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Maximal $\mathcal{J}$-semi-definite invariant subspaces of unbounded $\mathcal{J}$-selfadjoint operators in Krein spaces

Heinz Langer [a], Christiane Tretter [b,*]

[a] *Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8-10, 1040 Wien, Austria*
[b] *Mathematisches Institut, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*

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ABSTRACT

In this paper we establish conditions for the existence of maximal $\mathcal{J}$-semi-definite invariant subspaces of unbounded $\mathcal{J}$-selfadjoint operators. Our results allow for operators where all entries of the formal matrix representation induced by the indefinite metric are unbounded and they do not require any definiteness or $\mathcal{J}$-dissipativity assumptions. As a consequence of the existence of invariant subspaces, we obtain an unexpected result on the accumulation of non-real eigenvalues at the real axis which is of independent interest. An application to some dissipative two-channel Hamiltonians illustrates this new phenomenon.

1. Introduction

The existence of maximal $\mathcal{J}$-semi-definite invariant subspaces of $\mathcal{J}$-selfadjoint operators $\mathcal{A}$ in a Krein space was first proved by L.S. Pontrjagin [17] where the negative component of the Krein space was supposed to be finite dimensional, see also [10]. This result was generalized in [14], [15] to $\mathcal{J}$-selfadjoint operators with compact 'imaginary part' in arbitrary Krein spaces. Shortly after a different proof was given by M.G. Krein [12] for bounded 'imaginary part' satisfying some relative compactness condition. In the bounded case, the existence results for maximal $\mathcal{J}$-semi-definite invariant subspaces in [14] found applications in the spectral theory of quadratic eigenvalue problems, see [13]. In the unbounded case considered in [14], [15], [12], a strong assumption, called condition (L) in [2, Def. 3.1.5], had to be imposed which requires that the restriction of $\mathcal{A}$ to the positive component of the space is bounded.

The theory of operator matrices with all entries unbounded, see [23, Chapt. 2], has opened up a new way to attack the existence problem of maximal $\mathcal{J}$-semi-definite subspaces for much wider classes of operators. Our results allow for operators $\mathcal{A}$ that are $\mathcal{J}$-selfadjoint in a Krein space $\mathcal{H}$ and arise as closures of $\mathcal{J}$-

* Corresponding author.
E-mail addresses: heinz.langer@tuwien.ac.at (H. Langer), tretter@math.unibe.ch (C. Tretter).

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symmetric operators $\mathcal{A}_0$ in $\mathcal{H}$ admitting a matrix representation with respect to a canonical decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{J} = \mathrm{diag}\,(I_{\mathcal{H}_+}, -I_{\mathcal{H}_-})$. This means that

$$\mathcal{A}_0 = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}, \quad \mathrm{dom}\,\mathcal{A}_0 := \big(\mathrm{dom}\,A \cap \mathrm{dom}\,B^*\big) \oplus \big(\mathrm{dom}\,B \cap \mathrm{dom}\,D\big),$$

in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with selfadjoint operators A in $\mathcal{H}_+$, D in $\mathcal{H}_-$ and a densely defined closable operator B from $\mathcal{H}_-$ to $\mathcal{H}_+$. Condition (L) imposed in [14], [15], [12] requires that $\mathcal{H}_+ = \mathrm{dom}\,A \cap \mathrm{dom}\,B^*$ which necessitates both entries A and B^* in the first column, and hence also B, to be bounded; then $\mathcal{A}_0$ itself is $\mathcal{J}$-selfadjoint in the Krein space $\mathcal{H}$ and the only possibly unbounded entry in $\mathcal{A}_0$ is D.

In this paper *all* entries of $\mathcal{A}_0$ may be unbounded and thus, in general, the closure $\mathcal{A} = \overline{\mathcal{A}_0}$ may no longer have an operator matrix representation. We establish the existence of maximal $\mathcal{J}$-semi-definite invariant subspaces of $\mathcal{A}$ for three cases which differ in the assumptions on the domains of the entries of $\mathcal{A}_0$. The most general result requires that in one column the diagonal element is dominating columnwise, i.e. either $\mathrm{dom}\,D \subset \mathrm{dom}\,B$ or $\mathrm{dom}\,A \subset \mathrm{dom}\,B^*$. The two other results concern diagonally dominant operators on the one hand and corner dominant operators on the other hand. In the diagonally dominant case both diagonal entries dominate columnwise, i.e. $\mathrm{dom}\,D \subset \mathrm{dom}\,B$ and $\mathrm{dom}\,A \subset \mathrm{dom}\,B^*$, while in the corner dominant case one diagonal element dominates both column- and rowwise, i.e. $\mathrm{dom}\,|D|^{1/2} \subset \mathrm{dom}\,B$ or $\mathrm{dom}\,|A|^{1/2} \subset \mathrm{dom}\,B^*$. Our main tools are the Cayley transform $\mathcal{U}$ of $\mathcal{A}$ and its matrix representation in terms of the Schur complements of $\mathcal{A}_0$ and an invariant subspace theorem for $\mathcal{U}$ from [12].

The strongest assertions are obtained in the diagonally dominant case. Here we prove that if *either*

$$B \text{ is } D\text{-compact} \quad \textit{or} \quad B^* \text{ is } A\text{-compact}, \tag{1.1}$$

then there exists a pair of maximal $\mathcal{J}$-semi-definite invariant subspaces $\mathcal{L}_\pm$ of $\mathcal{A}$, i.e.

$$\mathcal{A}(\mathcal{L}_\pm \cap \mathrm{dom}\,\mathcal{A}) \subset \mathcal{L}_\pm,$$

with $\mathcal{L}_+$ maximal $\mathcal{J}$-non-negative, $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$ maximal $\mathcal{J}$-non-positive, and

$$\overline{\mathcal{L}_\pm \cap \mathrm{dom}\,\mathcal{A}} = \mathcal{L}_\pm, \quad P_+(\mathcal{L}_+ \cap \mathrm{dom}\,\mathcal{A}) = \mathrm{dom}\,A, \quad P_-(\mathcal{L}_- \cap \mathrm{dom}\,\mathcal{A}) = \mathrm{dom}\,D.$$

The existence of these invariant subspaces and the special way in which they can be chosen allows us to prove a result on the essential spectrum of $\mathcal{A}$ which is of independent interest. More precisely, we show that, if (1.1) is satisfied, then the essential spectrum of $\mathcal{A}$ satisfies $\sigma_{\mathrm{e}}(\mathcal{A}) = \sigma_{\mathrm{e}}(A) \cup \sigma_{\mathrm{e}}(D)$ and its non-real spectrum $\sigma(\mathcal{A}) \setminus \mathbb{R}$ can accumulate only at the *intersection* $\sigma_{\mathrm{e}}(A) \cap \sigma_{\mathrm{e}}(D)$ of the essential spectra of the diagonal entries. In particular, if $\sigma_{\mathrm{e}}(A) \cap \sigma_{\mathrm{e}}(D) = \emptyset$, e.g. because one of A or D has compact resolvent, then $\sigma(\mathcal{A}) \setminus \mathbb{R}$ does not have *any* finite accumulation point.

In the corner dominant case the situation is different from the diagonally dominant case in at least two respects. First, we can allow for even more freedom in one column under slightly stronger assumptions in the other column, but here $\mathcal{A}_0$ is only essentially selfadjoint in the Krein space $\mathcal{H}$. Secondly, in general, only one of the invariant subspaces of the Cayley transform $\mathcal{U}$ carries over to $\mathcal{A}$, namely $\mathcal{L}_+$ in the case where D dominates column- and rowwise and $\mathcal{L}_-$ in the case where A dominates column- and rowwise. Correspondingly, we obtain that the non-real spectrum of $\mathcal{A}$ can accumulate only at $\sigma_{\mathrm{e}}(A)$ in the former case, and in $\sigma_{\mathrm{e}}(D)$ in the latter case.

Note that, even in the special case considered in [15], [12] where only D was allowed to be unbounded, our results are stronger. We do not only obtain an invariant subspace $\mathcal{L}_+$ on which $\mathcal{A}$ reduces to a bounded operator, but also an invariant subspace $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$ on which $\mathcal{A}$ is only densely defined and unbounded if so is D. Moreover, our results on the essential spectrum yield stronger claims than Weyl's stability theorem under weaker assumptions. The latter applies if *both* B is D-compact *and* B^* is A-compact and yields that $\sigma_e(\mathcal{A}) = \sigma_e(A) \cup \sigma_e(D)$, i.e. the non-real eigenvalues accumulate at most at the *union* $\sigma_e(A) \cup \sigma_e(D)$. By contrast, we only need *one* relative compactness assumption and prove that the non-real eigenvalues accumulate at most at the *intersection* $\sigma_e(A) \cap \sigma_e(D)$. This improvement over Weyl's theorem enables us to prove that, for some dissipative two-channel Hamiltonians with essential spectrum $[0, \infty)$, the non-real spectrum does not have *any* finite accumulation points.

Finally, we mention another class of existence results for maximal $\mathcal{J}$-semi-definite invariant subspaces for unbounded operators in Krein spaces which all require that the diagonal elements A and D are separated by the imaginary axis. A.A. Shkalikov [21] considers $\mathcal{J}$-dissipative operators, which amounts to $A > 0$ and $D < 0$ in the $\mathcal{J}$-selfadjoint case, and uses an approach inspired by [14], while our results in [16], see also [23], for $\mathcal{J}$-sectorial operators and those by S.G. Pyatkov [18] for $\mathcal{J}$-dissipative operators use a different technique involving integration of the resolvent along the imaginary axis.

The paper is organized as follows. In Section 2 we introduce the $\mathcal{J}$-selfadjoint operators $\mathcal{A} = \overline{\mathcal{A}_0}$ and provide some basic results for their $\mathcal{J}$-unitary Cayley transforms $\mathcal{U}$, including the invariant subspace theorem from [12]. In Section 3 we prove a general result on the existence of invariant subspaces for $\mathcal{J}$-selfadjoint operators via the Cayley transform, see Theorem 3.1. The assumptions therein are as weak as possible to ensure wide applicability. In Section 4 we first establish a new criterion for the selfadjointness and $\mathcal{J}$-selfadjointness of operator matrices with unbounded entries. Then we use Theorem 3.1 to show the existence of pairs $\mathcal{L}_+$ and $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$ of maximal $\mathcal{J}$-non-negative and maximal $\mathcal{J}$-non-positive invariant subspaces in the diagonally dominant case under assumption (1.1). In Section 5, we show the existence of one of the invariant subspaces $\mathcal{L}_+$ or $\mathcal{L}_-$ in the corner dominant case. In Section 6 we apply the result for the essential and discrete spectrum to some dissipative two-channel Hamiltonians.

The following notation and basic facts will be used throughout the paper. Our main objects are Krein spaces and linear operators therein, see e.g. [2], [3]. In a Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with inner product $(\cdot, \cdot)$ which is the orthogonal sum of the Hilbert spaces $\mathcal{H}_+$, $\mathcal{H}_-$ the operator matrix $\mathcal{J} := \operatorname{diag}(I_{\mathcal{H}_+}, -I_{\mathcal{H}_-})$ in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, defines an indefinite inner product by

$$[x, y] := (\mathcal{J}x, y), \quad x, y \in \mathcal{H}.$$

Then $(\mathcal{H}, [\cdot, \cdot])$ is a *Krein space*. An element $x \in \mathcal{H}$ therein is called *$\mathcal{J}$-positive, $\mathcal{J}$-non-negative* or *$\mathcal{J}$-neutral* if $[x, x] > 0$, $[x, x] \geq 0$, or $[x, x] = 0$, respectively; a subspace $\mathcal{L} \subset \mathcal{H}$ is called *$\mathcal{J}$-positive, $\mathcal{J}$-non-negative* or *$\mathcal{J}$-neutral* if all its nonzero elements $x \in \mathcal{L} \setminus \{0\}$ are *$\mathcal{J}$-positive, $\mathcal{J}$-non-negative* or *$\mathcal{J}$-neutral*, respectively. Analogously, we define *$\mathcal{J}$-negative and $\mathcal{J}$-non-positive* elements and subspaces. A *maximal $\mathcal{J}$-non-negative subspace* of $\mathcal{H}$ is a $\mathcal{J}$-non-negative subspace that is not properly contained in any other $\mathcal{J}$-non-negative subspace, and analogously for *maximal $\mathcal{J}$-non-positive subspaces*. We recall, see [3, Sect. II.11], that for every maximal $\mathcal{J}$-non-negative subspace $\mathcal{L}$ there exists a contraction K from $\mathcal{H}_+$ into $\mathcal{H}_-$, i.e. $\|K\| \leq 1$, such that

$$\mathcal{L} = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_+ \right\}; \tag{1.2}$$

here K is called *angular operator* of the subspace $\mathcal{L}$. The $\mathcal{J}$-orthogonal complement $\mathcal{L}^{[\perp]} := \{x \in \mathcal{H}: [x, \mathcal{L}] = 0\}$ of the maximal $\mathcal{J}$-non-negative subspace $\mathcal{L}$ is a maximal $\mathcal{J}$-non-positive subspace and, if (1.2) holds, it has the form

$$\mathcal{L}^{[\perp]} = \left\{ \begin{pmatrix} K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_- \right\}. \tag{1.3}$$

If $\mathcal{T}$ is a densely defined linear operator in $\mathcal{H}$ with domain $\operatorname{dom} \mathcal{T}$ and $\mathcal{T}^*$ is its Hilbert space adjoint, the *$\mathcal{J}$-adjoint* (or *Krein space adjoint*) $\mathcal{T}^+$ of $\mathcal{T}$ is defined by $\mathcal{T}^+ := \mathcal{J}\mathcal{T}^*\mathcal{J}$, i.e.

$$[\mathcal{T}x, y] = [x, \mathcal{T}^+ y], \quad x \in \operatorname{dom} \mathcal{T},\ y \in \operatorname{dom} \mathcal{T}^+.$$

The operator $\mathcal{T}$ is called *$\mathcal{J}$-symmetric* if $\mathcal{T} \subset \mathcal{T}^+$, *$\mathcal{J}$-selfadjoint* if $\mathcal{T} = \mathcal{T}^+$, *essentially $\mathcal{J}$-selfadjoint* if $\mathcal{T}$ is closable and $\overline{\mathcal{T}}$ is $\mathcal{J}$-selfadjoint, *$\mathcal{J}$-isometric* if $\mathcal{T}^+\mathcal{T} = I_{\mathcal{H}}$, and *$\mathcal{J}$-unitary* if $\mathcal{T}^+\mathcal{T} = \mathcal{T}\mathcal{T}^+ = I_{\mathcal{H}}$. Clearly, $\mathcal{T}$ is $\mathcal{J}$-symmetric, $\mathcal{J}$-selfadjoint, essentially $\mathcal{J}$-selfadjoint, $\mathcal{J}$-isometric or $\mathcal{J}$-unitary if and only if $\mathcal{J}\mathcal{T}$ has the respective property in the Hilbert space $\mathcal{H}$. The *essential spectrum* of $\mathcal{T}$ is defined as

$$\sigma_{\mathrm{e}}(\mathcal{T}) := \{\lambda \in \mathbb{C} : \mathcal{T} - \lambda \text{ is not Fredholm}\},$$

see [9, Sect. XVII.2]; note that in [5, Sect. IX.1] this is the essential spectrum $\sigma_{\mathrm{e}3}(\mathcal{T})$. A closed subspace $\mathcal{L} \subset \mathcal{H}$ is called *invariant* for $\mathcal{T}$ if $\mathcal{T}(\operatorname{dom} \mathcal{T} \cap \mathcal{L}) \subset \mathcal{L}$; in this case the restriction $\mathcal{T}|_{\mathcal{L}}$ of $\mathcal{T}$ to $\mathcal{L}$ is well-defined with domain $\operatorname{dom} \mathcal{T}|_{\mathcal{L}} = \operatorname{dom} \mathcal{T} \cap \mathcal{L}$.

In order to compare unbounded operators, we need the notions of relative boundedness and relative compactness, see e.g. [11, Sect. IV.1.1]. If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces and $\mathcal{S}$ is a linear operator from $\mathcal{H}$ to $\mathcal{H}'$ with domain $\operatorname{dom} \mathcal{S}$, then $\mathcal{S}$ is called *$\mathcal{T}$-bounded* if $\operatorname{dom} \mathcal{T} \subset \operatorname{dom} \mathcal{S}$ and there exist constants a, $b \geq 0$ such that

$$\|\mathcal{S}x\|^2 \leq a^2\|x\|^2 + b^2\|\mathcal{T}x\|^2, \quad x \in \operatorname{dom} \mathcal{T};$$

the infimum $\delta_{\mathcal{T}} \geq 0$ of all b such that the above inequality holds for some $a \geq 0$ is called the *$\mathcal{T}$-bound of* $\mathcal{S}$. Moreover, $\mathcal{S}$ is called *$\mathcal{T}$-compact* if $\operatorname{dom} \mathcal{T} \subset \operatorname{dom} \mathcal{S}$ and, for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset \operatorname{dom} \mathcal{T}$ such that $(\mathcal{T}x_n)_{n\in\mathbb{N}}$ is bounded, $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ contains a convergent subsequence. Note that if $\mathcal{T}$ is closed and $\mathcal{S}$ is closable, then $\operatorname{dom} \mathcal{T} \subset \operatorname{dom} \mathcal{S}$ already implies that $\mathcal{S}$ is $\mathcal{T}$-bounded by the closed graph theorem. If there exists $\mu \in \varrho(\mathcal{T})$, then $\mathcal{S}$ is $\mathcal{T}$-bounded if and only if $\mathcal{S}(\mathcal{T} - \mu)^{-1}$ is bounded, and $\mathcal{S}$ is $\mathcal{T}$-compact if and only if $\mathcal{S}(\mathcal{T} - \mu)^{-1}$ is compact.

2. $\mathcal{J}$-selfadjoint operators and their Cayley transforms

In this section we introduce the classes of $\mathcal{J}$-selfadjoint operators $\mathcal{A}$ that we consider, establish conditions for the existence of their Cayley transforms $\mathcal{U}$ and provide the key tools for the next sections, in particular, the invariant subspace theorem for $\mathcal{U}$ from [12].

The operators $\mathcal{A}$ arise as closures of $\mathcal{J}$-symmetric operator matrices $\mathcal{A}_0$ in $\mathcal{H}$ admitting a matrix representation with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$\mathcal{A}_0 := \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}, \quad \operatorname{dom} \mathcal{A}_0 := \big(\operatorname{dom} A \cap \operatorname{dom} B^*\big) \oplus \big(\operatorname{dom} B \cap \operatorname{dom} D\big). \tag{2.1}$$

Here A and D are selfadjoint operators in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively, B is a closable operator from $\mathcal{H}_-$ to $\mathcal{H}_+$, and all entries A, B, D may be unbounded with dense domains. Moreover, we assume that $\operatorname{dom} \mathcal{A}_0$ is dense, i.e.

$$\overline{\operatorname{dom} A \cap \operatorname{dom} B^*} = \mathcal{H}_+, \quad \overline{\operatorname{dom} B \cap \operatorname{dom} D} = \mathcal{H}_-. \tag{2.2}$$

Clearly, $\mathcal{A}_0$ is $\mathcal{J}$-symmetric with respect to

$$\mathcal{J} := \operatorname{diag}\,(I_{\mathcal{H}_+}, -I_{\mathcal{H}_-}) = \begin{pmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & -I_{\mathcal{H}_-} \end{pmatrix}.$$

Since $\mathcal{J}\mathcal{A}_0$ is symmetric and hence closable, $\mathcal{A}_0 = \mathcal{J}^{-1}(\mathcal{J}\mathcal{A}_0)$ is closable as well, see [11, Prob. III.5.7]. The closure $\mathcal{A} := \overline{\mathcal{A}_0}$ is $\mathcal{J}$-symmetric, but it need not possess a matrix representation with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, see Proposition 2.1 below.

Important tools for the analysis of operator matrices (2.1) and their closures are the Schur complements S_1 and S_2, see [23, Sect. 2.2], given by

$$S_1(\lambda) := A - \lambda + B(D-\lambda)^{-1}B^*, \quad \lambda \in \varrho(D), \tag{2.3}$$
$$S_2(\lambda) := D - \lambda + B^*(A-\lambda)^{-1}B, \quad \lambda \in \varrho(A). \tag{2.4}$$

The values $S_1(\lambda)$ and $S_2(\lambda)$ are unbounded linear operators in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. While, in general, their domains may depend on λ, we have

$$\begin{aligned} \operatorname{dom}\big(S_1(\lambda)\big) &= \operatorname{dom}(A) \cap \operatorname{dom}(B^*) \quad &\text{if} \quad \operatorname{dom}(D) \subset \operatorname{dom}(B), \\ \operatorname{dom}\big(S_2(\lambda)\big) &= \operatorname{dom}(B) \cap \operatorname{dom}(D) \quad &\text{if} \quad \operatorname{dom}(A) \subset \operatorname{dom}(B^*), \end{aligned}$$

and hence, in these cases, since $\mathcal{A}_0$ is densely defined, so are $S_1(\lambda)$, $S_2(\lambda)$, respectively.

If $\mathcal{A}_0$ satisfies a domain inclusion for one diagonal element, the closure $\mathcal{A}$ of $\mathcal{A}_0$ can be described by means of the Schur complement associated with the other column, i.e. by S_1 if $\operatorname{dom} D \subset \operatorname{dom} B$ and by S_2 if $\operatorname{dom} A \subset \operatorname{dom} B^*$, comp. [23, Thm. 2.2.18].

Proposition 2.1.

i) *If* $\operatorname{dom} D \subset \operatorname{dom} B$, *then* $S_1(\mu)$ *is closable for all* $\mu \in \varrho(D)$ *and the closure* $\mathcal{A} = \overline{\mathcal{A}_0}$ *of* $\mathcal{A}_0$ *is given by*

$$\operatorname{dom} \mathcal{A} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_+ \oplus \mathcal{H}_- : x \in \operatorname{dom} \overline{S_1(\mu)},\ \overline{(D-\mu)^{-1}B^*}x + y \in \operatorname{dom} D \right\},$$
$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(\overline{S_1(\mu)} + \mu\big)\, x - B\big(\overline{(D-\mu)^{-1}B^*}\, x + y\big) \\ (D-\mu)\big(-\overline{(D-\mu)^{-1}B^*}\, x + y\big) + \mu\, y \end{pmatrix}.$$

ii) *If* $\operatorname{dom} A \subset \operatorname{dom} B^*$, *then* $S_2(\mu)$ *is closable for all* $\mu \in \varrho(A)$ *and the closure* $\mathcal{A} = \overline{\mathcal{A}_0}$ *of* $\mathcal{A}_0$ *is given by*

$$\operatorname{dom} \mathcal{A} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_+ \oplus \mathcal{H}_- : x + \overline{(A-\mu)^{-1}B}y \in \operatorname{dom} A,\ y \in \operatorname{dom} \overline{S_2(\mu)} \right\},$$
$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A-\mu)\big(x + \overline{(A-\mu)^{-1}B}\, y\big) + \mu\, x \\ B^*\big(x + \overline{(A-\mu)^{-1}B}\, y\big) + \big(\overline{S_2(\mu)} + \mu\big)\, y \end{pmatrix}.$$

Proof. We prove claim i); the proof of ii) is analogous. If $\operatorname{dom} D \subset \operatorname{dom} B$ and $\mu \in \varrho(D)$, then $B(D-\mu)^{-1}$ is bounded since B is D-bounded, see [11, Rem. IV.1.5], and $(D-\mu)^{-1}B^*$ is bounded with dense domain $\operatorname{dom} B^*$ since $(D-\mu)^{-1}B^* \subset (B(D-\overline{\mu})^{-1})^*$. Thus $\mathcal{A}_0$ satisfies the assumptions of [23, Thm. 2.2.18] with $C = -B^*$.

That $S_1(\mu)$ is closable for all $\mu \in \varrho(D)$ follows from [23, Thm. 2.2.18] and the fact that $\mathcal{A}_0$ is already known to be closable. It can also be seen directly from the inclusions

$$S_1(\mu) \subset (A-\overline{\mu})^* + (B(D-\overline{\mu})^{-1}B^*)^* \subset (A-\overline{\mu} + B(D-\overline{\mu})^{-1}B^*)^* = S_1(\overline{\mu})^* \tag{2.5}$$

for $\mu \in \varrho(D)$; here we have used the rules for adjoints of products and sums of unbounded operators, see e.g. [24, Satz 2.43, 2.45]. The representation of $\mathcal{A} = \overline{\mathcal{A}_0}$ claimed in i) follows from [23, Thm. 2.2.18]. □

If the $\mathcal{J}$-symmetric operator $\mathcal{A} = \overline{\mathcal{A}_0}$ has a complex conjugate pair z_0, $\overline{z_0} \in \varrho(\mathcal{A}) \setminus \mathbb{R}$, then the *Cayley transform* $\mathcal{U}$ of $\mathcal{A}$ (with respect to z_0) given by

$$\mathcal{U} := (\mathcal{A} - \overline{z_0})(\mathcal{A} - z_0)^{-1} = I_{\mathcal{H}} + (z_0 - \overline{z_0})(\mathcal{A} - z_0)^{-1} \tag{2.6}$$

is $\mathcal{J}$-unitary and $\mathcal{A}$ is $\mathcal{J}$-selfadjoint, see [3, Thm. VI.7.1, VI.7.2]. In this case, $\mathcal{U}-I_{\mathcal{H}}$ is injective, $\operatorname{ran}(\mathcal{U}-I_{\mathcal{H}}) = \operatorname{dom}\mathcal{A}$ and the *inverse Cayley transform* is given by

$$\mathcal{A} = (z_0\mathcal{U} - \overline{z_0})(\mathcal{U} - I_{\mathcal{H}})^{-1} = z_0 I_{\mathcal{H}} + (z_0 - \overline{z_0})(\mathcal{U} - I_{\mathcal{H}})^{-1}. \tag{2.7}$$

Unlike selfadjoint operators in a Hilbert space, $\mathcal{J}$-selfadjoint operators may have empty resolvent set. The next proposition provides sufficient conditions for the existence of z_0, $\overline{z_0} \in \varrho(\mathcal{A}) \setminus \mathbb{R}$ and matrix representations of the Cayley transform $\mathcal{U}$ of $\mathcal{A}$ which will be crucial in the sequel.

Proposition 2.2. i) *If* $\operatorname{dom} D \subset \operatorname{dom} B$ *and there exists* $z_0 \in \mathbb{C}\setminus\mathbb{R}$ *such that*

(R) $\operatorname{ran}\overline{S_1(z_0)} = \mathcal{H}_+$, $\operatorname{ran}\overline{S_1(\overline{z_0})} = \mathcal{H}_+$,

then $\overline{S_1(z_0)}$, $\overline{S_1(\overline{z_0})}$ *are boundedly invertible,* $\mathcal{A}$ *is* $\mathcal{J}$*-selfadjoint with* z_0, $\overline{z_0} \in \varrho(\mathcal{A})$*, and the Cayley transform* $\mathcal{U}$ *given by* (2.6) *has the form*

$$\mathcal{U}=\begin{pmatrix} I_{\mathcal{H}_+}+(z_0-\overline{z_0})\overline{S_1(z_0)}^{-1} & -(z_0-\overline{z_0})\overline{S_1(z_0)}^{-1}B(D-z_0)^{-1} \\ (z_0-\overline{z_0})\overline{(D-z_0)^{-1}B^*}\;\overline{S_1(z_0)}^{-1} & I_{\mathcal{H}_-}+(z_0-\overline{z_0})\big(I_{\mathcal{H}_-}-\overline{(D-z_0)^{-1}B^*}\;\overline{S_1(z_0)}^{-1}B\big)(D-z_0)^{-1} \end{pmatrix}.$$

ii) *If* $\operatorname{dom} A \subset \operatorname{dom} B^*$ *and there exists* $z_0 \in \mathbb{C}\setminus\mathbb{R}$ *such that*

(R′) $\operatorname{ran}\overline{S_2(z_0)} = \mathcal{H}_-$, $\operatorname{ran}\overline{S_2(\overline{z_0})} = \mathcal{H}_-$,

then $\overline{S_2(z_0)}$, $\overline{S_2(\overline{z_0})}$ *are boundedly invertible,* $\mathcal{A}$ *is* $\mathcal{J}$*-selfadjoint with* z_0, $\overline{z_0} \in \varrho(\mathcal{A})$*, and the Cayley transform* $\mathcal{U}$ *given by* (2.6) *has the form*

$$\mathcal{U}=\begin{pmatrix} I_{\mathcal{H}_+}+(z_0-\overline{z_0})\big(I_{\mathcal{H}_+}-\overline{(A-z_0)^{-1}B}\;\overline{S_2(z_0)}^{-1}B^*\big)(A-z_0)^{-1} & -(z_0-\overline{z_0})\overline{(A-z_0)^{-1}B}\;\overline{S_2(z_0)}^{-1} \\ (z_0-\overline{z_0})\overline{S_2(z_0)}^{-1}B^*(A-z_0)^{-1} & I_{\mathcal{H}_-}+(z_0-\overline{z_0})\overline{S_2(z_0)}^{-1} \end{pmatrix}.$$

Proof. i) Due to condition (R) and the inclusion (2.5), it follows that

$$\operatorname{ran}\overline{S_1(z_0)}^* = \operatorname{ran} S_1(z_0)^* = \mathcal{H}_+, \quad \operatorname{ran}\overline{S_1(\overline{z_0})}^* = \operatorname{ran} S_1(\overline{z_0})^* = \mathcal{H}_+,$$

and thus $\ker\overline{S_1(z_0)} = \{0\}$, $\ker\overline{S_1(\overline{z_0})} = \{0\}$. Together with (R), we see that $\overline{S_1(z_0)}$, $\overline{S_1(\overline{z_0})}$ are closed bijective operators and hence boundedly invertible. By [23, Thm. 2.3.3 ii)] applied with $C = -B^*$ therein, this implies that z_0, $\overline{z_0} \in \varrho(\mathcal{A})$, and hence $\mathcal{A}$ is $\mathcal{J}$-selfadjoint. The matrix representation of $\mathcal{U}$ follows from the second formula in (2.6) and from the matrix representation of the resolvent of $\mathcal{A}$ in [23, Thm. 2.3.3 ii)].

ii) The proof of ii) is analogous if we use [23, Thm. 2.3.3 i)] with $C = -B^*$. □

The next result on the existence of maximal $\mathcal{J}$-semi-definite invariant subspaces for the Cayley transform $\mathcal{U}$, which follows from [12, Thm. 3], is the starting point for our further investigations.

Proposition 2.3. *Assume either* $\operatorname{dom} D \subset \operatorname{dom} B$ *and there is* $z_0 \in \mathbb{C}\setminus\mathbb{R}$ *so that* (R) *holds and*

(C) $\overline{S_1(z_0)}^{-1}B(D - z_0)^{-1}$ *is compact,*

or assume that $\operatorname{dom} A \subset \operatorname{dom} B^*$ *and there exists* $z_0 \in \mathbb{C}\setminus\mathbb{R}$ *such that* (R′) *holds and*

(C′) $\overline{S_2(z_0)}^{-1}B^*(A - z_0)^{-1}$ *is compact.*

Then the non-unitary spectrum $\sigma(\mathcal{U}) \setminus \{z \in \mathbb{C} : |z| = 1\}$ *of the Cayley transform* $\mathcal{U}$ *in* (2.6) *of* $\mathcal{A}$ *is discrete and symmetric with respect to the unit circle. Moreover, for any disjoint decomposition*

$$\sigma(\mathcal{U}) \setminus \{z \in \mathbb{C} : |z| = 1\} = \widetilde{\sigma} \,\dot{\cup}\, \widetilde{\sigma}^{-*}, \quad \widetilde{\sigma}^{-*} := \{\overline{z}^{-1} \in \mathbb{C} : z \in \widetilde{\sigma}\},$$

there exists a maximal $\mathcal{J}$*-non-negative subspace* $\mathcal{L}_+$ *of* $\mathcal{H}$ *such that* $\mathcal{L}_+$ *and the maximal* $\mathcal{J}$*-non-positive subspace* $\mathcal{L}_- := \mathcal{L}_+^{[\perp]}$ *are invariant for* $\mathcal{U}$, $\mathcal{U}\mathcal{L}_\pm = \mathcal{L}_\pm$, *the non-unitary spectra of the restrictions* $\mathcal{U}|_{\mathcal{L}_\pm}$ *both coincide with* $\widetilde{\sigma}$, *and that all root subspaces* $\mathcal{S}_z$ *of* $\mathcal{U}$ *at eigenvalues in* $z \in \widetilde{\sigma}$ *are contained in* $\mathcal{L}_\pm$, *i.e.*

$$\sigma(\mathcal{U}|_{\mathcal{L}_\pm}) \setminus \{z \in \mathbb{C} : |z| = 1\} = \widetilde{\sigma}, \quad \mathcal{S}_z \subset \mathcal{L}_\pm, \quad z \in \widetilde{\sigma}. \tag{2.8}$$

Proof. If $\operatorname{dom} D \subset \operatorname{dom} B$, then assumption (C) ensures that the operator matrix representing the Cayley transform $\mathcal{U}$ of $\mathcal{A}$ in Proposition 2.2 i) has compact off-diagonal entries, and similarly if $\operatorname{dom} A \subset \operatorname{dom} B^*$ and (C′) holds. Thus the assumptions of [12, Thm. 3] are satisfied; note that the compactness of one off-diagonal entry of a $\mathcal{J}$-unitary operator implies the compactness of the other.

Now all claims follow from [12, Thm. 3] if we add the following proof of the last claim on the non-unitary spectrum of $\mathcal{U}|_{\mathcal{L}_\pm}$ and the associated root subspaces, comp. [15]. If e.g. $\widetilde{\sigma}$ is an infinite set, $\widetilde{\sigma} =: \{z_j : j \in \mathbb{N}\}$, an *arbitrary* maximal $\mathcal{J}$-non-negative invariant subspace $\mathcal{L}_+$ of $\mathcal{U}$ can be adapted as follows. Starting with $\mathcal{L}_+$, we inductively define a sequence $\mathcal{L}_j$, $j \in \mathbb{N}_0$, of maximal $\mathcal{J}$-non-negative invariant subspaces of $\mathcal{U}$ by

$$\mathcal{L}_0 := \mathcal{L}_+, \quad \mathcal{L}_j := \big(\mathcal{L}_{j-1} + (\mathcal{S}_{z_j} \dotplus \mathcal{S}_{\overline{z_j}^{-1}})\big) \cap \mathcal{S}_{z_j}^{[\perp]} = \big(\mathcal{L}_{j-1} \cap (\mathcal{S}_{z_j} \dotplus \mathcal{S}_{\overline{z_j}^{-1}})^{[\perp]}\big) \dotplus \mathcal{S}_{z_j}, \quad j \in \mathbb{N},$$

where $\mathcal{S}_z$ denotes the root subspace of $\mathcal{U}$ at an eigenvalue z of $\mathcal{U}$. By construction, the subspace $\mathcal{L}_j$ contains all the root subspaces $\mathcal{S}_{z_i}$, $i = 1, 2, \ldots, j$. Further, $\widehat{\mathcal{L}}_j := \mathcal{L}_{j-1} \cap (\mathcal{S}_{z_j} \dotplus \mathcal{S}_{\overline{z_j}^{-1}})^{[\perp]}$ is invariant under $\mathcal{U}$ and the points $z_j, \overline{z_j}^{-1}$ belong to the resolvent set of the restriction $\mathcal{U}|_{\widehat{\mathcal{L}}_j}$. Then the subspace $\mathcal{L}_j$ is $\mathcal{J}$-non-negative since $\widehat{\mathcal{L}}_{j-1}$ is $\mathcal{J}$-non-negative, $\mathcal{S}_{z_j}$ is $\mathcal{J}$-neutral and $\widehat{\mathcal{L}}_{j-1}$ is $\mathcal{J}$-orthogonal to $\mathcal{S}_{z_j}$. Since

$$\kappa_j := \dim \mathcal{S}_{z_j} = \dim \mathcal{S}_{\overline{z_j}^{-1}} = \dim \big(\mathcal{L}_{j-1} \cap (\mathcal{S}_{z_j} \dotplus \mathcal{S}_{\overline{z_j}^{-1}}\big),$$

the subspace $\mathcal{L}_j$ is maximal $\mathcal{J}$-non-negative. If we consider the limit $j \to \infty$, then a subsequence of the sequence of the angular operators $K_{\mathcal{L}_j}$ of $\mathcal{L}_j$, $j \in \mathbb{N}_0$, see (1.2), converges weakly to a contraction $\widetilde{K}$ from $\mathcal{H}_+$ into $\mathcal{H}_-$. The corresponding maximal $\mathcal{J}$-non-negative subspace $\widetilde{\mathcal{L}}_+$ is invariant under $\mathcal{U}$, and the non-unitary spectrum of the restriction $\mathcal{U}|_{\widetilde{\mathcal{L}}_+}$ is $\widetilde{\sigma}$, as required in (2.8). □

3. Existence of maximal $\mathcal{J}$-semi-definite invariant subspaces

The following theorem is our most general existence result for maximal $\mathcal{J}$-semi-definite invariant subspaces of $\mathcal{J}$-selfadjoint operators in this paper. Here we assume a domain inclusion for one diagonal element in (2.1), i.e. $\operatorname{dom} A \subset \operatorname{dom} B^*$ or $\operatorname{dom} D \subset \operatorname{dom} B$. This result is also the key ingredient for the invariant subspace theorems in the following two sections. There the domain assumptions are strengthened, while the sufficient conditions (R), (C) and (R′), (C′), respectively, below can be simplified considerably.

Theorem 3.1. i) *Suppose that* $\operatorname{dom} D \subset \operatorname{dom} B$ *and that there exists* $z_0 \in \mathbb{C} \setminus \mathbb{R}$ *such that*

(R) $\operatorname{ran} \overline{S_1(z_0)} = \mathcal{H}_+$, $\operatorname{ran} \overline{S_1(\overline{z_0})} = \mathcal{H}_+$,

(C) $\overline{S_1(z_0)}^{-1} B (D - z_0)^{-1}$ *is compact.*

Then $\mathcal{A} = \overline{\mathcal{A}_0}$ *is* $\mathcal{J}$*-selfadjoint and* $\mathcal{A}$ *has a maximal* $\mathcal{J}$*-non-negative invariant subspace* $\mathcal{L}_+$, *i.e.*

$$\mathcal{A}(\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}) \subset \mathcal{L}_+, \tag{3.1}$$

and

$$\overline{\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}} = \mathcal{L}_+, \quad P_+(\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}) = \operatorname{dom} \overline{S_1(z_0)}. \tag{3.2}$$

ii) *Suppose that* $\operatorname{dom} A \subset \operatorname{dom} B^*$ *and that there exists* $z_0 \in \mathbb{C} \setminus \mathbb{R}$ *such that*

(R′) $\operatorname{ran} \overline{S_2(z_0)} = \mathcal{H}_-$, $\operatorname{ran} \overline{S_2(\overline{z_0})} = \mathcal{H}_-$,
(C′) $\overline{S_2(z_0)}^{-1} B^*(A - z_0)^{-1}$ *is compact.*

Then $\mathcal{A} = \overline{\mathcal{A}_0}$ *is* $\mathcal{J}$*-selfadjoint and* $\mathcal{A}$ *has a maximal* $\mathcal{J}$*-non-positive invariant subspace* $\mathcal{L}_-$*, i.e.*

$$\mathcal{A}(\mathcal{L}_- \cap \operatorname{dom} \mathcal{A}) \subset \mathcal{L}_-, \tag{3.3}$$

and

$$\overline{\mathcal{L}_- \cap \operatorname{dom} \mathcal{A}} = \mathcal{L}_-, \quad P_-(\mathcal{L}_- \cap \operatorname{dom} \mathcal{A}) = \operatorname{dom} \overline{S_2(z_0)}. \tag{3.4}$$

If all four conditions in i) *and* ii) *are satisfied, then* $\mathcal{L}_-$ *can be chosen such that* $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$.

In each of the cases i) *and* ii)*, the non-real spectrum* $\sigma(\mathcal{A}) \setminus \mathbb{R}$ *of* $\mathcal{A}$ *is discrete, symmetric to* $\mathbb{R}$ *and for any disjoint decomposition*

$$\sigma(\mathcal{A}) \setminus \mathbb{R} = \widehat{\sigma} \,\dot\cup\, \widehat{\sigma}^*, \quad \widehat{\sigma}^* := \{\overline{z} \in \mathbb{C} : z \in \widehat{\sigma}\},$$

the invariant subspace $\mathcal{L}_+$ *in* i) *and the invariant subspace* $\mathcal{L}_-$ *in* ii)*, respectively, can be chosen such that the non-real spectrum of the restriction* $\mathcal{A}|_{\mathcal{L}_+}$ *in* i) *and* $\mathcal{A}|_{\mathcal{L}_-}$ *in* ii) *coincides with* $\widehat{\sigma}$*, and that all root subspaces* $\mathcal{S}_z$ *of* $\mathcal{A}$ *at eigenvalues* $z \in \widehat{\sigma}$ *are contained in* $\mathcal{L}_+$ *and in* $\mathcal{L}_-$*, respectively, i.e.*

$$\sigma(\mathcal{A}|_{\mathcal{L}_+}) \setminus \mathbb{R} = \widehat{\sigma}, \qquad \mathcal{S}_z \subset \mathcal{L}_+, \quad z \in \widehat{\sigma}, \qquad \textit{in case i)}, \tag{3.5}$$

$$\sigma(\mathcal{A}|_{\mathcal{L}_-}) \setminus \mathbb{R} = \widehat{\sigma}, \qquad \mathcal{S}_z \subset \mathcal{L}_-, \quad z \in \widehat{\sigma}, \qquad \textit{in case ii)}. \tag{3.6}$$

Remark 3.2. If $\overline{S_1(z)}^{-1} B(D - z)^{-1}$ is compact for some $z \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{ran} \overline{S_1(z)} = \mathcal{H}_+$, then $\overline{S_1(z')}^{-1} B(D - z')^{-1}$ is compact for all $z' \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{ran} \overline{S_1(z')} = \mathcal{H}_+$, see the proof below.

Proof of Theorem 3.1. By Proposition 2.3, in each of the two cases i) and ii), the Cayley transform $\mathcal{U}$ of $\mathcal{A}$ possesses a maximal $\mathcal{J}$-non-negative subspace $\mathcal{L}_+$ as well as a maximal $\mathcal{J}$-non-positive subspace $\mathcal{L}_- := \mathcal{L}_+^{[\perp]}$ of $\mathcal{H}$ such that $\mathcal{L}_\pm$ are both invariant for $\mathcal{U}$, more precisely, $\mathcal{U}\mathcal{L}_\pm = \mathcal{L}_\pm$. In the sequel we show that under the conditions in i) the subspace $\mathcal{L}_+$ is also invariant for $\mathcal{A}$ and satisfies (3.1), (3.2), while under the conditions in ii) the subspace $\mathcal{L}_-$ is invariant for $\mathcal{A}$ and satisfies (3.3), (3.4).

i) In order to show that $\mathcal{L}_+$ is an invariant subspace for $\mathcal{A}$ with (3.1), (3.2) if (R) and (C) hold, we distinguish two cases. First we assume that $\|B(D - z_0)^{-1}\| < 1$ and prove

$$(\mathcal{U} - I_{\mathcal{H}})\mathcal{L}_+ = \mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}. \tag{3.7}$$

The inclusion '$\subset$' in (3.7) is clear since $\mathcal{L}_+$ is invariant for $\mathcal{U}$ and $\operatorname{ran}(\mathcal{U} - I_{\mathcal{H}}) = \operatorname{dom} \mathcal{A}$. For the reverse inclusion '$\supset$' in (3.7) we first note that, with $\mu = z_0$ in Proposition 2.2 and using the angular operator representation (1.2) of $\mathcal{L}_+$, we have

$$\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A} = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \operatorname{dom} \overline{S_1(z_0)}, \ (\overline{(D - z_0)^{-1} B^*} + K)x \in \operatorname{dom} D \right\}$$

and

$$(\mathcal{U}-I_{\mathcal{H}})\mathcal{L}_+ = \left\{(\mathcal{U}-I_{\mathcal{H}})\begin{pmatrix} y \\ Ky \end{pmatrix} : y\in\mathcal{H}_+\right\} \subset \operatorname{dom}\mathcal{A}.$$

If we denote by $P_\pm:\mathcal{H}\to\mathcal{H}_\pm$ the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_\pm$, the matrix representation of $\mathcal{U}$ in Proposition 2.2 i) shows that

$$P_+(\mathcal{U}-I_{\mathcal{H}})P_+ = (z_0-\overline{z_0})\overline{S_1(z_0)}^{-1}, \tag{3.8}$$

$$P_+(\mathcal{U}-I_{\mathcal{H}})P_- = -(z_0-\overline{z_0})\overline{S_1(z_0)}^{-1}B(D-z_0)^{-1}. \tag{3.9}$$

If $\binom{x}{Kx}\in\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}$, then $x\in\operatorname{dom}\overline{S_1(z_0)}$. Since $\|B(D-z_0)^{-1}\|<1$ and K is a contraction, we can set

$$y := (z_0-\overline{z_0})^{-1}\big(I_{\mathcal{H}_+}-B(D-z_0)^{-1}K\big)^{-1}\overline{S_1(z_0)}x \in \mathcal{H}_+.$$

Then

$$P_+(\mathcal{U}-I_{\mathcal{H}})\begin{pmatrix} y \\ Ky \end{pmatrix} = (z_0-\overline{z_0})\overline{S_1(z_0)}^{-1}\big(I_{\mathcal{H}_+}-B(D-z_0)^{-1}K\big)y = x.$$

Since $(\mathcal{U}-I_{\mathcal{H}})\mathcal{L}_+\subset\mathcal{L}_+$ and $\mathcal{L}_+$ has the form (1.2), it follows that

$$P_-(\mathcal{U}-I_{\mathcal{H}})\begin{pmatrix} y \\ Ky \end{pmatrix} = KP_+(\mathcal{U}-I_{\mathcal{H}})\begin{pmatrix} y \\ Ky \end{pmatrix} = Kx,$$

and hence $\binom{x}{Kx} = (\mathcal{U}-I_{\mathcal{H}})\binom{y}{Ky}\in(\mathcal{U}-I_{\mathcal{H}})\mathcal{L}_+$, which completes the proof of (3.7).

Now we prove (3.1), (3.2). By (3.7) and (2.6), it follows that

$$\mathcal{L}_+\cap\operatorname{dom}\mathcal{A} = (z_0-\overline{z_0})(\mathcal{A}-z_0)^{-1}\mathcal{L}_+. \tag{3.10}$$

Applying $\mathcal{A}$ to (3.10), we obtain

$$\mathcal{A}(\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}) = (z_0-\overline{z_0})\mathcal{A}(\mathcal{A}-z_0)^{-1}\mathcal{L}_+ \subset (z_0-\overline{z_0})\big(\mathcal{L}_+ + z_0(\mathcal{A}-z_0)^{-1}\mathcal{L}_+\big) = \mathcal{L}_+,$$

and hence (3.1). Using the relations (3.7), (3.8), (3.9) and the fact that $I_{\mathcal{H}_+}-B(D-z_0)^{-1}K$ is bijective because $\|B(D-z_0)^{-1}\|<1$, we conclude that

$$P_+\big(\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}\big) = \overline{S_1(z_0)}^{-1}(I_{\mathcal{H}_+}-B(D-z_0)^{-1}K)\mathcal{H}_+ = \operatorname{dom}\overline{S_1(z_0)}, \tag{3.11}$$

$$P_-\big(\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}\big) = KP_+\big(\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}\big) = K\operatorname{dom}\overline{S_1(z_0)},$$

and thus the second claim in (3.2) is proved. Since $\mathcal{A}_0$ was assumed to be densely defined, see (2.2), $\operatorname{dom}A\cap\operatorname{dom}B^* = \operatorname{dom}S_1(z_0)$ is dense in $\mathcal{H}_+$. Then also $\operatorname{dom}\overline{S_1(z_0)}$ is dense in $\mathcal{H}_+ = P_+\mathcal{L}_+$. Because K is bounded, it follows that $K\operatorname{dom}\overline{S_1(z_0)}$ is dense in $K\mathcal{H}_+ = P_-\mathcal{L}_+$. This completes the proof that $\mathcal{L}_+\cap\operatorname{dom}\mathcal{A}$ is dense in $\mathcal{L}_+$, and hence of the first claim in (3.2).

Now suppose that $\|B(D-z_0)^{-1}\|\geq 1$. Then we choose $\mu > \|B(D-z_0)^{-1}\|$ and define

$$\mathcal{M}_\mu := \begin{pmatrix} \frac{1}{\sqrt{\mu}}I_{\mathcal{H}_+} & 0 \\ 0 & \sqrt{\mu}I_{\mathcal{H}_-} \end{pmatrix}, \quad \mathcal{U}_\mu := \mathcal{M}_\mu\mathcal{U}\mathcal{M}_\mu^{-1} = \begin{pmatrix} U_{11} & \frac{1}{\mu}U_{12} \\ \mu U_{21} & U_{22} \end{pmatrix}, \quad \mathcal{A}_\mu := \mathcal{M}_\mu\mathcal{A}\mathcal{M}_\mu^{-1} = \begin{pmatrix} A & \frac{1}{\mu}B \\ -\mu B^* & D \end{pmatrix}.$$

Note that, although $\mathcal{A}_\mu$ is no longer $\mathcal{J}$-symmetric and $\mathcal{U}$ no longer $\mathcal{J}$-unitary, they are still related by the formula for the Cayley transform and its inverse, i.e.

$$\begin{aligned}\mathcal{U}_\mu = \mathcal{M}_\mu(I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{A} - z_0)^{-1})\mathcal{M}_\mu^{-1} &= I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{M}_\mu \mathcal{A}\mathcal{M}_\mu^{-1} - z_0)^{-1} \\ &= I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{A}_\mu - z_0)^{-1}, \end{aligned} \tag{3.12}$$

$$\begin{aligned}\mathcal{A}_\mu = \mathcal{M}_\mu(z_0 I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{U} - I_\mathcal{H})^{-1})\mathcal{M}_\mu^{-1} &= z_0 I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{M}_\mu \mathcal{U}\mathcal{M}_\mu^{-1} - I_\mathcal{H})^{-1} \\ &= z_0 I_\mathcal{H} + (z_0 - \overline{z_0})(\mathcal{U}_\mu - I_\mathcal{H})^{-1}. \end{aligned} \tag{3.13}$$

Further, $\mathcal{U}\mathcal{L}_+ = \mathcal{L}_+$ implies that $\mathcal{L}_{+,\mu} := \mathcal{M}_\mu \mathcal{L}_+$ satisfies $\mathcal{U}_\mu \mathcal{L}_{+,\mu} = \mathcal{L}_{+,\mu}$. In the same way as in the proof of the first case, we can now prove the analogue of (3.7) for $\mathcal{U}_\mu$, $\mathcal{A}_\mu$ and $\mathcal{L}_{+,\mu}$, i.e.

$$(\mathcal{U}_\mu - I_\mathcal{H})\mathcal{L}_{+,\mu} = \mathcal{L}_{+,\mu} \cap \operatorname{dom} \mathcal{A}_\mu, \tag{3.14}$$

if we use that $(I_{\mathcal{H}_+} - \frac{1}{\mu}B(D - z_0)^{-1}K)^{-1}$ exists since $\mu > \|B(D - z_0)^{-1}\| \geq \|B(D - z_0)^{-1}K\|$. Now (3.14) and (3.12) show that $(z_0 - \overline{z_0})(\mathcal{A}_\mu - z_0)^{-1}\mathcal{L}_{+,\mu} = \mathcal{L}_{+,\mu} \cap \operatorname{dom} \mathcal{A}_\mu$ and hence

$$\mathcal{A}_\mu(\mathcal{L}_{+,\mu} \cap \operatorname{dom} \mathcal{A}_\mu) \subset \mathcal{L}_{+,\mu}. \tag{3.15}$$

If we apply $\mathcal{M}_\mu^{-1}$ from the left to (3.15) and use that $\mathcal{M}_\mu^{-1}\mathcal{A}_\mu = \mathcal{A}\mathcal{M}_\mu^{-1}$, $\operatorname{dom} \mathcal{A}_\mu = \mathcal{M}_\mu \operatorname{dom} \mathcal{A}$, $\mathcal{L}_{+,\mu} = \mathcal{M}_\mu \mathcal{L}_+$, we obtain (3.1).

In a similar way as above, we can show that $\overline{\mathcal{L}_{+,\mu} \cap \operatorname{dom} \mathcal{A}_\mu} = \mathcal{L}_{+,\mu}$ using that $I_{\mathcal{H}_+} - \mu B(D - z_0)^{-1}K$ is bijective by the choice of μ. Again, applying $\mathcal{M}_\mu^{-1}$ and noting $\mathcal{L}_{+,\mu} = \mathcal{M}_\mu \mathcal{L}_+$, $\operatorname{dom} \mathcal{A}_\mu = \mathcal{M}_\mu \operatorname{dom} \mathcal{A}$, we obtain the first claim in (3.2). Further, we can prove the analogue of (3.11) and note that the first Schur complement $S_{1,\mu}$ of $\mathcal{A}_\mu$ coincides with S_1,

$$S_{1,\mu}(z_0) = A - z_0 + \frac{1}{\mu}B(D - z_0)^{-1}\mu B^* = S_1(z_0),$$

to conclude that

$$\begin{aligned}P_+\big(\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}\big) &= \frac{1}{\sqrt{\mu}}P_+(\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}) = P_+\big(\mathcal{M}_\mu \mathcal{L}_+ \cap \mathcal{M}_\mu \operatorname{dom} \mathcal{A}\big) \\ &= P_+(\mathcal{L}_{+,\mu} \cap \operatorname{dom} \mathcal{A}_\mu) = \operatorname{dom} S_{1,\mu}(z_0) = \operatorname{dom} S_1(z_0),\end{aligned}$$

and hence the second claim in (3.2).

ii) In order to show that $\mathcal{L}_-$ is an invariant subspace for $\mathcal{A}$ with (3.3), (3.4) if (R′) and (C′) hold, we proceed in the same way as in case i) using Proposition 2.2 ii).

To prove the last claims in Theorem 3.1 for $\mathcal{L}_+$ in case i) and for $\mathcal{L}_-$ in case ii), we use the spectral mapping theorem for closed linear operators, see [5, Thm. IX.2.3 (i)]. This, together with the second identity in (2.6), implies that, for $\lambda \in \mathbb{C}$, $\lambda \neq z_0$,

$$\lambda \in \sigma\big(\mathcal{A}|_{\mathcal{L}_\pm}\big) \iff \frac{1}{\lambda - z_0} \in \sigma\big((\mathcal{A} - z_0)^{-1}|_{\mathcal{L}_\pm}\big) \iff \frac{\lambda - \overline{z_0}}{\lambda - z_0} \in \sigma(\mathcal{U}|_{\mathcal{L}_\pm}), \tag{3.16}$$

and analogously for $\mathcal{A}$ and $\mathcal{U}$. Now all claims follow from the corresponding claims for $\mathcal{U}$ in Proposition 2.3. □

Proof of Remark 3.2. Let z, $z' \in \mathbb{C} \setminus \mathbb{R}$ be such that $\operatorname{ran} \overline{S_1(z)} = \mathcal{H}_+$, $\operatorname{ran} \overline{S_1(z')} = \mathcal{H}_+$. Then, as was shown above, $\overline{S_1(z)}$, $\overline{S_1(z')}$ are boundedly invertible and we can write

$$\begin{aligned}
&\frac{1}{z-z'}\Big(\overline{S_1(z)}^{-1}B(D-z)^{-1}-\overline{S_1(z')}^{-1}B(D-z')^{-1}\Big)\\
&=\frac{1}{z-z'}\Big(\Big(\overline{S_1(z)}^{-1}-\overline{S_1(z')}^{-1}\Big)B(D-z)^{-1}+\overline{S_1(z')}^{-1}\big(B(D-z)^{-1}-B(D-z')^{-1}\big)\Big)\\
&=\overline{S_1(z')}^{-1}\Big(I_{\mathcal{H}_+}+B(D-z')^{-1}(D-z)^{-1}B^*\Big)\overline{S_1(z)}^{-1}B(D-z)^{-1}+\overline{S_1(z')}^{-1}B(D-z')^{-1}(D-z)^{-1}.
\end{aligned}$$

This shows that if $\overline{S_1(z)}^{-1}B(D-z)^{-1}$ is compact, then the operator

$$\begin{aligned}
\frac{1}{z-z'}&\overline{S_1(z')}^{-1}B(D-z')^{-1}+\overline{S_1(z')}^{-1}B(D-z')^{-1}(D-z)^{-1}\\
&=\frac{1}{z-z'}\overline{S_1(z')}^{-1}B(D-z')^{-1}\big(I_{\mathcal{H}_+}+(z-z')(D-z)^{-1}\big)\\
&=\frac{1}{z-z'}\overline{S_1(z')}^{-1}B(D-z')^{-1}(D-z')(D-z)^{-1}
\end{aligned}$$

is compact as well. Since the product $(D-z')(D-z)^{-1}$ is boundedly invertible, we conclude that $\overline{S_1(z')}^{-1}B(D-z')^{-1}$ is compact. □

In the following we consider the essential spectra of $\mathcal{A}$ and of its restrictions $\mathcal{A}|_{\mathcal{L}_+}$, $\mathcal{A}|_{\mathcal{L}_-}$. If the operator matrix $\mathcal{A}_0$ in (2.1) satisfies $\operatorname{dom} D\subset\operatorname{dom} B$ or $\operatorname{dom} A\subset\operatorname{dom} B^*$, the essential spectrum of $\mathcal{A}=\overline{\mathcal{A}_0}$ can always be described as

$$\begin{aligned}
\sigma_{\mathrm{e}}(\mathcal{A})&=\sigma_{\mathrm{e}}(D)\cup\big\{\lambda\in\varrho(D):0\in\sigma_{\mathrm{e}}\big(\overline{S_1(\lambda)}\big)\big\}\quad\text{if }\operatorname{dom} D\subset\operatorname{dom} B,\\
\sigma_{\mathrm{e}}(\mathcal{A})&=\sigma_{\mathrm{e}}(A)\cup\big\{\lambda\in\varrho(A):0\in\sigma_{\mathrm{e}}\big(\overline{S_2(\lambda)}\big)\big\}\quad\text{if }\operatorname{dom} A\subset\operatorname{dom} B^*;
\end{aligned}\tag{3.17}$$

this follows from the Schur-Frobenius factorization, see e.g. [23, (2.2.11), Thm. 2.4.6] for (3.17). The assumptions (R), (C) or (R′), (C′) in Theorem 3.1, ensure that $\sigma_{\mathrm{e}}(\mathcal{A})$ is real and outside of it $\sigma(\mathcal{A})$ is discrete. More can be said under an additional compactness condition.

Corollary 3.3. i) *Suppose that in Theorem* 3.1 i)*, in addition,* $\overline{S_1(z_0)}^{-1}-(A-z_0)^{-1}$ *is compact. Then* $\sigma_{\mathrm{e}}(\mathcal{A})=\sigma_{\mathrm{e}}(A)\cup\sigma_{\mathrm{e}}(D)\,(\subset\mathbb{R})$*, outside of this set* $\sigma(\mathcal{A})$ *is discrete, and if* $\mathcal{L}_+$ *is an invariant subspace of* $\mathcal{A}$ *as in Theorem* 3.1 i)*, then*

$$\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_+}\big)=\sigma_{\mathrm{e}}(A).$$

ii) *Suppose that in Theorem* 3.1 ii)*, in addition,* $\overline{S_2(z_0)}^{-1}-(D-z_0)^{-1}$ *is compact. Then* $\sigma_{\mathrm{e}}(\mathcal{A})=\sigma_{\mathrm{e}}(A)\cup\sigma_{\mathrm{e}}(D)\,(\subset\mathbb{R})$*, outside of this set* $\sigma(\mathcal{A})$ *is discrete, and if* $\mathcal{L}_-$ *is an invariant subspace of* $\mathcal{A}$ *as in Theorem* 3.1 ii)*, then*

$$\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_-}\big)=\sigma_{\mathrm{e}}(D).$$

Proof. We prove i); the proof of ii) is analogous. The first claim is immediate from (3.17) and the additional compactness assumption. Since $\sigma_{\mathrm{e}}(\mathcal{A})\subset\mathbb{R}$ and, due to assumption (R) in Theorem 3.1 i), there is $z_0\in\mathbb{C}\setminus\mathbb{R}$ with z_0, $\overline{z_0}\in\varrho(\mathcal{A})$, the second claim follows from [9, Thm. XVII.2.1].

Now we prove that $\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_+}\big)=\sigma_{\mathrm{e}}(A)$. The spectral mapping theorem for essential spectra, see [5, Thm. IX.2.3 (iii)], yields that, for $\lambda\in\mathbb{C}$, $\lambda\neq z_0$,

$$\lambda\in\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_+}\big)\iff\mu(\lambda):=\frac{\lambda-\overline{z_0}}{\lambda-z_0}\in\sigma_{\mathrm{e}}(\mathcal{U}|_{\mathcal{L}_+}),\tag{3.18}$$

comp. (3.16). Let $\mathcal{U} =: (U_{ij})_{i,j=1}^2$ be the matrix representation of $\mathcal{U}$ in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ according to Proposition 2.2 i). The operator $Q_+ : \mathcal{L}_+ \to \mathcal{H}_+$, $Q_+\binom{x}{Kx} := x$, $x \in \mathcal{H}_+$, is bijective, $\mathcal{U}\mathcal{L}_+ = \mathcal{L}_+$ and hence $Q_+\mathcal{U}|_{\mathcal{L}_+}Q_+^{-1} = U_{11} + U_{12}K$ in $\mathcal{H}_+$. This, the compactness of U_{12} by condition (C) and the additional compactness assumption above imply that

$$\sigma_e(\mathcal{U}|_{\mathcal{L}_+}) = \sigma_e(U_{11}+U_{12}K) = \sigma_e(U_{11}) = \sigma_e\big(I_{\mathcal{H}_+} + (z_0-\overline{z_0})\overline{S_1(z_0)}^{-1}\big)$$
$$= \sigma_e\big(I_{\mathcal{H}_+} + (z_0-\overline{z_0})(A-z_0)^{-1}\big) = \sigma_e(U_A)$$

where U_A is the Cayley transform of the selfadjoint operator A in $\mathcal{H}_+$. Now the claim for $\sigma_e(\mathcal{A}|_{\mathcal{L}_+})$ follows from (3.18) and the analogous mapping property for $\sigma_e(A)$ and $\sigma_e(U_A)$. □

4. Diagonally dominant case

In this section we consider *diagonally dominant* operator matrices $\mathcal{A}_0$ introduced in [22], see also [23, Def. 2.2.1], i.e. we assume that $\mathcal{A}_0$ in (2.1) satisfies

$$\operatorname{dom} A \subset \operatorname{dom} B^* \quad \text{and} \quad \operatorname{dom} D \subset \operatorname{dom} B, \tag{4.1}$$

and hence $\operatorname{dom}\mathcal{A}_0 = \operatorname{dom} A \oplus \operatorname{dom} D$. In this case we establish more explicit and elegant assumptions to ensure the existence of maximal $\mathcal{J}$-semi-definite invariant subspaces of $\mathcal{A} = \overline{\mathcal{A}_0}$. Moreover, here we can show the existence of both a maximal $\mathcal{J}$-non-negative and a maximal $\mathcal{J}$-non-positive invariant subspace $\mathcal{J}$-orthogonal to each other. The latter allows us to prove a new result on the accumulation of non-real eigenvalues at the real axis.

To achieve the optimal form of our assumptions, we need the following general criterion for symmetric and $\mathcal{J}$-symmetric operator matrices to be selfadjoint and $\mathcal{J}$-selfadjoint, respectively; it extends [23, Thm. 2.6.6], where $\mathcal{A}_0$ was assumed to be diagonally dominant of order $\delta := \max\{\delta_A, \delta_D\} < 1$, to the case where only $\delta_A\delta_D < 1$.

Proposition 4.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and let $\mathbb{A}$ be a diagonally dominant operator matrix in $\mathcal{H}_1 \oplus \mathcal{H}_2$, see (4.1), of the form*

$$\mathbb{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad C = B^* \text{ or } C = -B^*, \quad \operatorname{dom}\mathbb{A} := \operatorname{dom} A \oplus \operatorname{dom} D, \tag{4.2}$$

with selfadjoint operators A and D in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and a closable operator B from $\mathcal{H}_1$ to $\mathcal{H}_2$. If the A-bound δ_A of B^* and the D-bound δ_D of B satisfy

$$\delta_A\delta_D < 1, \tag{4.3}$$

then $\mathbb{A}$ is selfadjoint if $C = B^*$, and $\mathbb{A}$ is $\mathcal{J}$-selfadjoint if $C = -B^*$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ with respect to $\mathcal{J} :=$ $\operatorname{diag}(I_{\mathcal{H}_1}, -I_{\mathcal{H}_2})$; this holds, in particular, if one relative bound is 0, i.e. $\delta_A = 0$ or $\delta_D = 0$, or if either B is D-compact or B^* is A-compact.

Moreover, if $C = -B^*$, the spectrum of $\mathbb{A}$ lies in a hyperbolic region around the real axis, i.e. there exist α, $\beta \geq 0$ such that

$$\sigma(\mathbb{A}) \subset \big\{z \in \mathbb{C} : |\operatorname{Im} z|^2 \leq \alpha + \beta(\operatorname{Re} z)^2\big\}. \tag{4.4}$$

Proof. Since $\mathbb{A}$ is symmetric if $C = B^*$ and $\mathcal{J}$-symmetric if $C = -B^*$, it suffices to show (4.4). Here, in the $\mathcal{J}$-symmetric case, (4.4) implies that the Cayley transform $\mathcal{U}$ of $\mathbb{A}$ given by (2.6) exists and is $\mathcal{J}$-unitary and hence $\mathbb{A}$ is $\mathcal{J}$-selfadjoint, see [3, Thm. VI.7.1, VI.7.2].

If (4.3) is satisfied, there are two cases. If both $\delta_A < 1$ and $\delta_D < 1$, then $\mathbb{A}$ is diagonally dominant of order < 1 and (4.4) follows from [4, Thm. 2.1 i)]. Otherwise, either (i) $\delta_A > 1$ and $\delta_D < 1$ or (ii) $\delta_A < 1$ and $\delta_D > 1$. We consider case (i), the proof for case (ii) is analogous. By (4.3), in case (i) we can choose $\mu > 0$ such that

$$\delta_D < \mu < \frac{1}{\delta_A} \tag{4.5}$$

and define the transformed (not symmetric or $\mathcal{J}$-symmetric) operator matrix $\mathbb{A}_\mu$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ by

$$\mathbb{A}_\mu := \mathcal{M}_\mu \mathbb{A} \mathcal{M}_\mu^{-1} = \begin{pmatrix} A & \frac{1}{\mu}B \\ \pm\mu B^* & D \end{pmatrix}, \quad \mathcal{M}_\mu := \begin{pmatrix} \frac{1}{\sqrt{\mu}} I_{\mathcal{H}_1} & 0 \\ 0 & \sqrt{\mu} I_{\mathcal{H}_2} \end{pmatrix}. \tag{4.6}$$

Then, clearly, $z \in \varrho(\mathbb{A})$ if and only if $z \in \varrho(\mathbb{A}_\mu)$. Condition (4.5) ensures that the A-bound of μB^* and the D-bound of $\frac{1}{\mu}B$ satisfy $\mu\delta_A < 1$ and $\frac{1}{\mu}\delta_D < 1$. Hence $\mathbb{A}_\mu$ is diagonally dominant of order < 1 and (4.4) follows from [4, Thm. 2.1 i)].

It remains to be noted that, obviously, (4.3) is satisfied if either $\delta_A = 0$ or $\delta_D = 0$ and, further, that a linear operator between Hilbert spaces that is relatively compact with respect to a closable (here even closed) operator is relatively bounded with relative bound 0, see [5, Cor. III.7.7]. □

Theorem 4.2. *Let $\mathcal{A}_0$ be diagonally dominant, i.e.* $\operatorname{dom} A \subset \operatorname{dom} B^*$ *and* $\operatorname{dom} D \subset \operatorname{dom} B$*, and suppose that either*

$$B \text{ is } D\text{-compact} \quad \text{or} \quad B^* \text{ is } A\text{-compact}. \tag{4.7}$$

Then $\mathcal{A} = \mathcal{A}_0$ is $\mathcal{J}$-selfadjoint and $\mathcal{A}$ has invariant subspaces $\mathcal{L}_+$ and $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$, i.e.

$$\mathcal{A}(\mathcal{L}_\pm \cap \operatorname{dom} \mathcal{A}) \subset \mathcal{L}_\pm, \tag{4.8}$$

such that $\mathcal{L}_+$ is maximal $\mathcal{J}$-non-negative, $\mathcal{L}_-$ is maximal $\mathcal{J}$-non-positive, and

$$\overline{\mathcal{L}_\pm \cap \operatorname{dom} \mathcal{A}} = \mathcal{L}_\pm, \quad P_+(\mathcal{L}_+ \cap \operatorname{dom} \mathcal{A}) = \operatorname{dom} A, \quad P_-(\mathcal{L}_- \cap \operatorname{dom} \mathcal{A}) = \operatorname{dom} D. \tag{4.9}$$

Moreover, $\sigma_{\mathrm{e}}(\mathcal{A}) = \sigma_{\mathrm{e}}(A) \cup \sigma_{\mathrm{e}}(D)$, outside of this set $\sigma(\mathcal{A})$ is discrete, $\mathcal{L}_\pm$ satisfy (3.5), (3.6) for any decomposition $\sigma(\mathcal{A}) \setminus \mathbb{R} = \widehat{\sigma} \,\dot\cup\, \widehat{\sigma}^*$, and $\sigma(\mathcal{A}) \setminus \mathbb{R}$ may accumulate only at the intersection $\sigma_{\mathrm{e}}(A) \cap \sigma_{\mathrm{e}}(D)$.

Remark 4.3. If, in Theorem 4.2, at least one of A or D has compact resolvent, then $\sigma(\mathcal{A}) \setminus \mathbb{R}$ has no finite accumulation point.

Remark 4.4. If $\mathcal{A}_0$ in Theorem 4.2 satisfies the condition (L), i.e. A and B are bounded operators, $\mathcal{A} = \mathcal{A}_0$ is clearly diagonally dominant with $\operatorname{dom} \mathcal{A} = \operatorname{dom} \mathcal{A}_0 = \mathcal{H}_+ \oplus \operatorname{dom} D$. In this special case, the existence of the maximal $\mathcal{J}$-non-negative invariant subspace $\mathcal{L}_+$ of $\mathcal{A}$ with bounded restriction $\mathcal{A}|_{\mathcal{L}_+}$ was proved in [14], [12], [2]. Theorem 4.2 shows that there also exists a maximal $\mathcal{J}$-non-positive invariant subspace $\mathcal{L}_-$ of $\mathcal{A}$ for which the restriction $\mathcal{A}|_{\mathcal{L}_-}$ is only densely defined and unbounded if so is D, see (4.9).

Proof of Theorem 4.2. By assumption (4.7) and Proposition 4.1, $\mathcal{A}_0$ is $\mathcal{J}$-selfadjoint and hence, in particular, closed, i.e. $\mathcal{A} = \mathcal{A}_0$. In this case, $S_1(z)$ is closed with $\operatorname{dom} S_1(z) = \operatorname{dom} A$ and $S_2(z)$ is closed with $\operatorname{dom} S_2(z) = \operatorname{dom} D$, see [23, Rem. 2.2.13].

It suffices to consider the case that B is D-compact; the proof in the case that B^* is A-compact is completely analogous. If B is D-compact, then B is D-bounded with D-bound 0, see [5, Cor. II.7.7]. Since

$\operatorname{dom} A \subset \operatorname{dom} B^*$ and A is closed, B^* is A-bounded. Hence the two relative bounds δ_D of B and δ_A of B^* satisfy

$$\delta_D = \lim_{|\xi|\to\infty} \|B(D-\mathrm{i}\xi)^{-1}\| = 0, \quad \delta_A = \lim_{|\xi|\to\infty} \|B^*(A-\mathrm{i}\xi)^{-1}\| < \infty, \tag{4.10}$$

see [24, Satz 9.1].

Let $\delta \in (0,1)$ be arbitrary. By (4.10), there exists $\nu_0 > 0$ such that for $z = \pm\mathrm{i}\nu$ with $\nu > \nu_0$,

$$\|B(D-z)^{-1}\| < \min\left\{1, \frac{1}{\delta_A + \delta}\right\}, \quad \|B^*(A-z)^{-1}\| < \delta_A + \delta. \tag{4.11}$$

Then $\|B(D-z)^{-1}\|\,\|B^*(A-z)^{-1}\| < 1$ and therefore

$$\overline{S_1(z)} = S_1(z) = \big(I_{\mathcal{H}_+} + B(D-z)^{-1}B^*(A-z)^{-1}\big)(A-z), \quad \operatorname{dom} \overline{S_1(z)} = \operatorname{dom} A, \tag{4.12}$$

$$\overline{S_2(z)} = S_2(z) = \big(I_{\mathcal{H}_-} + B^*(A-z)^{-1}B(D-z)^{-1}\big)(D-z), \quad \operatorname{dom} \overline{S_2(z)} = \operatorname{dom} D, \tag{4.13}$$

are boundedly invertible for $z = \pm\mathrm{i}\nu$ with $\nu > \nu_0$ and hence both range conditions (R) and (R′) are satisfied for $z_0 = \mathrm{i}\nu$ with $\nu > \nu_0$. Since B is D-compact, $\overline{S_1(z_0)}^{-1} B(D-z_0)^{-1}$ is compact, as required in assumption (C). This proves that $\mathcal{A}$ satisfies all conditions of Theorem 3.1 i).

Since $\mathcal{A}_0$ is diagonally dominant, i.e. $\operatorname{dom} A \subset \operatorname{dom} B^*$, $\operatorname{dom} D \subset \operatorname{dom} B$, we can use both matrix representations of $\mathcal{U}$ in Proposition 2.2. Comparing their left lower entries, we find that

$$\overline{S_2(z_0)}^{-1} B^*(A-z_0)^{-1} = (D-z_0)^{-1} B^* \overline{S_1(z_0)}^{-1}. \tag{4.14}$$

Since $\overline{S_1(z_0)}^{-1}$ is bounded and $(D-z_0)^{-1}B^* \subset (B(D-\overline{z_0})^{-1})^*$ is compact by assumption, (4.14) shows that $\overline{S_2(z_0)}^{-1} B^*(A-z_0)^{-1}$ is compact as well, as required in assumption (C′). This proves that $\mathcal{A}$ also satisfies the conditions of Theorem 3.1 ii). Now the existence of both $\mathcal{L}_+$ and $\mathcal{L}_-$ and all claims for them follow from Theorem 3.1 i) and ii) if we observe the domain equalities in (4.12), (4.13) for the last two identities in (4.9).

To prove the claims for the essential spectrum and the spectrum of $\mathcal{A}$ we first show that the additional assumption in Corollary 3.3 is satisfied, i.e. that $\overline{S_1(z_0)}^{-1} - (A-z_0)^{-1}$ is compact. For $z_0 \in \mathbb{C} \setminus \mathbb{R}$ chosen as above, $\overline{S_1(z_0)}^{-1} = S_1(z_0)^{-1}$ and by (4.12) we can write, with $L := (A-z_0)^{-1}B(D-z_0)^{-1}B^*$,

$$S_1(z_0)^{-1} - (A-z_0)^{-1} = \big((I_{\mathcal{H}_+} + L)^{-1} - I_{\mathcal{H}_+}\big)(A-z_0)^{-1} = \big(I_{\mathcal{H}_+} + L\big)^{-1} L (A-z_0)^{-1}. \tag{4.15}$$

Since $\mathcal{A}$ is diagonally dominant and assumption (4.7) holds, one of the factors $B(D-z_0)^{-1}$ or $B^*(A-z_0)^{-1}$ is compact while the other is bounded. This shows that L is compact and hence, by (4.15), $S_1(z_0)^{-1} - (A-z_0)^{-1}$ is compact, as required. Now the first two claims on the essential spectrum and the spectrum follow directly from Corollary 3.3.

In order to prove the claim on the accumulation of the non-real spectrum $\sigma(\mathcal{A}) \setminus \mathbb{R}$, which is symmetric to $\mathbb{R}$, it is sufficient to consider the spectrum of $\mathcal{A}$ e.g. in the open upper half-plane $\mathbb{C}_+$. Combining the first identity in (3.5) with $\widehat{\sigma} = \sigma(\mathcal{A}) \cap \mathbb{C}_+$ for $\mathcal{L}_+$ and $\mathcal{L}_-$, we conclude that

$$\sigma(\mathcal{A}) \cap \mathbb{C}_+ = \sigma\big(\mathcal{A}|_{\mathcal{L}_+}\big) \cap \mathbb{C}_+ = \sigma\big(\mathcal{A}|_{\mathcal{L}_-}\big) \cap \mathbb{C}_+. \tag{4.16}$$

For the set in the middle in (4.16), Corollary 3.3 shows that $\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_+}\big) = \sigma_{\mathrm{e}}(A) \subset \mathbb{R}$. By (4.16), this implies that $\sigma_{\mathrm{e}}(\mathcal{A}) \cap \mathbb{C}_+ = \emptyset$ and that $\sigma(\mathcal{A}) \cap \mathbb{C}_+ = \sigma\big(\mathcal{A}|_{\mathcal{L}_+}\big) \cap \mathbb{C}_+$ may accumulate at most at $\sigma_{\mathrm{e}}(A)$. Analogously, using (3.6), we can prove that $\sigma_{\mathrm{e}}\big(\mathcal{A}|_{\mathcal{L}_-}\big) = \sigma_{\mathrm{e}}(D) \subset \mathbb{R}$. Then, by (4.16), $\sigma_{\mathrm{e}}(\mathcal{A}) \cap \mathbb{C}_+ = \emptyset$ and so $\sigma(\mathcal{A}) \cap \mathbb{C}_+ = \sigma\big(\mathcal{A}|_{\mathcal{L}_-}\big) \cap \mathbb{C}_+$ may accumulate at most at $\sigma_{\mathrm{e}}(D)$, and thus, by what was shown above, in $\sigma_{\mathrm{e}}(A) \cap \sigma_{\mathrm{e}}(D)$. $\square$

5. Corner dominant case

In this section we consider operator matrices $\mathcal{A}_0$ of the form (2.1) which we call *corner dominant*. This means that one of the diagonal elements dominates both column- and rowwise which, in our case, amounts to

$$\operatorname{dom}|A|^{1/2} \subset \operatorname{dom} B^* \quad \text{or} \quad \operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B;$$

the term 'corner dominant' refers to the property that, if e.g. $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$, the product $B(D-\lambda)^{-1}B^*$ is bounded on $\operatorname{dom} B^*$, see the proof of Theorem 5.1. Unlike the diagonally dominant case, even if the dominance is strong, $\mathcal{A}_0$ with its original domain need not be closed.

Theorem 5.1. *Let $\mathcal{A}_0$ be corner dominant, i.e.* $\operatorname{dom}|A|^{1/2} \subset \operatorname{dom} B^*$ *or* $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$.

i) *If* $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$*, suppose that* $\operatorname{dom} B^* \cap \operatorname{dom} A$ *is a core of A and that*

$$B \text{ is } D\text{-compact.} \tag{5.1}$$

Then $\mathcal{A} = \overline{\mathcal{A}_0}$ is $\mathcal{J}$-selfadjoint and $\mathcal{A}$ has a maximal $\mathcal{J}$-non-negative invariant subspace $\mathcal{L}_+$, i.e.

$$\mathcal{A}(\mathcal{L}_+ \cap \operatorname{dom}\mathcal{A}) \subset \mathcal{L}_+, \tag{5.2}$$

and

$$\overline{\mathcal{L}_+ \cap \operatorname{dom}\mathcal{A}} = \mathcal{L}_+, \quad P_+(\mathcal{L}_+ \cap \operatorname{dom}\mathcal{A}) = \operatorname{dom} A. \tag{5.3}$$

Moreover, the non-real spectrum $\sigma(\mathcal{A}) \setminus \mathbb{R}$ is discrete and $\mathcal{L}_+$ can be chosen such that (3.5) holds for any decomposition $\sigma(\mathcal{A}) \setminus \mathbb{R} = \widehat{\sigma} \,\dot\cup\, \widehat{\sigma}^*$. If B is even $|D|^{1/2}$-compact, then $\sigma_e(\mathcal{A}) = \sigma_e(A) \cup \sigma_e(D)$, outside of this set $\sigma(\mathcal{A})$ is discrete, and $\sigma(\mathcal{A}) \setminus \mathbb{R}$ may only accumulate at $\sigma_e(A)$.

ii) *If* $\operatorname{dom}|A|^{1/2} \subset \operatorname{dom} B^*$*, suppose that* $\operatorname{dom} B \cap \operatorname{dom} D$ *is a core of D and that*

$$B^* \text{ is } A\text{-compact.} \tag{5.4}$$

Then $\mathcal{A} = \overline{\mathcal{A}_0}$ is $\mathcal{J}$-selfadjoint and $\mathcal{A}$ has a maximal $\mathcal{J}$-non-positive invariant subspace $\mathcal{L}_-$, i.e.

$$\mathcal{A}(\mathcal{L}_- \cap \operatorname{dom}\mathcal{A}) \subset \mathcal{L}_-,$$

and

$$\overline{\mathcal{L}_- \cap \operatorname{dom}\mathcal{A}} = \mathcal{L}_-, \quad P_-(\mathcal{L}_- \cap \operatorname{dom}\mathcal{A}) = \operatorname{dom} D. \tag{5.5}$$

Moreover, the non-real spectrum $\sigma(\mathcal{A}) \setminus \mathbb{R}$ is discrete and $\mathcal{L}_-$ can be chosen such that (3.6) holds for any decomposition $\sigma(\mathcal{A}) \setminus \mathbb{R} = \widehat{\sigma} \,\dot\cup\, \widehat{\sigma}^*$. If B^* is even $|A|^{1/2}$-compact, then $\sigma_e(\mathcal{A}) = \sigma_e(A) \cup \sigma_e(D)$, outside of this set $\sigma(\mathcal{A})$ is discrete, and $\sigma(\mathcal{A}) \setminus \mathbb{R}$ may only accumulate at $\sigma_e(D)$.

Remark 5.2. i) If B is $|D|^{1/2}$-compact, $\operatorname{dom} B^* \cap \operatorname{dom} A$ is a core of A and A has compact resolvent, then $\sigma(\mathcal{A}) \setminus \mathbb{R}$ has no finite accumulation point.

ii) If B^* is $|A|^{1/2}$-compact, $\operatorname{dom} B \cap \operatorname{dom} D$ is a core of D and D has compact resolvent, then $\sigma(\mathcal{A}) \setminus \mathbb{R}$ has no finite accumulation point.

Proof of Theorem 5.1. In both cases i) and ii) $\mathcal{J}\mathcal{A}_0$ is essentially selfadjoint due to [23, Prop. 2.3.6] and its analogue in case i). Since $\mathcal{J}$ is boundedly invertible, it follows that $\mathcal{J}\mathcal{A} = \mathcal{J}\overline{\mathcal{A}_0} = \overline{\mathcal{J}\mathcal{A}_0}$ is selfadjoint, i.e. $\mathcal{A} = \overline{\mathcal{A}_0}$ is $\mathcal{J}$-selfadjoint.

i) First we show that the condition $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$ implies that $\overline{S_1(z_0)}$, $\overline{S_1(\overline{z_0})}$ are boundedly invertible for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$. To this end, let $z \in \mathbb{C} \setminus \mathbb{R}$. Using the generalized polar decomposition of $(D-z)^{-1}$, see [8, Thm. 2.7] with $\alpha = 1/2$, there exists a unitary operator V_z in $\mathcal{H}_-$ such that we can write

$$B(D-z)^{-1}B^* = B\,|(D-\overline{z})^{-1}|^{1/2}V_z|(D-z)^{-1}|^{1/2}B^* = B\,|D-z|^{-1/2}V_z|D-z|^{-1/2}B^*; \tag{5.6}$$

here we have used the functional calculus for D with $f_z(t) := |t-z|^{-1/2} = f_{\overline{z}}(t)$, $t \in \mathbb{R}$, see [20, Sect. IX.128]. It is not difficult to see, e.g. using the spectral theorem, that $\operatorname{dom}|D-z|^{1/2} = \operatorname{dom}|D|^{1/2}$. Hence the assumption $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$ implies that $B\,|D-z|^{-1/2}$ is bounded and hence so is $|D-z|^{-1/2}B^* \subset (B\,|D-\overline{z}|^{-1/2})^* = (B\,|D-z|^{-1/2})^*$ on $\operatorname{dom} B^*$. Since $\operatorname{dom} B^* \cap \operatorname{dom} A$ is a core of A by assumption, it follows that, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\overline{S_1(z)} = A - z + \overline{B(D-z)^{-1}B^*} = \big(I_{\mathcal{H}_+} + \overline{B(D-z)^{-1}B^*}(A-z)^{-1}\big)(A-z), \quad \operatorname{dom}\overline{S_1(z)} = \operatorname{dom} A. \tag{5.7}$$

In order to bound $\|B(D-z)^{-1}B^*\|$, we fix some $\xi_0 > 0$, use that B is $|D|^{1/2}$-bounded by assumption and estimate

$$\big\|B\,|D-z|^{-1/2}\big\| \le \big\|B\,(|D|^{1/2} - \mathrm{i}\xi_0)^{-1}\big\|\,\big\|(|D|^{1/2} - \mathrm{i}\xi_0)|D-z|^{-1/2}\big\|. \tag{5.8}$$

It is not difficult to check that, for $z = \mathrm{i}\nu$ with $\nu \in \mathbb{R} \setminus \{0\}$,

$$\big\|(|D|^{1/2} - \mathrm{i}\xi_0)|D-z|^{-1/2}\big\| = \sup_{t\in\sigma(D)} \frac{\big||t|^{1/2} - \mathrm{i}\xi_0\big|}{|t-\mathrm{i}\nu|^{1/2}} = \sup_{t\in\sigma(D)} \frac{(|t| + \xi_0^2)^{1/2}}{(t^2+\nu^2)^{1/4}} \le \Big(1 + \frac{\xi_0^4}{\nu^2}\Big)^{1/4}; \tag{5.9}$$

here the supremum is attained at $|t| = \nu^2/\xi_0^2$. As a function of $|\nu| \in [0,\infty)$, the right hand side of (5.9) is monotonically decreasing to its limit 1 at ∞ and thus, by (5.6), (5.8) and (5.9),

$$\big\|\overline{B(D-z)^{-1}B^*}\big\| \le \big\|B(|D|^{1/2} - \mathrm{i}\xi_0)^{-1}\big\|^2\Big(1 + \frac{\xi_0^4}{\nu^2}\Big)^{1/2} \le \big\|B(|D|^{1/2} - \mathrm{i}\xi_0)^{-1}\big\|^2\big(1+\xi_0^2\big)^{1/2}, \quad z = \mathrm{i}\nu, \ |\nu| \ge \xi_0.$$

If we now choose $\nu_0 > \xi_0$ so that $\nu_0 > \big\|B\,(|D|^{1/2} - \mathrm{i}\xi_0)^{-1}\big\|^2\big(1+\xi_0^2\big)^{1/2}$, we conclude that

$$\big\|\overline{B(D-z)^{-1}B^*}(A-z)^{-1}\big\| \le \big\|B\,(|D|^{1/2} - \mathrm{i}\xi_0)^{-1}\big\|^2\big(1+\xi_0^2\big)^{1/2}\frac{1}{|\nu|} < 1, \quad z = \mathrm{i}\nu, \ |\nu| \ge \nu_0 \ (> \xi_0).$$

Thus, by (5.7), $\overline{S_1(z_0)}$, $\overline{S_1(\overline{z_0})}$ are boundedly invertible for $z_0 = \mathrm{i}\nu$ with $|\nu| \ge \nu_0$ and hence condition (R) in Theorem 3.1 i) holds.

Clearly, since B is D-compact, the product $\overline{S_1(z_0)}^{-1}B(D-z_0)^{-1}$ is compact for $z_0 = \mathrm{i}\nu$ with $|\nu| \ge \nu_0$ and hence condition (C) in Theorem 3.1 i) holds as well. Now Theorem 3.1 i) yields the existence of $\mathcal{L}_+$, its properties and also that $\sigma(\mathcal{A}) \setminus \mathbb{R}$ is discrete; here we obtain the second identity in (5.3) from (3.2) using the domain equality in (5.7).

To prove the further claims on the spectrum of $\mathcal{A}$ we assume the stronger condition that B is even $|D|^{1/2}$-compact, which implies both that $\operatorname{dom}|D|^{1/2} \subset \operatorname{dom} B$ and that B is D-compact. First we show that the assumption that B is $|D|^{1/2}$-compact implies that $B|D-z_0|^{-1/2}$ is compact for all $z_0 \in \mathbb{C} \setminus \mathbb{R}$. In fact, for any $z \in \mathbb{C} \setminus \mathbb{R}$, we can write

$$B|D-z_0|^{-1/2} = B(|D|^{1/2} - z)^{-1}(|D|^{1/2} - z)|D-z_0|^{-1/2}. \tag{5.10}$$

Since $\operatorname{dom}|D-z_0|^{1/2}=\operatorname{dom}|D|^{1/2}$, the product $(|D|^{1/2}-z)|D-z_0|^{-1/2}$ is closed and everywhere defined, whence bounded. Thus, if $B(|D|^{1/2}-z)^{-1}$ is compact, so is $B|D-z_0|^{-1/2}$ by (5.10).

Next we show that the additional assumption in Corollary 3.3, i.e. that $\overline{S_1(z_0)}^{-1}-(A-z_0)^{-1}$ is compact, is satisfied. By (5.7),

$$\overline{S_1(z_0)}-(A-z_0)=\overline{B(D-z_0)^{-1}B^*}\Big|_{\operatorname{dom}A}$$

is bounded, densely defined and compact by (5.6) since B is $|D-z_0|^{1/2}$-compact. This implies that also the difference of inverses

$$\overline{S_1(z_0)}^{-1}-(A-z_0)^{-1}=\overline{S_1(z_0)}^{-1}\big(A-z_0-\overline{S_1(z_0)}\,\big)(A-z_0)^{-1}$$

is compact, as required. Now the two assertions on the essential spectrum and the spectrum follow directly from Corollary 3.3.

In order to prove the claim on the accumulation of the non-real spectrum $\sigma(\mathcal{A})\backslash\mathbb{R}$, we use the symmetry of $\sigma(\mathcal{A})$ with respect to $\mathbb{R}$, (3.5) with $\widehat{\sigma}=\sigma(\mathcal{A})\cap\mathbb{C}_+$ and Corollary 3.3.

ii) If assumption (5.4) is satisfied, we reverse the ordering of the components of $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ and apply i). □

6. Example: dissipative two-channel Hamiltonians

In this section we illustrate Theorem 4.2 by an application to Hamiltonians of a non-relativistic two-channel potential scattering model in $\mathbb{R}^d$ with dissipation.

The Hamiltonian $H_c=-\Delta+a|x|-\frac{b}{|x|}$, a, $b>0$, in the confined channel which governs the relative motion between two permanently confined particles, e.g. a quark and an antiquark, is considered with a 'Coulomb-plus-linear' potential, see [1, Sect. 3.4], while $H_s=-\Delta$ in the scattering channel is assumed to be the free Schrödinger operator. The communication between the two channels is represented by the off-diagonal complex-valued potentials V_{12}, V_{21} for which we assume that $V_{21}=-\overline{V_{12}}$. This condition makes the two-channel Hamiltonian non-selfadjoint and, in the special case that V_{12} is real-valued, accretive and hence $-\mathrm{i}\mathscr{H}$ dissipative, see [7, Sect. 4.3] for the scalar case.

In the Hilbert space $L^2(\mathbb{R}^d,\mathbb{C})\oplus L^2(\mathbb{R}^d,\mathbb{C})$, we consider the operator $\mathscr{H}$ given by

$$\mathscr{H}:=\begin{pmatrix}H_c & V_{12}\\ -\overline{V_{12}} & H_s\end{pmatrix}=\begin{pmatrix}-\Delta+a|x|-\dfrac{b}{|x|} & V_{12}\\ -\overline{V_{12}} & -\Delta\end{pmatrix},\quad \operatorname{dom}\mathscr{H}=\big(\operatorname{dom}H_c\cap\operatorname{dom}V_{12}\big)\oplus\big(\operatorname{dom}H_s\cap\operatorname{dom}V_{12}\big).$$

Here all masses and the Planck constant $\hbar$ have been set to unity, $|x|=|(x_i)_{i=1}^d|:=\big(\sum_{i=1}^d|x_i|^2\big)^{1/2}$, $x\in\mathbb{R}^d$, and V_{12}, $-\overline{V_{12}}$ also denote the corresponding multiplication operators in $L^2(\mathbb{R}^d,\mathbb{C})$ with maximal domains. The Hamiltonians H_c, H_s are selfadjoint operators in $L^2(\mathbb{R}^d,\mathbb{C})$ defined via quadratic forms. More precisely, one can introduce $H_{c,0}:=-\Delta\dot{+}a|x|\geq 0$ as the operator form sum in $L^2(\mathbb{R}^d,\mathbb{C})$, which has separated domain

$$\operatorname{dom}H_{c,0}=\{y\in W^{2,2}(\mathbb{R}^d,\mathbb{C}):x\mapsto|x|y(x)\in L^2(\mathbb{R}^d,\mathbb{C})\},$$

see e.g. [6], and hence coincides with the operator sum. The multiplication operator by the Coulomb potential $M_C:=-\frac{b}{|x|}\cdot$ is $(-\Delta)$-compact, comp. Remark 6.2 iii) below, whence $(-\Delta)$-bounded with relative bound 0 and thus $(-\Delta)$-form-bounded with form-bound 0. Since $a>0$, M_C is also $H_{c,0}$-form-bounded with form-bound 0. Therefore $H_c=H_{c,0}\dot{+}\big(-\frac{b}{|x|}\big)$ can be defined as the operator form sum, see [11, Thm. VI.3.4], and $\operatorname{dom}H_{c,0}\subset W^{2,2}(\mathbb{R}^d,\mathbb{C})\subset\operatorname{dom}M_C$ implies that

$$\operatorname{dom} H_c = \{y \in W^{2,2}(\mathbb{R}^d, \mathbb{C}) : x \mapsto |x|y(x) \in L^2(\mathbb{R}^d, \mathbb{C})\} \subset W^{2,2}(\mathbb{R}^d, \mathbb{C}) = \operatorname{dom} H_s.$$

This inclusion yields that if $W^{2,2}(\mathbb{R}^d, \mathbb{C}) \subset \operatorname{dom} V_{12}$, then the two-channel Hamiltonian $\mathscr{H}$ is diagonally dominant, see (4.1). Moreover, Proposition 4.1 shows that if

$$\Big(\lim_{\xi\to\infty} \Big\| V_{12}\Big(-\Delta + a|x| - \frac{b}{|x|} - \mathrm{i}\xi\Big)^{-1}\Big\|\Big)\Big(\lim_{|\xi|\to\infty} \Big\| V_{12}(-\Delta - \mathrm{i}\xi)^{-1}\Big\|\Big) < 1, \tag{6.1}$$

then $\mathscr{H}$ is $\mathcal{J}$-selfadjoint in $L^2(\mathbb{R}^d, \mathbb{C}) \oplus L^2(\mathbb{R}^d, \mathbb{C})$ with respect to $\mathcal{J} = \operatorname{diag}\big(I_{L^2(\mathbb{R}^d,\mathbb{C})}, -I_{L^2(\mathbb{R}^d,\mathbb{C})}\big)$; note that (6.1) is equivalent to condition (4.3) by [24, Satz 9.1]. The more explicit conditions (6.2) or (6.3) below ensure, in particular, that (6.1) holds and hence that $\mathscr{H}$ is $\mathcal{J}$-selfadjoint.

The following result on the non-accumulation of the non-real spectrum of $\mathscr{H}$ seems to be new. It relies on our existence result of semi-definite invariant subspaces, Theorem 4.2, rather than on perturbation theory. Therefore we *neither* require the uncoupled diagonal Hamiltonian $\mathscr{H}_{\mathrm{free}} = \operatorname{diag}(-\Delta + a|x| - \frac{b}{|x|}, -\Delta)$ to have discrete spectrum *nor* the coupling to be relatively compact with respect to $\mathscr{H}_{\mathrm{free}}$, see Remark 6.2 i) and ii) below.

Theorem 6.1. *If the coupling V_{12} satisfies either*

$$V_{12} \in L^\infty(\mathbb{R}^d, \mathbb{C}) \quad \textit{or} \quad V_{12} \textit{ is } \Delta\textit{-compact}, \tag{6.2}$$

then the two-channel Hamiltonian $\mathscr{H}$ is $\mathcal{J}$-selfadjoint, $\sigma_{\mathrm{e}}(\mathscr{H}) = [0, \infty)$, $\sigma(\mathscr{H}) \setminus [0, \infty)$ is discrete, and the non-real spectrum $\sigma(\mathscr{H}) \setminus \mathbb{R}$ has no finite accumulation point.

Proof. The operator $H_{c,0} = -\Delta \dot{+} a|x|$ defined above has compact resolvent by [19, Thm. XIII.67] since $a > 0$. Because the Coulomb part $M_C = -\frac{b}{|x|}$ is $H_{c,0}$-form-bounded with form-bound 0, see above, the Hamiltonian $H_c = H_{c,0} \dot{+} (-\frac{b}{|x|})$ has compact resolvent by [11, Thm. VI.3.4]. Thus the first assumption $V_{12} \in L_\infty(\mathbb{R}^d, \mathbb{C})$ in (6.2) implies that V_{12} is H_c-compact. Altogether we see that (6.2) ensures that condition (4.7) is satisfied for $\mathscr{H}$. Now the claims follow from Theorem 4.2 if we note that $\sigma_{\mathrm{e}}(H_c) = \emptyset$, $\sigma_{\mathrm{e}}(H_s) = [0, \infty)$ and hence $\sigma_{\mathrm{e}}(H_c) \cap \sigma_{\mathrm{e}}(H_s) = \emptyset$. □

Remark 6.2. i) Condition (6.2) does *not* imply that the off-diagonal coupling is relatively compact with respect to the uncoupled diagonal Hamiltonian $\mathscr{H}_{\mathrm{free}} = \operatorname{diag}(-\Delta + a|x| - \frac{b}{|x|}, -\Delta)$, so the claims on the essential and discrete spectrum do not follow from classical perturbation theory.

ii) Even if the off-diagonal coupling were relatively compact with respect to $\mathscr{H}_{\mathrm{free}}$, the claim on the non-accumulation of the non-real spectrum $\sigma(\mathscr{H}) \setminus \mathbb{R}$ would not follow from classical perturbation theory since $\sigma_{\mathrm{e}}(\mathscr{H}_{\mathrm{free}}) = [0, \infty)$.

iii) There are various sufficient conditions for V_{12} to be Δ-compact; e.g. by [19, Ex. XIII.4.6, Probl. 41] and [7, p. 209] a sufficient condition for (6.2) is that V_{12} satisfies

$$V_{12} \in L^\infty(\mathbb{R}^d, \mathbb{C}) \quad \text{or} \quad V_{12} \in L^p(\mathbb{R}^d, \mathbb{C}) + L^\infty_\varepsilon(\mathbb{R}^d, \mathbb{C}) \ \text{ with } \begin{cases} p \geq \max\{\frac{d}{2}, 2\} & \text{if } d \neq 4, \\ p > 2 & \text{if } d = 4, \end{cases} \tag{6.3}$$

with $L^p(\mathbb{R}^d, \mathbb{C}) + L^\infty_\varepsilon(\mathbb{R}^d, \mathbb{C}) := \big\{ f : \mathbb{R}^n \to \mathbb{C} : f \text{ measurable}, \forall \varepsilon > 0\ \exists f_1 \in L^p(\mathbb{R}^d, \mathbb{C}), f_2 \in L^\infty(\mathbb{R}^d, \mathbb{C}),\ \|f_2\|_\infty < \varepsilon, f = f_1 + f_2 \big\}$.

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