The monotonicity of Ulam floating body transform (which easily follows from Proposition 2.1 of [27]) implies that, after normalising by volume, \mathcal{E}_α becomes monotone, namely,

$$\mathcal{E}_{\alpha/V_d(K)}(K) \subseteq \mathcal{E}_{\alpha/V_d(L)}(L), \quad 0 \leq \alpha \leq V_d(K),$$

if $K \subseteq L$.

If the family \mathcal{P}_e in (6.5) consists of a single measure ν , we obtain a convex body $\mathcal{E}_{f_\varphi}(K)$ generated by the spectral sublinear expectation e_{f_φ} , where φ is the spectral function related to ν by (2.10). Recall that the maximum extension of the average quantile is a spectral sublinear expectation, see Example 2.8.

Example 6.7. Consider the sublinear expectation $e_1^{\vee m}$ given by (2.14). Note that

$$\max(\langle u, \xi_1 \rangle, \dots, \langle u, \xi_m \rangle) = h(P_m, u),$$

where $P_m = \text{conv}(\xi_1, \dots, \xi_m)$ is the convex hull of independent copies of ξ . Then $\mathbf{E}h(P_m, u)$ is the support function of the expectation $\mathbf{E}P_m$ of the random polytope P_m , see [40, Sec. 2.1]. Therefore, $\mathcal{E}_1^{\vee m}(K) = \mathbf{E}P_m$. Asymptotic properties of these expected polytopes and their relation to floating bodies have been studied in [19], see also [18].

If $m = 1$, then $\mathcal{E}_1(K) = \{x_K\}$ is the barycentre of K . The calculation in Example 2.8 yields that

$$\begin{aligned} \mathbf{E}P_m &= \mathcal{E}_1^{\vee m}(K) = m(m-1) \int_{(0,1]} \mathcal{E}_\alpha(K) \alpha(1-\alpha)^{m-2} d\alpha \\ &= m(m-1) \int_{(0,1]} M_{\alpha V_d(K)}(K) \alpha(1-\alpha)^{m-2} d\alpha. \end{aligned}$$

Hence, the expected random polytope equals the weighted integral of Ulam floating bodies.

More generally, $\mathcal{E}_\alpha^{\vee m}(K)$ is obtained by applying (2.15) as follows

$$\begin{aligned} \mathcal{E}_\alpha^{\vee m}(K) &= \frac{m(m-1)}{\alpha} \int_0^{1-(1-\alpha)^{1/m}} t(1-t)^{m-2} \mathcal{E}_t(K) dt \\ &\quad + \frac{m}{\alpha} (1-\alpha)^{(m-1)/m} (1-(1-\alpha)^{1/m}) \mathcal{E}_{1-(1-\alpha)^{1/m}}(K). \end{aligned}$$

6.3. Centroid bodies and the expectile transform

If $e_{p,a}$ is defined by (2.11) for $p \in [1, \infty)$, then the corresponding floating-like body $\mathcal{E}_{p,a}(K)$ has the support function

$$h(\mathcal{E}_{p,a}(K), u) = \langle x_K, u \rangle + a \left(\mathbf{E}(\langle \xi - x_K, u \rangle_+^p) \right)^{1/p}, \tag{6.6}$$

where ξ is uniformly distributed on K and $x_K = \mathbf{E}\xi$ is the barycentre of K .

If K is origin-symmetric, then $x_K = 0$ and

$$\mathcal{E}_{p,1}(K) = c\Gamma_p K,$$

where $c > 0$ is an explicit constant depending on p and dimension and $\Gamma_p K$ is the L^p -centroid body of K , see [35] for $p = 1$ and [37] for general p . For a not necessarily origin symmetric K , this convex body is defined as

$$h(\Gamma_p K, u) = \left(\frac{1}{c_{d,p} V_d(K)} \int_K |\langle u, y \rangle|^p dy \right)^{1/p},$$

where $c_{d,p}$ is a constant chosen to ensure that this transformation does not change the unit Euclidean ball, see [47, Eq. (10.72)]. For $p = 1$, $a = 1$, and an origin symmetric K ,

$$\mathcal{E}_{1,1}(K) = \frac{1}{2}\Gamma K,$$

where ΓK is the classical *centroid body* of K , see [47, Eq. (10.67)] and [35]. The dual representation of $\mathbf{e}_{1,1}$ from Example 2.6 yields that

$$\Gamma K = 2\mathcal{E}_{1,1}(K) = \text{conv}\{\mathbf{E}(\gamma\xi) : \gamma \in [0, 2]\}.$$

The right-hand side is the expectation of the random convex body $[0, 2\xi]$ being the segment in \mathbb{R}^d with end-points at the origin and 2ξ , see [40, Sec. 2.1].

The *asymmetric L^p -moment body* $M_p^+ K$ introduced in [34] (see also [47, Eq. (10.76)]) has the support function proportional to

$$\int_K \langle (x, y) \rangle_+^p dy.$$

Thus,

$$\mathcal{E}_{p,a}(K) = x_K + c_1 a M_p^+(K - x_K)$$

for a constant c_1 depending on $p \in [1, \infty)$ and dimension.

Corollary 3.5 and the dual representation of $\mathbf{e}_{p,a}$ from Example 2.6 (see also [13, p. 46]) yield that the asymmetric L^p -moment bodies with $p \in [1, \infty)$ can be represented in terms of

$$\mathcal{E}_{p,a}(K) = x_K + a \text{cl} \{ \mathbf{E}((\gamma - \mathbf{E}\gamma)\xi) : \gamma \in L^q(\mathbb{R}_+), \|\gamma\|_q \leq 1 \}.$$

Furthermore, Corollary 6.5 shows that each L^p -centroid body of an origin symmetric K equals the convex hull of a family of integrated Ulam floating bodies of K . This representation can be made very explicit in case $p = 1$; it follows from Theorem 2.4 combined with the results presented in Example 2.6. Namely,

$$\mathcal{E}_{1,a}(K) = x_K + a \text{conv} \bigcup_{t \in [0,1]} t \mathcal{E}_t(K - x_K). \tag{6.7}$$

The following result specialises the above relationship for centroid bodies.

Corollary 6.8. *If K is an origin symmetric convex body, then its centroid body ΓK satisfies*

$$\Gamma K = \frac{2}{V_d(K)} \text{conv} \bigcup_{t \in [0, V_d(K)]} t M_t(K). \tag{6.8}$$

Since K is origin symmetric, $t\mathcal{E}_t(K) = \int_0^t D_s(K) ds$ grows in $t \in (0, 1/2]$, see Corollary 6.4. Thus, the union in (6.8) can be reduced to $t \in [V_d(K)/2, 1]$.

Example 6.9. The definition of the Orlicz centroid bodies from [36] can be also incorporated in our setting using the sublinear expectation

$$e(\beta) = \inf\{\lambda > 0 : \mathbf{E}\psi(\beta/\lambda) \leq 1\},$$

where $\psi : \mathbb{R} \rightarrow [0, \infty)$ is a convex function with $\psi(0) = 0$ and such that ψ is strictly increasing on the positive half-line or strictly decreasing on the negative half-line. This sublinear expectation is the norm of β in the corresponding Orlicz space.

Example 6.10. Consider the expectile $e_{[\tau]}$ defined in Example 2.7 with parameter $\tau \in (0, 1/2]$. In view of the results presented in Example 2.7, the corresponding floating-like body $\mathcal{E}_{[\tau]}(K)$ can be represented as

$$\mathcal{E}_{[\tau]}(K) = x_K + \text{conv} \bigcup_{t \in [0, V_d(K)]} \frac{t(2\tau - 1)}{t(2\tau - 1) + (1 - \tau)V_d(K)} (M_t(K) - x_K). \tag{6.9}$$

Representations (6.8) and (6.9) suggest looking at the transform of convex bodies given by

$$K \mapsto x_K + \text{conv} \bigcup_{t \in [0, V_d(K)]} \psi(t) (M_t(K) - x_K)$$

for a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. As demonstrated above, this transform relates the centroid body transform and the expectile transform to the Ulam floating body transform.

6.4. Open problems related to the sublinear transform

Several calculated examples suggest that $\mathcal{E}_\alpha(K + L) \subseteq \mathcal{E}_\alpha(K) + \mathcal{E}_\alpha(L)$, and we conjecture that this is the case. It is easy to see that this holds on the line for a general sublinear transform.

It was shown in [23] that the equality of two symmetric p -centroid bodies for p not being an even integer yields the equality of the corresponding sets. This question is open for Ulam floating bodies, see [27], not to say also for general floating-like bodies.

It is obvious that $\mathcal{E}_e(K)$ is a dilate of K if K is an ellipsoid. This question has been explored for convex floating bodies, see [57] and references therein. However, the case of Ulam floating body seems to be open, as well as the case of general sublinear transforms.

There is a substantial theory of conditional (dynamic) sublinear expectations, e.g., constructed using backwards stochastic differential equations, see [44]. By applying conditional sublinear expectations to ξ uniformly distributed in K , one comes up with stochastic processes whose values are convex bodies. Further investigation of such processes is left for future work.

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