# From distributive $\ell$-monoids to $\ell$-groups, and back again 

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We prove that an inverse-free equation is valid in the variety LG of lattice-ordered groups ( $\ell$-groups) if and only if it is valid in the variety DLM of distributive lattice-ordered monoids (distributive $\ell$-monoids). This contrasts with the fact that, as proved by Repnitskiǐ, there exist inverse-free equations that are valid in all Abelian $\ell$-groups but not in all commutative distributive $\ell$-monoids, and, as we prove here, there exist inverse-free equations that are valid in all totally ordered groups but not in all totally ordered monoids. We also prove that DLM has the finite model property and a decidable equational theory, establish a correspondence between the validity of equations in DLM and the existence of certain right orders on free monoids, and provide an effective method for reducing the validity of equations in LG to the validity of equations in DLM.
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## 1. Introduction

A lattice-ordered group ( $\ell$-group) is an algebraic structure $\left\langle L, \wedge, \vee, \cdot,{ }^{-1}, \mathrm{e}\right\rangle$ such that $\left\langle L, \cdot,^{-1}, \mathrm{e}\right\rangle$ is a group, $\langle L, \wedge, \vee\rangle$ is a lattice, and the group multiplication preserves the lattice order, i.e., $a \leq b$ implies $c a d \leq c b d$ for all $a, b, c, d \in L$, where $a \leq b: \Longleftrightarrow a \wedge b=a$. The class of $\ell$-groups forms a variety (equational class) LG and admits the following Cayley-style representation theorem:

Theorem 1.1 (Holland [6]). Every $\ell$-group embeds into an $\ell$-group $\operatorname{Aut}(\langle\Omega, \leq\rangle)$ consisting of the group of order-automorphisms of a totally ordered set (chain) $\langle\Omega, \leq\rangle$ equipped with the pointwise lattice order.

Holland's theorem has provided the foundations for the development of a rich and extensive theory of $\ell$-groups (see [2,11] for details). In particular, it was proved by Holland [7] that an equation is valid in LG if and only if it is valid in $\operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle)$, and by Holland and McCleary [8] that the equational theory of LG is decidable.

The inverse-free reduct of any $\ell$-group is a distributive lattice-ordered monoid (distributive $\ell$-monoid): an algebraic structure $\langle M, \wedge, \vee, \cdot, \mathrm{e}\rangle$ such that $\langle M, \cdot, \mathrm{e}\rangle$ is a monoid, $\langle M, \wedge, \vee\rangle$ is a distributive lattice, and the monoid multiplication distributes over the lattice operations, i.e., for all $a, b, c, d \in M$,

$$
a(b \vee c) d=a b d \vee a c d \quad \text { and } \quad a(b \wedge c) d=a b d \wedge a c d
$$

The class of distributive $\ell$-monoids also forms a variety DLM and admits a Cayley-style (or Holland-style) representation theorem:

Theorem 1.2 (Anderson and Edwards [1]). Every distributive $\ell$-monoid embeds into a distributive $\ell$-monoid $\operatorname{End}(\langle\Omega, \leq\rangle)$ consisting of the monoid of order-endomorphisms of a chain $\langle\Omega, \leq\rangle$ equipped with the pointwise lattice order.

Despite the obvious similarity of Theorem 1.2 to Theorem 1.1, the precise nature of the relationship between the varieties of distributive $\ell$-monoids and $\ell$-groups has remained unclear. It was proved by Repnitskiy̆ in [13] that the variety of commutative distributive $\ell$-monoids does not have the same equational theory as the class of inverse-free reducts of Abelian $\ell$-groups, but the decidability of its equational theory remains an open problem. In this paper, we prove the following results for the general (noncommutative) case:

Theorem 2.3. The variety of distributive $\ell$-monoids has the finite model property. ${ }^{3}$ More precisely, an equation is valid in all distributive $\ell$-monoids if and only if it is valid in all distributive $\ell$-monoids of order-endomorphisms of a finite chain.

[^1]Corollary 2.4. The equational theory of distributive $\ell$-monoids is decidable.

Theorem 2.9. An inverse-free equation is valid in the variety of $\ell$-groups if and only if it is valid in the variety of distributive $\ell$-monoids.

Theorem 2.9 shows, by way of Birkhoff's variety theorem [3], that distributive $\ell$ monoids are precisely the homomorphic images of the inverse-free subreducts of $\ell$-groups. It also allows us, using a characterization of valid $\ell$-group equations given in [4], to relate the validity of equations in distributive $\ell$-monoids to the existence of certain right orders on free monoids. As a notable consequence of this correspondence, we obtain:

Corollary 3.4. Every right order on the free monoid over a set $X$ extends to a right order on the free group over $X$.

To check whether an equation is valid in all distributive $\ell$-monoids, it suffices, by Theorem 2.9, to check the validity of this same equation in all $\ell$-groups. We prove here that a certain converse also holds, namely:

Theorem 4.2. Let $\varepsilon$ be any $\ell$-group equation with variables in a set $X$. A finite set of inverse-free equations $\Sigma$ with variables in $X \cup Y$ for some finite set $Y$ can be effectively constructed such that $\varepsilon$ is valid in all $\ell$-groups if and only if the equations in $\Sigma$ are valid in all distributive $\ell$-monoids.

Finally, we turn our attention to totally ordered groups and totally ordered monoids, that is, $\ell$-groups and distributive $\ell$-monoids with a total lattice order. We show that the variety generated by the class of totally ordered monoids can be axiomatized relative to DLM by a single equation (Proposition 5.4). However, analogously to the case of commutative distributive $\ell$-monoids and unlike the case of DLM, we prove:

Theorem 5.7. There is an inverse-free equation that is valid in all totally ordered groups, but not in all totally ordered monoids.

We also exhibit an inverse-free equation that is valid in all finite totally ordered monoids, but not in the ordered group of the integers (Proposition 5.8), witnessing the failure of the finite model property for the variety of commutative distributive $\ell$-monoids and the varieties generated by totally ordered monoids and inverse-free reducts of totally ordered groups (Corollary 5.9).

## 2. From distributive $\ell$-monoids to $\ell$-groups

In this section, we establish the finite model property for the variety DLM of distributive $\ell$-monoids (Theorem 2.3) and the decidability of its equational theory (Corollary 2.4). We then prove that an inverse-free equation is valid in DLM if and only if it is
valid in the variety LG of $\ell$-groups (Theorem 2.9). The key tool for obtaining these results is the notion of a total preorder on a set of monoid terms that is preserved under right multiplication, which bears some similarity to the notion of a diagram employed in [8]. In particular, the existence of such a preorder satisfying a given finite set of inequalities is related to the validity of a corresponding inverse-free equation in DLM or LG.

Let $X$ be any set. We denote by $\mathbf{T}_{m}(X), \mathbf{T}_{g}(X), \mathbf{T}_{d}(X)$, and $\mathbf{T}_{\ell}(X)$ the term algebras over $X$ for monoids, groups, distributive $\ell$-monoids, and $\ell$-groups, respectively, and by $\mathbf{F}_{m}(X), \mathbf{F}_{g}(X), \mathbf{F}_{d}(X)$, and $\mathbf{F}_{\ell}(X)$, the corresponding free algebras, assuming for convenience that $F_{m}(X) \subseteq T_{m}(X), F_{g}(X) \subseteq T_{g}(X), F_{d}(X) \subseteq T_{d}(X)$, and $F_{\ell}(X) \subseteq T_{\ell}(X)$. Given a set of ordered pairs of monoid terms $S \subseteq F_{m}(X)^{2}$, we define the set of initial subterms of $S$ :

$$
\text { is }(S):=\left\{u \in F_{m}(X) \mid \exists s, t \in F_{m}(X):\langle u s, t\rangle \in S \text { or }\langle s, u t\rangle \in S\right\} .
$$

Note in particular that $s, t \in \operatorname{is}(S)$ for each $\langle s, t\rangle \in S$.
Recall now that a preorder $\preceq$ on a set $P$ is a binary relation on $P$ that is reflexive and transitive. We write $a \prec b$ to denote that $a \preceq b$ and $b \npreceq a$, and call $\preceq$ total if $a \preceq b$ or $b \preceq a$ for all $a, b \in P$. Let $\preceq$ be a preorder on a set of monoid terms $P \subseteq F_{m}(X)$. We say that $\preceq$ is right- $X$-invariant if for all $x \in X$, whenever $u \preceq v$ and $u x, v x \in P$, also $u x \preceq v x$, and strictly right- $X$-invariant if it is right $X$-invariant and for all $x \in X$, whenever $u \prec v$ and $u x, v x \in P$, also $u x \prec v x$.

Following standard practice for $\ell$-groups, we write $(p) f$ for the value of a (partial) map $f: \Omega \rightarrow \Omega$ defined at $p \in \Omega$. As a notational aid, we also often write $\varphi_{r}$ to denote the value of a (partial) map $\varphi$ defined for some element $r$.

Lemma 2.1. Let $S \subseteq F_{m}(X)^{2}$ be a finite set of ordered pairs of monoid terms and let $\preceq$ be a total right- $X$-invariant preorder on is $(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$.
(a) There exists a chain $\langle\Omega, \leq\rangle$ satisfying $|\Omega| \leq|\operatorname{is}(S)|$, a homomorphism $\varphi: \mathbf{T}_{d}(X) \rightarrow$ $\operatorname{End}(\langle\Omega, \leq\rangle)$, and some $p \in \Omega$ such that $(p) \varphi_{s}<(p) \varphi_{t}$ for each $\langle s, t\rangle \in S$.
(b) If $\preceq$ is also strictly right- $X$-invariant, then there exists a homomorphism $\psi: \mathbf{T}_{\ell}(X)$ $\rightarrow \boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$ and some $q \in \mathbb{Q}$ such that $(q) \psi_{s}<(q) \psi_{t}$ for each $\langle s, t\rangle \in S$.

Proof. For (a), we let $[u]:=\{v \in \operatorname{is}(S) \mid u \preceq v$ and $v \preceq u\}$ for each $u \in$ is $(S)$ and define $\Omega:=\{[u] \mid u \in$ is $(S)\}$, noting that $|\Omega| \leq|\operatorname{is}(S)|$. If $[u]=\left[u^{\prime}\right],[v]=\left[v^{\prime}\right]$, and $u \preceq v$, then $u^{\prime} \preceq v^{\prime}$, so we can define for $[u],[v] \in \Omega$,

$$
[u] \leq[v]: \Longleftrightarrow u \preceq v
$$

Clearly, $\leq$ is a total order on $\Omega$ and $[s]<[t]$ for each $\langle s, t\rangle \in S$. Moreover, if $[u],[v] \in \Omega$, $x \in X$, and $u x, v x \in \operatorname{is}(S)$, then, using the right- $X$-invariance of $\preceq$,

$$
[u] \leq[v] \Longrightarrow[u x] \leq[v x] .
$$

In particular, if $[u]=[v] \in \Omega, x \in X$, and $u x, v x \in \operatorname{is}(S)$, then $[u x]=[v x]$. Hence for each $x \in X$, we obtain a partial order-endomorphism $\tilde{\varphi}_{x}: \Omega \rightarrow \Omega$ of $\langle\Omega, \leq\rangle$ by defining $([u]) \tilde{\varphi}_{x}:=[u x]$ whenever $[u] \in \Omega$ and $u x \in \operatorname{is}(S)$. Moreover, each of these partial maps $\tilde{\varphi}_{x}$ extends to an order-endomorphism $\varphi_{x}: \Omega \rightarrow \Omega$ of $\langle\Omega, \leq\rangle$. Now let $\varphi: \mathbf{T}_{d}(X) \rightarrow \mathbf{E n d}(\langle\Omega, \leq\rangle)$ be the homomorphism extending the assignment $x \mapsto \varphi_{x}$. Then $([\mathrm{e}]) \varphi_{u}=[u]$ for every $u \in \operatorname{is}(S)$ and hence $([\mathrm{e}]) \varphi_{s}<([\mathrm{e}]) \varphi_{t}$ for each $\langle s, t\rangle \in S$.

For (b), note that the set $\Omega$ defined in (a) is finite and, assuming that $\preceq$ is strictly right- $X$-invariant, the partial order-endomorphisms $\tilde{\varphi}_{x}: \Omega \rightarrow \Omega$ of $\langle\Omega, \leq\rangle$ for $x \in X$ are injective. Hence $\langle\Omega, \leq\rangle$ can be identified with a subchain of $\langle\mathbb{Q}, \leq\rangle$ and each $\tilde{\varphi}_{x}$ can be extended to an order-automorphism $\psi_{x}: \mathbb{Q} \rightarrow \mathbb{Q}$ of $\langle\mathbb{Q}, \leq\rangle$. As in (a), we obtain a homomorphism $\psi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$ extending the assignment $x \mapsto \psi_{x}$ such that $([\mathrm{e}]) \varphi_{s}<([\mathrm{e}]) \varphi_{t}$ for each $\langle s, t\rangle \in S$.

For $s, t \in T_{\ell}(X)$, we write $s \leq t$ as an abbreviation for the equation $s \wedge t \approx s$, noting that $s \approx t$ is valid in an $\ell$-monoid or $\ell$-group $\mathbf{L}$ if and only if $s \leq t$ and $t \leq s$ are valid in $\mathbf{L}$. It is easily seen that every $\ell$-group (or $\ell$-monoid) term is equivalent in LG (or DLM) to both a join of meets of group (monoid) terms and a meet of joins of group (monoid) terms. It follows that to check the validity of an (inverse-free) equation in LG (or DLM), it suffices to consider equations of the form $\bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}$ where $s_{j}, t_{i} \in F_{g}(X)$ (or $\left.s_{j}, t_{i} \in F_{m}(X)\right)$ for $1 \leq i \leq n, 1 \leq j \leq m$. The next lemma relates the validity of an inverse-free equation of this form in LG or DLM to the existence of a total (strictly) right- $X$-invariant preorder on a corresponding set of initial subterms.

Lemma 2.2. Let $\varepsilon=\left(\bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}\right)$ where $s_{j}, t_{i} \in F_{m}(X)$ for $1 \leq i \leq n, 1 \leq j \leq m$, and let $S:=\left\{\left\langle s_{j}, t_{i}\right\rangle \in F_{m}(X)^{2} \mid 1 \leq i \leq n, 1 \leq j \leq m\right\}$.
(a) $\operatorname{DLM} \models \varepsilon$ if and only if there is no total right- $X$-invariant preorder $\preceq$ on is $(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$.
(b) LG $\models \varepsilon$ if and only if there is no total strictly right- $X$-invariant preorder $\preceq$ on is $(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$.

Proof. For the left-to-right direction of (a), suppose contrapositively that there exists a total right- $X$-invariant preorder $\preceq$ on is $(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$. By Lemma 2.1(a), there exist a chain $\langle\Omega, \leq\rangle$, a homomorphism $\varphi: \mathbf{T}_{d}(X) \rightarrow \boldsymbol{E n d}(\langle\Omega, \leq\rangle)$, and some $p \in \Omega$ such that $(p) \varphi_{s}<(p) \varphi_{t}$ for each $\langle s, t\rangle \in S$. So $(p) \varphi_{\bigwedge_{i=1}^{n} t_{i}}>(p) \varphi_{\bigvee_{j=1}^{m} s_{j}}$, and hence DLM $\not \vDash \varepsilon$. Similarly, for the left-to-right direction of (b), there exist, by Lemma 2.1(b), a homomorphism $\psi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$ and some $q \in \mathbb{Q}$ such that (q) $\psi_{\bigwedge_{i=1}^{n} t_{i}}>(q) \psi_{\bigvee_{j=1}^{m} s_{j}}$ and hence LG $\not \models \varepsilon$.

For the right-to-left direction of (a), suppose contrapositively that DLM $\neq \varepsilon$. By Theorem 1.2, there exist a chain $\langle\Omega, \leq\rangle$, a homomorphism $\varphi: \mathbf{T}_{d}(X) \rightarrow \boldsymbol{\operatorname { E n d }}(\langle\Omega, \leq\rangle)$, and some $p \in \Omega$ such that $\bigwedge_{i=1}^{n}(p) \varphi_{t_{i}}>\bigvee_{j=1}^{m}(p) \varphi_{s_{j}}$. Then $(p) \varphi_{t}>(p) \varphi_{s}$ for each $\langle s, t\rangle \in S$ and we define for $u, v \in$ is $(S)$,

$$
u \preceq v: \Longleftrightarrow(p) \varphi_{u} \leq(p) \varphi_{v}
$$

Clearly $\preceq$ is a total preorder satisfying $s \prec t$ for each $\langle s, t\rangle \in S$. Moreover, since $\varphi$ is a homomorphism, $\preceq$ is right- $X$-invariant on is $(S)$.

For the right-to-left direction of (b), suppose that LG $\notin \varepsilon$. By Theorem 1.1, there exist a chain $\langle\Omega, \leq\rangle$, a homomorphism $\psi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{\operatorname { A u t }}(\langle\Omega, \leq\rangle)$, and $q \in \Omega$ such that $\bigwedge_{i=1}^{n}(q) \psi_{t_{i}}>\bigvee_{j=1}^{m}(q) \psi_{s_{j}}$. The proof then proceeds exactly as in the case of (a), except that we may observe finally that $\preceq$ is strictly right- $X$-invariant on is $(S)$, using the fact that $\psi_{u}$ is bijective for each $u \in \operatorname{is}(S)$.

We now combine the first parts of the preceding lemmas to obtain:

Theorem 2.3. The variety of distributive $\ell$-monoids has the finite model property. More precisely, an equation is valid in all distributive $\ell$-monoids if and only if it is valid in all distributive $\ell$-monoids of order-endomorphisms of a finite chain.

Proof. It suffices to establish the result for an equation $\varepsilon=\left(\bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}\right)$, where $s_{1}, \ldots, s_{m}, t_{1}, \ldots, t_{n} \in F_{m}(X)$. Suppose that DLM $\not \vDash \varepsilon$ and let $S:=\left\{\left\langle s_{j}, t_{i}\right\rangle \mid 1 \leq i \leq\right.$ $n, 1 \leq j \leq m\}$. Combining Lemmas 2.2(a) and 2.1(a), there exist a finite chain $\langle\Omega, \leq\rangle$, a homomorphism $\varphi: \mathbf{T}_{d}(X) \rightarrow \mathbf{E n d}(\langle\Omega, \leq\rangle)$, and some $p \in \Omega$ such that $(p) \varphi_{s}<(p) \varphi_{t}$ for each $\langle s, t\rangle \in S$. But then $(p) \varphi_{\bigwedge_{i=1}^{n} t_{i}}>(p) \varphi_{\bigvee_{j=1}^{m} s_{j}}$, so $\operatorname{End}(\langle\Omega, \leq\rangle) \not \vDash \varepsilon$.

Since DLM is a finitely axiomatized variety, we also obtain:

Corollary 2.4. The equational theory of distributive $\ell$-monoids is decidable.

Similarly, the second parts of Lemmas 2.1 and 2.2 can be used to show that an inversefree equation is valid in all $\ell$-groups if and only if it is valid in $\operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle)$. Indeed, this correspondence is known to hold for all equations.

Theorem 2.5 ([7]). An equation is valid in all $\ell$-groups if and only if it is valid in $\boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$.

Lemma 2.7 below provides the key ingredient for showing that an inverse-free equation is valid in LG if and only if it is valid in DLM. First, we illustrate the rather involved construction in the proof of this lemma with a simple example.

Example 2.6. Let $\operatorname{End}(\mathbf{2})$ be the distributive $\ell$-monoid of order-endomorphisms of the two-element chain $\mathbf{2}=\langle\{0,1\}, \leq\rangle$, and let $\left\langle k_{0}, k_{1}\right\rangle$ denote the member of $\operatorname{End}(\mathbf{2})$ with $0 \mapsto k_{0}$ and $1 \mapsto k_{1}$. The equation $y x y \leq x y x$ fails in $\operatorname{End}(\mathbf{2})$, since for the homomorphism $\varphi: \mathbf{T}_{d}(\{x, y\}) \rightarrow \mathbf{E n d}(\mathbf{2})$ extending the assignment $x \mapsto \varphi_{x}=\langle 0,0\rangle$ and $y \mapsto \varphi_{y}=\langle 1,1\rangle$, we obtain


Fig. 1. The paths for $x y x=(1,0,1,0)$ and $y x y=(1,1,0,1)$.

$$
(1) \varphi_{y x y}=\left(\left((1) \varphi_{y}\right) \varphi_{x}\right) \varphi_{y}=1>0=\left(\left((1) \varphi_{x}\right) \varphi_{y}\right) \varphi_{x}=(1) \varphi_{x y x} \text {. }
$$

Let $S:=\{\langle x y x, y x y\rangle\}$. Then $\varphi$ yields a total right- $\{x, y\}$-invariant preorder $\preceq$ on is $(S)=$ $\{\mathrm{e}, x, y, x y, y x, x y x, y x y\}$ given by $x \sim y x \sim x y x \prec \mathrm{e} \sim y \sim x y \sim y x y$, since (1) $\varphi_{x}=$ (1) $\varphi_{y x}=(1) \varphi_{x y x}=0<1=(1) \varphi_{\mathrm{e}}=(1) \varphi_{y}=(1) \varphi_{x y}=(1) \varphi_{y x y}$. Note that $\preceq$ is not strictly right- $\{x, y\}$-invariant, since $x \prec \mathrm{e}$, but $x y \sim y$; this corresponds to the fact that $\varphi_{y}$ is not a partial bijective map on $\{0,1\}$, as $0<1$ and $(0) \varphi_{y}=(1) \varphi_{y}$.

We describe a total strictly right- $\{x, y\}$-invariant preorder $\unlhd$ on is $(S)$ such that $\prec \subseteq \triangleleft$. This corresponds to constructing partial bijections $\widehat{\varphi}_{x}$ and $\widehat{\varphi}_{y}$ on is $(S)$ that extend $\varphi_{x}$ and $\varphi_{y}$, respectively. The relation $\triangleleft$ can be computed directly using the definition given in Lemma 2.7, but to provide both a simpler description and intuition for the construction, we identify each element $x_{k} \cdots x_{1}$ of is $(S)$ with the sequence $\left((1) \varphi_{\mathrm{e}},(1) \varphi_{x_{k}}, \ldots,(1) \varphi_{x_{k} \cdots x_{1}}\right)$, so $\mathrm{e}=(1), x=(1,0), y=(1,1), x y=(1,0,1)$, $y x=(1,1,0), x y x=(1,0,1,0)$, and $y x y=(1,1,0,1)$. Note that these are the paths of elements of $\{0,1\}$ involved in the successive computation steps for each term at the point $p=1$ and can be visualized as indicated in Fig. 1.

The relation $\triangleleft$ on these paths is simply the reverse lexicographic order:

$$
(1,0) \triangleleft(1,0,1,0) \triangleleft(1,1,0) \triangleleft(1) \triangleleft(1,0,1) \triangleleft(1,1,0,1) \triangleleft(1,1),
$$

where the first three elements serve as copies of 0 and the last four as copies of 1 , so via the above identification we obtain

$$
x \triangleleft x y x \triangleleft y x \triangleleft \mathrm{e} \triangleleft x y \triangleleft y x y \triangleleft y .
$$

It can be verified that this is a total strictly right- $\{x, y\}$-invariant (pre)order, or, more easily, that the corresponding partial order-endomorphisms $\widehat{\varphi}_{x}$ and $\widehat{\varphi}_{y}$ are partial bijections as shown in Fig. 2.

Lemma 2.7. For any $S \subseteq F_{m}(X)^{2}$ and total right- $X$-invariant preorder $\preceq$ on is $(S)$, there exists a total strictly right- $X$-invariant preorder $\unlhd$ on is $(S)$ such that $\prec \subseteq \triangleleft$.

Proof. We define the following relations on is $(S)$ :

$$
\begin{aligned}
u \sim v: \Longleftrightarrow & u \preceq v \text { and } v \preceq u ; \\
x_{k} \cdots x_{1} \triangleleft y_{l} \cdots y_{1}: \Longleftrightarrow & \exists j \leq l+1: x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \text { for all } i<j \text { and } \\
& \left(x_{k} \cdots x_{j} \prec y_{l} \cdots y_{j} \text { or } j=k+2\right) ;
\end{aligned}
$$



Fig. 2. The partial bijections $\widehat{\varphi}_{x}$ and $\widehat{\varphi}_{y}$ and the evaluation of $\widehat{\varphi}_{x y x}$ at $(1)=\mathrm{e}$.

$$
\begin{aligned}
x_{k} \cdots x_{1} \equiv y_{l} \cdots y_{1} & : \Longleftrightarrow k=l \text { and } x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \text { for each } i \leq k \\
u \unlhd v & : \Longleftrightarrow u \triangleleft v \text { or } u \equiv v
\end{aligned}
$$

assuming that $x_{k} \cdots x_{i}$ is the empty product e for $i>k$.
Observe that setting $j=1$ in the definition of $\triangleleft$ yields $\prec \subseteq \triangleleft$. Also $u \triangleleft v$ implies $u \not \equiv v$. The irreflexivity of $\triangleleft$ follows directly from the fact that $\prec$ is irreflexive. For the transitivity of $\triangleleft$, we consider $u, v, w \in \operatorname{is}(S)$ satisfying $u=x_{k} \cdots x_{1}, v=y_{l} \cdots y_{1}$, $w=z_{m} \cdots z_{1}, u \triangleleft v$, and $v \triangleleft w$. By definition, there exists a $j_{1} \leq l+1$ such that $x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i}$ for all $i<j_{1}$, and either $x_{k} \cdots x_{j_{1}} \prec y_{l} \cdots y_{j_{1}}$ or $j_{1}=k+2$, and there exists a $j_{2} \leq m+1$ such that $y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{2}$, and either $y_{l} \cdots y_{j_{2}} \prec z_{m} \cdots z_{j_{2}}$ or $j_{2}=l+2$. There are four cases to check:

1. $x_{k} \cdots x_{j_{1}} \prec y_{l} \cdots y_{j_{1}}$ and $y_{l} \cdots y_{j_{2}} \prec z_{m} \cdots z_{j_{2}}$. If $j_{2} \leq j_{1}$, then $x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \sim$ $z_{m} \cdots z_{i}$ for all $i<j_{2}$ and $x_{k} \cdots x_{j_{2}} \sim y_{l} \cdots y_{j_{2}} \prec z_{m} \cdots z_{j_{2}}$, so (since $\sim$ and $\preceq$ are transitive), $u \triangleleft w$. If $j_{1}<j_{2}$, then $j_{1} \leq m+1$ and $x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{1}$ and $x_{k} \cdots x_{j_{1}} \prec y_{l} \cdots y_{j_{1}} \sim z_{m} \cdots z_{j_{1}}$, so $u \triangleleft w$.
2. $x_{k} \cdots x_{j_{1}} \prec y_{l} \cdots y_{j_{1}}$ and $j_{2}=l+2$. Then $j_{1} \leq l+1<j_{2} \leq m+1$, so $x_{k} \cdots x_{i} \sim$ $y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{1}$ and $x_{k} \cdots x_{j_{1}} \prec y_{l} \cdots y_{j_{1}} \sim z_{m} \cdots z_{j_{1}}$. Hence $u \triangleleft w$.
3. $j_{1}=k+2$ and $y_{l} \cdots y_{j_{2}} \prec z_{m} \cdots z_{j_{2}}$. If $j_{2}<j_{1}$, then $x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{2}$ and $x_{k} \cdots x_{j_{2}} \sim y_{l} \cdots y_{j_{2}} \prec z_{m} \cdots z_{j_{2}}$, so $u \triangleleft w$. If $j_{1} \leq j_{2}$, then $j_{1} \leq m+1, x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{1}$, and $j_{1}=k+2$, so $u \triangleleft w$.
4. $j_{1}=k+2$ and $j_{2}=l+2$. Then $j_{1} \leq m+1$ and $x_{k} \cdots x_{i} \sim y_{l} \cdots y_{i} \sim z_{m} \cdots z_{i}$ for all $i<j_{1}$. Hence $u \triangleleft w$.

For the transitivity of $\unlhd$, there are also several cases to check. Clearly, if $u \triangleleft v$ and $v \triangleleft w$, then $u \triangleleft w$, by the transitivity of $\triangleleft$. If $u \triangleleft v$ and $v \equiv w$, then $u \triangleleft w$, using the definition of $\triangleleft$ and $\equiv$ and the transitivity of $\sim$ and $\prec$. Similarly, if $u \equiv v$ and $v \triangleleft w$, then $u \triangleleft w$. Finally, if $u \equiv v$ and $v \equiv w$, then $u \equiv w$, by the transitivity of $\sim$. Moreover, $\unlhd$ is reflexive, since $u \equiv u$ for any $u \in$ is $(S)$, so $\unlhd$ is a preorder. Since $\preceq$ is total, $u \nrightarrow v$ and $v \nexists u$ implies $u \equiv v$; so $\unlhd$ is total. Note also that $u \triangleleft v$ if and only if $u \unlhd v$ and $v \nsupseteq u$ as suggested by the notation.

To prove that $\unlhd$ is strictly right- $X$-invariant on is $(S)$, consider $x \in X$ and $u, v \in$ is $(S)$ such that $u \unlhd v$ and $u x, v x \in \operatorname{is}(S)$. Suppose first that $u \equiv v$, so $u$ and $v$ have the same length and $u \sim v$. Then $u x$ and $v x$ have the same length and, since $\preceq$ is right- $X$-invariant, $u x \sim v x$. So $u x \equiv v x$ and hence $u x \unlhd v x$. Now suppose that $u \triangleleft v$. If $u x \prec v x$, then $u x \triangleleft v x$. Also, if $u x \sim v x$, then, since $u \triangleleft v$, the definition of $\triangleleft$ gives $u x \triangleleft v x$. Finally, suppose towards a contradiction that $u x \npreceq v x$. Since $\preceq$ is right- $X$-invariant, $u \npreceq v$. But then, since $\preceq$ is total, $v \prec u$ and so $v \triangleleft u$, contradicting $u \triangleleft v$.

Proposition 2.8. An inverse-free equation is valid in all distributive $\ell$-monoids if and only if it is valid in $\operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle)$.

Proof. The left-to-right direction follows directly from the fact that the inverse-free reduct of $\operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle)$ is a distributive $\ell$-monoid. For the converse, suppose without loss of generality that DLM $\not \models \bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}$, where $s_{j}, t_{i} \in F_{m}(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and let $S:=\left\{\left\langle s_{j}, t_{i}\right\rangle \mid 1 \leq i \leq n, 1 \leq j \leq m\right\}$. By Lemma 2.2(a), there exists a total right- $X$-invariant preorder $\preceq$ on is $(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$. By Lemma 2.7, there exists a total strictly right-X-invariant preorder $\unlhd$ on is $(S)$ such that $\prec \subseteq \triangleleft$. In particular, $s \triangleleft t$ for each $\langle s, t\rangle \in S$. Hence, by Lemma 2.1(b), there exist a homomorphism $\psi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{A u t}(\langle\mathbb{Q}, \leq\rangle)$ and $q \in \mathbb{Q}$ such that $(q) \psi_{s_{j}}<(q) \psi_{t_{i}}$ for $1 \leq i \leq n, 1 \leq j \leq m$. So $\boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle) \not \vDash \bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}$.

The main result of this section now follows directly from Proposition 2.8 and the fact that the inverse-free reduct of any $\ell$-group is a distributive $\ell$-monoid.

Theorem 2.9. An inverse-free equation is valid in the variety of $\ell$-groups if and only if it is valid in the variety of distributive $\ell$-monoids.

It follows by Birkhoff's variety theorem [3] that DLM is generated as a variety by the class of inverse-free reducts of $\ell$-groups and hence that distributive $\ell$-monoids are precisely the homomorphic images of the inverse-free subreducts of $\ell$-groups.

Since the equational theories of the varieties of distributive lattices [10] and $\ell$ groups [5] are co-NP-complete, we also obtain the following complexity result:

Corollary 2.10. The equational theory of distributive $\ell$-monoids is co-NP-complete.

The correspondence between $\ell$-groups and distributive $\ell$-monoids established in Theorem 2.9 does not extend to inverse-free quasiequations. In particular, the quasiequation $x z \approx y z \Longrightarrow x \approx y$, describing right cancellativity, is valid in all $\ell$-groups, but not in the distributive $\ell$-monoid $\operatorname{End}(\mathbf{2})$. A further example is the quasiequation $x y \approx \mathrm{e} \Longrightarrow y x \approx \mathrm{e}$, which is clearly valid in all $\ell$-groups, but not in the distributive $\ell$-monoid $\operatorname{End}(\langle\mathbb{N}, \leq\rangle)$. To see this, define $f, g \in \operatorname{End}(\langle\mathbb{N}, \leq\rangle)$ by $(n) f:=n+1$ and $(n) g:=\max (n-1,0)$; then $(n) f g=n$ for all $n \in \mathbb{N}$, but $(0) g f=1$. Let us also remark, however, that this quasiequation is valid in any finite distributive $\ell$-monoid $\mathbf{L}$.

If $a b=\mathrm{e}$ for some $a, b \in L$, then, by finiteness, $a^{n}=a^{n+k}$ for some $n, k \in \mathbb{N}^{>0}$, so $\mathrm{e}=a^{n} b^{n}=a^{n+k} b^{n}=a^{k}$ and $b a=a^{k} b a=a^{k-1} a b a=a^{k}=\mathrm{e}$. Hence the variety of distributive $\ell$-monoids does not have the strong finite model property.

## 3. Right orders on free groups and free monoids

In this section, we use Theorem 2.9 and a characterization of valid $\ell$-group equations in LG given in [4] to relate the existence of a right order on a free monoid satisfying some finite set of inequalities to the validity of an equation in DLM (Theorem 3.3). In particular, it follows that any right order on the free monoid over a set $X$ extends to a right order on the free group over $X$ (Corollary 3.4).

Recall first that a right order on a monoid (or group) $\mathbf{M}$ is a total order $\leq$ on $M$ such that $a \leq b$ implies $a c \leq b c$ for any $a, b, c \in M$; in this case, $\mathbf{M}$ is said to be right-orderable. Left orders and left-orderability are defined symmetrically.

The following result of [4] establishes a correspondence between the validity of an equation in LG and the existence of a right order on a free group with a negative cone (or, by duality, a positive cone) containing certain elements.

Theorem 3.1 ([4, Theorem 2]). Let $s_{1}, \ldots, s_{m} \in F_{g}(X)$. Then $\mathrm{LG} \models \mathrm{e} \leq \bigvee_{j=1}^{m} s_{j}$ if and only if there is no right order $\leq$ on $\mathbf{F}_{g}(X)$ satisfying $s_{j}<\mathrm{e}$ for $1 \leq j \leq m$.

Combining this result with Theorem 2.9, we obtain a correspondence between the validity of an equation in DLM and the existence of a right order on a free monoid satisfying certain corresponding inequalities.

Proposition 3.2. Let $\varepsilon=\left(\bigwedge_{i=1}^{n} t_{i} \leq \bigvee_{j=1}^{m} s_{j}\right)$ where $s_{j}, t_{i} \in F_{m}(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then DLM $\models \varepsilon$ if and only if there is no right order $\leq$ on $\mathbf{F}_{m}(X)$ satisfying $s_{j}<t_{i}$ for $1 \leq i \leq n, 1 \leq j \leq m$.

Proof. For the left-to-right direction, suppose contrapositively that there exists a right order $\leq$ on $\mathbf{F}_{m}(X)$ satisfying $s_{j}<t_{i}$ for $1 \leq i \leq n, 1 \leq j \leq m$. Then DLM $\not \models \varepsilon$ by Lemma 2.2(a). For the converse, suppose contrapositively that DLM $\not \models \varepsilon$. By Theorem 2.9, also $\mathrm{LG} \not \vDash \varepsilon$ and, rewriting the equation,

$$
\mathrm{LG} \not \models \mathrm{e} \leq \bigvee\left\{s_{j} t_{i}^{-1} \mid 1 \leq i \leq n, 1 \leq j \leq m\right\}
$$

By Theorem 3.1, there exists a right order $\leq$ on $\mathbf{F}_{g}(X)$ such that $s_{j} t_{i}^{-1}<\mathrm{e}$, or equivalently $s_{j}<t_{i}$, for $1 \leq i \leq n, 1 \leq j \leq m$. The restriction of $\leq$ to $F_{m}(X)$ therefore provides the required right order on $\mathbf{F}_{m}(X)$.

Proposition 3.2 relates the validity of an equation in DLM to the existence of a right order extending an associated set of inequalities on a free monoid. However, it does not relate the existence of a right order on a free monoid extending a given set of inequalities
to the validity of some equation in DLM. The next result establishes such a relationship via the introduction of finitely many new variables.

Theorem 3.3. Let $s_{1}, t_{1} \ldots, s_{n}, t_{n} \in F_{m}(X)$. The following are equivalent:
(1) There exists a right order $\leq$ on $\mathbf{F}_{g}(X)$ satisfying $s_{i}<t_{i}$ for $1 \leq i \leq n$.
(2) There exists a right order $\leq$ on $\mathbf{F}_{m}(X)$ satisfying $s_{i}<t_{i}$ for $1 \leq i \leq n$.
(3) $\mathrm{DLM} \not \models \bigwedge_{i=1}^{n} t_{i} y_{i} \leq \bigvee_{i=1}^{n} s_{i} y_{i}$ for any distinct $y_{1}, \ldots, y_{n} \notin X$.

Proof. $(1) \Rightarrow(2)$. This follows directly from the fact that if $\leq$ is a right order on $\mathbf{F}_{g}(X)$, then the restriction of $\leq$ to $F_{m}(X)$ is a right order on $\mathbf{F}_{m}(X)$.
$(2) \Rightarrow(3)$. Let $\leq$ be a right order on $\mathbf{F}_{m}(X)$ satisfying $s_{i}<t_{i}$ for $1 \leq i \leq n$, assuming without loss of generality that $X$ is finite. By Lemma 2.1(b), there exists a homomor$\operatorname{phism} \psi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$ and $q \in \mathbb{Q}$ such that $(q) \psi_{s_{i}}<(q) \psi_{t_{i}}$ for $1 \leq i \leq n$. So $\operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle) \not \vDash \mathrm{e} \leq \bigvee_{i=1}^{n} s_{i} t_{i}^{-1}$ and clearly LG $\not \vDash \bigwedge_{i=1}^{n} t_{i} t_{i}^{-1} \leq \bigvee_{i=1}^{n} s_{i} t_{i}^{-1}$. But then for any distinct $y_{1}, \ldots, y_{n} \notin X$, we have LG $\not \models \bigwedge_{i=1}^{n} t_{i} y_{i} \leq \bigvee_{i=1}^{n} s_{i} y_{i}$ and therefore also DLM $\not \models \bigwedge_{i=1}^{n} t_{i} y_{i} \leq \bigvee_{i=1}^{n} s_{i} y_{i}$.
$(3) \Rightarrow(1)$. Suppose that DLM $\not \models \bigwedge_{i=1}^{n} t_{i} y_{i} \leq \bigvee_{i=1}^{n} s_{i} y_{i}$ for some distinct $y_{1}, \ldots, y_{n} \notin$ $X$. By Theorem 2.9, also LG $\not \models \bigwedge_{i=1}^{n} t_{i} y_{i} \leq \bigvee_{i=1}^{n} s_{i} y_{i}$ and, by multiplying by the inverse of the left side, LG $\not \models \mathrm{e} \leq\left(\bigvee_{i=1}^{n} s_{i} y_{i}\right)\left(\bigvee_{i=1}^{n} y_{i}^{-1} t_{i}^{-1}\right)$. But then, since LG $\models \bigvee_{i=1}^{n} s_{i} t_{i}^{-1} \leq$ $\left(\bigvee_{i=1}^{n} s_{i} y_{i}\right)\left(\bigvee_{i=1}^{n} y_{i}^{-1} t_{i}^{-1}\right)$, it follows that LG $\not \models \mathrm{e} \leq \bigvee_{i=1}^{n} s_{i} t_{i}^{-1}$. Hence, by Theorem 3.1, there exists a right order $\leq$ on $\mathbf{F}_{g}(X)$ satisfying $s_{i} t_{i}^{-1}<\mathrm{e}$, or equivalently $s_{i}<t_{i}$, for $1 \leq i \leq n$.

For any group $\mathbf{G}$ and $N \subseteq G$, there exists a right order $\leq$ on $\mathbf{G}$ satisfying $a<\mathrm{e}$ for all $a \in N$ if and only if for every finite subset $N^{\prime} \subseteq N$, there exists a right order $\leq^{\prime}$ on G satisfying $a<\mathrm{e}$ for all $a \in N^{\prime}$ (see, e.g., [11, Chapter 5, Lemma 1]). Theorem 3.3 therefore yields the following corollary:

Corollary 3.4. Every right order on the free monoid over a set $X$ extends to a right order on the free group over $X$.

Note also that by left-right duality, every left order on the free monoid over a set $X$ extends to a left order on the free group over $X$.

We conclude this section with a brief discussion of the relationship between distributive $\ell$-monoids and right-orderable monoids. It was proved in [9] that a group is right-orderable if and only if it is a subgroup of the group reduct of an $\ell$-group, and claimed in [1] that an analogous theorem holds in the setting of distributive $\ell$-monoids. Indeed, any monoid $\mathbf{M}$ that admits a right order $\leq$ embeds into the monoid reduct of the distributive $\ell$-monoid $\operatorname{End}(\langle M, \leq\rangle)$ by mapping each $a \in M$ to the order-endomorphism $x \mapsto x a$. However, contrary to the claim made in [1], it is not the case that every submonoid of the monoid reduct of a distributive $\ell$-monoid is right-orderable.

Proposition 3.5. The monoid reduct of $\operatorname{End}(\langle\Omega, \leq\rangle)$ is not right-orderable for any chain $\langle\Omega, \leq\rangle$ with $|\Omega| \geq 3$.

Proof. We first prove the claim for the distributive $\ell$-monoid $\operatorname{End}(\mathbf{3})$ of orderendomorphisms of the three-element chain $\mathbf{3}=\langle\{0,1,2\}, \leq\rangle$, using the same notation for endomorphisms as in Example 2.6. Assume towards a contradiction that $\operatorname{End}(\mathbf{3})$ admits a right order $\leq$. Note that for any $a, b, c \in \operatorname{End}(\mathbf{3})$, if $b a<c a$, then $b<c$, since otherwise $c \leq b$ would yield $c a \leq b a$. Suppose first that $\langle 0,0,2\rangle<\langle 0,1,1\rangle$. Then

$$
\langle 0,0,1\rangle=\langle 0,0,2\rangle \circ\langle 0,1,1\rangle \leq\langle 0,1,1\rangle \circ\langle 0,1,1\rangle=\langle 0,1,1\rangle
$$

and $\langle 0,0,1\rangle \circ\langle 0,1,1\rangle=\langle 0,0,1\rangle<\langle 0,1,1\rangle=\langle 0,1,2\rangle \circ\langle 0,1,1\rangle$. So $\langle 0,0,1\rangle<\langle 0,1,2\rangle$, yielding $\langle 0,0,0\rangle=\langle 0,0,1\rangle \circ\langle 0,0,1\rangle \leq\langle 0,1,2\rangle \circ\langle 0,0,1\rangle=\langle 0,0,1\rangle$. But $\langle 0,0,2\rangle<\langle 0,1,1\rangle$ also implies $\langle 0,0,1\rangle=\langle 0,0,2\rangle \circ\langle 0,0,1\rangle \leq\langle 0,1,1\rangle \circ\langle 0,0,1\rangle=\langle 0,0,0\rangle$. Hence $\langle 0,0,1\rangle=$ $\langle 0,0,0\rangle$, a contradiction. By replacing $<$ with $>$ in the above argument, $\langle 0,0,2\rangle>\langle 0,1,1\rangle$ implies $\langle 0,0,1\rangle=\langle 0,0,0\rangle$, also a contradiction. So the monoid reduct of $\operatorname{End}(\mathbf{3})$ is not right-orderable.

Now let $\langle\Omega, \leq\rangle$ be any chain with $|\Omega| \geq 3$. Without loss of generality we can assume that $\mathbf{3}$ is a subchain of $\langle\Omega, \leq\rangle$. We define a map $\varphi: \operatorname{End}(\mathbf{3}) \rightarrow \operatorname{End}(\langle\Omega, \leq\rangle)$ by fixing for each $q \in \Omega$,

$$
(q) \varphi_{f}:= \begin{cases}(\lfloor q\rfloor) f & \text { if } 0 \leq q \\ q & \text { if } q<0\end{cases}
$$

where $\lfloor q\rfloor:=\max \{k \in\{0,1,2\} \mid k \leq q\}$. Observe that $\lfloor\cdot\rfloor$ is order-preserving, so $\varphi_{f} \in$ $\operatorname{End}(\langle\Omega, \leq\rangle)$ for every $f \in \operatorname{End}(\mathbf{3})$. Also $\varphi$ is injective, since $\varphi_{f}$ restricted to $\mathbf{3}$ is $f$ for each $f \in \operatorname{End}(\mathbf{3})$. Let $f, g \in \operatorname{End}(\mathbf{3})$ and $q \in \Omega$. If $q<0$, then $(q) \varphi_{f \circ g}=q=(q)\left(\varphi_{f} \circ \varphi_{g}\right)$. Otherwise $0 \leq q$, so $(q) \varphi_{f \circ g}=((\lfloor q\rfloor) f) g=(\lfloor(\lfloor q\rfloor) f\rfloor) g=(q)\left(\varphi_{f} \circ \varphi_{g}\right)$. Hence $\varphi$ is a semigroup embedding. Finally, since the monoid reduct of $\operatorname{End}(\mathbf{3})$ is not right-orderable, it follows that the monoid reduct of $\operatorname{End}(\langle\Omega, \leq\rangle)$ is not right-orderable.

Note that, although a group is left-orderable if and only if it is right-orderable, this is not the case in general for monoids, even when they are submonoids of groups [15]. Nevertheless, a very similar argument to the one given in the proof of Proposition 3.5 shows that also the monoid of endomorphisms of any chain with at least three elements cannot be left-orderable.

## 4. From $\ell$-groups to distributive $\ell$-monoids

The validity of an equation in the variety of Abelian $\ell$-groups is equivalent to the validity of the inverse-free equation obtained by multiplying on both sides to remove inverses. Although this method fails for LG, we show here that inverses can still be
effectively eliminated from equations, while preserving validity, via the introduction of new variables. Hence, by Theorem 2.9, the validity of an equation in LG is equivalent to the validity of finitely many effectively constructed inverse-free equations in DLM (Theorem 4.2).

The following lemma shows how to remove one occurrence of an inverse from an equation while preserving validity in LG.

Lemma 4.1. Let $r, s, t, u, v \in T_{\ell}(X)$ and $y \notin X$.
(a) $\mathrm{LG} \vDash \mathrm{e} \leq v \vee s t \Longleftrightarrow \mathrm{LG} \models \mathrm{e} \leq v \vee s y \vee y^{-1} t$.
(b) $\mathrm{LG} \models u \leq v \vee s r^{-1} t \Longleftrightarrow \mathrm{LG} \models r y u \leq r y v \vee r y s y u \vee t$.

Proof. The left-to-right direction of (a) follows from the validity in LG of the quasiequation $\mathrm{e} \leq x y \vee z \Longrightarrow \mathrm{e} \leq x \vee y \vee z$ (cf. [5, Lemma 3.3]). For the converse, suppose that LG $\not \vDash \mathrm{e} \leq v \vee s t$. Then $\boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle) \not \vDash \mathrm{e} \leq v \vee s t$, by Theorem 2.5. Hence there exist a homomorphism $\varphi: \mathbf{T}_{\ell}(X) \rightarrow \boldsymbol{\operatorname { A u t }}(\langle\mathbb{Q}, \leq\rangle)$ and $q \in \mathbb{Q}$ such that $(q) \varphi_{v}<q$ and $(q) \varphi_{s t}<q$. Consider $p_{1}, p_{2} \in \mathbb{Q}$ with $p_{1}<q<p_{2}$. Since $(q) \varphi_{s}<(q) \varphi_{t^{-1}}$ and $p_{1}<p_{2}$, there exists a partial order-embedding on $\mathbb{Q}$ mapping $(q) \varphi_{s}$ to $p_{1}$ and $(q) \varphi_{t^{-1}}$ to $p_{2}$ that extends to an order-preserving bijection $\widehat{\varphi}_{y} \in \operatorname{Aut}(\langle\mathbb{Q}, \leq\rangle)$. Now let also $\widehat{\varphi}_{x}:=\varphi_{x}$ for each $x \in X$ to obtain a homomorphism $\widehat{\varphi}: \mathbf{T}_{\ell}(X \cup\{y\}) \rightarrow \boldsymbol{A u t}(\langle\mathbb{Q}, \leq\rangle)$ satisfying $q>(q) \widehat{\varphi}_{v}, q>(q) \widehat{\varphi}_{s y}$, and $q>(q) \widehat{\varphi}_{y^{-1} t}$. Hence LG $\not \models \mathrm{e} \leq v \vee s y \vee y^{-1} t$ as required.

For (b), we apply (a) to obtain

$$
\begin{aligned}
\mathrm{LG} \models u \leq v \vee s r^{-1} t & \Longleftrightarrow \mathrm{LG} \models \mathrm{e} \leq v u^{-1} \vee s r^{-1} t u^{-1} \\
& \Longleftrightarrow \mathrm{LG} \models \mathrm{e} \leq v u^{-1} \vee s y \vee y^{-1} r^{-1} t u^{-1} \\
& \Longleftrightarrow \mathrm{LG} \models r y u \leq r y v \vee r y s y u \vee t . \quad \square
\end{aligned}
$$

Eliminating variables as described in the proof of Lemma 4.1 yields an inverse-free equation that is valid in LG if and only if it is valid in DLM.

Theorem 4.2. Let $\varepsilon$ be any $\ell$-group equation with variables in a set $X$. A finite set of inverse-free equations $\Sigma$ with variables in $X \cup Y$ for some finite set $Y$ can be effectively constructed such that $\varepsilon$ is valid in all $\ell$-groups if and only if the equations in $\Sigma$ are valid in all distributive $\ell$-monoids.

Proof. Let $\varepsilon$ be any equation with variables in a set $X$. Since LG $\models s \approx t$ if and only if LG $\models \mathrm{e} \leq s^{-1} t \wedge s t^{-1}$ and every $\ell$-group term is equivalent in LG to a meet of joins of group terms, we may assume that $\varepsilon$ has the form $\mathrm{e} \leq u_{1} \wedge \cdots \wedge u_{k}$ for some joins of group terms $u_{1}, \ldots, u_{k}$. Suppose now that for each $i \in\{1, \ldots, k\}$, a finite set of inverse-free equations $\Sigma_{i}$ with variables in $X \cup Y_{i}$ for some finite set $Y_{i}$ can be effectively constructed such that $\mathrm{e} \leq u_{i}$ is valid in all $\ell$-groups if and only if the equations in $\Sigma_{i}$ are valid
in all distributive $\ell$-monoids. Then $\Sigma:=\Sigma_{1} \cup \cdots \cup \Sigma_{k}$ with variables in $X \cup Y$, where $Y:=Y_{1} \cup \cdots \cup Y_{k}$ is the finite set of inverse-free equations required by the theorem.

Generalizing slightly for the sake of the proof, it therefore suffices to define an algorithm that given as input any $t_{0} \in T_{m}(X)$ and $t_{1}, \ldots, t_{n} \in T_{g}(X)$ constructs $s_{0}, s_{1}, \ldots, s_{m} \in T_{m}(X \cup Y)$ for some finite set $Y$ such that

$$
\mathrm{LG} \models t_{0} \leq t_{1} \vee \cdots \vee t_{n} \Longleftrightarrow \mathrm{DLM} \models s_{0} \leq s_{1} \vee \cdots \vee s_{m} .
$$

If $t_{0} \leq t_{1} \vee \cdots \vee t_{n}$ is an inverse-free equation, then the algorithm outputs the same equation, which satisfies the equivalence by Theorem 2.9. Otherwise, suppose without loss of generality that $t_{1}=u x^{-1} v$. By Lemma 4.1, for any $y \notin X$,

$$
\mathrm{LG} \models t_{0} \leq t_{1} \vee \cdots \vee t_{n} \Longleftrightarrow \mathrm{LG} \models x y t_{0} \leq x y u y t_{0} \vee v \vee x y t_{2} \vee \cdots \vee x y t_{n}
$$

The equation $x y t_{0} \leq x^{\prime} u x t_{0} \vee v \vee x y t_{2} \vee \cdots \vee x y t_{n}$ contains fewer inverses than $t_{0} \leq$ $t_{1} \vee \cdots \vee t_{n}$, so iterating this procedure produces an inverse-free equation after finitely many steps.

Since the variety DLM has the finite model property (Theorem 2.3), the algorithm given in the proof of Theorem 4.2 provides an alternative proof of the decidability of the equational theory of $\ell$-groups, first established in [8].

## 5. Totally ordered monoids

In this section, we turn our attention to totally ordered monoids and groups, that is, distributive $\ell$-monoids and $\ell$-groups where the lattice order is total. We show that the variety generated by the class OM of totally ordered monoids can be axiomatized relative to DLM by a single equation (Proposition 5.4), and that there exist inverse-free equations that are valid in the class OG of totally ordered groups but not in OM (Theorem 5.7). We also prove that there is an inverse-free equation that is valid in all finite totally ordered monoids, but not in the ordered group of the integers (Proposition 5.8), showing that the variety of commutative distributive $\ell$-monoids and the varieties generated by totally ordered monoids and inverse-free reducts of totally ordered groups do not have the finite model property (Corollary 5.9). The proofs of these results build on earlier work on distributive $\ell$-monoids by Merlier [12] and Repnitskiĭ [13,14].

We begin by establishing a subdirect representation theorem for distributive $\ell$ monoids. Note first that since every distributive $\ell$-monoid $\mathbf{M}$ has a distributive lattice reduct, prime ideals of its lattice reduct exist. For a prime (lattice) ideal $I$ of a distributive $\ell$-monoid $\mathbf{M}$ and $a, b \in M$, define

$$
\frac{I}{a}:=\{\langle c, d\rangle \in M \times M \mid c a d \in I\} \quad \text { and } \quad a \sim_{I} b: \Longleftrightarrow \frac{I}{a}=\frac{I}{b} .
$$

Proposition 5.1 ([12]). Let $\mathbf{M}$ be a distributive $\ell$-monoid and let $I$ be a prime lattice ideal of $\mathbf{M}$. Then $\sim_{I}$ is an $\ell$-monoid congruence and the quotient $\mathbf{M} / I:=\mathbf{M} / \sim_{I}$ is a distributive $\ell$-monoid. Moreover, for any $a, b \in M$,

$$
[a]_{\sim_{I}} \leq[b]_{\sim_{I}} \Longleftrightarrow \frac{I}{b} \subseteq \frac{I}{a}, \quad \frac{I}{a \vee b}=\frac{I}{a} \cap \frac{I}{b}, \quad \text { and } \quad \frac{I}{a \wedge b}=\frac{I}{a} \cup \frac{I}{b} .
$$

In particular, $\mathbf{M} / I$ is totally ordered if and only if $\left\langle\left\{\left.\frac{I}{a} \right\rvert\, a \in M\right\}, \subseteq\right\rangle$ is a chain.
Proposition 5.2. Every distributive $\ell$-monoid $\mathbf{M}$ is a subdirect product of all the distributive $\ell$-monoids of the form $\mathbf{M} / I$, where $I$ is a prime ideal of $\mathbf{M}$.

Proof. Let $\mathcal{I}$ be the set of all prime lattice ideals of M. By Proposition 5.1, there exists a natural surjective homomorphism $\nu_{I}: \mathbf{M} \rightarrow \mathbf{M} / I ; a \mapsto[a]_{\sim_{I}}$ for each $I \in \mathcal{I}$. Combining these maps, we obtain a homomorphism

$$
\nu: \mathbf{M} \rightarrow \prod_{I \in \mathcal{I}} \mathbf{M} / I ; \quad a \mapsto\left(\nu_{I}(a)\right)_{I \in \mathcal{I}}
$$

It remains to show that $\nu$ is injective. Let $a, b \in M$ with $a \neq b$. By the prime ideal separation theorem for distributive lattices, there exists an $I \in \mathcal{I}$ such that, without loss of generality, $a \in I$ and $b \notin I$, yielding $\langle\mathrm{e}, \mathrm{e}\rangle \in \frac{I}{a}$ and $\langle\mathrm{e}, \mathrm{e}\rangle \notin \frac{I}{b}$. But then $\nu_{I}(a) \neq \nu_{I}(b)$ and $\nu(a) \neq \nu(b)$. So $\nu$ is a subdirect embedding.

The following lemma provides a description of the prime lattice ideals $I$ of a distributive $\ell$-monoid $\mathbf{M}$ such that $\mathbf{M} / I$ is a totally ordered monoid.

Lemma 5.3. Let $\mathbf{M}$ be a distributive $\ell$-monoid and let $I$ be a prime lattice ideal of $\mathbf{M}$. Then $\mathbf{M} / I$ is totally ordered if and only if for all $b_{1}, b_{2}, c_{1}, c_{2}, d_{1}, d_{2} \in M$,

$$
c_{1} b_{1} c_{2} \in I \quad \text { and } d_{1} b_{2} d_{2} \in I \Longrightarrow c_{1} b_{2} c_{2} \in I \text { or } d_{1} b_{1} d_{2} \in I .
$$

Proof. Suppose first that $\mathbf{M} / I$ is totally ordered and hence, by Proposition 5.1, that $\frac{I}{b_{1}} \subseteq \frac{I}{b_{2}}$ or $\frac{I}{b_{2}} \subseteq \frac{I}{b_{1}}$ for all $b_{1}, b_{2} \in M$. Then $c_{1} b_{1} c_{2} \in I$ (i.e., $\left\langle c_{1}, c_{2}\right\rangle \in \frac{I}{b_{1}}$ ) and $d_{1} b_{2} d_{2} \in I$ (i.e., $\left\langle d_{1}, d_{2}\right\rangle \in \frac{I}{b_{2}}$ ) must entail $c_{1} b_{2} c_{2} \in I$ (i.e., $\left\langle c_{1}, c_{2}\right\rangle \in \frac{I}{b_{2}}$ ) or $d_{1} b_{1} d_{2} \in I$ (i.e., $\left\langle d_{1}, d_{2}\right\rangle \in \frac{I}{b_{1}}$ ) as required. For the converse, suppose that $\mathbf{M} / I$ is not totally ordered. By Proposition 5.1, there exist $b_{1}, b_{2} \in M$ such that $\frac{I}{b_{1}} \nsubseteq \frac{I}{b_{2}}$ and $\frac{I}{b_{2}} \nsubseteq \frac{I}{b_{1}}$. That is, there exist $c_{1}, c_{2}, d_{1}, d_{2} \in M$ such that $c_{1} b_{1} c_{2} \in I$ and $d_{1} b_{2} d_{2} \in I$, but $c_{1} b_{2} c_{2} \notin I$ and $d_{1} b_{1} d_{2} \notin I$, as required.

An $\ell$-group or a distributive $\ell$-monoid is called representable if it is isomorphic to a subdirect product of members of OG or OM, respectively. The following result provides a characterization of representable distributive $\ell$-monoids in terms of their prime lattice ideals, and an equation axiomatizing the variety of these algebras relative to DLM.

Proposition 5.4. The following are equivalent for any distributive $\ell$-monoid $\mathbf{M}$ :
(1) $\mathbf{M}$ is representable.
(2) $\mathbf{M} \models\left(x_{1} \leq x_{2} \vee z_{1} y_{1} z_{2}\right) \&\left(x_{1} \leq x_{2} \vee w_{1} y_{2} w_{2}\right) \Longrightarrow x_{1} \leq x_{2} \vee z_{1} y_{2} z_{2} \vee w_{1} y_{1} w_{2}$.
(3) $\mathbf{M} \models z_{1} y_{1} z_{2} \wedge w_{1} y_{2} w_{2} \leq z_{1} y_{2} z_{2} \vee w_{1} y_{1} w_{2}$.
(4) For any prime lattice ideal I of $\mathbf{M}$, the quotient $\mathbf{M} / I$ is totally ordered.

Proof. $(1) \Rightarrow(2)$. Since quasiequations are preserved by taking direct products and subalgebras, it suffices to prove that (2) holds for the case where $\mathbf{M}$ is a totally ordered monoid. Let $a_{1}, a_{2}, b_{1}, b_{2}, c_{1}, c_{2}, d_{1}, d_{2} \in M$ satisfy $a_{1} \leq a_{2} \vee c_{1} b_{1} c_{2}$ and $a_{1} \leq a_{2} \vee d_{1} b_{2} d_{2}$. Since $\mathbf{M}$ is totally ordered, we can assume without loss of generality that $b_{1} \leq b_{2}$. It follows that $c_{1} b_{1} c_{2} \leq c_{1} b_{2} c_{2}$ and therefore $a_{1} \leq a_{2} \vee c_{1} b_{1} c_{2} \leq a_{2} \vee c_{1} b_{2} c_{2} \leq a_{2} \vee c_{1} b_{2} c_{2} \vee d_{1} b_{1} d_{2}$ as required.
$(2) \Rightarrow(3)$. Let $s_{1}:=z_{1} y_{1} z_{2}, s_{2}:=w_{1} y_{2} w_{2}, t_{1}:=z_{1} y_{2} z_{2}$, and $t_{2}:=w_{1} y_{1} w_{2}$, and suppose that $\mathbf{M} \models\left(x_{1} \leq x_{2} \vee s_{1}\right) \&\left(x_{1} \leq x_{2} \vee s_{2}\right) \Longrightarrow x_{1} \leq x_{2} \vee t_{1} \vee t_{2}$. Since $\mathbf{M} \models s_{1} \wedge s_{2} \leq t_{1} \vee t_{2} \vee s_{1}$ and $\mathbf{M} \models s_{1} \wedge s_{2} \leq t_{1} \vee t_{2} \vee s_{2}$, it follows that $\mathbf{M} \models s_{1} \wedge s_{2} \leq t_{1} \vee t_{2}$ as required.
$(3) \Rightarrow(4)$. Assume (3) and suppose that $c_{1} b_{1} c_{2} \in I$ and $d_{1} b_{2} d_{2} \in I$ for some $b_{1}, b_{2}, c_{1}, c_{2}, d_{1}, d_{1} \in M$. Since $I$ is a lattice ideal, $c_{1} b_{1} c_{2} \vee d_{1} b_{2} d_{2} \in I$. By (3) and the downwards closure of $I$, also $c_{1} b_{2} c_{2} \wedge d_{1} b_{1} d_{2} \in I$. But then, since $I$ is prime, it must be the case that either $c_{1} b_{2} c_{2} \in I$ or $d_{1} b_{1} d_{2} \in I$. Hence, by Lemma 5.3, the quotient $\mathbf{M} / I$ is totally ordered.
$(4) \Rightarrow(1)$. By (4), $\mathbf{M} / I$ is totally-ordered when $I$ is a prime ideal of $\mathbf{M}$, so representability follows by Proposition 5.2.

It follows directly from Propositions 5.2 and 5.4 that the class of representable distributive $\ell$-monoids is the variety generated by the class OM of totally ordered monoids. Similarly, it follows from these results that the class of representable $\ell$-groups is the variety generated by the class OG of totally ordered groups and is axiomatized relative to LG by $z_{1} y_{1} z_{2} \wedge w_{1} y_{2} w_{2} \leq z_{1} y_{2} z_{2} \vee w_{1} y_{1} w_{2}$. (Just observe that if the inverse-free reduct of an $\ell$-group $\mathbf{L}$ is a subdirect product of totally ordered monoids, then each component is a homomorphic image of $\mathbf{L}$ and hence a totally ordered group.) Hence, an equation is valid in these varieties if and only if it is valid in their totally ordered members.

We also obtain the following known fact:

Corollary 5.5 ([12, Corollary 2]). Commutative distributive $\ell$-monoids are representable.

Proof. By Proposition 5.4, it suffices to note that for any commutative distributive $\ell$ $\operatorname{monoid} \mathbf{M}$ and $b_{1}, b_{2}, c_{1}, c_{2}, d_{1}, d_{2} \in M$,

$$
\begin{aligned}
c_{1} b_{1} c_{2} \wedge d_{1} b_{2} d_{2} & =c_{1} c_{2} b_{1} \wedge d_{1} d_{2} b_{2} \\
& \leq\left(c_{1} c_{2} \vee d_{1} d_{2}\right) b_{1} \wedge\left(c_{1} c_{2} \vee d_{1} d_{2}\right) b_{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\left(c_{1} c_{2} \vee d_{1} d_{2}\right)\left(b_{1} \wedge b_{2}\right) \\
& =c_{1} c_{2}\left(b_{1} \wedge b_{2}\right) \vee d_{1} d_{2}\left(b_{1} \wedge b_{2}\right) \\
& \leq c_{1} c_{2} b_{2} \vee d_{1} d_{2} b_{1} \\
& =c_{1} b_{2} c_{2} \vee d_{1} b_{1} d_{2} .
\end{aligned}
$$

It is shown in [13] that there are inverse-free equations that are valid in all totally ordered Abelian groups, but not in all totally ordered commutative monoids. We make use here of just one of these equations.

Lemma 5.6 ([13, Lemma 7]). The following equation is valid in all totally ordered Abelian groups, but not in all totally ordered commutative monoids:

$$
x_{1} x_{2} x_{3} \wedge x_{4} x_{5} x_{6} \wedge x_{7} x_{8} x_{9} \leq x_{1} x_{4} x_{7} \vee x_{2} x_{5} x_{8} \vee x_{3} x_{6} x_{9}
$$

We use this result to show that the same discrepancy holds when comparing the equational theories of OM and OG.

Theorem 5.7. There is an inverse-free equation that is valid in all totally ordered groups, but not in all totally ordered monoids.

Proof. Consider the inverse-free equation $t_{1} \wedge t_{2} \leq s_{1} \vee s_{2}$, where

$$
\begin{aligned}
& t_{1}:=x_{1} x_{2} x_{3} \wedge x_{5} x_{4} x_{6} \wedge x_{9} x_{7} x_{8} ; \quad s_{1}:=x_{1} x_{4} x_{7} \vee x_{5} x_{2} x_{8} \vee x_{9} x_{6} x_{3} \\
& t_{2}:=x_{1} x_{3} x_{2} \wedge x_{5} x_{6} x_{4} \wedge x_{9} x_{8} x_{7} ; \quad s_{2}:=x_{1} x_{7} x_{4} \vee x_{5} x_{8} x_{2} \vee x_{9} x_{3} x_{6}
\end{aligned}
$$

Clearly $t_{1} \approx t_{2}$ and $s_{1} \approx s_{2}$ are valid in all totally ordered commutative monoids, so $t_{1} \wedge t_{2} \leq s_{1} \vee s_{2}$ fails in some totally ordered monoid by Lemma 5.6. It remains to show that this equation, or equivalently e $\leq\left(t_{1}^{-1} \vee t_{2}^{-1}\right)\left(s_{1} \vee s_{2}\right)$, is valid in every totally ordered group. Recall first that (cf. [5, Lemma 3.3])

$$
\begin{equation*}
\mathrm{LG} \models \mathrm{e} \leq x y \vee z \Longrightarrow \mathrm{e} \leq x \vee y \vee z \tag{1}
\end{equation*}
$$

Since LG $\models \mathrm{e} \leq \mathrm{e} \vee x_{8} x_{3}^{-1} x_{8}^{-1} x_{3}$, it follows using (1) that

$$
\begin{equation*}
\mathrm{LG} \models \mathrm{e} \leq x_{3}^{-1} x_{8} x_{3} x_{8}^{-1} \vee x_{8} x_{3}^{-1} x_{8}^{-1} x_{3} . \tag{2}
\end{equation*}
$$

An application of (1) with (2) as premise yields

$$
\begin{equation*}
\mathrm{LG} \models \mathrm{e} \leq x_{3}^{-1} x_{8} x_{6}^{-1} x_{7} \vee x_{7}^{-1} x_{6} x_{3} x_{8}^{-1} \vee x_{8} x_{3}^{-1} x_{8}^{-1} x_{3}, \tag{3}
\end{equation*}
$$

and then another application of (1) with (3) as premise yields

$$
\begin{equation*}
\mathrm{LG} \models \mathrm{e} \leq x_{3}^{-1} x_{8} x_{6}^{-1} x_{7} \vee x_{7}^{-1} x_{6} x_{3} x_{8}^{-1} \vee x_{8} x_{3}^{-1} x_{7} x_{6}^{-1} \vee x_{6} x_{7}^{-1} x_{8}^{-1} x_{3} . \tag{4}
\end{equation*}
$$

For any ordered group $\mathbf{L}$ and $a, b, c \in L$, if $\mathrm{e} \leq a b \vee c$, then either $\mathrm{e} \leq c$, or $a^{-1} \leq b$ and hence $\mathrm{e} \leq b a$, so $\mathrm{e} \leq b a \vee c$. Hence

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq x y \vee z \Longrightarrow \mathrm{e} \leq y x \vee z \tag{5}
\end{equation*}
$$

We apply (5) four times with (4) as the first premise to obtain

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq x_{7} x_{3}^{-1} x_{8} x_{6}^{-1} \vee x_{8}^{-1} x_{7}^{-1} x_{6} x_{3} \vee x_{3}^{-1} x_{7} x_{6}^{-1} x_{8} \vee x_{7}^{-1} x_{8}^{-1} x_{3} x_{6} \tag{6}
\end{equation*}
$$

For convenience, let

$$
\begin{array}{lll}
u_{1}:=x_{3}^{-1} x_{2}^{-1} x_{4} x_{7} ; & u_{2}:=x_{6}^{-1} x_{4}^{-1} x_{2} x_{8} ; & u_{3}:=x_{8}^{-1} x_{7}^{-1} x_{6} x_{3} \\
u_{4}:=x_{2}^{-1} x_{3}^{-1} x_{7} x_{4} ; & u_{5}:=x_{4}^{-1} x_{6}^{-1} x_{8} x_{2} ; & u_{6}:=x_{7}^{-1} x_{8}^{-1} x_{3} x_{6}
\end{array}
$$

An application of (1) with (6) as premise yields

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq x_{7} x_{3}^{-1} x_{2}^{-1} x_{4} \vee x_{4}^{-1} x_{2} x_{8} x_{6}^{-1} \vee u_{3} \vee x_{3}^{-1} x_{7} x_{6}^{-1} x_{8} \vee u_{6} \tag{7}
\end{equation*}
$$

Applying (5) twice with (7) as the first premise, we obtain

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq u_{1} \vee u_{2} \vee u_{3} \vee x_{3}^{-1} x_{7} x_{6}^{-1} x_{8} \vee u_{6} . \tag{8}
\end{equation*}
$$

Another application of (1) with (8) as premise yields

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq u_{1} \vee u_{2} \vee u_{3} \vee x_{3}^{-1} x_{7} x_{4} x_{2}^{-1} \vee x_{2} x_{4}^{-1} x_{6}^{-1} x_{8} \vee u_{6} . \tag{9}
\end{equation*}
$$

Applying (5) twice with (9) as the first premise, we obtain

$$
\begin{equation*}
\mathrm{OG} \models \mathrm{e} \leq u_{1} \vee u_{2} \vee u_{3} \vee u_{4} \vee u_{5} \vee u_{6} \tag{10}
\end{equation*}
$$

Observe now that for some joins of group terms $u^{\prime}, u^{\prime \prime}$,

$$
\mathrm{OG} \models t_{1}^{-1} s_{1} \approx u_{1} \vee u_{2} \vee u_{3} \vee u^{\prime} \quad \text { and } \quad \mathrm{OG} \models t_{2}^{-1} s_{2} \approx u_{4} \vee u_{5} \vee u_{6} \vee u^{\prime \prime}
$$

Hence, since OG $\models\left(t_{1}^{-1} \vee t_{2}^{-1}\right)\left(s_{1} \vee s_{2}\right) \approx t_{1}^{-1} s_{1} \vee t_{1}^{-1} s_{2} \vee t_{2}^{-1} s_{1} \vee t_{2}^{-1} s_{2}$, by (10),

$$
\mathrm{OG} \models \mathrm{e} \leq\left(t_{1}^{-1} \vee t_{2}^{-1}\right)\left(s_{1} \vee s_{2}\right)
$$

In [13], it is proved that the variety generated by the class of inverse-free reducts of Abelian $\ell$-groups is not finitely based and can be axiomatized relative to DLM by the set of inverse-free equations $s_{1} \wedge \cdots \wedge s_{n} \leq t_{1} \vee \cdots \vee t_{n}$ such that $s_{1}, \ldots, s_{n}, t_{1}, \ldots, t_{n} \in T_{m}(X)$
and $s_{1} \cdots s_{n} \approx t_{1} \cdots t_{n}$ is valid in all commutative monoids. It is not known, however, if the variety generated by the class of inverse-free reducts of totally ordered groups is finitely based. Decidability in each case of the equational theories of commutative distributive $\ell$-monoids, totally ordered monoids, and inverse-free reducts of totally ordered groups is also open. The following result shows, at least, that unlike DLM, the varieties generated by these classes do not have the finite model property.

Proposition 5.8. There is an equation that is valid in every finite totally ordered monoid, but not in $\mathbf{Z}=\langle\mathbb{Z}, \min , \max ,+, 0\rangle$.

Proof. Consider the equation $x y^{2} \leq e \vee x^{2} y^{3}$. Note that $\mathbf{Z} \not \vDash x y^{2} \leq e \vee x^{2} y^{3}$, since $(-3)+2+2=1>0=0 \vee((-3)+(-3)+2+2+2)$. We show that this equation holds in every finite totally ordered monoid $\mathbf{M}$. Suppose towards a contradiction that $a b^{2}>e \vee a^{2} b^{3}$ for some $a, b \in M$, i.e., $a b^{2}>e$ and $a b^{2}>a^{2} b^{3}$.

Observe first that, inductively, $a b^{2}>a^{2+n} b^{3+n}$ for each $n \in \mathbb{N}$. The base case $n=0$ holds by assumption, and for $n>0$, assuming $a b^{2}>a^{2+n-1} b^{3+n-1}$ yields $a b^{2}>a^{2} b^{3}=$ $a\left(a b^{2}\right) b \geq a\left(a^{2+n-1} b^{3+n-1}\right) b=a^{2+n} b^{3+n}$. Also, inductively, $a^{n} b^{2 n} \geq a b^{2}$ for each $n \in$ $\mathbb{N}^{>0}$. The base case $n=1$ is clear, and for $n>1$, assuming $a^{n-1} b^{2 n-2} \geq a b^{2}$ yields (recalling that $a b^{2}>e$ ),

$$
a^{n} b^{2 n}=a^{n-1}\left(a b^{2}\right) b^{2 n-2} \geq a^{n-1} e b^{2 n-2}=a^{n-1} b^{2 n-2} \geq a b^{2} .
$$

Finally, since $\mathbf{M}$ is finite and totally ordered, $a^{n+1}=a^{n}$ and $b^{n+1}=b^{n}$ for some $n \in \mathbb{N}$. But then $a b^{2}>a^{2+n} b^{3+n}=a^{n} b^{n}=a^{n} b^{2 n} \geq a b^{2}$, a contradiction.

Corollary 5.9. The variety of commutative distributive $\ell$-monoids and varieties generated by the classes of totally ordered monoids and inverse-free reducts of totally ordered groups do not have the finite model property.

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[^1]:    ${ }^{3}$ Recall that a variety V has the (strong) finite model property if an equation (respectively, quasiequation) is valid in V if and only if it is valid in the finite members of V .

