



Chiral extrapolation of hadronic vacuum polarization

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ABSTRACT

We study the pion-mass dependence of the two-pion channel in the hadronic-vacuum-polarization (HVP) contribution to the anomalous magnetic moment of the muon a_μ^{HVP} , by using an Omnès representation for the pion vector form factor with the phase shift derived from the inverse-amplitude method (IAM). Our results constrain the dominant isospin-1 part of the isospin-symmetric light-quark contribution, and should thus allow one to better control the chiral extrapolation of a_μ^{HVP} , required for lattice-QCD calculations performed at larger-than-physical pion masses. In particular, the comparison of the one- and two-loop IAM allows us to estimate the associated systematic uncertainties and show that these are under good control.

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1. Introduction

Currently the biggest uncertainty in the Standard-Model prediction for the anomalous magnetic moment of the muon [1–28]

$$a_\mu^{\text{SM}} = 116\,591\,810(43) \times 10^{-11} \quad (1)$$

resides in the HVP contribution, which, when derived from $e^+e^- \rightarrow \text{hadrons}$ cross-section data [1,6–12]

$$a_\mu^{\text{HVP}}|_{e^+e^-} = 6931(40) \times 10^{-11}, \quad (2)$$

leads to a 4.2σ difference to experiment [29–33]

$$a_\mu^{\text{exp}} = 116\,592\,061(41) \times 10^{-11}. \quad (3)$$

Improving the (time-like) data-driven evaluation of HVP (2) relies on new data, most crucially for the $e^+e^- \rightarrow 2\pi$ channel [34,35], while a space-like measurement would be possible at the MUonE experiment [36,37].

Alternatively, the precision of the HVP contribution evaluated in lattice QCD is getting closer to the data-driven determination, with an average [1] (based on Refs. [38–46])

$$a_\mu^{\text{HVP}}|_{\text{lattice average}} = 7\,116(184) \times 10^{-11}, \quad (4)$$

and a subsequent first sub-percent result [47]

$$a_\mu^{\text{HVP}} = 7075(55) \times 10^{-11}. \quad (5)$$

In this Letter, we do not address the 2.1σ tension with the data-driven approach,¹ see Refs. [56–60], but instead focus on the potential source of systematic uncertainty in lattice calculations that may arise if the simulation is performed at unphysical values of the quark masses.

This effect is most relevant for the isospin-symmetric ud correlator, both because its contribution is by far the largest, and because it is the lightest quarks that make simulations at the physical point expensive. Often, the required quark-mass extrapolation can be controlled using chiral perturbation theory (ChPT), at least for sufficiently small masses, but the analysis of Ref. [61] showed that for the HVP contribution this does not seem to be the case. On the one hand, the presence of a mass scale lighter than M_π , namely the muon mass, makes the pure chiral expansion of practical use only for $M_\pi \ll m_\mu$ [61]. Physically, it is well known that the 2π contribution to HVP is dominated by the $\rho(770)$ meson, see, e.g., Ref. [62] for the implication for lattice calculations, and that controlling the quark-mass dependence of its parameters requires information beyond ChPT. On the other hand, one would not expect the quark-mass dependence of the $\rho(770)$ mass, for example, to be so complicated that it could not be described by a simple parameterization. If this is the case, it is not clear why a simple

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¹ In contrast, there is good agreement between data-driven and lattice-QCD evaluations for hadronic light-by-light scattering, as further corroborated by recent work [48–55].

parameterization of the quark-mass dependence of the 2π contribution to HVP should not be possible, and even allow for a controlled chiral extrapolation of good precision (in fact, finite-volume corrections have been addressed using ChPT methods [63]). Given the high computational cost of simulations at the physical quark masses this is a question of current high interest, which can be addressed from a ChPT/phenomenological point of view and deserves the detailed investigation we aim to provide in this Letter.

Our approach here is to follow Ref. [64] and combine an Omnès description [65] of the pion vector form factor (VFF) with the inverse-amplitude method (IAM) [66–73], to capture the quark-mass dependence of the dominant two-pion intermediate states. To this end, we employ the one- and two-loop IAM to describe the pion-mass dependence of the $\pi\pi$ P -wave phase shift [74], leading to a representation that guides the chiral extrapolation of the $I = 1$ component of the isospin-symmetric ud contribution to a_μ^{HVP} . We stress that our goal is not to show that the IAM is able to predict to high precision the quark-mass dependence of the $\pi\pi$ P -wave phase shift, but rather whether it is able to describe it, and we trust that the analysis in Ref. [74] provides a positive answer to this question.

$I = 0$ and isospin-breaking terms are much smaller in size, in such a way that the systematic uncertainty in their extrapolation becomes less critical. Further, effects from inelastic states (mainly 4π) are sufficiently small that standard polynomial extrapolations should be sufficient. For the dominant 2π contribution, which we can capture with the IAM, the comparison of one- and two-loop extrapolations provides a measure of the systematic uncertainty, and thus allows the complete quantification of uncertainties that arise when ensembles at heavier-than-physical pion masses are included in the analysis.

We establish the formalism in Sec. 2, reviewing the relevant aspects of HVP, Omnès methods for the pion VFF, and the IAM. For the applications described in Sec. 3, we will first impose the physical point from data and show the resulting quark-mass dependence of the HVP integral, before turning to strategies how the corresponding constraints could be implemented in lattice analyses in Secs. 4 and 5. We conclude in Sec. 6.

2. Formalism

In the data-driven approach the HVP contribution is calculated as [75,76]

$$\begin{aligned} a_\mu^{\text{HVP}} &= \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{s_{\text{thr}}}^\infty ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s), \\ R_{\text{had}}(s) &= \frac{3s}{4\pi\alpha^2} \sigma(e^+e^- \rightarrow \text{hadrons}), \end{aligned} \quad (6)$$

where $\hat{K}(s)$ is a known kernel function and the hadronic cross section is photon inclusive. In contrast, lattice QCD does not proceed via the R -ratio, but instead employs a representation [77–79]

$$a_\mu^{\text{HVP}} = \left(\frac{\alpha}{\pi} \right)^2 \int_0^\infty dt \tilde{K}(t) G(t), \quad (7)$$

where $\tilde{K}(t)$ is another analytically known kernel function and $G(t)$ is determined by the correlator of two electromagnetic currents j_μ^{em}

$$\begin{aligned} G(t) &= -\frac{a^3}{3} \sum_{k=1}^3 \sum_{\mathbf{x}} G_{kk}(t, \mathbf{x}), \\ G_{\mu\nu}(\mathbf{x}) &= \langle 0 | j_\mu^{\text{em}}(\mathbf{x}) j_\nu^{\text{em}}(0) | 0 \rangle, \end{aligned} \quad (8)$$

where a is the lattice spacing and the limit $a \rightarrow 0$ implied in the end. This shows that in this approach the contributions of particular channels in $R_{\text{had}}(s)$ cannot be resolved, while instead the calculation is organized in a flavor decomposition, separated into an isospin-symmetric ud contribution, other quarks flavor-by-flavor, and isospin-breaking corrections (both electromagnetic and from the quark-mass difference $m_u - m_d$). For that reason, information on the quark-mass dependence of a particular hadronic channel, in general, does not translate to an extrapolation prescription for lattice calculations. However, the $I = 1$ component of the isospin-symmetric ud correlator does correspond predominantly to two-pion intermediate states, with effects from other possible states, such as 4π , appreciably suppressed. Since, in addition, the lightest states are expected to be most affected by non-trivial features of the chiral extrapolation, we will assume that such subleading effects can be adequately described by a polynomial, with the chiral behavior of $a_\mu^{\text{HVP}}[\pi\pi]$ thus a proxy for that of $a_\mu^{\text{HVP}}[ud, I = 1]$.

The two-pion contribution to $R_{\text{had}}(s)$ can be expressed in terms of the pion VFF $F_\pi^V(s)$

$$\langle \pi^\pm(p') | j_\mu^{\text{em}}(0) | \pi^\pm(p) \rangle = \pm (p' + p)^\mu F_\pi^V((p' - p)^2), \quad (9)$$

with

$$\sigma(e^+e^- \rightarrow \pi^+\pi^-) = \frac{\pi\alpha^2}{3s} \sigma_\pi^3(s) |F_\pi^V(s)|^2, \quad (10)$$

and $\sigma_\pi(s) = \sqrt{1 - 4M_\pi^2/s}$. $F_\pi^V(s)$ is then strongly constrained by $\pi\pi$ scattering, as reflected by the fact that up to a polynomial the combination of analyticity and unitarity equates the elastic contributions to the VFF with the Omnès factor [65]

$$\Omega_1^1(s) = \exp \left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s' - s)} \right\}, \quad (11)$$

where $\delta_1^1(s)$ is the P -wave $\pi\pi$ scattering phase shift. This connection between the VFF and $\pi\pi$ scattering has been employed in numerous works in the literature, see, e.g., Refs. [80–88], and also forms the basis for the present analysis. In general, inelastic and isospin-breaking corrections need to be considered for a phenomenologically viable description [8,81,82]

$$F_\pi^V(s) = \Omega_1^1(s) G_\omega(s) G_{\text{in}}(s), \quad (12)$$

including factors $G_\omega(s)$ and $G_{\text{in}}(s)$ that account for 3π and 4π intermediate states, respectively, with the former dominated by ρ - ω mixing and the latter by the $\omega\pi$ channel, which justifies the expansion in a conformal polynomial.

For the quark-mass extrapolation of $a_\mu^{\text{HVP}}[ud]$ this representation can be simplified in several ways. First, since isospin-breaking effects are booked elsewhere in lattice calculations, we can set $G_\omega(s) = 1$ and ignore final-state-radiation corrections to the cross section. The effects of inelastic states on the 2π -channel below 1 GeV are small and well described by a conformal polynomial of low degree [8]: we will truncate the integral in Eq. (6) at $\Lambda = 1$ GeV (with the threshold at $s_{\text{thr}} = 4M_\pi^2$). In order to simplify the analysis of its quark-mass dependence we will first replace $G_{\text{in}}(s)$ by a polynomial and consider its coefficients as parameters in the lattice analysis, meant to subsume inelastic effects. For a linear polynomial, $G_{\text{in}}(s) = 1 + \beta s$, the free parameter is related to the pion charge radius via

$$\langle r_\pi^2 \rangle = 6 \frac{dF_\pi^V(s)}{ds} \Big|_{s=0} = 6[\beta + \dot{\Omega}_1^1(0)], \quad (13)$$

where $\dot{\Omega}_1^1(0)$ denotes the derivative of the Omnès factor at $s = 0$. At the physical point all parameters are then determined via Eqs. (11) and (13), using input for $\delta_1^1(s)$ and $\langle r_\pi^2 \rangle$ from Ref. [8] (derived from a fit to the data sets of Refs. [89–102], including constraints from $\pi\pi$ Roy equations [103–106] and the Eidelman–Łukaszuk bound [107,108]). The final representation for the VFF in the isospin limit then reads

$$F_\pi^V(s)|_{\epsilon_\omega=0} = \left[1 + \left(\frac{\langle r_\pi^2 \rangle}{6} - \dot{\Omega}_1^1(0) \right) s \right] \Omega_1^1(s), \quad (14)$$

where, by including information on the charge radius, we have incorporated the dominant inelastic effects. We will show below that the switch from a polynomial in s to one in a conformal variable does not change the results of our analysis.

The quark-mass dependence of the resulting $a_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]$ is taken from the IAM, using the analytic expressions from Ref. [74]. The phase shift $\delta_1^1(s)$ is expressed in terms of the pion decay constant F in the chiral limit, the pion mass M_π (including quark-mass renormalization), and a set of low-energy constants (LECs): at next-to-leading order (NLO) $l_2^r - 2l_1^r$, at next-to-next-to-leading order (NNLO) $l_{1,2,3}^r$, $r_{a,b,c}$, and, in both cases, potentially l_4^r (plus r_F^r at NNLO) to convert F to the physical-point F_π . Here, we will illustrate the resulting quark-mass dependence using the lattice results for $\pi\pi$ scattering from Ref. [109], but these LECs could also become free parameters of the lattice analysis. As final ingredient we need the quark-mass dependence of $\langle r_\pi^2 \rangle$, which is also known at two-loop order [110]

$$\begin{aligned} \langle r_\pi^2 \rangle &= \frac{1}{16\pi^2 F^2} \left[R_4 + \frac{M_\pi^2}{16\pi^2 F^2} R_6 \right], \\ R_4 &= -96\pi^2 l_6^r - 1 - L, \quad L = \log \frac{M_\pi^2}{\mu^2}, \\ R_6 &= 6(16\pi^2)^2 r_{V1}^r + \frac{52\pi^2 - 181}{48} + \left[\frac{19}{6} - 96\pi^2 (2l_1^r - l_2^r) \right] L. \end{aligned} \quad (15)$$

At NLO the only new LEC, l_6^r , is determined from the physical-point $\langle r_\pi^2 \rangle = 0.429(4) \text{ fm}^2$ [8],² while the quark-mass dependence of the Omnès function and its derivative is given by the IAM. At NNLO a new LEC, r_{V1}^r , enters, as discussed in more detail below.

3. Chiral extrapolation of $I = 1$ contribution

As phenomenological reference point we start from [8]

$$a_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}] = 494.8(1.4)(2.1) \times 10^{-10}, \quad (16)$$

which gives the two-pion contribution to Eq. (6) up to a cutoff $\Lambda = 1 \text{ GeV}$ and includes final-state radiation in the point-like approximation. Numerically, this dominant, infrared-enhanced contribution increases the HVP integral by 4.2×10^{-10} [113]. In addition, we need to remove the impact of $\rho-\omega$ mixing as the second important isospin-breaking effect, which can be done by setting the corresponding mixing parameter ϵ_ω in $G_\omega(s)$ to zero, amounting to a shift of 4.3×10^{-10} . In total, we then arrive at

$$\begin{aligned} \bar{a}_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}] &\equiv a_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]|_{\text{no FSR}, \epsilon_\omega=0} \\ &= 486.3(1.4)(2.1) \times 10^{-10} \end{aligned} \quad (17)$$

² This value is in agreement with Refs. [85,89,111,112], and, at the quoted precision, is insensitive to $\rho-\omega$ mixing, whose relative effect is suppressed by $\epsilon_\omega \sim 2 \times 10^{-3}$.

for the two-pion contribution to $a_\mu^{\text{HVP}}[ud, I = 1]$. As a first step, we may compare to the result if δ_1^1 is solely determined via the IAM fits to the lattice data of Ref. [109] (and the physical pion decay constant $F_\pi = 92.28(10) \text{ MeV}$ [112]), which gives

$$\begin{aligned} \bar{a}_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]|_{[109]}^{\text{NLO}} &= 458.6(1.9)(14.9)(7.2) \times 10^{-10}, \\ \bar{a}_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]|_{[109]}^{\text{NNLO}} &= 508.0(28.8)(1.6)(8.5) \times 10^{-10}, \end{aligned} \quad (18)$$

where the first error derives from the fit parameters, the second one gives the truncation error in the chiral expansion, estimated for an observable X as [74,114]

$$\begin{aligned} \Delta X_{\text{NLO}} &= \alpha X_{\text{NLO}}, \quad \alpha = \frac{M_\pi^2}{M_\rho^2}, \\ \Delta X_{\text{NNLO}} &= \max \left\{ \alpha^2 X_{\text{NLO}}, \alpha |X_{\text{NLO}} - X_{\text{NNLO}}| \right\}, \end{aligned} \quad (19)$$

and the third one propagates the uncertainty in $\langle r_\pi^2 \rangle$. The level of agreement between Eqs. (17) and (18) reflects the extent to which the extrapolations of the lattice fits to the physical point via the one- and two-loop IAM reproduce the physical phase shift, see Fig. 1 in Ref. [74].

Next, we consider a variant of the IAM fits that includes the physical δ_1^1 from Ref. [8] (using 20 equidistant data points between 0.35 GeV and 1.15 GeV [115]). As expected, this reduces the uncertainties substantially

$$\begin{aligned} \bar{a}_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]|_{[8,109]}^{\text{NLO}} &= 460.4(0.3)(14.9)(7.2) \times 10^{-10}, \\ \bar{a}_\mu^{\text{HVP}}[\pi\pi, \leq 1 \text{ GeV}]|_{[8,109]}^{\text{NNLO}} &= 482.4(0.1)(0.7)(8.0) \times 10^{-10}, \end{aligned} \quad (20)$$

especially in the two-loop fit. In this case, the central value moves close to Eq. (17), mainly, because the functional form has the necessary freedom to reconcile the expected asymptotic behavior of the phase shift $\delta_1^1 \xrightarrow{s \rightarrow \infty} \pi$ with the resonant line shape of the $\rho(770)$. The remaining uncertainty originates from the input for the pion charge radius, which in turn is dominated by inelastic effects.³ Accordingly, the IAM representation reproduces the full result up to the level at which uncertainties from inelastic channels begin to matter, but provides a reliable implementation of the elastic $\pi\pi$ effects. This points the way towards the application in the chiral extrapolation of lattice HVP results: the pion-mass dependence of the $\pi\pi$ physics can be controlled with the IAM, and only the estimate of the pion-mass dependence of inelastic effects needs to rely on a parameterization that is not controlled by effective field theory.

For larger-than-physical pion masses the $\rho(770)$ resonance also moves higher in energy, so that a fixed cutoff at $\Lambda = 1 \text{ GeV}$ may not be equally well motivated for all pion masses. With the IAM representation successfully benchmarked against phenomenology, we will thus take $\Lambda \rightarrow \infty$ in the following, which changes Eq. (20) to

$$\begin{aligned} \bar{a}_\mu^{\text{HVP}}[\pi\pi]|_{[8,109]}^{\text{NLO}} &= 468.8(0.3)(15.2)(7.6) \times 10^{-10}, \\ \bar{a}_\mu^{\text{HVP}}[\pi\pi]|_{[8,109]}^{\text{NNLO}} &= 490.8(0.1)(0.7)(8.4) \times 10^{-10}, \end{aligned} \quad (21)$$

as starting point for a study of the pion-mass dependence. The increase of 8.4×10^{-10} from Eq. (20) to Eq. (21) is slightly smaller than the 11.8×10^{-10} obtained when extrapolating the central fit from Eq. (17) to $\Lambda = \infty$. In both cases, the effect is close to

³ The value $\langle r_\pi^2 \rangle = 0.429(4) \text{ fm}^2$ [8] is derived from fits to $e^+e^- \rightarrow 2\pi$ data via a sum rule, which implies a sensitivity to $\text{Im } F_\pi^V(s)$ beyond 1 GeV , where inelastic effects become important.

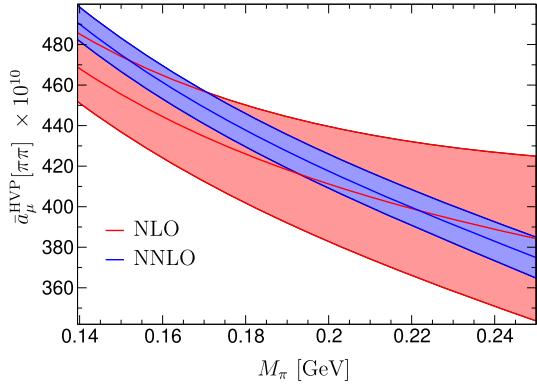


Fig. 1. Pion-mass dependence of $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$ from the NLO (red) and NNLO (blue) IAM, with parameters determined as described in the main text. The NNLO band is dominated by the uncertainty in the input for $\langle r_\pi^2 \rangle$, the NLO band by the truncation of the chiral expansion.

$a_\mu^{\text{HVP}}[\pi\pi, [1, 1.8]\text{ GeV}] = 10.4 \times 10^{-11}$ [1,10,11], but we stress that this extrapolation is not constrained by $e^+e^- \rightarrow 2\pi$ data, and, in the case of the IAM representation, simply serves as a convenient reference point compared to which we will study the quark-mass dependence. In practice, this subsumes some inelastic effects, but their contribution will need to be included as an additional term in any case, see Sec. 5.

To obtain the pion-mass dependence of $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$, we need to determine the free parameters in Eq. (15). At NLO, only l_6^r enters, which is then eliminated by imposing $\langle r_\pi^2 \rangle$ at the physical point. At NNLO, however, a new LEC (r_{V1}^r) arises, which describes the quark-mass dependence of $\langle r_\pi^2 \rangle$ and is therefore essentially inaccessible to phenomenology.⁴ Instead, we turn to the estimate of r_{V1}^r via resonance saturation [110,116,117]

$$r_{V1}^r = \frac{2\sqrt{2}f_V F_\pi^2}{M_V^2}, \quad (22)$$

with parameters that can be determined from $\rho \rightarrow e^+e^-$, $\rho \rightarrow \pi\pi$, and $K^* \rightarrow K\pi$

$$\Gamma[\rho \rightarrow e^+e^-] = \frac{(4\pi\alpha)^2 M_\rho f_V^2}{12\pi}, \quad (23)$$

$$\Gamma[\rho \rightarrow \pi\pi] = \frac{M_\rho^2 (M_\rho^2 - 4M_\pi^2)^{3/2}}{48\pi F_\pi^4} \left(g_V + 2\sqrt{2}f_\chi \frac{2M_\pi^2}{M_\rho^2} \right)^2,$$

$$\Gamma[K^* \rightarrow K\pi] = \frac{\lambda^{3/2}(M_{K^*}^2, M_K^2, M_\pi^2)}{48\pi F_K^4 M_{K^*}} \left(g_V + 2\sqrt{2}f_\chi \frac{M_\pi^2 + M_K^2}{M_{K^*}^2} \right)^2,$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$. Numerically, we find

$$f_V = 0.20, \quad g_V = 0.084, \quad f_\chi = 2.5 \times 10^{-3}, \quad (24)$$

and thus

$$r_{V1}^r = 2.0 \times 10^{-5}, \quad (25)$$

where the difference to Ref. [110] originates from using the kaon decay constant F_K instead of F_π in $\Gamma[K^* \rightarrow K\pi]$ to minimize SU(3) breaking effects. Alternatively, and in the future hopefully more reliably, one could constrain the pion-mass dependence of

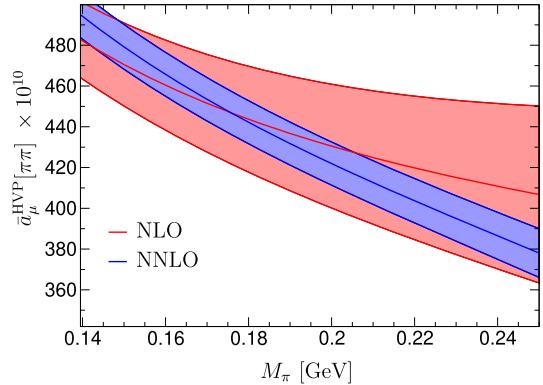


Fig. 2. Same as Fig. 1, but using a conformal polynomial in Eq. (14).

$\langle r_\pi^2 \rangle$ directly from lattice QCD [118,119], e.g., at $M_\pi = 340.9\text{ MeV}$ ChPT predicts

$$\begin{aligned} \langle r_\pi^2 \rangle &|_{M_\pi=340.9\text{ MeV}}^{\text{NLO}} = 0.373(4)(72)\text{ fm}^2, \\ \langle r_\pi^2 \rangle &|_{M_\pi=340.9\text{ MeV}}^{\text{NNLO}} = 0.350(4)(13)(7)\text{ fm}^2, \end{aligned} \quad (26)$$

where the first error is propagated from the physical-point $\langle r_\pi^2 \rangle$, the second one estimates the truncation error by the prescription in Eq. (19), and the third one arises when assigning a 100% uncertainty to Eq. (25). In comparison, Ref. [118] quotes

$$\langle r_\pi^2 \rangle |_{M_\pi=340.9\text{ MeV}}^{[118]} = 0.3485(27)\text{ fm}^2, \quad (27)$$

in good agreement with Eq. (26). We will thus continue to use Eq. (25) in the following (including a 100% uncertainty).

The resulting pion-mass dependence of $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$ is shown in Fig. 1. The NLO band is dominated by the truncation of the chiral expansion, while at NNLO this error stays below 5×10^{-10} up to $M_\pi = 0.25\text{ GeV}$, with the main effect thus from the uncertainty in the physical $\langle r_\pi^2 \rangle$ (and, for higher pion masses, increasingly r_{V1}^r). In particular, we observe good consistency between the NLO and NNLO trajectories.

We also considered a variant in which the polynomial in Eq. (14) is replaced in favor of a conformal polynomial,

$$G_{\text{in}}(s) = 1 + \sum_{i=1}^2 c_i ([z(s)]^i - [z(0)]^i), \quad (28)$$

with conformal variable

$$z(s) = \frac{\sqrt{s_{\text{in}} - s_c} - \sqrt{s_{\text{in}} - s}}{\sqrt{s_{\text{in}} - s_c} + \sqrt{s_{\text{in}} - s}}, \quad s_{\text{in}} = (M_\pi + M_\omega)^2, \quad (29)$$

furthermore $s_c = -1\text{ GeV}^2$, $c_1 = -2c_2$ to remove the S -wave singularities, and the remaining parameter again determined via $\langle r_\pi^2 \rangle$. The pion-mass dependence of M_ω is taken from Ref. [120]. The resulting bands, shown in Fig. 2 on the same scale as in Fig. 1, are well consistent with the parameterization in terms of a linear polynomial, especially the NNLO result is very stable under the change of parameterization. In both cases, one could also include a second free parameter in the (conformal) polynomial, to be identified with the curvature $c_\pi = \frac{1}{2} \frac{d^2 F_\pi^V(s)}{ds^2}|_{s=0}$, as the only new LEC, r_{V2} , would be determined by c_π at the physical point.

4. Space-like formulation

The results shown so far do not indicate any conceptual issues with the pion-mass dependence at least of the two-pion contribution, in contrast to the negative conclusions reached in Ref. [61].

⁴ Separating r_{V1}^r and l_6^r from the momentum dependence of $F_\pi^V(s)$ is possible, in principle, indirectly via loop effects, but comes at the cost of introducing difficult-to-control systematic errors especially in the determination of l_6^r .

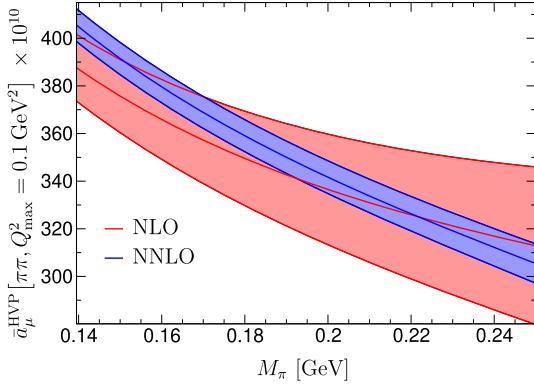


Fig. 3. Same as Fig. 1, but including a space-like cutoff at $Q_{\max}^2 = 0.1 \text{ GeV}^2$.

In order to better understand the relation between the two approaches, we will now consider the space-like HVP master formula, as it was used in Ref. [61] (after applying a cutoff Q_{\max}). Starting from [78,121]

$$a_\mu^{\text{HVP}} = -4\alpha^2 \int_0^{Q_{\max}^2} \frac{dQ^2}{Q^2} w(Q^2) \bar{\Pi}(-Q^2), \quad (30)$$

with the subtracted vacuum-polarization function $\bar{\Pi}(s) = \Pi(s) - \Pi(0)$ and weight function $w(Q^2)$, the $\pi\pi$ contribution can be evaluated by inserting a dispersion relation for $\bar{\Pi}(s)$ and retaining the imaginary part produced by $\pi\pi$ intermediate states. In the end, this leads to a modification of the time-like master formula (6) by a weight function

$$\theta(s, Q_{\max}^2) = \frac{\int_0^{Q_{\max}^2} dQ^2 \frac{w(Q^2)}{s+Q^2}}{\int_0^\infty dQ^2 \frac{w(Q^2)}{s+Q^2}}, \quad (31)$$

reminiscent of the definition of Euclidean time windows [40]. The result for $Q_{\max}^2 = 0.1 \text{ GeV}^2$ shown in Fig. 3 looks very similar to Fig. 1, only with a lower overall scale given that part of the integral has been removed by the space-like cutoff. In Ref. [61] this cutoff is required because ChPT is used directly for the vector correlator [122,123], which does include effects beyond the $\pi\pi$ channel, but restricts the domain of validity in the momentum integral. The approach we are suggesting here does not rely on any cutoffs and amounts to instead concentrating on the $\pi\pi$ channel (including some inelastic effects via the pion charge radius), since the pion-mass dependence can then be controlled via the IAM, with only the remainder to be described by simpler parameterizations.

In the space-like region, the integrand $\bar{\Pi}(-Q^2)$ becomes sufficiently smooth that simple descriptions in terms of a few parameters become possible, e.g., we have checked that the $\pi\pi$ contribution to $\bar{\Pi}(-Q^2)$ can be represented by the ansatz

$$\frac{\bar{\Pi}(-Q^2)}{Q^2} = \frac{a + bQ^2}{1 + cQ^2 + dQ^4}, \quad (32)$$

with an accuracy below 10^{-3} for $M_\pi \in [0.14, 0.25] \text{ GeV}$ and $Q^2 \in [0, 10] \text{ GeV}^2$. We can thus express the IAM prediction for the pion-mass dependence of $\bar{\Pi}(-Q^2)$ in terms of the fit coefficients in Eq. (32), see Fig. 4. We tried several ansätze

$$\begin{aligned} f_1(M_\pi^2) &= x + yM_\pi^2 + zM_\pi^4, \\ f_2(M_\pi^2) &= x \log M_\pi^2 + y + zM_\pi^2, \\ f_3(M_\pi^2) &= \frac{x}{M_\pi^2} + y + zM_\pi^2, \end{aligned}$$

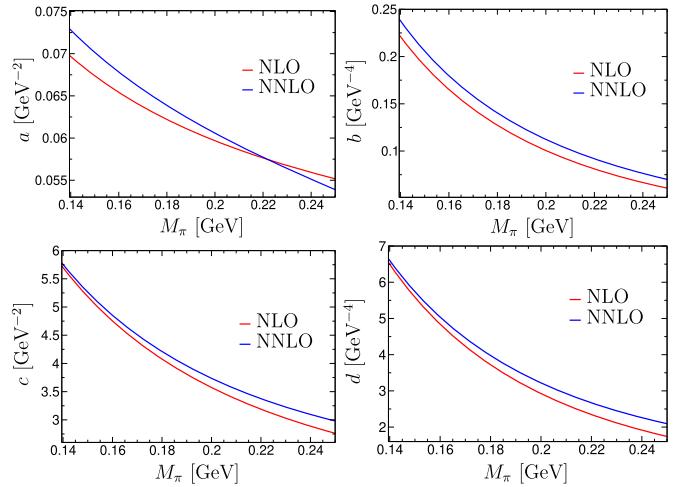


Fig. 4. Pion-mass dependence of the fit coefficients $\{a, b, c, d\}$ in Eq. (32) as predicted by the NLO (red) and NNLO (blue) IAM, with the same input for the VFF as in Fig. 1.

$$f_4(M_\pi^2) = \frac{x}{M_\pi^2} + y \log M_\pi^2 + z, \quad (33)$$

cf. also Ref. [61], for the pion-mass dependence of $\{a, b, c, d\}$ at NLO and NNLO, with the result that f_1 and in most cases f_2 are (strongly) disfavored, indicating that an M_π^{-2} term is necessary. The NLO result for a displays some preference for an additional $\log M_\pi^2$ term, but for all other coefficients, and also for a at NNLO, f_3 and f_4 describe the pion-mass dependence from the IAM equally well. Extending these empirical fits to include pion masses below the physical point increases the sensitivity to the infrared singularities, with results that suggest the presence of a $\log M_\pi^2$ in addition to the M_π^{-2} term.

In principle, a similar strategy could also be pursued for the integrand $G(t)$ in the time-momentum representation (7), but due to its more complicated behavior an accurate description in terms of few parameters in analogy to Eq. (32) is difficult to find. In contrast, the pion-mass dependence of $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$ is sufficiently smooth to be described by f_{2-4} with an error below 1×10^{-10} for $M_\pi \in [0.14, 0.25] \text{ GeV}$ when fitting the central curves. Again, these empirical fits favor the presence of an M_π^{-2} term in the extrapolation, and when including pion masses below the physical point we also see indications for the presence of an additional $\log M_\pi^2$ singularity. We emphasize that these findings are purely empirical, to describe the IAM results in a finite range of M_π , and we do not claim that either fit function represent an analytic approximation to the full IAM.

5. Possible implementation strategies

The above results suggest two main strategic approaches to the chiral extrapolation of lattice results for HVP, each of which could then be implemented following different variants. The first would explicitly rely on the description of the dominant $\pi\pi$ contribution in terms of the P -wave phase shift, whose quark-mass dependence has been shown to be well described by the IAM [74], and use the LECs appearing in that description directly as fit parameters. Possible variants for the implementation of this strategy include:

1. In the most constrained scenario, the free parameters of the IAM could be taken from an independent lattice calculation of the $\pi\pi$ P -wave and the pion decay constant, leaving only the parameter β as a free parameter. In fact, to use the maximum amount of chiral input, the charge radius $\langle r_\pi^2 \rangle$ at the physical point could be identified as fit parameter (instead of β),

- given that this input yields the dominant uncertainty in predicting the pion-mass dependence of $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$, and further lattice input could constrain r_{V1}^c .
2. Relaxing chiral constraints, (some of) the LECs could be left free as additional fit parameters.
 3. Since $\bar{a}_\mu^{\text{HVP}}[\pi\pi]$ does not saturate $a_\mu^{\text{HVP}}[ud, I = 1]$, it needs to be supplemented by an additional term, in the simplest case $a_\mu^{\text{HVP}}[ud, I = 1, \text{non-}\pi\pi] = \zeta + M_\pi^2 \xi$, without further chiral constraints on the parameters. It is clear that higher intermediate states can only affect the singularity structure of a_μ^{HVP} towards the chiral limit at higher chiral orders. We thus consider it highly unlikely that these higher intermediate states would affect the parameterization we adopted to describe the chiral extrapolation in any perceivable way, so that a simple phenomenological description should be sufficient for any practical purposes.

We stress that the bands in Fig. 1 appear broad compared to the sub-percent target precision, but also that, crucially, at NNLO the uncertainty is dominated by external input quantities (most notably $\langle r_\pi^2 \rangle$), with the chiral convergence well under control. This implies that in the opposite direction, extrapolating HVP values at larger-than-physical pion masses towards the physical point, the intrinsic uncertainty to be assigned to the $\pi\pi$ component should be small, with the dominant sources of uncertainty likely from the required LECs and the chiral extrapolation of the non- $\pi\pi$ contribution. In particular, the LECs that enter the extrapolation can be constrained from independent lattice calculations for δ_1^1 , F_π , and $\langle r_\pi^2 \rangle$.

The second strategic approach to the chiral extrapolation of the HVP contribution only indirectly relies on the description of the two-pion contribution in terms of the P -wave phase shift. The latter is used only to show that in the space-like region the polarization function due to the $\pi\pi$ contribution is sufficiently smooth to allow for an accurate representation in terms of just a few fit parameters, whose quark-mass dependence is again well described by simple parameterizations. One would then adopt one (or a few) of the parameterizations proposed and tested here and fit its parameters to the lattice data to perform the chiral extrapolation. Here the possible variants would depend on the amount and precision of lattice data and would boil down to choosing among the different possible parameterizations discussed here or on further refinements thereof.

6. Conclusions

In this Letter we studied potential strategies to control the quark-mass dependence of the HVP contribution to the anomalous magnetic moment of the muon based on effective field theory. In particular, we focused on the $I = 1$ component of the isospin-symmetric ud correlator, which receives its by far dominant contribution from $\pi\pi$ intermediate states, with 4π and other non- $\pi\pi$ contributions appreciably suppressed. A direct application of ChPT is not possible due to the limited range of convergence [61], ultimately, because information on the ρ meson needs to be provided. Here, we argued that this can be achieved based on the IAM, with one- and two-loop implementations allowing one to verify the chiral convergence and thus assess the corresponding systematic uncertainties.

To illustrate the formalism, we addressed the opposite problem, to predict the quark-mass dependence starting from the physical point, with the main result shown in Fig. 1, based on input from the $\pi\pi$ P -wave phase shift δ_1^1 and the pion charge radius $\langle r_\pi^2 \rangle$ at the physical point [8], combined with a lattice calculation of δ_1^1 and the pion decay constant F_π [109] to determine the ChPT parameters. We found that the IAM representation indeed allows

one to reproduce the expected HVP value, and suggested strategies how it could be used to constrain the chiral extrapolation of HVP calculations in lattice QCD performed at larger-than-physical quark masses.

Moreover, we found that in the space-like region the IAM results are sufficiently smooth that the correlator can be described by simple parameterizations. We could successfully reproduce their quark-mass dependence by simple fit functions, with the result that the presence of an infrared singularity as strong as M_π^{-2} seems to be preferred empirically. Altogether, we see no conceptual issues in controlling the quark-mass dependence of HVP in lattice calculations along these lines using effective field theory, with several opportunities to constrain the extrapolation with independent lattice input. Of course, the precision that can be reached in extrapolations to the physical point, or even interpolations around it, will strongly depend on the details of each lattice calculation, but the possibility to achieve high precision even if working away from the physical point and reaching it by an extrapolation stays open.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.physletb.2021.136852>.

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