Semigroup Forum (2013) 86:41–58
DOI 10.1007/s00233-012-9386-2

RESEARCH ARTICLE

A semigroup approach to the Gnedenko system with single vacation of a repairman

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Received: 15 October 2011 / Accepted: 27 February 2012 / Published online: 14 March 2012

Abstract We investigate the Gnedenko system with one repairman who can take vacations. Our main focus is on the time asymptotic behaviour of the system. Using C_0-semigroup theory for linear operators we first prove the well-posedness of the system and the existence of a unique positive dynamic solution given an initial value. Then by analysing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator we show that the dynamic solution converges strongly to the steady state solution. Thus we obtain asymptotic stability of the dynamic solution.

Keywords Gnedenko system · Well-posedness · Positive C_0-semigroup · Irreducibility · Asymptotic stability

1 Introduction

Repairable systems are one of the main objects studied in reliability theory. "Repairable" means that if a failure in the system occurs it can be repaired and then the system works normally again, see [1] for more details. In general the system consists of some units under the supervision of one or more repairmen. If a unit fails then it

Communicated by Abdelaziz Rhandi.

This work was supported by the Foundation of Scientific Research Key Project of Universities Research Plan of Xinjiang Uighur Autonomous Region (No. XJEDU2010I02).

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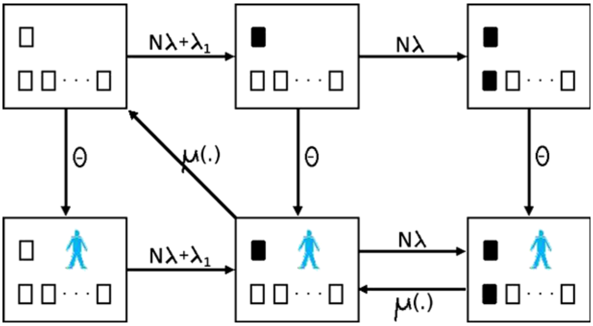

Fig. 1 State transition diagram—the *upper row* refers to those states where the repairman is in vacation while the *bottom row* refers to states in which the repairman is available. The small boxes represent the working units. The single unit in the upper row of each box is the standby unit. If a box is black then the unit is out of order

is sent to repair. However, the repairman might not be available due to other work he has to execute. So it is important to study systems where the repairman can take vacations, see [2].

The Gnedenko system is one of the classical repairable systems in reliability theory. This system is an N-unit series system supported by a warm standby unit and a single repairman. In [3], Gnedenko first considered the system and obtained the reliability function and some limit theorems for the system. In [4], Subramanian discussed the system under the assumption that the operating units and the standby unit have the same constant failure rates as the system is down, and obtained the system availability. In [5], Cao Jinhua studied the stochastic behavior of this system and obtained the explicit formula of the system availability and failure frequency using Takfics' method and a Markov renewal process.

Let us explain the system we investigate in this paper in more detail. The system consists of N working units in series, one repairman and one standby unit. If one of the working units fails then it is immediately replaced by the standby unit. The failed unit is repaired as soon as the repairman is available and then serves as a standby unit after repair. If another unit fails and the standby unit has not yet been repaired, then the whole system is down, i.e., all the other $N-1$ units stop working as well. The system will resume work once there are again N functioning units. After having repaired everything the repairman takes a vacation. If the repairman returns from a vacation and there are no failed units he remains in the system until a failure occurs.

The N working units are assumed to have the same constant failure rate $\lambda > 0$ whereas the failure rate of the standby unit is $\lambda_1 > 0$. It can happen that the standby unit fails while the system is working. Thus, for each working unit the probability that a failure occurs within time t is $1 - e^{-\lambda t}$ and for the standby unit it is $1 - e^{-\lambda_1 t}$, respectively. The probability that the repair time is less than t is

$$1 - e^{-\int_0^t \mu(x)dx}$$

where μ is an arbitrary positive continuous function. For the exact assumptions on μ see General Assumption 1.1 below. Finally, the vacation time is exponentially distributed with parameter $\theta > 0$. The state transition diagram of this model is given in Fig. 1.

To describe the system by differential equations we introduce functions p_{ij}, $i = 0, 1$, $j = 0, 1, 2$. The first index i refers to the availability of the repairman while the second index j gives information on the number of working units.

$i=0$	The repairman is in vacation.
$i=1$	The repairman is immediately available.
$j=0$	All units including the standby unit are working.
$j=1$	One unit is broken.
$j=2$	Two units are broken.

We introduce two time parameters $t \in [0,\infty)$ and $x \in [0,\infty)$. The parameter t refers to the total elapsed time whereas x counts the time during which the repairman is busy. It is reset to 0 after each repair.

Then $p_{0j}(t)$, $j=0,1,2$, gives the probability that at time t the repairman is in vacation and the number of failed units is j. Moreover, $p_{10}(t)$ represents the probability that all units are working at time t and the repairman is available. Finally, $p_{1j}(t,x)dx$, $j=1,2$, is the probability that j units have failed and the elapsed repair time lies in $(x, x+dx]$.

This leads to the following description of the system via partial differential equations, (see [6]).

$$(R)\begin{cases} \dfrac{dp_{00}(t)}{dt} = -(N\lambda+\lambda_1+\theta)p_{00}(t) + \displaystyle\int_0^\infty \mu(x)p_{11}(t,x)dx, \\ \dfrac{dp_{01}(t)}{dt} = -(N\lambda+\theta)p_{01}(t) + (N\lambda+\lambda_1)p_{00}(t), \\ \dfrac{dp_{02}(t)}{dt} = -\theta p_{02}(t) + N\lambda p_{01}(t), \\ \dfrac{dp_{10}(t)}{dt} = -(N\lambda+\lambda_1)p_{10}(t) + \theta p_{00}(t), \\ \dfrac{\partial p_{11}(t,x)}{\partial t} + \dfrac{\partial p_{11}(t,x)}{\partial x} = -(N\lambda+\mu(x)p_{11}(t,x), \\ \dfrac{\partial p_{12}(t,x)}{\partial t} + \dfrac{\partial p_{12}(t,x)}{\partial x} = -\mu(x)p_{12}(t,x) + N\lambda p_{11}(t,x), \end{cases}$$

with the boundary condition

$$(BC)\begin{cases} p_{11}(t,0) = \displaystyle\int_0^\infty p_{12}(t,x)\mu(x)dx + \theta p_{01}(t) + (N\lambda+\lambda_1)p_{10}(t), \\ p_{12}(t,0) = \theta p_{02}(t), \end{cases}$$

and the initial condition

$$(IC)\begin{cases} p_{00}(0) = 1, \\ p_{0i}(0) = 0,\ i=1,2, \\ p_{10}(0) = 0, \\ p_{1j}(0,x) = 0,\ j=1,2. \end{cases}$$

In [6], the authors analysed the system using the supplementary variables approach and generalized Markov process method and obtained some reliability expressions such as the Laplace transform of the reliability, the mean time to the first failure, the availability and the failure frequency of the system based on the following hypotheses:

Hypothesis 1. The system has a unique positive dynamic solution $p(t,x)$.
Hypothesis 2. The dynamic solution $p(t,x)$ converges to the steady state solution $p(x)$ as time tends to infinity, where

$$p(t,x) = \big(p_{00}(t), p_{01}(t), p_{02}(t), p_{10}(t), p_{11}(t,x), p_{12}(t,x)\big),$$
$$p(x) = \big(p_{00}, p_{01}, p_{02}, p_{10}, p_{11}(x), p_{12}(x)\big).$$

Motivated by this, we prove the existence of a unique positive dynamic solution of the system and study the asymptotic stability of the dynamic solution. So it turns out that for the system above these hypotheses are fulfilled.

Our main focus in this paper is on the asymptotic behaviour of the system. We closely follow the approach from [7] and [8]. We investigate the system using the theory of positive strongly continuous semigroups. It is well-known that semigroup theory provides a useful tool to study queueing and reliability problems, see for example [9, 10] and [11]. Even though the ideas we use in this paper are not new and we adapt only the results from [7] and [8], these ideas have—to the best of our knowledge—not yet been applied to the Gnedenko system and they lead to some new insights in the description of the asymptotic behaviour of this system.

For background reading on semigroup theory we refer to [12] and [13] or [14] for a brief introduction. To apply this theory we first transform the system above into an abstract Cauchy problem in Sect. 2. Then in Sect. 3 we investigate the spectrum of the operators involved and show in Sect. 4 the well-posedness of the system. Using the information on the spectrum from Sect. 3 combined with the irreducibility of the semigroup generated by the system operator we finally obtain our main results on the asymptotic behaviour. We prove that the dynamic solution converges strongly to the steady state solution which is the eigenfunction corresponding to eigenvalue 0 of the system operator. Thus we obtain the asymptotic stability of the dynamic solution of this system.

Throughout the paper we require the following assumption for the failure rate function $\mu(x)$.

General Assumption 1.1 *The function* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *is measurable and bounded such that* $\lim_{x\to\infty} \mu(x)$ *exists and*

$$\mu_\infty = \lim_{x\to\infty} \mu(x) > 0.$$

2 The problem as an abstract Cauchy problem

To apply semigroup theory we transform in this section the system (R), (BC), (IC) into an abstract Cauchy problem [12, Def. II.6.1] on the Banach space $(X, \|\cdot\|)$ where

$$X = \mathbb{C}^4 \times \left(L^1[0,\infty)\right)^2$$

and

$$\|p\| = \sum_{i=0}^{2} |p_{0i}| + |p_{10}| + \sum_{j=1}^{2} \|p_{1j}\|_{L^1[0,\infty)},$$

$$p = \left(p_{00}, p_{01}, p_{02}, p_{10}, p_{11}(x), p_{12}(x)\right)^t \in X.$$

The space $(X, \|\cdot\|)$ will also be called *state space*.

In the following we need the functional

$$\psi : L^1[0,\infty) \to \mathbb{C}, \qquad f \mapsto \psi(f) = \int_0^\infty \mu(x) f(x) dx.$$

and the operators

$$\tilde{D}_1 : W^{1,1}[0,\infty) \to L^1[0,\infty), \qquad f \mapsto -\frac{d}{dx} f - \left(N\lambda + \mu(x)\right) f,$$

$$\tilde{D}_2 : W^{1,1}[0,\infty) \to L^1[0,\infty), \qquad f \mapsto -\frac{d}{dx} f - \mu(x) f.$$

In a first step we introduce a *maximal operator* $(A_m, D(A_m))$ describing only (R). It is given by

$$A_m = \begin{pmatrix} -(N\lambda+\lambda_1+\theta) & 0 & 0 & 0 & \psi & 0 \\ N\lambda+\lambda_1 & -(N\lambda+\theta) & 0 & 0 & 0 & 0 \\ 0 & N\lambda & -\theta & 0 & 0 & 0 \\ \theta & 0 & 0 & -(N\lambda+\lambda_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{D}_1 & 0 \\ 0 & 0 & 0 & 0 & N\lambda & \tilde{D}_2 \end{pmatrix},$$

$$D(A_m) = \mathbb{C}^4 \times \left(W^{1,1}[0,\infty)\right)^2.$$

It does not yet take into account the boundary conditions (BC). To this purpose we introduce the *boundary space*

$$\partial X := \mathbb{C}^2,$$

and then define *boundary operators* L and Φ by

$$L : D(A_m) \to \partial X, \qquad \begin{pmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11}(x) \\ p_{12}(x) \end{pmatrix} \mapsto L \begin{pmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11}(x) \\ p_{12}(x) \end{pmatrix} := \begin{pmatrix} p_{11}(0) \\ p_{12}(0) \end{pmatrix},$$

and

$$\Phi : X \to \partial X,$$

$$\begin{pmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11}(x) \\ p_{12}(x) \end{pmatrix} \mapsto \Phi \begin{pmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11}(x) \\ p_{12}(x) \end{pmatrix} := \begin{pmatrix} 0 & \theta & 0 & N\lambda + \lambda_1 & 0 & \psi \\ 0 & 0 & \theta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11}(x) \\ p_{12}(x) \end{pmatrix}.$$

We have that $L \in \mathcal{L}([D(A_m)], \partial X)$ where $[D(A_m)]$ denotes the space $D(A_m)$ endowed with the graph norm induced by A_m and $\Phi \in \mathcal{L}(X, \partial X)$. Now the *system operator* $(A, D(A))$ on X given by

$$Ap := A_m p, \qquad D(A) := \{p \in D(A_m) \mid Lp = \Phi p\}.$$

describes the system completely, i.e. the equations (R), (BC) and (IC) are equivalent to the abstract Cauchy problem

$$\begin{cases} \dfrac{dp(t)}{dt} = Ap(t), \quad t \in [0, \infty), \\ p(0) = (1, 0, 0, 0, 0, 0)^t \in X. \end{cases} \tag{ACP}$$

Note that $(A, D(A))$ is a closed operator. In fact, if $(x_n) \subseteq D(A)$, $x_n \to x \in X$ and $Ax_n \to y \in X$, then $x \in D(A_m)$ and $A_m x = y$, since A_m is a closed operator. Moreover, by the continuity of the operators L and Φ we have $Lx_n \to Lx$ and $\Phi x_n \to \Phi x$. Since $Lx_n = \Phi x_n$ we conclude that $Lx = \Phi x$. Hence, $x \in D(A)$, $Ax = A_m x = y$ and thus A is closed.

3 Boundary spectrum

In this section we investigate the spectrum $\sigma(A)$ of A. In particular, the boundary of $\sigma(A)$ is of interest to us. We obtain information on the spectrum of A by a *characteristic equation* (see Characteristic Equation 3.5 below) which relates $\sigma(A)$ to the spectrum of an operator on the boundary space ∂X. Clearly, it is in general much easier to determine the spectrum of an operator in the boundary space than to determine $\sigma(A)$ directly. The abstract background can be found in [15]. It has been widely used since then, see [16, 17] or [7] to mention a few applications.

Before we can give the explicit form of the characteristic equation we have to introduce the relevant operators. We start from the operator $(A_0, D(A_0))$ defined by

$$D(A_0) := \{p \in D(A_m) \mid Lp = 0\},$$
$$A_0 p := A_m p.$$

We give the representation of the resolvent of A_0 needed in Section 5 to prove the irreducibility of the semigroup generated by A.

Lemma 3.1 *For the set* $S := \{\gamma \in \mathbb{C} \mid \Re\gamma > -\mu_\infty\} \setminus \{-\theta, -(N\lambda+\theta), -(N\lambda+\lambda_1+\theta), -(N\lambda+\lambda_1)\}$ *we have*

$$S \subseteq \rho(A_0).$$

Moreover, if $\gamma \in S$, *then*

$$R(\gamma, A_0) = \begin{pmatrix} s_{1,1} & 0 & 0 & 0 & s_{1,5} & 0 \\ s_{2,1} & s_{2,2} & 0 & 0 & s_{2,5} & 0 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & s_{3,5} & 0 \\ s_{4,1} & 0 & 0 & s_{4,4} & s_{4,5} & 0 \\ 0 & 0 & 0 & 0 & s_{5,5} & 0 \\ 0 & 0 & 0 & 0 & s_{6,5} & s_{6,6} \end{pmatrix}, \tag{1}$$

where

$$s_{1,1} = \frac{1}{\gamma + N\lambda + \lambda_1 + \theta},$$

$$s_{1,5} = \frac{1}{\gamma + N\lambda + \lambda_1 + \theta} \psi R(\gamma, \tilde{D}_{1,0}),$$

$$s_{2,1} = \frac{N\lambda + \lambda_1}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)},$$

$$s_{2,2} = \frac{1}{\gamma + N\lambda + \theta},$$

$$s_{2,5} = \frac{(N\lambda + \lambda_1)}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)} \psi R(\gamma, \tilde{D}_{1,0}),$$

$$s_{3,1} = \frac{N\lambda(N\lambda + \lambda_1)}{(\gamma + \theta)(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)},$$

$$s_{3,2} = \frac{N\lambda}{(\gamma + \theta)(\gamma + N\lambda + \theta)},$$

$$s_{3,3} = \frac{1}{\gamma + \theta},$$

$$s_{3,5} = \frac{N\lambda(N\lambda + \lambda_1)}{(\gamma + \theta)(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)} \psi R(\gamma, \tilde{D}_{1,0}),$$

$$s_{4,1} = \frac{\theta}{(\gamma + N\lambda + \lambda_1)(\gamma + N\lambda + \lambda_1 + \theta)},$$

$$s_{4,4} = \frac{1}{\gamma + N\lambda + \lambda_1},$$

$$s_{4,5} = \frac{\theta}{(\gamma + N\lambda + \lambda_1)(\gamma + N\lambda + \lambda_1 + \theta)} \psi R(\gamma, \tilde{D}_{1,0}),$$

$$s_{5,5} = R(\gamma, \tilde{D}_{1,0}),$$

$$s_{6,5} = N\lambda R(\gamma, \tilde{D}_{2,0}) R(\gamma, \tilde{D}_{1,0}),$$

$$s_{6,6} = R(\gamma, \tilde{D}_{2,0}).$$

The resolvent operators of the differential operators $\tilde{D}_{j,0}$ *where* $\tilde{D}_{j,0} = \tilde{D}_j$ *with domain* $D(\tilde{D}_{j,0}) = \{u \in W^{1,1}(0,\infty) \, : \, u(0) = 0\}$, $j = 1, 2$, *are given by*

$$\big(R(\gamma, \tilde{D}_{1,0})p\big)(x) = e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} \int_0^x e^{(\gamma+N\lambda)s + \int_0^s \mu(\xi)d\xi} p(s)ds,$$

$$\big(R(\gamma, \tilde{D}_{2,0})p\big)(x) = e^{-\gamma x - \int_0^x \mu(\xi)d\xi} \int_0^x e^{\gamma s + \int_0^s \mu(\xi)d\xi} p(s)ds, \quad \textit{for } p \in L^1[0,\infty).$$

Proof Combining [18, Prop. 2.1] and [19, Thm. 2.4] we see that

$$\rho(A_0) \supseteq S$$

is satisfied. Moreover, we can use the formula for the inverse of operator matrices given in [19, Thm. 2.4] to check the representation (1) of the resolvent of A_0.

Clearly, knowing the operator matrix in (1), we can directly compute that it represents the resolvent of A_0. □

For future reference we state the following corollary. It will be useful later to locate the boundary spectrum of A.

Corollary 3.2 *The imaginary axis belongs to the resolvent set of* A_0, *i.e.*,

$$i\mathbb{R} \subseteq \rho(A_0).$$

In a next step we consider the restriction of the boundary operator L to $\ker(\gamma - A_m)$, $\gamma \in \mathbb{C}$. By a direct computation we obtain the explicit form of the elements in $\ker(\gamma - A_m)$ as follows.

Lemma 3.3 *For* $\gamma \in \rho(A_0)$, *we have*

$$p \in \ker(\gamma - A_m) \tag{2}$$

$$\Leftrightarrow \quad p = \big(p_{00}, p_{01}, p_{02}, p_{10}, p_{11}(x), p_{12}(x)\big)^t \in D(A_m), \quad \textit{with}$$

$$p_{00} = \frac{c_1}{\gamma + N\lambda + \lambda_1 + \theta} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx, \tag{3}$$

$$p_{01} = \frac{c_1(N\lambda + \lambda_1)}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx, \tag{4}$$

$$p_{02} = \frac{c_1 N\lambda(N\lambda + \lambda_1)}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)(\gamma + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx, \tag{5}$$

$$p_{10} = \frac{c_1\theta}{(\gamma + N\lambda + \lambda_1)(\gamma + N\lambda + \lambda_1 + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx, \tag{6}$$

$$p_{11}(x) = c_1 e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi}, \tag{7}$$

$$p_{12}(x) = \left[c_1\left(1 - e^{-N\lambda x}\right) + c_2\right] e^{-\gamma x - \int_0^x \mu(\xi)d\xi}, \tag{8}$$

where $c_1, c_2 \in \mathbb{C}$.

Using [15, Lemma 1.2], for $\gamma \in \rho(A_0)$ the domain $D(A_m)$ of the maximal operator A_m decomposes as

$$D(A_m) = D(A_0) \oplus \ker(\gamma - A_m). \tag{9}$$

Moreover, since L is surjective,

$$L|_{\ker(\gamma - A_m)} : \ker(\gamma - A_m) \to \partial X$$

is invertible for each $\gamma \in \rho(A_0)$, see [15, Lemma 1.2]. We denote its inverse by

$$D_\gamma := (L|_{\ker(\gamma - A_m)})^{-1} : \partial X \longrightarrow \ker(\gamma - A_m)$$

and call it "Dirichlet operator".

We can give the explicit form of D_γ as follows.

Lemma 3.4 *For each* $\gamma \in \rho(A_0)$, *the operator* D_γ *has the form*

$$D_\gamma = \begin{pmatrix} d_{1,1} & 0 \\ d_{2,1} & 0 \\ d_{3,1} & 0 \\ d_{4,1} & 0 \\ d_{5,1} & 0 \\ d_{6,1} & d_{6,2} \end{pmatrix},$$

where

$$d_{1,1} = \frac{1}{\gamma + N\lambda + \lambda_1 + \theta} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx,$$

$$d_{2,1} = \frac{N\lambda + \lambda_1}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx,$$

$$d_{3,1} = \frac{N\lambda(N\lambda + \lambda_1)}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)(\gamma + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx,$$

$$d_{4,1} = \frac{\theta}{(\gamma + N\lambda + \lambda_1)(\gamma + N\lambda + \lambda_1 + \theta)} \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx,$$

$$d_{5,1} = e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi},$$

$$d_{6,1} = \left(1 - e^{-N\lambda x}\right) e^{-\gamma x - \int_0^x \mu(\xi)d\xi},$$

$$d_{6,2} = e^{-\gamma x - \int_0^x \mu(\xi)d\xi}.$$

We are now ready to state the characteristic equation for the spectrum of A. It relates the spectrum of A to the spectrum of operators in the boundary space $\mathbb{C}^2$. We omit the proof. It can be found in [17, Prop. 3.3].

Characteristic Equation 3.5 *Let* $\gamma \in \rho(A_0)$. *Then*

(i)

$$\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(\Phi D_\gamma).$$

(ii) *If, in addition, there exists* $\gamma_0 \in \mathbb{C}$ *such that* $1 \notin \sigma(\Phi D_{\gamma_0})$, *then*

$$\gamma \in \sigma(A) \iff 1 \in \sigma(\Phi D_\gamma).$$

In our situation the operator ΦD_γ, $\gamma \in \rho(A_0)$, is a 2×2-matrix

$$\Phi D_\gamma = \begin{pmatrix} a_{1,1}^\gamma & a_{1,2}^\gamma \\ a_{2,1}^\gamma & 0 \end{pmatrix},$$

where

$$\begin{aligned} a_{1,1}^\gamma = {} & \left[\frac{(N\lambda + \lambda_1)\theta}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)} + \frac{(N\lambda + \lambda_1)\theta}{(\gamma + N\lambda + \lambda_1)(\gamma + N\lambda + \lambda_1 + \theta)} \right] \\ & \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx \\ & + \int_0^\infty \mu(x) \left(1 - e^{-N\lambda x}\right) e^{-\gamma x - \int_0^x \mu(\xi)d\xi} dx, \\ a_{1,2}^\gamma = {} & \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi)d\xi} dx, \\ a_{2,1}^\gamma = {} & \frac{N\lambda(N\lambda + \lambda_1)\theta}{(\gamma + N\lambda + \theta)(\gamma + N\lambda + \lambda_1 + \theta)(\gamma + \theta)} \\ & \times \int_0^\infty \mu(x) e^{-(\gamma+N\lambda)x - \int_0^x \mu(\xi)d\xi} dx. \end{aligned}$$

By a straightforward calculation we see that ΦD_0 is column stochastic and thus $1 \in \sigma_p(\Phi D_0)$. Applying the Characteristic Equation 3.5 (i) immediately yields the following lemma.

Lemma 3.6 *For the operator* $(A, D(A))$ *we have* $0 \in \sigma_p(A)$.

Again with the help of the Characteristic Equation 3.5 we can even show that 0 is the only spectral value of A on the imaginary axis.

Lemma 3.7 *Under the General Assumption* 1.1, *the spectrum* $\sigma(A)$ *of* A *satisfies*

$$\sigma(A) \cap i\mathbb{R} = \{0\}.$$

Proof By the Characteristic Equation 3.5 it suffices to show that $1 \notin \sigma(\Phi D_{ir})$ for all $r \in \mathbb{R} \setminus \{0\}$.

Clearly, this is satisfied if the entries of ΦD_{ir} fulfill

$$\begin{aligned} \left|a_{1,1}^{ir}\right| + \left|a_{2,1}^{ir}\right| &< 1, \\ \left|a_{1,2}^{ir}\right| &< 1, \end{aligned}$$

for every $r \in \mathbb{R} \setminus \{0\}$. The same computation as in [7, Lemma 3.7] now shows that this is indeed true. Note that in this step we make use of the General Assumption 1.1. □

4 Well-posedness of the system

The main goal in this section is to prove the well-posedness of the system in the sense of [12, Def. II.6.8]. This is equivalent to $(A, D(A))$ being the generator of a C_0-semigroup, see [12, Thm. II.6.7]. We will prove using Phillips' theorem [13, Thm. C-II 1.2] that $(A, D(A))$ in fact generates a positive contraction C_0-semigroup $(T(t))_{t\geq 0}$. To this purpose let us check that $(A, D(A))$ fulfills the conditions in Phillips' theorem, namely that $D(A)$ is dense in X, $(A, D(A))$ is dispersive and $\gamma - A$ is surjective for some $\gamma > 0$.

The following lemma in particular shows the surjectivity of $\gamma - A$ for any $\gamma > 0$.

Lemma 4.1 *Let* $\gamma > 0$. *Then* $\gamma \in \rho(A)$.

Proof Let $\gamma \in \mathbb{R}, \gamma > 0$. Then all the entries of ΦD_γ are positive and one can show using only elementary calculations that both column sums are strictly less than 1. Hence, $\|\Phi D_\gamma\| < 1$, and thus $1 \notin \sigma(\Phi D_\gamma)$. Using the Characteristic Equation 3.5 we conclude that $\gamma \in \rho(A)$. □

Lemma 4.2 *The operator* $(A, D(A))$ *is closed and densely defined.*

Proof The closedness of $(A, D(A))$ has already been observed at the end of Sect. 2. Moreover, for $\gamma \in \rho(A_0) \cap \rho(A)$, one has $\gamma - A = (\gamma - A_0)(Id - D_\gamma \Phi)$, where $Id - D_\gamma \Phi \in \mathcal{L}(X)$ is an isomorphism, see [15, Lemma 1.4]. Since $D(A_0)$ is dense in X, this implies that $D(A) = (Id - D_\gamma \Phi)^{-1} D(A_0)$ is dense as well. □

If X' denotes the dual space of X, then

$$X' = \mathbb{C}^4 \times \left(L^\infty[0,\infty)\right)^2.$$

It is obvious that X' is a Banach space endowed with the norm

$$\|q\| := \max\left(|q_{00}|, |q_{01}|, |q_{02}|, |q_{10}|, \|q_{11}\|_{L^\infty[0,\infty)}, \|q_{12}\|_{L^\infty[0,\infty)}\right),$$

where $q = (q_{00}, q_{01}, q_{02}, q_{10}, q_{11}(x), q_{12}(x))^t \in X'$.

Lemma 4.3 *The operator* $(A, D(A))$ *is dispersive.*

Proof For $p = (p_{00}, p_{01}, p_{02}, p_{10}, p_{11}(x), p_{12}(x))^t \in D(A)$, we define

$$q = \left(q_{00}, q_{01}, q_{02}, q_{10}, q_{11}(x), q_{12}(x)\right)^t \in X',$$

where

$$q_{0i} = \|p\| \operatorname{sgn}_+(p_{0i}), \quad i = 0, 1, 2, \qquad q_{10} = \|p\| \operatorname{sgn}_+(p_{10}),$$
$$q_{1j}(x) = \|p\| \operatorname{sgn}_+\left(p_{1j}(x)\right), \quad j = 1, 2$$

and

$$\operatorname{sgn}_+(p_{0i}) = \begin{cases} 1 & \text{if } p_{0i} > 0, \\ 0 & \text{if } p_{0i} \le 0, \end{cases} \quad i = 0, 1, 2, \qquad \operatorname{sgn}_+(p_{10}) = \begin{cases} 1 & \text{if } p_{10} > 0, \\ 0 & \text{if } p_{10} \le 0, \end{cases}$$
$$\operatorname{sgn}_+\left(p_{1j}(x)\right) = \begin{cases} 1 & \text{if } p_{1j}(x) > 0, \\ 0 & \text{if } p_{1j}(x) \le 0, \end{cases} \quad j = 1, 2.$$

Moreover, for a real number r we use the notation

$$[r]^+ = \begin{cases} r & \text{if } r > 0, \\ 0 & \text{if } r \le 0. \end{cases}$$

If we define $L_{1j} = \{x \in [0,\infty) | p_{1j}(x) > 0\}$ and $M_{1j} = \{x \in [0,\infty) | p_{1j}(x) \le 0\}$ for $j = 1, 2$, then we have

$$\begin{aligned} &\int_0^\infty \frac{dp_{1j}(x)}{dx} \operatorname{sgn}_+\left(p_{1j}(x)\right) dx \\ &\quad = \int_{L_{1j}} \frac{dp_{1j}(x)}{dx} \operatorname{sgn}_+\left(p_{1j}(x)\right) dx + \int_{M_{1j}} \frac{dp_{1j}(x)}{dx} \operatorname{sgn}_+\left(p_{1j}(x)\right) dx \\ &\quad = \int_{L_{1j}} \frac{dp_{1j}(x)}{dx} \operatorname{sgn}_+\left(p_{1j}(x)\right) dx \\ &\quad = \int_{L_{1j}} \frac{dp_{1j}(x)}{dx} dx \end{aligned}$$

$$= \int_0^\infty \frac{d[p_{1j}(x)]^+}{dx} dx$$
$$= -\left[p_{1j}(0)\right]^+, \quad j = 1, 2, \tag{10}$$

$$\begin{aligned} &p_{00} \operatorname{sgn}_+(p_{01}) \le [p_{00}]^+, \qquad p_{00} \operatorname{sgn}_+(p_{10}) \le [p_{00}]^+, \\ &p_{01} \operatorname{sgn}_+(p_{02}) \le [p_{01}]^+, \end{aligned} \tag{11}$$

$$\int_0^\infty \mu(x) p_{11}(x) \operatorname{sgn}_+(p_{00}) dx \le \int_0^\infty \mu(x)\left[p_{11}(x)\right]^+ dx, \tag{12}$$

$$\int_0^\infty \mu(x) p_{ij}(x) \operatorname{sgn}_+(p_{1j}) dx \le \int_0^\infty \mu(x)\left[p_{1j}(x)\right]^+ dx, \quad j = 1, 2, \tag{13}$$

$$\begin{aligned} \sum_{j=1}^{2}\left[p_{1j}(0)\right]^+ &= \left[\int_0^\infty p_{12}(x)\mu(x)dx + \theta p_{01} + (N\lambda + \lambda_1)p_{10}\right]^+ + [\theta p_{02}]^+ \\ &\le \theta[p_{01}]^+ + \theta[p_{02}]^+ + (N\lambda + \lambda_1)[p_{10}]^+ \\ &\quad + \int_0^\infty \mu(x)\left[p_{12}(x)\right]^+ dx. \end{aligned} \tag{14}$$

Using (10), (11), (12), (13), (14) and the boundary conditions on $p \in D(A)$ we obtain that

$$\begin{aligned} \langle Ap, q\rangle &= \left[-(N\lambda + \lambda_1 + \theta)p_{00} + \int_0^\infty \mu(x)p_{11}(x)dx\right]\|p\| \operatorname{sgn}_+(p_{00}) \\ &\quad + \left[-(N\lambda + \theta)p_{01} + (N\lambda + \lambda_1)p_{00}\right]\|p\| \operatorname{sgn}_+(p_{01}) \\ &\quad + [-\theta p_{02} + N\lambda p_{01}]\|p\| \operatorname{sgn}_+(p_{02}) \\ &\quad + \left[-(N\lambda + \lambda_1)p_{10} + \theta p_{00}\right]\|p\| \operatorname{sgn}_+(p_{10}) \\ &\quad + \int_0^\infty \left[-\frac{dp_{11}(x)}{dx} - \big(N\lambda + \mu(x)\big)p_{11}(x)\right]\|p\| \operatorname{sgn}_+\big(p_{11}(x)\big)dx \\ &\quad + \int_0^\infty \left[-\frac{dp_{12}(x)}{dx} - \mu(x)p_{12}(x) + N\lambda p_{11}(x)\right]\|p\| \operatorname{sgn}_+\big(p_{12}(x)\big)dx \\ &= \|p\|\Bigg\{\left[-(N\lambda + \lambda_1 + \theta)p_{00} \operatorname{sgn}_+(p_{00}) + \int_0^\infty \mu(x)p_{11}(x) \operatorname{sgn}_+(p_{00})dx\right] \\ &\quad + \left[-(N\lambda + \theta)p_{01} \operatorname{sgn}_+(p_{01}) + (N\lambda + \lambda_1)p_{00} \operatorname{sgn}_+(p_{01})\right] \\ &\quad + \left[-\theta p_{02} \operatorname{sgn}_+(p_{02}) + N\lambda p_{01} \operatorname{sgn}_+(p_{02})\right] \\ &\quad + \left[-(N\lambda + \lambda_1)p_{10} \operatorname{sgn}_+(p_{10}) + \theta p_{00} \operatorname{sgn}_+(p_{10})\right] \\ &\quad + \int_0^\infty \Bigg[-\frac{dp_{11}(x)}{dx} \operatorname{sgn}_+\big(p_{11}(x)\big) \end{aligned}$$

$$
\begin{aligned}
&- \big(N\lambda + \mu(x)\big) p_{11}(x)\, \mathrm{sgn}_+\big(p_{11}(x)\big)\Bigg]dx \\
&+ \int_0^\infty \Bigg[-\frac{dp_{12}(x)}{dx}\, \mathrm{sgn}_+\big(p_{12}(x)\big) - \mu(x) p_{12}(x)\, \mathrm{sgn}_+\big(p_{12}(x)\big) \\
&+ N\lambda p_{11}(x)\, \mathrm{sgn}_+\big(p_{12}(x)\big)\Bigg]dx \\
\le \|p\| \Bigg\{ &\Bigg[-(N\lambda + \lambda_1 + \theta)[p_{00}]^+ + \int_0^\infty \mu(x)\big[p_{11}(x)\big]^+ dx\Bigg] \\
&+ \big[-(N\lambda + \theta)[p_{01}]^+ + (N\lambda + \lambda_1)[p_{00}]^+\big] + \big[-\theta[p_{02}]^+ + N\lambda[p_{01}]^+\big] \\
&+ \big[-(N\lambda + \lambda_1)[p_{10}]^+ + \theta[p_{00}]^+\big] + \int_0^\infty \big[-\big(N\lambda + \mu(x)\big)\big[p_{11}(x)\big]^+\big]dx \\
&+ \int_0^\infty \big[-\mu(x)\big[p_{12}(x)\big]^+ + N\lambda\big[p_{11}(x)\big]^+\big]dx + \sum_{j=1}^{2}\big[p_{1j}(0)\big]^+ \Bigg\} \\
= \|p\| \Bigg\{ &-\theta[p_{01}]^+ - \theta[p_{02}]^+ - (N\lambda + \lambda_1)[p_{10}]^+ - \int_0^\infty \mu(x)\big[p_{12}(x)\big]^+ dx \\
&+ \sum_{j=1}^{2}\big[p_{1j}(0)\big]^+ \Bigg\} \le 0.
\end{aligned}
$$

By [13, p. 249] we obtain that $(A, D(A))$ is a dispersive operator. □

From Lemma 4.1, Lemma 4.2 and Lemma 4.3 we see that all the conditions in Phillips' theorem [13, Thm. C-II 1.2] are fulfilled and thus we obtain the following result.

Theorem 4.4 *The operator $(A, D(A))$ generates a positive contraction C_0-semigroup $(T(t))_{t\ge 0}$.*

Using [12, Cor. II.6.9] we thus have well-posedness of the underlying problem.

Theorem 4.5 *The abstract Cauchy problem (ACP) associated to A is well-posed.*

We return now to our original problem (R), (BC) and (IC) and formulate our main result.

Theorem 4.6 *The system (R), (BC) and (IC) has a unique positive solution $p(t,x)$ which satisfies $\|p(t,\cdot)\| = 1,\ t \in [0,\infty)$.*

Proof From Theorem 4.4, Theorem 4.5 and [12, Prop. II.6.2] we obtain that the associated abstract Cauchy problem (ACP) has a unique positive dynamic solution $p(x,t)$

which can be expressed as

$$p(t,x) = T(t)p(0) = T(t)(1,0,0,0,\dots). \tag{15}$$

Let $P(t) = p(t,x) = (p_{00}(t), p_{01}(t), p_{02}(t), p_{10}(t), p_{11}(t,x), p_{12}(t,x))$, then $P(t)$ satisfies the system of equations:

$$\frac{dp_{00}(t)}{dt} = -(N\lambda + \lambda_1 + \theta)p_{00}(t) + \int_0^\infty \mu(x)p_{11}(t,x)dx, \tag{16}$$

$$\frac{dp_{01}(t)}{dt} = -(N\lambda + \theta)p_{01}(t) + (N\lambda + \lambda_1)p_{00}(t), \tag{17}$$

$$\frac{dp_{02}(t)}{dt} = -\theta p_{02}(t) + N\lambda p_{01}(t), \tag{18}$$

$$\frac{dp_{10}(t)}{dt} = -(N\lambda + \lambda_1)p_{10}(t) + \theta p_{00}(t), \tag{19}$$

$$\frac{\partial p_{11}(t,x)}{\partial t} = -\frac{\partial p_{11}(t,x)}{\partial x} - \big(N\lambda + \mu(x)\big)p_{11}(t,x), \tag{20}$$

$$\frac{\partial p_{12}(t,x)}{\partial t} = -\frac{\partial p_{12}(t,x)}{\partial x} - \mu(x)p_{12}(t,x) + N\lambda p_{11}(t,x), \tag{21}$$

$$p_{11}(t,0) = \int_0^\infty p_{12}(t,x)\mu(x)dx + \theta p_{01}(t) + (N\lambda + \lambda_1)p_{10}(t), \tag{22}$$

$$p_{12}(t,0) = \theta p_{02}(t), \tag{23}$$

$$P(0) = (1,0,0,0,0,\dots). \tag{24}$$

Since

$$\int_0^\infty \frac{\partial p_{1j}(t,x)}{\partial x}dx = p_{1j}(t,\infty) - p_{1j}(t,0) = -p_{1j}(t,0), \quad j = 1,2. \tag{25}$$

Using (16)–(25) we compute

$$\begin{aligned}
\frac{d\|P(t)\|}{dt} &= \sum_{i=0}^{2}\frac{dp_{0i}(t)}{dt} + \frac{dp_{10}(t)}{dt} + \sum_{j=1}^{2}\int_0^\infty \frac{\partial p_{1j}(t,x)}{\partial t}dx \\
&= -(N\lambda + \lambda_1 + \theta)p_{00}(t) + \int_0^\infty \mu(x)p_{11}(t,x)dx \\
&\quad - (N\lambda + \theta)p_{01}(t) + (N\lambda + \lambda_1)p_{00}(t) \\
&\quad - \theta p_{02}(t) + N\lambda p_{01}(t) \\
&\quad - (N\lambda + \lambda_1)p_{10}(t) + \theta p_{00}(t) \\
&\quad + \int_0^\infty \left[-\frac{\partial p_{11}(t,x)}{\partial x} - \big(N\lambda + \mu(x)\big)p_{11}(t,x)\right]dx \\
&\quad + \int_0^\infty \left[-\frac{\partial p_{12}(t,x)}{\partial x} - \mu(x)p_{12}(t,x) + N\lambda p_{11}(t,x)\right]dx
\end{aligned}$$

$$= -\int_0^\infty p_{12}(t,x)\mu(x)dx - \theta p_{01}(t) - (N\lambda + \lambda_1)p_{10}(t)$$

$$- \theta p_{02}(t) + \sum_{j=1}^{2} p_1 j(t,0)$$

$$= -\sum_{j=1}^{2} p_{1j}(t,0) + \sum_{j=1}^{2} p_{1j}(t,0) = 0. \tag{26}$$

By (15) and (26) we obtain

$$\frac{d\|P(t)\|}{dt} = \frac{d\|T(t)P(0)\|}{dt} = 0.$$

Therefore,

$$\|T(t)P(0)\| = \|P(t)\| = \|P(0)\| = 1.$$

This shows $\|p(t,\cdot)\| = 1, \ \forall t \in [0,\infty)$. □

5 Asymptotic stability of the solution

In this section, we will investigate the asymptotic stability of the dynamic solution of the system. We apply results from the theory of positive operators and semigroups that can be found in [20] and [13]. We use the same notation as in these books.

We obtain from slightly modifying [15, Lemma 1.4] or from [17, Prop. 3.3] that we can express the resolvent of A in terms of the resolvent of A_0, the Dirichlet operator D_γ and the boundary operator Φ in the following way.

Lemma 5.1 *Let* $\gamma \in \rho(A_0) \cap \rho(A)$. *Then*

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma (Id - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0).$$

We have given the explicit form of the operators in Lemma 5.1 in Sect. 3. Note also that, for $\gamma > 0$, $(Id - \Phi D_\gamma)^{-1}$ can be written via the Neumann series, since we know from the proof of Lemma 4.1 that $\|\Phi D_\gamma\| < 1$, i.e. $(Id - \Phi D_\gamma)^{-1} = \sum_{n=0}^{\infty} (\Phi D_\gamma)^n$. One can now see as in [7, Lemma 3.9] that $R(\gamma, A)$ transforms any positive vector $p \in X$ into a strictly positive vector:

$$p \in X, \ p > 0 \quad \Longrightarrow \quad R(\gamma, A)p \gg 0.$$

By [13, Def. C-III 3.1] this is equivalent to the irreducibility of the semigroup $(T(t))_{t \geq 0}$ generated by A. We thus have the following theorem.

Theorem 5.2 *The semigroup* $(T(t))_{t\geq 0}$ *generated by* $(A, D(A))$ *is irreducible.*

With this at hand one can then show the convergence of the semigroup to a one-dimensional equilibrium point, see [7, Thm. 3.11].

Theorem 5.3 *The space X can be decomposed into the direct sum*

$$X = X_1 \oplus X_2$$

where $X_1 = \mathrm{fix}(T(t))_{t\geq 0} = \ker A$ *is one-dimensional and spanned by a strictly positive eigenvector* $\tilde{p} \in \ker A$ *of* A. *In addition, the restriction* $(T(t)|_{X_2})_{t\geq 0}$ *is strongly stable.*

Corollary 5.4 *There exists* $p' \in X'$, $p' \gg 0$, *such that for all* $p \in X$

$$\lim_{t\to\infty} T(t)p = \langle p', p\rangle \tilde{p},$$

where $\ker A = \langle \tilde{p} \rangle$, $\tilde{p} \gg 0$.

Since the semigroup gives the solutions of the original system, we obtain our final result.

Corollary 5.5 *The dynamic solution of the system* (R), (BC) *and* (IC) *converges strongly to the steady-state solution as time tends to infinity, that is*, $\lim_{t\to\infty} p(t,\cdot) = \alpha\tilde{p}$, *where* $\alpha > 0$ *and* $\tilde{p}$ *as in Corollary* 5.4.

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