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Space filling with metric measure spaces

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Abstract We show a sharp relationship between the existence of space filling mappings with an upper gradient in a Lorentz space and the Poincaré inequality in a general metric setting. As key examples, we consider these phenomena in Cantor diamond spaces and the Heisenberg groups.

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1 Introduction

The classical Hahn–Mazurkiewicz Theorem states that a topological space Y is the continuous image of the unit cube $[0, 1]^n$, $n \geq 1$, if and only if it is compact, connected, locally connected, and metrizable. The theory of analysis on metric spaces has allowed for a differentiable version of this result. Sobolev mappings with metric space targets are now ubiquitous and well understood, and they provide the language for the following modern version of the Hahn–Mazurkiewicz theorem [8].

Theorem 1.1 (Hajłasz-Tyson) *Let Y be a length-compact metric space. If $n \geq 2$, then there is a continuous surjection $f : [0, 1]^n \to Y$ in the Sobolev class $W^{1,n}([0, 1]^n, Y)$.*

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A metric space Y is said to be length-compact if it is a compact metric space when equipped with the associated path distance. This additional condition, though not fully necessary, can be considered as a differentiable version of the connectedness conditions imposed in the classical Hahn–Mazurkiewicz.

In [28], we gave the following version of Theorem 1.1, which uses the Lorentz scale for measuring the magnitude of the gradient. This provides a sharper picture of space-filling phenomena by giving a complementary rigidity result for dimension.

Theorem 1.2 *Let Y be a length-compact metric space. If $n \geq 2$ and $1 < q \leq n$, then there is a continuous surjection $f\colon [0,1]^n \to Y$ in the Sobolev-Lorentz class $W^{1,n,q}([0,1]^n; Y)$. On the other hand, if there is a continuous surjection $f\colon [0,1]^n \to Y$ in the class $W^{1,n,1}([0,1]^n; Y)$, then the Hausdorff dimension of Y is at most n.*

This paper examines similar issues when the domain is a general metric space rather than the cube $[0,1]^n$; a key example being the Heisenberg group. In this general setting, we consider a class of Sobolev-Lorentz mappings based on the concept of an upper gradient, which serves as a generalization of the modulus of the gradient of a Sobolev mapping on a Euclidean space. An analogue of the first part of the Theorem 1.2 holds in great generality. The assumption that a space X be upper Q-regular at a point heuristically means that the space is at least Q-dimensional near that point; precise definitions are given in Sect. 2.

Theorem 1.3 *Let (X, d, μ) be a locally compact metric measure space, let Y be any length-compact metric space, and let $1 < q \leq Q$. Suppose that there is a non-empty set $P \subseteq X$ that has no isolated points and compact closure, and that X is upper Q-regular at each point of P. Then there is a continuous surjection $f\colon X \to Y$ that has an upper gradient in the Lorentz space $L^{Q,q}(X)$.*

The mapping $f\colon X \to Y$ produced in Theorem 1.3 has several nice features in addition to the regularity of an upper gradient. The mapping f itself is integrable in a strong sense, which we describe in Sect. 3.3 below. Moreover, the local Lipschitz constant of f is finite off a set of Hausdorff dimension 0. This condition is related to differentiability via Stepanov-type theorems in a quite general setting [1,4,14].

The condition that a mapping $f\colon X \to Y$ have an upper gradient with some specified regularity is vacuous if X contains no rectifiable curves. Thus some condition on the plentitude of curves in X is needed to prove a result analogous to the second part of Theorem 1.2. We employ an appropriate Poincaré inequality.

Theorem 1.4 *Let $Q \geq 1$, and suppose that (X, d, μ) is a complete and doubling metric measure space that is Q-regular on small scales and supports a Q-Poincaré inequality. Let Y be any metric space. If $f\colon X \to Y$ is a continuous surjection with an upper gradient in the Lorentz space $L^{Q,1}(X)$, then the Hausdorff dimension of Y is at most Q.*

We note that the class of length-compact metric spaces includes even infinite-dimensional spaces such as the Hilbert Cube. Thus, the following statement shows that the Poincaré inequality condition in Theorem 1.4 cannot be relaxed.

Theorem 1.5 *For any $\epsilon > 0$, there is a compact Ahlfors 2-regular metric space X which supports a $(2+\epsilon)$-Poincaré inequality with the following property: for any $1 \leq p < 2+\epsilon$, and any length-compact metric space Y, there is a continuous surjection $f\colon X \to Y$ that is constant off a set of finite measure and has an upper gradient in the space $L^p(X)$. In particular, there is a continuous and integrable surjection $f\colon X \to Y$ with an upper gradient in the space $L^{2,1}(X)$.*

The proof of Theorem 1.3 is modelled on the proof of Theorem 1.1 and has two main components. First, we show that if a metric space X contains a set with no isolated points, and each point of that set is a zero set for a certain capacity, then there is a continuous surjection from X to any length-compact space with an upper gradient in a space corresponding to the capacity. This step is based on a construction in [8], originally due to Kaufman [12], and we employ an abstract approach. The second part of the proof shows that if a space is upper Q-regular at some given point, then that point is a zero set for the continuous (Q, q)-Lorentz capacity. Theorem 1.5 is proven by constructing a space that is 2-regular and supports an appropriate Poincaré inequality, but contains a set with no isolated points, each point of which is a zero set for the continuous (2, 1)-Lorentz capacity.

The proof of Theorem 1.4 relies on the following principle, noted by Stein [25] and more recently explored by Kauhanen, Koskela, and Malý [13]: A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ with a weak gradient whose norm is in the Lorentz space $L^{n,1}(\mathbb{R}^n)$ enjoys many of the properties of mappings in the Sobolev space $W^{1,1}(\mathbb{R})$, while the weaker condition that the norm of the weak gradient be in $L^n(\mathbb{R}^n)$ does not guarantee this. This principle has been recently extended to the abstract metric setting [21,22]. The crucial property for this paper is Lusin's condition N.

Our results can be extended to provide continuous surjections onto non-compact spaces in certain circumstances. A modification of Theorem 1.3 yields the following result regarding the Heisenberg groups $\mathbb{H}^n$. Compare with [8, Corollary 1.5].

Corollary 1.6 *For each $n \geq 1$, and each $1 < q \leq 4$, there is a continuous surjection $f : \mathbb{H}^1 \to \mathbb{H}^n$ that is constant off a set of finite measure, has finite local Lipschitz constant off a set of Hausdorff dimension 0, and has an upper gradient in the space $L^{4,q}(\mathbb{H}^1)$. On the other hand, if $f : \mathbb{H}^1 \to \mathbb{H}^n$ is a continuous mapping with an upper gradient in the space $L^{4,1}(\mathbb{H}^1)$, then the image of f has Hausdorff dimension at most 4.*

Section 2 introduces the metric setting. In Sect. 3, we discuss mappings with an upper gradient satisfying an abstract integrability condition. The properties of such a mapping depend on the structure of the underlying space. To quantify this, in Sect. 4, we introduce an abstract notion of the capacity of a point and study it in a variety of concrete cases. Section 5 links the capacity of a point to space filling phenomena. Finally, Sect. 6 explores the properties of a mapping from a Q-dimensional space X that has an upper gradient in the space $L^{Q,1}(X)$, and proves Theorem 1.4.

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2 Notation and basic definitions

Given a metric space (X, d), we denote the open ball centered at a point $x \in X$ of radius $r > 0$ by

$$B_X(x, r) = \{y \in X : d(x, y) < r\},$$

and the corresponding closed ball by

$$\overline{B}_X(x, r) = \{y \in X : d(x, y) \leq r\}.$$

When there is no danger of confusion, we often write $B(x, r)$ in place of $B_X(x, r)$. A similar convention will be used for all objects that depend implicitly on the ambient space. Given a subset A of X and a number $\epsilon > 0$, we notate the ϵ-neighborhood of A by

$$\mathcal{N}(A, \epsilon) = \{x \in X : \mathrm{dist}(A, x) < \epsilon\}.$$

Given an open ball $B = B(x, r)$ and a parameter $\lambda > 0$, we set $\lambda B = B(x, \lambda r)$.

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and μ is a measure on X. For our purposes, a *measure* is a nonnegative countably subadditive set function defined on all subsets of a measure space that gives the value 0 to the empty set. We further assume that measures are Borel inner and outer regular.

The metric measure space (X, d, μ) is *doubling* if balls have finite and positive measure, and there is a constant $C \geq 1$ such $\mu(2B) \leq C\mu(B)$ for any open ball B in X. It follows from the definitions that if (X, d, μ) is a doubling metric measure space, then the metric space (X, d) enjoys the following property, also called *doubling*: there is a number $n \in \mathbb{N}$ such that any ball in X of radius $r > 0$ can be covered by at most n balls of radius $r/2$. It is easy to see that a doubling metric space is complete if and only if it is proper, i.e., closed and bounded sets are compact.

Doubling metric spaces are precisely those of finite Assouad dimension [9, Chap. 10]. However, this notion of dimension is not uniform; a doubling metric space may have some parts or scales where the space appears to be of lower dimension than is actually the case. We will have occasion to be more precise. The metric measure space (X, d, μ) is called *Q-regular at a point $a \in X$* if there exists a constant $C \geq 1$ and a radius $r_0 > 0$ such that if $0 < r < r_0$, then

$$\frac{r^Q}{C} \leq \mu(B(a, r)) \leq Cr^Q. \tag{2.1}$$

If only the first inequality is assumed to hold, then X is called *lower Q-regular at a*, and if only the second is assumed to hold, then X is called *upper Q-regular at a*. If X is Q-regular at every point $a \in X$, and the constant C and radius r_0 may be chosen uniformly, then X is said to be *Q-regular on small scales*. We define the terms *upper* and *lower Q-regular on small scales* in a similar way. Finally, we say that X is *Ahlfors Q-regular* if there is a constant $C \geq 1$ such that (2.1) holds for all points and all radii less than $2 \operatorname{diam} X$. We will occasionally only need (2.1) to hold only on some sequence of radii tending to zero rather than all sufficiently small radii; such generalizations are left to the reader.

For $Q \geq 0$, we denote the Q-dimensional Hausdorff measure by $\mathcal{H}^Q$, and the corresponding premeasures by $\mathcal{H}^{Q,\epsilon}$, where $\epsilon > 0$.

Let $f : X \to Y$ be a mapping between metric spaces. An *upper gradient* of f is a Borel function $g : X \to [0, \infty]$ such that for each rectifiable path $\gamma : [0, 1] \to X$,

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma g \, ds.$$

If X contains no rectifiable curves, then the constant function with value 0 is an upper gradient of any mapping. If f is locally Lipschitz, then the local Lipschitz constant of f, defined by

$$\mathrm{Lip}(f)(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{d_Y(f(x), f(y))}{r},$$

is an upper gradient of f [4, Proposition 1.11].

A key idea in theory of analysis on metric spaces is to measure the plentitude of curves in a given space. Fundamental work has resulted in an analytic condition which guarantees the presence of "many" rectifiable curves in a metric space [10]. Let $p \geq 1$, and let f and

g be measurable functions on a metric measure space (X, d, μ). The pair (f, g) *satisfies a p-Poincaré inequality* with constant $C > 0$ and dilation factor $\sigma > 0$ if for each ball B in X,

$$\fint_B |f - f_B|\, d\mu \le C(\operatorname{diam} B) \left(\fint_{\sigma B} g^p \, d\mu \right)^{\frac{1}{p}}. \tag{2.2}$$

The space (X, d, μ) *supports a p-Poincaré* inequality if there is a constant $C > 0$ and a dilation factor $\sigma > 0$ such that for each measurable function f on X and each upper gradient g of f, the pair (f, g) satisfies a p-Poincaré inequality with constant C and dilation factor σ.

A deep theorem of Keith and Zhong states that the Poincaré inequality is an open ended condition [15, Theorem 1.0.1].

Theorem 2.1 (Keith-Zhong) *Let $p > 1$ and let (X, d, μ) be a complete and doubling metric measure space that supports a p-Poincaré inequality with constant C and dilation factor σ. Then there exists $1 \le q < p$ such that (X, d, μ) supports a q-Poincaré inequality, with constant and dilation factor depending only on C, σ, and the doubling constant.*

3 Generalized Sobolev classes of mappings between metric spaces

Classical Sobolev functions on Euclidean spaces are defined by two conditions: a Lebesgue integrability condition on the weak gradient of the mapping, and a Lebesgue integrability condition on the mapping itself. If the domain of the mapping is a metric space containing sufficiently many rectifiable curves, then the concept of an upper gradient has proven to be a suitable generalization of the modulus of the weak gradient [10]. The Newtonian spaces of Shanmugalingam are Sobolev spaces based on the integrability of upper gradients [24]. This approach has been expanded to include Banach space valued (and hence metric space valued) mappings [11]. Variants such as Sobolev-Orlicz spaces have also been studied [27]. As we will employ yet another generalization, we will proceed in a rather abstract fashion, using the language of Banach function spaces.

3.1 Banach function spaces

Let (X, μ) be a totally σ-finite and complete measure space. We denote by $\mathcal{M}(X)$ the set of measurable functions on X, and by $\mathcal{M}^+(X)$ the set of measurable functions $f : X \to [0, \infty]$.

Definition 3.1 A Banach function norm on X is a function $\mathcal{G} : \mathcal{M}^+(X) \to [0, \infty]$ such that for $f, g, f_1, f_2, \ldots \in \mathcal{M}^+$, all $c \ge 0$, and all measurable subsets $E \subseteq X$, the following properties hold:

(A1) $\mathcal{G}(f) = 0 \iff f = 0$ a.e., $\mathcal{G}(cf) = c\mathcal{G}(f)$, and $\mathcal{G}(f + g) \le \mathcal{G}(f) + \mathcal{G}(g)$,
(A2) if $g \le f$ a.e., then $\mathcal{G}(g) \le \mathcal{G}(f)$,
(A3) if $f_n \nearrow f$ a.e., then $\mathcal{G}(f_n) \nearrow \mathcal{G}(f)$,
(A4) if $\mu(E) < \infty$, then $\mathcal{G}(\chi_E) < \infty$,
(A5) if $\mu(E) < \infty$, then $\int_E f \, d\mu \le C_E \mathcal{G}(f)$, where $0 < C_E < \infty$ depends only on E and not on f.

Definition 3.2 A Banach function space is the collection

$$L^{\mathcal{G}}(X) = \{f \in \mathcal{M}(X) : \mathcal{G}(|f|) < \infty\},$$

where $\mathcal{G}$ is a Banach function norm.

A Banach function space $L^{\mathcal{G}}(X)$ is indeed a Banach space when equipped with the norm $||f||_{\mathcal{G}} := \mathcal{G}(|f|)$, after the usual identifications [2, Chapter 1.1]. Banach function spaces have properties often associated with the familiar Lebesgue spaces $L^p(X)$, $1 \leq p \leq \infty$, which are prototypical examples. Other examples include the Orlicz spaces, and most important to this paper, the Lorentz spaces.

3.2 Lorentz spaces

We now define and discuss the Lorentz norms, a family of Banach function norms. For a measurable function $f \in \mathcal{M}^+(X)$, we define the *distribution function* $\omega_f\colon [0,\infty) \to [0,\infty]$ of f by

$$\omega_f(\alpha) = \mu(\{x \in X : f(x) > \alpha\}).$$

The *non-increasing rearrangement* $f^*\colon [0,\infty) \to [0,\infty]$ is given by

$$f^*(t) = \inf\{\alpha \geq 0 : \omega_f(\alpha) \leq t\}.$$

Definition 3.3 Let $1 \leq q \leq Q$. The Lorentz function norm $\mathcal{G}_{Q,q}\colon \mathcal{M}^+ \to [0,\infty]$ is defined by

$$\mathcal{G}_{Q,q}(f) = \left(\int_0^\infty t^{-1} \left(t^{1/Q} f^*(t) \right)^q \, dt \right)^{1/q}. \tag{3.1}$$

By [2, Theorem 4.4.3], $\mathcal{G}_{Q,q}$ is a Banach function norm. We denote the corresponding Banach function space by $L^{Q,q}(X)$, equipped with the norm

$$||f||_{L^{Q,q}} := \mathcal{G}_{Q,q}(|f|).$$

The following statement gives the basic relationships between the Lorentz spaces [2, Propositions 2.1.8 and 4.4.2].

Proposition 3.4 *For all $1 \leq r \leq q \leq Q$, there is a constant c, depending only on r, q, and Q, such that for all measurable functions $f\colon X \to \mathbb{R}$,*

$$||f||_{L^{Q,q}} \leq c||f||_{L^{Q,r}}.$$

In particular, there is a bounded embedding $L^{Q,r}(X) \hookrightarrow L^{Q,q}(X)$. Moreover, $L^{Q,Q}(X) = L^Q(X)$ and

$$||f||_{L^{Q,Q}} = ||f||_Q.$$

Finally, if $p > Q$ and X has finite total measure, then for every $1 \leq q \leq Q$, there is a bounded embedding $L^p(X) \hookrightarrow L^{Q,q}(X)$.

Corollary 3.5 *If $1 \leq q \leq Q$, then there is a bounded embedding $L^{Q,q}(X) \hookrightarrow L^Q(X)$.*

We now discuss a characterization of Lorentz spaces given in [13]. We say that a *gauge* is a non-negative non-increasing function $\phi\colon (0,\infty) \to [0,\infty)$. Given $1 \leq q \leq Q$ and a gauge ϕ, we define functions $T_\phi^{Q,q}, F_\phi^{Q,q}\colon [0,\infty) \to [0,\infty)$ by

$$T_\phi^{Q,q}(s) = \begin{cases} s^{q-1}\phi^{q/Q}(s) & s > 0, \\ 0 & s = 0, \end{cases} \quad \text{and} \quad F_\phi^{Q,q}(s) = \begin{cases} s^q \phi^{(q-Q)/Q}(s) & s > 0, \\ 0 & s = 0. \end{cases}$$

A gauge is *(Q,q)-admissible* if

$$\int_0^\infty T_\phi^{Q,q}(s)\,ds < \infty.$$

We denote the set of (Q,q)-admissible gauges by $\mathcal{A}^{Q,q}$.

The following theorem states that the Lorentz spaces are determined by a family of Orlicz conditions [13, Corollary 2.4].

Theorem 3.6 (Kauhanen-Koskela-Malý) *A measurable function $f\colon X \to \mathbb{R}$ is in $L^{Q,q}(X)$ if and only if there is $\phi \in \mathcal{A}^{Q,q}$ such that $\phi(|f(x)|) > 0$ for almost every $x \in X$ with $|f(x)| > 0$, and*

$$\int_X F_\phi^{Q,q}(|f(x)|)\,d\mu(x) < \infty.$$

In addition, there is a constant C depending only on ϕ, Q, and q such that

$$||f||_{L^{Q,q}}^Q \le C \int_X F_\phi^{Q,q}(|f(x)|)\,d\mu(x). \tag{3.2}$$

3.3 Integrability conditions for metric space valued mappings

Standing Assumption 3.7 For the remainder of the paper, we denote by X a locally compact metric measure space and by Y any metric space.

As mentioned above, classical Sobolev functions are themselves required to satisfy integrability conditions. The main purpose of this is to guarantee that a sensible norm may be defined for such functions, and that the resulting Sobolev space is a Banach space. Typically, the integrability of metric space valued mappings is defined via isometric embeddings of the target into a Banach space. Recall that any metric space Y may be isometrically embedded in the Banach space $\ell^\infty(Y)$ [9, Page 99]. Moreover, if Y is separable, it may even be isometrically embedded in the cannonical space $\ell^\infty(\mathbb{N})$. The Bochner integral then provides a framework for Banach function spaces of Banach space valued mappings. See [11] for an example of how this works using the Lebesgue scale. However, the fact that there are many possible isometric embeddings of a given metric space in a Banach space means that the "function norm" resulting from this process is not canonical. For our purposes, it suffices to consider an intrinsic notion of local integrability for metric space valued mappings.

Recall that a mapping $f\colon X \to Y$ is said to be *Bochner measurable* if it is measurable in the usual sense and *essentially separably valued*, meaning that there is a set $N \subseteq X$ of measure 0 such that $f(X\backslash N)$ is a separable subset of Y.

Definition 3.8 A mapping $f\colon X \to Y$ is in the class $L^1_{\mathrm{loc}}(X;Y)$, i.e., it is said to be *locally integrable*, if it is Bochner measurable and there exists a point $z \in Y$ such that the function $x \mapsto d_Y(f(x), z)$ is in the space $L^1_{\mathrm{loc}}(X)$.

The following proposition, the elementary proof of which we leave to the reader, shows that this agrees with the corresponding notion for Banach space valued mappings. Namely, if V is a Banach space, then $f\colon X \to V$ is said to be *locally Bochner integrable* if f is Bochner measurable and $||f||_V \in L^1_{\mathrm{loc}}(X)$.

Proposition 3.9 *Let $f: X \to Y$ be a Bochner measurable mapping. Then $f \in L^1_{\mathrm{loc}}(X; Y)$ if and only if for every Banach space V and every isometric embedding $\iota: Y \hookrightarrow V$, the mapping $\iota \circ f$ is locally Bochner integrable.*

Most of the mappings we construct have much stronger integrability properties than just local integrability. Often, they satisfy the hypotheses of the following statement.

Proposition 3.10 *Let $f: X \to Y$ be a measurable, essentially bounded, and essentially separably valued mapping. If f takes the value $z \in Y$ except on a set of finite measure, then for any Banach function norm $\mathcal{G}$, the mapping $x \mapsto d_Y(f(x), z)$ is in the space $L^{\mathcal{G}}(X)$. Moreover, if $\iota: Y \hookrightarrow V$ is an isometric embedding into a Banach space V, then $\mathcal{G}(||\iota \circ f||_V) < \infty$.*

Proof Define $d_z f: X \to [0, \infty)$ by $d_z f(x) = d_Y(f(x), z)$. We have assumed that there is a set $A \subseteq X$ of finite measure such that $f(x) = z$ for all $x \in X \setminus A$. By properties (A1), (A2), and (A4) of the definition of a Banach function norm, we have

$$\mathcal{G}(d_z f) = \mathcal{G}(d_z f \cdot \chi_A) \leq \mathcal{G}(\chi_A)\,(\text{ess sup}\, d_z f) < \infty,$$

as desired. The second statement is shown similarly. □

3.4 Mappings with an upper gradient in a Banach function space

Due to the difficulty in defining Banach function spaces of mappings with metric space targets, in this paper we choose not to consider Newtonian "spaces" of metric space valued mappings, though such objects are sensible. Let $\mathcal{G}$ be a Banach function norm. Our simplified philosophy is to consider a mapping $f: X \to Y$ of metric spaces to be a *$\mathcal{G}$-Newtonian mapping* if it is locally integrable and has an upper gradient in the space $L^{\mathcal{G}}(X)$, though often the mappings we construct will have stronger integrability properties, as in Proposition 3.10.

The following statement provides the completeness properties that, in the Banach space valued setting, would come from the completeness of Newtonian spaces. The proof, which is essentially Fuglede's lemma, is standard.

Proposition 3.11 *Let $\mathcal{G}$ be a Banach function norm. Suppose that the sequence of mappings $\{f_n: X \to Y\}_{n\in\mathbb{N}}$ converges pointwise to a mapping $f: X \to Y$, and that the sequence of functions $\{g_n: X \to [0, \infty]\}_{n\in\mathbb{N}}$ converges in $L^{\mathcal{G}}(X)$ to a function $g: X \to [0, \infty]$. If for each $n \in \mathbb{N}$, the function g_n is an upper gradient of f_n, then there is an upper gradient of f in every $L^{\mathcal{G}}(X)$-neighborhood of g.*

Proof There is a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$,

$$||g_{n_k} - g||_{\mathcal{G}} \leq 2^{-2k}.$$

Let $\rho_k = |g_{n_k} - g|$, and set

$$\Gamma = \left\{ \gamma: [0, 1] \to X : \lim_{k\to\infty} \int_\gamma \rho_k \, ds \neq 0 \right\}.$$

If $\gamma: [0, 1] \to X$ is a rectifiable path not in the family Γ, then

$$d_Y(f(\gamma(0)), f(\gamma(1))) = \lim_{k\to\infty} d_Y(f_{n_k}(\gamma(0)), f_{n_k}(\gamma(1))) \leq \lim_{k\to\infty} \int_\gamma g_{n_k} \, ds = \int_\gamma g \, ds,$$

and so g satisfies the upper gradient inequality for f on the path γ.

On the other hand, if $\gamma \in \Gamma$, then for all $j \in \mathbb{N}$ there is some integer $k \geq j$ such that

$$\int_\gamma \rho_k \, ds > 2^{-k}.$$

Thus, for all $j \in \mathbb{N}$, the function

$$\widetilde{\rho}_j = \sum_{k=j}^{\infty} 2^k \rho_k$$

satisfies

$$\int_\gamma \widetilde{\rho}_j \, ds \geq 1, \quad \text{and} \quad ||\widetilde{\rho}_j||_{\mathcal{G}} \leq \sum_{k=j}^{\infty} 2^{-k} = 2^{-j+1}.$$

We claim that for any $i \in \mathbb{N}$, the function

$$\widetilde{g}_i = g + \sum_{j=i}^{\infty} \widetilde{\rho}_j$$

is an upper gradient of f. Since $\widetilde{g}_i \geq g$, it suffices to show that $\widetilde{g}_i$ satisfies the upper gradient inequality on any path $\gamma \in \Gamma$. For such a path, we see that

$$\int_\gamma \widetilde{g}_i \, ds \leq \int_\gamma g \, ds + \sum_{j=i}^{\infty} \int_\gamma \widetilde{\rho}_j \, ds = \infty,$$

and so the upper gradient inequality is trivially satisfied. Moreover, by the basic properties of Banach function spaces [2, Chapter 1.1],

$$||\widetilde{g}_i - g||_{\mathcal{G}} = || \lim_{l \to \infty} \sum_{j=i}^{l} \widetilde{\rho}_j ||_{\mathcal{G}} = \lim_{l \to \infty} || \sum_{j=i}^{l} \widetilde{\rho}_j ||_{\mathcal{G}} \leq \lim_{l \to \infty} \sum_{j=i}^{l} ||\widetilde{\rho}_j||_{\mathcal{G}} \leq 2^{-i+2}.$$

As i may be chosen to be arbitrarily large, this shows that $\widetilde{g}_i$ may be chosen to lie in an arbitrary $L^{\mathcal{G}}$-neighborhood of g. □

We will also need the following simple pasting lemma for upper gradients. Much more sophisticated versions are available, as discussed in [27].

Lemma 3.12 *Let $U_1, \ldots, U_n$ be disjoint Borel sets in X, let $U_0 = X \backslash (\cup_i U_i)$, and let $f_0, \ldots, f_n \colon X \to Y$ be mappings with upper gradients $g_0, \ldots, g_n \colon X \to [0, \infty]$ respectively. Suppose, for $i = 1, \ldots, n$, the restriction $f_i|_{X \backslash U_i}$ is constant with value $y_i \in Y$, and the restriction $f_0|_{U_i}$ is constant with value y_i. Then the mapping $f \colon X \to Y$ defined by*

$$f(x) = \begin{cases} f_0(x) & x \notin \bigcup_{i=1}^n U_i, \\ f_i(x) & x \in U_i, \end{cases}$$

has an upper gradient defined by $g = \sum_{i=0}^n g_i$.

Proof Let $\iota \colon Y \to \ell^\infty(Y)$ be an isometric embedding. Our assumptions imply that for each $x \in X$,

$$\iota \circ f(x) = \iota \circ f_0(x) + \sum_{i=1}^{n} (\iota \circ f_i(x) - \iota(y_i)).$$

For $i = 1, \dots, n$, the mapping $y \mapsto \iota(y) - \iota(y_i)$ is an isometry, and so g_i is also an upper gradient of the mapping $x \mapsto \iota \circ f_i(x) - \iota(y_i)$. Thus g is an upper gradient of $\iota \circ f$, and hence of f. □

4 The capacity of a point

The strength of the condition that a given mapping $f\colon X \to Y$ has an upper gradient in the space $L^{\mathcal{G}}(X)$ depends on the underlying structure of the metric space X. To help understand this phenomena, we introduce a variational-type capacity condition. A much more involved capacity theory can be developed, as in [27] and [24].

Definition 4.1 A point $a \in X$ has *zero continuous $\mathcal{G}$-capacity* if for all $\epsilon > 0$, there is a continuous function $\eta\colon X \to [0, \infty)$ such that

(i) the support $\operatorname{supp} \eta$ is a compact subset of $B(a, \epsilon)$,
(ii) there exists $\delta > 0$ such that $\eta(x) \geq 1$ for all $x \in B(a, \delta)$,
(iii) there is an upper gradient g of η such that $||g||_{\mathcal{G}} < \epsilon$.

If in addition, the function η may be chosen to be Lipschitz, we say that $a \in X$ has *zero Lipschitz $\mathcal{G}$-capacity*.

Remark 4.2 In Definition 4.1 it is equivalent to require that for all $\epsilon > 0$ and $0 < \tau \leq 1$, there is a function $\eta\colon X \to [0, \tau]$ satisfying conditions (i), (iii), and the following modified version of condition (ii):

(ii)' there exists $\delta > 0$ such that $\eta(x) = \tau$ for all $x \in B(a, \delta)$.

To see this, choose a function η satisfying the requirements of Definition 4.1, and consider the continuous function $\widetilde{\eta}\colon X \to [0, \tau]$ defined by

$$\widetilde{\eta}(x) = \tau \min\{\eta(x), 1\}.$$

Then for all $x, y \in X$,

$$|\widetilde{\eta}(x) - \widetilde{\eta}(y)| \leq \tau|\eta(x) - \eta(y)| \leq |\eta(x) - \eta(y)|,$$

and so g is also an upper gradient of $\widetilde{\eta}$.

The continuous L^p-capacity of a point has been studied in a general setting. The following result can be deduced from the proof of [16, Theorem 3.4].

Theorem 4.3 (Korte) *Let (X, d, μ) be a doubling metric measure space and let $Q > 1$. If X is upper Q-regular at a point $a \in X$ and $1 \leq p < Q$, then the point a has zero continuous L^p-capacity. On the other hand, if X is lower Q-regular at the point a, and X supports a Q-Poincaré inequality, then for every $p > Q$, the point a does not have zero continuous L^p-capacity.*

4.1 The Lorentz capacity

The Lorentz capacity has been studied in detail in the Euclidean setting [5]. In this section we establish a version of Theorem 4.3 that employs the Lorentz scale in the borderline case. Let $1 \leq q \leq Q$. We say that a point $a \in X$ has zero Lipschitz (Q, q)-Lorentz capacity if it has zero Lipschitz $\mathcal{G}_{Q,q}$-capacity where $\mathcal{G}_{Q,q}$ is defined by (3.1).

Theorem 4.4 *Suppose that X is upper Q-regular at a point $a \in X$. Then for all $1 < q \leq Q$, the point a has zero Lipschitz (Q, q)-Lorentz capacity.*

Proof By Proposition 3.4, it suffices to consider the case that $1 < q < Q$.

For $0 < s < \infty$, define $\widetilde{\eta}_s : [0, \infty) \to [0, \infty)$ by

$$\widetilde{\eta}_s(r) = \begin{cases} 1 & 0 \leq r \leq e^{-e^{s+1}}, \\ \log\log\left(\frac{1}{r}\right) - s & e^{-e^{s+1}} \leq r \leq e^{-e^s}, \\ 0 & e^{-e^s} \leq r < \infty. \end{cases}$$

For all $e^{-e^{s+1}} < r < e^{-e^s}$, the function $\widetilde{\eta}_s$ is smooth at r, and we have

$$(\widetilde{\eta}_s)'(r) = \frac{-1}{r\log\left(\frac{1}{r}\right)}.$$

Let $a \in X$, and for $0 < r < R < \infty$ denote closed annuli centered at $a \in X$ by

$$A_a(r, R) := \{x \in X : r \leq d(a, x) \leq R\}.$$

Moreover, for $0 < s < \infty$, define $\eta_{s,a} : X \to [0, \infty)$ by

$$\eta_{s,a}(x) = \widetilde{\eta}_s(d(a, x)).$$

It is not hard to see that $\widetilde{\eta}_s$ and $\eta_{s,a}$ are Lipschitz functions, and that for any $x \in X$, the local Lipschitz constant of $\eta_{s,a}$ at x satisfies

$$\mathrm{Lip}(\eta_{s,a})(x) \leq \begin{cases} \frac{1}{d(a,x)\log\left(\frac{1}{d(a,x)}\right)} & x \in A_a(e^{-e^{s+1}}, e^{-e^s}), \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\epsilon > 0$. Recalling that X is assumed to be locally compact, it is clear that for sufficiently large $s > 0$, the function $\eta_{s,a}$ satisfies conditions (i) and (ii) of Definition 4.1. We now show that the third condition holds for sufficiently large $s > 0$. Recall that the local Lipschitz constant of a Lipschitz function is an upper gradient of that function.

By assumption, we may find $C \geq 1$ and $r_0 > 0$ such that if $0 < r < r_0$, then

$$\mu(B(a, r)) \leq Cr^Q.$$

Since $1 < q < Q$, there is a number α such that

$$\frac{Q - q}{q} < \alpha < Q - 1.$$

Define a gauge

$$\psi(s) = \begin{cases} s^{-Q}\log^{\alpha Q/(q-Q)}(e + s) & s \geq 1, \\ \log^{\alpha Q/(q-Q)}(e + 1) & s \leq 1. \end{cases}$$

An easy calculation shows that the assumption $\alpha > (Q - q)/q$ implies that $\psi \in \mathcal{A}^{Q,q}$.

For ease of notation let $t = s + 1$. It follows from the definitions that

$$\int_X F_\psi^{Q,q} \circ \mathrm{Lip}(\eta_{s,a})(x)\, d\mu(x) \leq \int_{A_a(e^{-e^t}, e^{-e^s})} \frac{\log^\alpha\left(e + \frac{1}{d(a,x)\log(d(a,x)^{-1})}\right)}{\left(d(a, x)\log(d(a, x)^{-1})\right)^Q}\, d\mu(x). \quad (4.1)$$

Let k be the unique positive integer satisfying $k \leq e^t - e^s < k+1$, and for $j \in \{0, \ldots, k\}$, set

$$A_j := A_a(e^{-e^t+j}, e^{-e^t+j+1}).$$

Since the function $r \mapsto r\log(1/r)$ is increasing for $r < e^{-1}$, the integrand on the right hand side of (4.1) is a decreasing function of $d(a,x)$. If s is so large that $e^{-e^s+1} \leq r_0$, then a trivial estimate and the volume growth assumption show that

$$\begin{aligned}\int\limits_{A_j} F_\psi^{Q,q} \circ \mathrm{Lip}(\eta_{s,a})(x)\, d\mu(x) &\leq \frac{\log^\alpha\left(e + \frac{e^{e^t-j}}{e^t-j}\right)}{(e^t-j)^Q} \frac{\mu(B(a, e^{-e^t+j+1}))}{(e^{-e^t+j})^Q} \\ &\leq Ce^Q(e^t-j)^{-Q}\log^\alpha\left(e + \frac{e^{e^t-j}}{e^t-j}\right) \\ &\leq Ce^Q 2^\alpha (e^t-j)^{\alpha-Q} \leq Ce^Q 2^\alpha \int\limits_{e^t-j-1}^{e^t-j} u^{\alpha-Q}\, du.\end{aligned}$$

As no three of the sets $\{A_a(e^{-e^t+j}, e^{-e^t+j+1})\}_{j=0}^k$ intersect, we see that

$$\begin{aligned}\int\limits_X F_\psi^{Q,q} \circ \mathrm{Lip}(\eta_{s,a})(x)\, d\mu(x) &\leq Ce^Q 2^{\alpha+1} \sum_{j=0}^k \int\limits_{e^t-j-1}^{e^t-j} u^{\alpha-Q}\, du \\ &\leq Ce^Q 2^{\alpha+1} \int\limits_{e^s-1}^{e^{s+1}} u^{\alpha-Q}\, du.\end{aligned}$$

Since $\alpha < Q-1$, the final term above tends to zero as s tends to infinity. That condition (iii) holds for sufficiently large s now follows from the characterization of $L^{Q,q}(X)$ provided by Theorem 3.6. □

For the negative result, we assume more about the growth of the space than is assumed in Theorem 4.3. More precise versions of this result can likely be deduced from [21].

Theorem 4.5 *Suppose that (X, d, μ) is complete, doubling, supports a Q-Poincaré inequality, and is Q-regular at small scales. Then no point of X has zero continuous $(Q, 1)$-Lorentz capacity.*

We defer the proof of this theorem to Sect. 6, in which the properties of mappings with an upper gradient in the space $L^{Q,1}(X)$ are studied.

4.2 The Cantor diamond sets

Though there are Ahlfors Q-regular metric spaces that contain points of zero continuous $(Q, 1)$-Lorentz capacity, Theorem 4.5 shows that such spaces cannot support a Q-Poincaré inequality. This subsection is devoted to a class of examples that will show that this relationship is sharp.

We first need some notation regarding Cantor sets. For $0 < \lambda < 1$, let E_λ be the middle interval Cantor set with the following properties. At stage i, there are 2^{i-1} removed open intervals $\{U_{i,j}\}_{j\in J_i}$ of length $(\lambda/2)^{i-1}(1-\lambda)$, and 2^i remaining closed intervals $\{U_i^k\}_{k\in K_i}$ of length $(\lambda/2)^i$. Then the Hausdorff dimension of E_λ is $\log_{2/\lambda} 2$. In this notation, the "standard middle-third Cantor set" is given by $E_{2/3}$.

For convenience, we denote the center point of $U_{i,j}$ by $u_{i,j}$, and we set

$$w_i = \frac{(\lambda/2)^{i-1}(1-\lambda)}{2},$$

so that $U_{i,j} = (u_{i,j} - w_i, u_{i,j} + w_i)$. Similarly, we write $U_i^k = [u_i^k - v_i, u_i^k + v_i]$, where $2v_i = (\lambda/2)^i$. We may assume that for each positive integer i, the intervals $\{U_i^k\}_{k\in K_i}$ are ordered so that the right endpoint of U_i^k is less than the left endpoint of U_i^{k+1}, and similarly for the intervals $\{U_{i,j}\}_{j\in J_i}$. Note that the notation established above depends implicitly on the parameter λ.

We now define the spaces that will be used in our example. For each $i \in \mathbb{N}$ and $j \in J_i$, we define the "diamond" $D_{i,j}$ by

$$D_{i,j} := \{(x, y) \in \mathbb{R}^2 : |x - u_{i,j}| \le w_i - |y|\}.$$

The λ-*Cantor diamond set* is defined by

$$X_\lambda = \left(\bigcup_{i\in\mathbb{N},\, j\in J_i} D_{i,j} \right) \cup (E_\lambda \times \{0\}).$$

To the best of our knowledge, these spaces were introduced in [17]. See Fig. 1.

Recall that an s-*similarity*, $s > 0$, is a mapping $\phi\colon X \to Y$ that satisfies, for all $x, y \in X$,

$$d_Y(\phi(x), \phi(y)) = s d_X(x, y).$$

The Cantor set E_λ is self-similar in the following sense. For each positive integer i and each $k \in K_i$, there is a $(\lambda/2)^i$-similarity $\phi_i^k : \mathbb{R} \to \mathbb{R}$ which maps E_λ bijectively onto $E_\lambda \cap U_i^k$. The space X_λ inherits self-similarities from the Cantor set E_λ. Namely, for each $i \in \mathbb{N}$ and $k \in K_i$, the map $\phi_i^k : \mathbb{R} \to \mathbb{R}$ extends to a $(\lambda/2)^i$-similarity $\Phi_i^k : \mathbb{R}^2 \to \mathbb{R}^2$ that maps X_λ bijectively onto $X_\lambda \cap (U_i^k \times [0, 1])$. Recall that if $A \subseteq \mathbb{R}^2$ is a Borel set and $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an s-similarity, then

$$\mathcal{H}^2(\Phi(A)) = s^2 \mathcal{H}^2(A).$$

We endow X_λ with the metric inherited from the plane $\mathbb{R}^2$ and the two-dimensional Hausdorff measure $\mathcal{H}^2$. The next two statements give the basic properties of the space X_λ. We leave the proof of the first to the reader, and the second can be found at [17, Theorem 3.1].

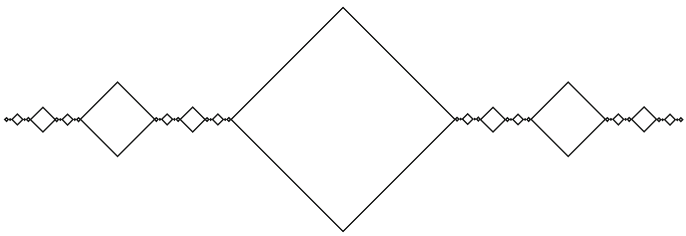

Fig. 1 The Cantor diamond set with $\lambda = 2/3$

Proposition 4.6 *The metric measure space X_λ is Ahlfors 2-regular with a constant depending only on λ.*

Theorem 4.7 (Koskela-MacManus) *The metric measure space X_λ satisfies a p-Poincaré inequality for all $p > p_\lambda$, where*

$$p_\lambda = 2 - \frac{\log 2}{\log \lambda} > 2.$$

We denote the set of *non-endpoints* of X_λ by

$$\widetilde{X_\lambda} = X_\lambda \setminus \left(\bigcup_{i \in \mathbb{N}, j \in J_i} D_{i,j} \right) = \left(E_\lambda \setminus \bigcup_{i \in \mathbb{N}, j \in J_i} \{u_{i,j} - w_i, u_{i,j} + w_i\} \right) \times \{0\}.$$

Note that the set $\widetilde{X_\lambda}$ has no isolated points.

Proposition 4.8 *Each point of $\widetilde{X_\lambda}$ has zero Lipschitz L^p-capacity for any $1 \leq p < p_\lambda$.*

Before we begin the proof, we define piecewise linear approximations to a version of the Cantor function that is supported on a fixed interval $U_{i_0}^{k_0}$. Let $i \in \mathbb{N}$, and define $c_{i_0;i}^{k_0} : U_{i_0}^{k_0} \to [0, 1]$ to be the piecewise linear continuous function given by

$$c_{i_0;i}^{k_0}(x) = \int_{u_{i_0}^{k_0} - v_{i_0}}^{x} \rho_{i_0;i}^{k_0}(t)\, dt,$$

where $\rho_{i_0;i}^{k_0} : U_{i_0}^{k_0} \to [0, \infty)$ is given by

$$\rho_{i_0;i}^{k_0}(t) = \begin{cases} \frac{2^{i_0}}{\lambda^{i_0+i}} & t \in \bigcup_{k \in K_i} \phi_{i_0}^{k_0}(U_i^k), \\ 0 & \text{otherwise.} \end{cases}$$

The indices i_0 and k_0 give the location of the support of the function $c_{i_0;i}^{k_0}$, while the index i determines how closely the function approximates the Cantor function. See Fig. 2.

A simple computation shows that

$$c_{i_0;i}^{k_0}(u_{i_0}^{k_0} - v_{i_0}) = 0 \quad \text{and} \quad c_{i_0;i}^{k_0}(u_{i_0}^{k_0} + v_{i_0}) = 1.$$

Moreover, the absolute continuity of the integral implies that for all $t \in U_{i_0}^{k_0}$,

$$\mathrm{Lip}(c_{i_0;i}^{k_0})(t) = \rho_{i_0;i}^{k_0}(t).$$

Proof of Proposition 4.8 Let $(a, 0) \in \widetilde{X_\lambda}$, and let $\epsilon > 0$. Since a is a "non-endpoint" of E_λ, we may find $i_0 \in \mathbb{N}$ and $k_l, k_r \in K_{i_0}$ such that

$$u_{i_0}^{k_l} + v_{i_0} < a < u_{i_0}^{k_r} - v_{i_0} \quad \text{and} \quad (U_{i_0}^{k_l} \cup U_{i_0}^{k_r}) \subseteq (a - \epsilon, a + \epsilon).$$

Then $\delta := \mathrm{dist}(a, U_{i_0}^{k_l} \cup U_{i_0}^{k_r}) > 0$. The condition that $p < p_\lambda$ implies that $\lambda^{2-p} < 2$, and hence we may find $i \in \mathbb{N}$ so that

$$2\mathcal{H}^2(X_\lambda) \left(\frac{2}{\lambda} \right)^{(p-2)i_0} \left(\frac{\lambda^{2-p}}{2} \right)^i < \epsilon^p.$$

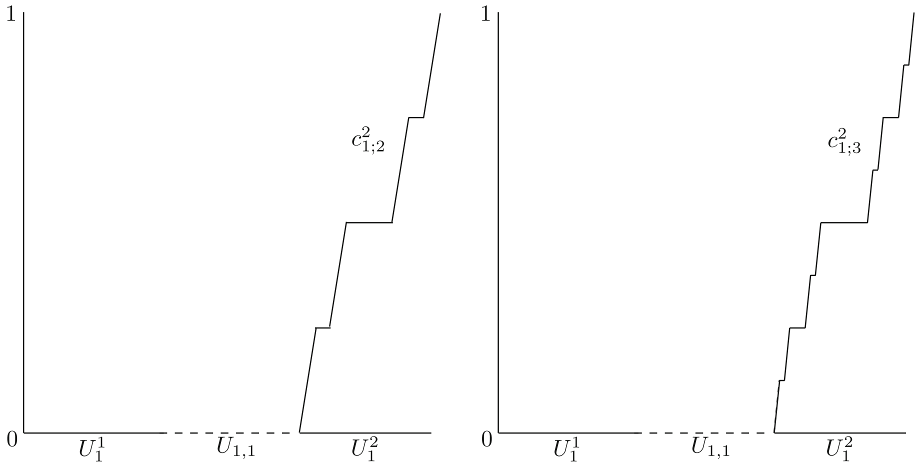

Fig. 2 Graphs of the functions $c^2_{1;2}$ and $c^2_{1;3}$, when $\lambda = 2/3$

Define $\eta\colon X_\lambda \to [0, 1]$ by

$$\eta(x, y) = \begin{cases} 1 & x \in [u^{k_l}_{i_0} + v_{i_0}, u^{k_r}_{i_0} - v_{i_0}], \\ c^{k_l}_{i_0;i}(x) & x \in U^{k_l}_{i_0}, \\ 1 - c^{k_r}_{i_0;i}(x) & x \in U^{k_r}_{i_0}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definitions that η has compact support contained in $B((a, 0), \epsilon)$, is identically one on $B((a, 0), \delta)$, and satisfies

$$\operatorname{Lip} \eta(x, y) = \begin{cases} \frac{2^{i_0}}{\lambda^{i_0+i}} & x \in \bigcup_{k \in K_i} (\phi^{k_l}_{i_0}(U^k_i) \cup \phi^{k_r}_{i_0}(U^k_i)), \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathcal{H}^2(\{(x, y) \in X_\lambda : x \in \bigcup_{k \in K_i} (\phi^{k_l}_{i_0}(U^k_i) \cup \phi^{k_r}_{i_0}(U^k_i))\}) = 2^{i+1}\mathcal{H}^2(\{(x, y) \in X_\lambda : x \in \phi^{k_l}_{i_0}(U^1_i)\}).$$

By the self-similarity of X_λ, there is some $k' \in K_{i_0+i}$ such that

$$\{(x, y) \in X_\lambda : x \in \phi^{k_l}_{i_0}(U^1_i)\} = \Phi^{k'}_{i_0+i}(X_\lambda).$$

This implies that

$$\mathcal{H}^2(\{(x, y) \in X_\lambda : x \in \phi^{k_l}_{i_0}(U^1_i)\}) = \left(\frac{\lambda}{2}\right)^{2(i_0+i)} \mathcal{H}^2(X_\lambda).$$

As a result, we see that

$$\begin{aligned} ||\operatorname{Lip} \eta||^p_{L^p} &= \left(\frac{2^{i_0}}{\lambda^{i_0+i}}\right)^p 2^{i+1} \left(\frac{\lambda}{2}\right)^{2(i_0+i)} \mathcal{H}^2(X_\lambda) \\ &= 2\mathcal{H}^2(X_\lambda) \left(\frac{2}{\lambda}\right)^{(p-2)i_0} \left(\frac{\lambda^{2-p}}{2}\right)^i < \epsilon^p. \end{aligned}$$

Recalling that $\operatorname{Lip} \eta$ is an upper gradient of η, this completes the proof. □

Combining this result with Proposition 3.4 yields the following statement.

Corollary 4.9 *Each point of* $\widetilde{X_\lambda}$ *has zero Lipschitz* $(2, 1)$*-Lorentz capacity in* X_λ.

We may deduce from this and the work of Keith and Zhong that the number p_λ given in Theorem 4.7 is sharp.

Corollary 4.10 *The space* X_λ *does not support a* p_λ*-Poincaré inequality.*

Proof Suppose that X_λ does support a p_λ-Poincaré inequality. Theorem 2.1 implies that X_λ-satisfies a p-Poincaré inequality for some $2 < p < p_\lambda$. Proposition 4.8 produces a point with zero Lipschitz L^p-capacity, yielding a contradiction by Proposition 4.6 and Theorem 4.3. □

5 Space filling with generalized Newtonian maps

This section is based on [8, Sect. 3], where the same construction is done in the setting of Reshetnyak-Sobolev spaces on the unit cube $[0, 1]^n$. The spirit of the construction is due to Kaufman [12]. We connect the capacity condition of the previous section to the construction of space filling mappings with controlled upper gradients. This allows us to prove Theorem 1.3 and Corollary 1.6.

5.1 The compact case

Theorem 5.1 *Let* $\mathcal{G}$ *be a Banach function norm, and suppose that there is a non-empty set* $P \subseteq X$ *that has no isolated points and compact closure, and such that each point of* P *has zero continuous* $\mathcal{G}$*-capacity. Then for any length-compact metric space* (Y, d_Y)*, any point* $z \in Y$*, and any* $\epsilon > 0$*, there is a continuous surjection* $f : X \to Y$ *that takes the value* z *outside the* ϵ*-neighborhood of* P*, and has an upper gradient* $g : X \to [0, \infty]$ *satisfying* $||g||_{\mathcal{G}} \le \epsilon$*.*

Proof To produce the desired mapping, we construct a uniform Cauchy sequence of continuous mappings from X to Y, such that the mappings cover finer and finer nets in Y. The limit mapping is seen to be a continuous surjection, and we use Proposition 3.11 to show that it has an upper gradient in the desired space.

The assumption that Y is length-compact implies that for each non-negative integer n, we may find a finite set $Y_n = \{y_i^n\}_{i=1}^{k_n}$ with the property that each $y \in Y$ can be connected to a point in Y_n by a path of length no greater than 2^{-n}. Then $\bigcup_n Y_n$ is dense in Y. We may assume the diameter of Y with respect to the path metric is 1, and hence we may assume that $y_1^0 = z$.

For each integer $n \ge 1$, we may partition Y_n into k_{n-1} sets $\mathcal{C}(y_i^{n-1})$ so that if $y_j^n \in \mathcal{C}(y_i^{n-1})$, then there is a 1-Lipschitz path $\gamma_j^n : [0, 2^{-(n-1)}] \to Y$ satisfying $\gamma_j^n(0) = y_i^{n-1}$ and $\gamma_j^n(1) = y_j^n$.

Let $f_0 : X \to Y$ be the constant mapping $f_0(x) = z$ for all $x \in X$. Clearly, the constant function $g_0(x) = 0$ is an upper gradient of f_0.

As P is non-empty and has no isolated points, it is infinite, and so we may find a collection $\mathcal{C}(x_1^0)$ of k_1 distinct points $\{x_i^1\}_{i=1}^{k_1} \subseteq P$. Choose $0 < \epsilon_1 < \epsilon/(2^1 k_1)$ so small that the balls $\{B(x_i^1, \epsilon_1)\}_{i=1}^{k_1}$ are pairwise disjoint.

By the capacity assumption and Remark 4.2, we may find a number $\delta_1 > 0$ and continuous functions $\eta_i^1 : X \to [0, 1]$ for $i = 1, \ldots, k_1$ satisfying

(i) $\operatorname{supp} \eta_i^1$ is a compact subset of $B(x_i^1, \epsilon_1)$,
(ii) $\eta_i^1(x) = 1$ for all $x \in B(x_i^1, \delta_1)$,
(iii) there is an upper gradient g_i^1 of η_i^1 such that $||g_i^1||_{\mathcal{G}} < \epsilon_1$.

As the collection $\{B(x_i^1, \epsilon_1)\}_{i=1}^{k_1}$ is pairwise disjoint, we may define the mapping $f_1\colon X \to Y$ by

$$f_1(x) = \begin{cases} f_0(x) & x \notin \bigcup_{i=1}^{k_1} B_X(x_i^1, \epsilon_1), \\ \gamma_i^1 \circ \eta_i^1(x) & x \in B_X(x_i^1, \epsilon_1). \end{cases}$$

We note that condition (ii) above implies that

$$f_1(B_X(x_i^1, \delta_1)) = \{y_i^1\}, \tag{5.1}$$

and so the image of f_1 contains Y_1. To see that f_1 is continuous, recall that $\gamma_i^1(0) = y_1^0$, and so if $x \notin B(x_i^1, \epsilon_1)$, then

$$\gamma_i^1 \circ \eta_i^1(x) = y_1^0 = f_0(x).$$

Moreover, Lemma 3.12 shows that $g_1 := g_0 + \sum_{i=1}^{k_1} g_i^1$ is an upper gradient of f_1, and

$$||g_1 - g_0||_{\mathcal{G}} = ||g_1||_{\mathcal{G}} \leq k_1 \epsilon_1 < \epsilon 2^{-1}.$$

Since $\mathrm{length}(\gamma_i^1) \leq 1$ for each $i = 1, \ldots, k_1$, we see that for all $x \in X$,

$$d_Y(f_1(x), f_0(x)) \leq 1.$$

We now consider the net $Y_2 = \{y_i^2\}_{i=1}^{k_2}$. Since P is non-empty and has no isolated points, we may find distinct points $\{x_j^2\}_{j=1}^{k_2} \subseteq P$ with the following properties. First, there is a partition of these points into k_1 collections, labelled $\mathcal{C}(x_i^1)$, so that

$$x_j^2 \in \mathcal{C}(x_i^1) \iff y_j^2 \in \mathcal{C}(y_i^1). \tag{5.2}$$

Second, we may find $0 < \epsilon_2 < \epsilon/(2^2 k_2)$ so small that the balls $\{B_X(x_j^2, \epsilon_2)\}_{j=1}^{k_2}$ are disjoint, and that if $x_j^2 \in \mathcal{C}(x_i^1)$, then

$$\overline{B}_X(x_j^2, \epsilon_2) \subseteq B_X(x_i^1, \delta_1) \backslash \{x_i^1\}. \tag{5.3}$$

By the capacity assumption and Remark 4.2, we may find $\delta_2 > 0$ and continuous functions $\eta_j^2\colon X \to [0, 2^{-1}]$ for $j = 1, \ldots, k_2$ satisfying

(i) $\operatorname{supp} \eta_j^2$ is a compact subset of $B(x_j^2, \epsilon_2)$,
(ii) $\eta_j^2(x) = 2^{-1}$ for all $x \in B(x_j^2, \delta_2)$,
(iii) there is an upper gradient g_j^2 of η_j^2 such that $||g_j^2||_{\mathcal{G}} < \epsilon_2$.

Define $f_2\colon X \to Y$ by

$$f_2(x) = \begin{cases} f_1(x) & x \notin \bigcup_{j=1}^{k_2} B_X(x_j^2, \epsilon_2), \\ \gamma_j^2 \circ \eta_j^2(x) & x \in B_X(x_j^2, \epsilon_2). \end{cases}$$

As in the first stage, the mapping f_2 is continuous. Moreover, for any $j = 1, \ldots, k_2$,

$$f_2(B_X(x_j^2, \delta_2)) = \{y_j^2\},$$

while (5.3) implies that for any $i = 1, \ldots, k_1$,

$$f_2(x_i^1) = f_1(x_i^1) = y_i^1.$$

We may apply Lemma 3.12 again to show that $g_2 := g_1 + \sum_{j=1}^{k_2} g_j^1$ is an upper gradient of f_2, and

$$||g_2 - g_1||_{\mathcal{G}} \leq k_2 \epsilon_2 < \epsilon 2^{-2}.$$

Finally, as length$(\gamma_j^2) \leq 2^{-1}$ for each $j = 1, \ldots, k_2$, we see that

$$d_Y(f_2(x), f_1(x)) \leq 2^{-1}.$$

Continuing in this fashion, for each $n \in \mathbb{N}$ we may find a continuous mapping $f_n : X \to Y$, an upper gradient g_n of f_n, and a set $\{x_i^n\}_{i=1}^{k_n} \subseteq P$ such that

- $d_Y(f_{n+1}(x), f_n(x)) \leq 2^{-n}$ for all $x \in X$,
- for all integers $m \geq n \geq 1$ and $i = 1, \ldots, k_n$, it holds that $f_m(x_i^n) = y_i^n$,
- $||g_{n+1} - g_n||_{\mathcal{G}} < \epsilon 2^{-(n+1)}$,
- $f_n(x) = z$ for all $x \notin \bigcup_{i=1}^{k_1} B(x_i^1, \epsilon_1)$.

The first point above shows that $\{f_n : X \to Y\}$ is a Cauchy sequence of mappings in the supremum norm. Since Y is length-compact, it is compact, and hence $\{f_n\}$ converges uniformly to a continuous function $f : X \to Y$. The second point shows that $\cup_n Y_n \subseteq f(P)$. Since P has compact closure in X and the set $\cup_n Y_n$ is dense in the compact space Y, it follows that $f(X) = Y$.

The third point above implies that $\{g_n\}$ is a Cauchy sequence in the Banach function space $L^{\mathcal{G}}(X)$, and that it converges to a function $g : X \to [0, \infty]$ satisfying $||g||_{\mathcal{G}} < \epsilon$. Thus Proposition 3.11 implies that f has an upper gradient $\widetilde{g}$ that also satisfies $||\widetilde{g}||_{\mathcal{G}} < \epsilon$. The fourth point above, along with the details of the construction, shows that f takes the value z outside the ϵ-neighborhood of P. □

With slightly stronger hypotheses, we can produce a continuous surjection with Lipschitz properties.

Theorem 5.2 *If it is additionally assumed in the hypotheses of Theorem 5.1 that each point of P has zero Lipschitz $\mathcal{G}$-capacity, then the mapping $f : X \to Y$ produced by Theorem 5.1 may be chosen so that it also satisfies* $\text{Lip}(f)(x) < \infty$ *for all $x \in X \backslash E$, where E is a compact subset of X with Hausdorff dimension* 0.

Proof Using the notation established in the proof of Theorem 5.1, for $n \in \mathbb{N}$, set

$$B_n = \bigcup_{i=1}^{k_n} B(x_i^n, \epsilon_n).$$

The construction shows that for each $n \in \mathbb{N}$, the closure of B_{n+1} is a subset of B_n. It follows that $E = \bigcap_{n \in \mathbb{N}} B_n$ is closed. Since the sequence $\{\epsilon_n\}$ tends to 0, the set $P \cap E$ is dense in E. Thus E is a subset of the closure of P, and hence compact. By choosing the sequence $\{\epsilon_n\}$ to tend to 0 sufficiently fast, we may assume that E has Hausdorff dimension 0.

Let $x \in X \backslash E$. The nesting of the sets $\{B_n\}_{n \in \mathbb{N}}$ and the fact that E is closed implies that there is open neighborhood U of x and some $n \in \mathbb{N}$ such that

$$U \subseteq (X \backslash B_n)$$

It follows from the construction that $f|_U = f_{n-1}|_U$. With the additional assumption that each point of P has zero Lipschitz $\mathcal{G}$-capacity, the proof of Theorem 5.1 shows that there is a constant $L_n \geq 1$ such that the mapping f_{n-1} is L_n-Lipschitz. From this, we may conclude that $\text{Lip}\, f(x) < L_n$.

We now have all the tools needed to prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3 We assume that there is a non-empty set $P \subseteq X$ that has no isolated points and compact completion, and that X is upper Q-regular at each point of P. By Theorem 4.4, each point of P has zero Lipschitz (Q, q)-Lorentz capacity. Theorem 5.1 now completes the proof. □

Proof of Theorem 1.5 Let $\epsilon > 0$. By Proposition 4.7, we may find $0 < \lambda < 1$ so small that the Cantor diamond space X_λ satisfies a $(2+\epsilon)$-Poincaré inequality. By Proposition 4.6, the space X_λ is Ahlfors 2-regular, and it is clearly compact. By Theorem 4.8, each point of the set $\widetilde{X_\lambda}$, which has compact closure and no isolated points, has zero Lipschitz L^p-capacity for any $1 \leq p < 2+\epsilon$. Theorem 5.1 now completes the proof. □

5.2 The non-compact case

Theorem 5.3 *Suppose that X contains a collection $\{P_i\}_{i\in\mathbb{N}}$ of non-empty subsets such that for each $i \in \mathbb{N}$,*

(i) *the set P_i has no isolated points and has compact closure,*
(ii) *each point of P_i has zero continuous $\mathcal{G}$-capacity,*
(iii) *there is a number $r_i > 0$ so that the resulting collection $\{\mathcal{N}(P_i, r_i)\}_{i\in\mathbb{N}}$ is pairwise disjoint.*

Moreover, suppose that Y is a metric space that may be written as a countable union of length-compact subsets with non-empty intersection. Then there is a continuous surjection $F: X \to Y$ with an upper gradient $G: X \to [0, \infty]$ in $L^{\mathcal{G}}(X)$.

Proof Write $Y = \bigcup_{i\in\mathbb{N}} Y_i$, where for each $i \in \mathbb{N}$ the subset Y_i is length-compact, and let $z \in \bigcap_{i\in\mathbb{N}} Y_i$. By applying Theorem 5.1 with $\epsilon < \min\{r_i, 2^{-i}\}$, we may find a continuous surjection $f_i: X \to Y_i$ that takes the value z off of the set $\mathcal{N}(P_i, r_i)$, and has an upper gradient $g_i: X \to [0, \infty]$ satisfying $||g_i||_{\mathcal{G}} < 2^{-i}$.

For each $k \in \mathbb{N}$, define $F_k: X \to Y$ by

$$F_k(x) = \begin{cases} f_i(x) & x \in \mathcal{N}(P_i, r_i) \text{ for some } 1 \leq i \leq k, \\ z & \text{otherwise.} \end{cases}$$

Then F_k converges pointwise to the continuous surjection $F: X \to Y$ defined by

$$F(x) = \begin{cases} f_i(x) & x \in \mathcal{N}(P_i, r_i), \text{ for some } i \in \mathbb{N}, \\ z & \text{otherwise.} \end{cases}$$

By Lemma 3.12, the function $G_k: X \to [0, \infty]$ defined by

$$G_k(x) = \sum_{i=1}^{k} g_i(x),$$

is an upper gradient of F_k. As $||g_i||_{\mathcal{G}} < 2^{-i}$ for each $i \in \mathbb{N}$, the sequence $\{G_k\}_{k\in\mathbb{N}}$ forms a Cauchy sequence in the Banach space $L^{\mathcal{G}}(X)$. Thus, by Proposition 3.11, F has an upper gradient G in $L^{\mathcal{G}}(X)$. □

Remark 5.4 If

$$\sum_{i\in\mathbb{N}} \mu\left(\mathcal{N}(P_i, r_i)\right) < \infty,$$

then the mapping F constructed above takes the value z off of a set of finite measure. In any case, it is clear that F is locally integrable in the sense of Definition 3.8.

Remark 5.5 Suppose that in the statement of Theorem 5.3, each point of $\cup_i P_i$ has zero Lipschitz $\mathcal{G}$-capacity, and that the set $\cup\overline{P_i}$ is closed. By Theorem 5.2, for each mapping $f_i\colon X \to Y$ in the above construction, we may find a compact set $E_i \subseteq \overline{P_i}$ of Hausdorff dimension 0 such that $\mathrm{Lip}(f_i)(x) < \infty$ for each $x \in X\backslash E_i$. Then the set $E = \cup E_i$ is closed and has Hausdorff dimension 0. Then the mapping F constructed in the proof of Theorem 5.3 can be chosen so that it has finite local Lipschitz constant except on E, as follows.

The proof of Theorem 5.1 shows that f_i takes the value z off the (finitely many) balls employed at the first stage of the construction of f_i, which have radius $\epsilon_i < r_i$. Adding the centers of these balls to the set E_i if necessary, it follows that the set $\mathcal{N}(E_i, 2r_i)$ contains these balls. If $x \in X\backslash E$, there is an open neighborhood U of x such that $\mathrm{dist}(U, E) > 0$. We may assume without loss of generality that the sequence $\{r_i\}$ tends to zero. Hence there is a finite number $N \in \mathbb{N}$ such that $\mathcal{N}(E_i, 2r_i) \cap U = \emptyset$ for all $i \geq N$. Thus, by the above discussion and the definition of F, we see that

$$\mathrm{Lip}(F)(x) \leq \max_{1\leq i\leq N} \mathrm{Lip}(f_i)(x) < \infty.$$

We conclude this section with the proof of Corollary 1.6. For basic information regarding the Heisenberg groups, see, for example, [3].

Proof of Corollary 1.6 We consider the Heisenberg space $\mathbb{H}^n$, $n \geq 1$, to be equipped with the standard Carnot-Carathéodory metric $d_{\mathbb{H}^n}$ and $(2n + 2)$-dimensional Hausdorff measure. Recall that $\mathbb{H}^n$ is an Ahlfors $(2n + 2)$-regular, complete, and geodesic metric space that supports a 1-Poincaré inequality. We will verify the hypotheses of Theorem 5.3 and Remarks 5.4 and 5.5.

We note that the x-axis A_x in $\mathbb{H}^1$ is isometric to $\mathbb{R}^1$. For an integer $i \geq 1$, let

$$P_i \subseteq [i + (1/4), i + (3/4)] \subseteq A_x$$

be a standard Cantor set of diameter 2^{-i}. Then P_i is compact and has no isolated points. Since the standard Euclidean metric on $\mathbb{R}^3$ is majorized by the Heisenberg distance $d_{\mathbb{H}^1}$, the collection $\{\mathcal{N}_{\mathbb{H}^1}(P_i, 1/2)\}$ is pairwise disjoint, and hence $\cup P_i$ is closed. By Theorem 4.4, each point of $\mathbb{H}^1$ has zero Lipschitz $(4, q)$-capacity for any $q > 1$. Moreover, there is a universal constant $c \geq 1$ such that

$$\sum_{i=1}^{\infty} \mathcal{H}^4(\mathcal{N}_{\mathbb{H}^1}(P_i, 2^{-i})) \leq c \sum_{i=1}^{\infty} (\mathrm{diam}_{\mathbb{H}^1} P_i + 2^{-i})^4 < \infty.$$

The metric $d_{\mathbb{H}^n}$ is proper and geodesic. Thus the collection $\{\overline{B}_{\mathbb{H}^n}(0, i)\}$ is an exhaustion of $\mathbb{H}^n$ by length-compact sets each containing the origin. We may now apply Theorem 5.3 and Remarks 5.4 and 5.5 to produce a continuous surjection $F\colon \mathbb{H}^1 \to \mathbb{H}^n$ that is constant off a set of finite measure, has finite local Lipschitz constant off a set of Hausdorff dimension 0, and has an upper gradient in $L^{4,q}(\mathbb{H}^1)$, as desired. The final statement of Corollary 1.6 follows directly from Theorem 1.4, which is proven independently in Sect. 6 □

6 Mappings with an upper gradient in $L^{Q,1}(X)$

In this section, we describe the properties of a mapping $f\colon X \to Y$ that has an upper gradient in the Lorentz space $L^{Q,1}(X)$. The results here are mostly based on [20] and [13]. For the purposes of this paper, the most important property is Lusin's condition N. The source of this property, and several others, is the Rado-Reichelderfer condition.

6.1 The Rado-Reichelderfer condition and its consequences

Definition 6.1 Let $\Theta \in L^1_{\mathrm{loc}}(X)$ be a non-negative function and $\sigma \geq 1$. A mapping $f\colon X \to Y$ satisfies the *Q-Rado-Reichelderfer condition on small scales* with weight Θ and scaling factor $\sigma \geq 1$ if there is a radius $r_0 > 0$ such that for any ball B of radius less than r_0 with compact closure in X,

$$(\operatorname{diam} f(B))^Q \leq \int_{\sigma B} \Theta \, d\mu.$$

Definition 6.2 Let $Q > 0$. A mapping $f\colon X \to Y$ satisfies *Lusin's condition* N_Q if every set $E \subseteq X$ satisfying $\mu(E) = 0$ also satisfies $\mathcal{H}^Q(f(E)) = 0$.

Theorem 6.3 *Let $Q > 0$, and assume that (X, d, μ) is doubling. Then each continuous mapping $f\colon X \to Y$ that satisfies the Q-Rado-Reichelderfer condition on small scales also satisfies Lusin's condition N_Q.*

Proof Iterating the doubling condition, we may find $0 < \beta < \infty$ and $C \geq 1$ such that if $B(y, r) \subseteq B(x, R)$ are nested balls in X, then

$$\frac{\mu(B(x, R))}{\mu(B(y, r))} \leq C \left(\frac{R}{r}\right)^{\beta}. \tag{6.1}$$

Let $E \subseteq X$ be a set satisfying $\mu(E) = 0$. As X is doubling, it is separable, and our standing assumptions states that X is locally compact. As a result, the set E has a countable open cover by sets with compact closure. By the countable sub-additivity of $\mathcal{H}^Q$, we may assume that E itself is contained in an open set U with compact closure.

We assume that X satisfies the Q-Rado-Reichelderfer condition with weight $\Theta \in L^1_{\mathrm{loc}}(X)$ and scaling factor $\sigma \geq 1$ on balls of radius smaller than $r_0 > 0$. We will show that $\mathcal{H}^Q(f(E)) = 0$ by splitting E in two pieces based on the behavior of the weight Θ. Let $\alpha > \beta$, and let G denote the set of points $x \in E$ such that there is a sequence of positive numbers $\{r_i\}$ tending to 0 satisfying

$$\int_{B(x,\sigma r_i)} \Theta \, d\mu \leq (5\sigma)^{\alpha} \int_{B(x, r_i/5)} \Theta \, d\mu. \tag{6.2}$$

We first show that $\mathcal{H}^Q(f(G)) = 0$. Fix $\epsilon > 0$. As μ is assumed to be Borel outer regular, we may find an open set $U_\epsilon \subseteq U$ containing E that satisfies $\mu(U_\epsilon) < \epsilon$. As U has compact closure, the mapping f is uniformly continuous on U_ϵ. Hence there is a number $0 < \delta < \epsilon$ such that if $B \subseteq U_\epsilon$ is a ball of radius less than δ, then $\operatorname{diam} f(B) < \epsilon$.

For each point $x \in G$, we may find a ball $B_x \subseteq U_\epsilon$ of diameter less than $\min\{\delta, r_0\}$ such that (6.2) holds. The separability of X and the standard covering lemma [9, Theorem 1.2] now imply that there is a countable cover $\{B_n\}$ of G consisting of such balls with the additional property that the collection $\{(1/5)B_n\}$ is disjoint. Then $\{f(B_n)\}$ is a cover of $f(G)$ by sets of diameter less than ϵ. Hence, applying the Rado-Reichelderfer condition and (6.2),

$$\begin{aligned}\mathcal{H}^{Q,\epsilon}(f(G)) \leq \sum_{n\in\mathbb{N}} (\operatorname{diam} f(B_n))^Q &\leq \sum_{n\in\mathbb{N}} \int_{\sigma B_n} \Theta\, d\mu \\ &\leq (5\sigma)^\alpha \sum_{n\in\mathbb{N}} \int_{(1/5)B_n} \Theta\, d\mu \\ &\leq (5\sigma)^\alpha \int_{U_\epsilon} \Theta\, d\mu.\end{aligned}$$

As Θ is in $L^1(U)$, letting ϵ tend to zero shows that $\mathcal{H}^Q(f(G)) = 0$.

Consider a point $x \in E\backslash G$. This implies that there is a scale $r_x > 0$ such that if $0 < r < \sigma r_x$, then

$$\int_{B(x,r)} \Theta\, d\mu \leq (5\sigma)^{-\alpha} \int_{B(x,5\sigma r)} \Theta\, d\mu. \tag{6.3}$$

We may also assume that $B(x, \sigma r_x) \subseteq U$. Let $0 < r < r_x$, and find $i \in \mathbb{N}$ such that

$$(5\sigma)^i \sigma r < \sigma r_x \leq (5\sigma)^{i+1} \sigma r. \tag{6.4}$$

Repeatedly applying (6.3) and using (6.4), we see that

$$(\operatorname{diam} f(B(x,r))^Q \leq \int_{B(x,\sigma r)} \Theta\, d\mu \leq (5\sigma)^{-\alpha i} \int_U \Theta\, d\mu \leq \left(\frac{5\sigma r}{r_x}\right)^\alpha \int_U \Theta\, d\mu. \tag{6.5}$$

By the countable sub-additivity of $\mathcal{H}^Q$, it suffices to show that the sets

$$E_n = \{x \in E\backslash G : r_x > 1/n\}$$

satisfy $\mathcal{H}^Q(f(E_n)) = 0$ for each $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N}$ and $\epsilon > 0$. Define $\delta > 0$ as before. Since E is contained in a compact set, we may find a ball B of radius $R > 0$ such that $\mathcal{N}(E_n, 1/n) \subseteq B$. Let $r < \min\{\delta, \sigma/n\}$. The doubling condition implies that we may find a finite maximal r-separated set $\{x_1, \ldots, x_N\} \subseteq E_n$. Then $\{B(x_j, r)\}_{j=1}^N$ covers E_n, while $\{B(x_j, r/2)\}_{j=1}^N$ is disjoint. By (6.1), for each $j = 1, \ldots, N$,

$$1 \leq C\left(\frac{2R}{r}\right)^\beta \frac{\mu(B(x_j, r/2))}{\mu(B)}.$$

Hence, by disjointness,

$$N \leq C\left(\frac{2R}{r}\right)^\beta \sum_{j=1}^N \frac{\mu(B(x_j, r/2))}{\mu(B)} \leq C\left(\frac{2R}{r}\right)^\beta.$$

Using this, (6.5), and the assumption that $r_x > 1/n$, we estimate that

$$\begin{aligned}\mathcal{H}^{Q,\epsilon}(f(E_n)) \leq \sum_{j=1}^N (\operatorname{diam} f(B(x_j, r)))^Q &\leq N\,(5\sigma n r)^\alpha \int_U \Theta\, d\mu \\ &\leq C(2R)^\beta\,(5\sigma n)^\alpha\, r^{\alpha-\beta} \int_U \Theta\, d\mu.\end{aligned}$$

Letting r tend to zero shows that $\mathcal{H}^{Q,\epsilon}(f(E_n)) = 0$. Letting ϵ tend to zero now implies the desired result. □

In appropriate circumstances, the Rado-Reichelderfer condition also implies that the mapping in question has finite Lipschitz constant almost everywhere.

Proposition 6.4 *Assume that (X, d, μ) is doubling and Q-regular at small scales. If $f : X \to Y$ satisfies the Q-Rado-Reichelderfer condition on small scales, then* $\operatorname{Lip} f(x) < \infty$ *for almost every $x \in X$.*

Proof Let $r_0 > 0$ be a scale below which both the Q-regularity and Q-Rado-Reichelderfer conditions hold. Let $\Theta \in L^1_{\text{loc}}(X)$ and $\sigma \geq 1$ be the weight and scaling factor from the Q-Rado-Reichelderfer condition, and suppose that x is a Lebesgue point of f. If $0 < r < r_0$, then

$$\frac{\operatorname{diam} f(B(x,r))}{r} \leq \left(r^{-Q} \int_{B(x,\sigma r)} \Theta \, d\mu \right)^{1/Q} \leq C \left(\fint_{B(x,\sigma r)} \Theta \, d\mu \right)^{1/Q},$$

where C is a number depending only on σ and the constant from the Q-regularity condition. The Lebesgue differentiation theorem now implies that $\operatorname{Lip} f(x) < \infty$ for almost every $x \in X$. □

6.2 The space $L^{Q,1}(X)$ and the Rado-Reichelderfer condition

We now establish the Rado-Reichelderfer condition for mappings with an upper gradient in an appropriate Lorentz space. In the Euclidean setting, this was done in [13]. In the metric setting, closely related results have been given in [22] and [21]. Our proof follows the outline of [13], and hence we occasionally skip a few details. The interested reader can find a full presentation in [29]. We first consider real-valued mappings.

The following lemma, proven in the Euclidean setting in [18], shows that if a pair (f, g) satisfies a 1-Poincaré inequality, then the Riesz potential of g provides a pointwise estimate on the oscillation of f.

Lemma 6.5 *Let $Q > 1$, and assume that (X, d, μ) is doubling and Ahlfors Q-regular on small scales. Suppose that f and g are in the space $L^1_{\text{loc}}(X)$, and that the pair (f, g) satisfies a 1-Poincaré inequality. Then there exists a quantity $C \geq 1$, a radius $r_0 > 0$, and a scaling factor $\sigma > 0$, each depending only the constants associated to the assumptions on X and (f, g), such that*

$$|f(z) - f_B| \leq C \int_{\sigma B} \frac{d(z,y)}{\mu(B(z, d(z,y)))} g(x) \, d\mu(y). \tag{6.6}$$

for almost every point $z \in X$ and any ball B of radius less than r_0 that contains z.

Proof We assume that there is a radius $r_0 > 0$, quantities $C, K \geq 1$, and a scaling factor $\sigma \geq 1$ such that if any ball $B = B(x,r)$ in X,

$$\fint_B |f - f_B| \, d\mu \leq C(\operatorname{diam} B) \fint_{\sigma B} g \, d\mu, \tag{6.7}$$

and if $0 < r < r_0$, then

$$\frac{r^Q}{K} \leq \mu(B) \leq K r^Q. \tag{6.8}$$

In this proof only, we let $c \geq 1$ denote a quantity, possibly varying at each instance, that depends only on Q, C, K, and σ.

By Lebesgue's differentiation theorem [9, Theorem 1.8], almost every point of X is a Lebesgue point. Let $z \in X$ be such a point, and let $B = B(x, r)$ be a ball containing z such that $r < r_0/(4\sigma)$. For integers $k \geq -1$, define $B_k = B(z, 2^{-k}r)$. Since $B_{k+1} \subseteq B_k$ for $k \geq -1$, and $B \subseteq B_{-1}$, the inequalities (6.7) and (6.8) imply that

$$\begin{aligned}
|f(z) - f_B| &\leq \left(\sum_{k=-1}^{\infty} |f_{B_k} - f_{B_{k+1}}|\right) + |f_{B_{-1}} - f_B| \\
&\leq c \sum_{k=-1}^{\infty} 2^{-k} r \fint_{\sigma B_k} g \, d\mu \\
&\leq c \sum_{k=-1}^{\infty} \int_{\sigma B_k} (2^{-k}r)^{1-Q} g \, d\mu \\
&\leq c \sum_{k=-1}^{\infty} \sum_{m=0}^{\infty} \int_{\sigma B_{k+m} \setminus \sigma B_{k+m+1}} (2^{-k}r)^{1-Q} g \, d\mu.
\end{aligned}$$

For integers $k \geq -1$ and $m \geq 0$, a point $y \in \sigma B_{k+m}$ satisfies

$$\left(\frac{d(z, y)}{\sigma 2^{-m}}\right)^{1-Q} \geq (2^{-k}r)^{1-Q}.$$

Hence, changing the order of summation and again using (6.8),

$$\begin{aligned}
|f(z) - f_B| &\leq c \sum_{k=-1}^{\infty} \sum_{m=0}^{\infty} 2^{(1-Q)m} \int_{\sigma B_{k+m} \setminus \sigma B_{k+m+1}} d(z, y)^{1-Q} g \, d\mu(y) \\
&\leq c \sum_{m=0}^{\infty} 2^{(1-Q)m} \int_{\sigma B_{m-1}} d(z, y)^{1-Q} g \, d\mu(y) \\
&\leq c \sum_{m=0}^{\infty} 2^{(1-Q)m} \int_{\sigma B_{-1}} d(z, y)^{1-Q} g \, d\mu(y) \\
&\leq c \int_{\sigma B_{-1}} \frac{d(z, y)}{\mu(B(z, d(z, y)))} \, d\mu(y) \\
&\leq c \int_{B(x,(2\sigma+1)r)} \frac{d(z, y)}{\mu(B(z, d(z, y)))} \, d\mu(y),
\end{aligned}$$

as desired. □

Theorem 6.6 *Let $Q \geq 1$, and assume that X is complete, doubling, Q-regular on small scales, and supports a Q-Poincaré inequality. Let $f : X \to \mathbb{R}$ be a locally integrable function* *with an upper gradient g in the Lorentz space $L^{Q,1}(X)$. Then there is a continuous representative of f that satisfies the Rado-Reichelderfer condition on small scales with weight and scaling factor depending only on g and the constants associated to the assumptions on X.*

Proof Throughout this proof, we refer to the quantities associated with the Poincaré inequality and the doubling and Q-regularity conditions as the data. We also denote by C a quantity, possibly varying at each instance, that depends only on the data. We first consider the case that $Q > 1$.

By Theorem 2.1, there is $\epsilon > 0$, depending only on the data, such that X supports a $(Q-\epsilon)$-Poincaré inequality. Define a perturbed maximal function of g by

$$\widetilde{g}(x) = \left(\sup_{r>0} \fint_{B(x,r)} g^{Q-\epsilon}\, d\mu\right)^{1/(Q-\epsilon)}. \tag{6.9}$$

By [7, Theorem 3.2], there is a constant $c \geq 1$ depending only on the data such that the preturbed maximal function

$$\widetilde{g}(x) = c\left(\sup_{r>0} \fint_{B(x,r)} g^{Q-\epsilon}\, d\mu\right)^{1/(Q-\epsilon)}$$

is a *Hajłasz upper gradient* of f, meaning that for almost every $x, y \in X$,

$$|f(x) - f(y)| \leq d(x, y)(\widetilde{g}(x) + \widetilde{g}(y)).$$

The standard Hardy-Littlewood maximal function theorem [9, Theorem 2.2] and the Marcinkiewicz Interpolation Theorem [2, Theorem IV.4.13] imply that

$$||\widetilde{g}||_{L^{Q,1}} \leq C||g||_{L^{Q,1}} < \infty.$$

It is shown in [6, Sect. 9] that the pair $(f, \widetilde{g})$ satisfies a 1-Poincaré inequality with constant and scaling factor that depend only on the data. Let N be a set of measure zero such that each point of $X\backslash N$ is a point of validity of the conclusions of Lemma 6.5. Hence, there is a radius $r_0 > 0$ and a scaling factor $\sigma > 0$, each depending only on the data, such that if B is a fixed ball of radius less than r_0, then we may find a point $z \in B\backslash N$ such that

$$\operatorname{diam} f(B\backslash N) \leq 3|f(z) - f_B| \leq C \int_{\sigma B} \frac{d(z, x)}{\mu(B(z, d(z, x)))}\widetilde{g}(x)\, d\mu(x). \tag{6.10}$$

A straight-forward generalization of [13, Theorem 3.1] to our setting shows, after possibly shrinking r_0 by a factor depending only on the data, that for any gauge ϕ,

$$\left(\int_{\sigma B} \frac{d(z, x)}{\mu(B(z, d(z, x)))}\widetilde{g}(x)\, d\mu(x)\right)^{Q} \leq C\left(\int_0^{\infty} \phi^{\frac{1}{Q}}(t)\, dt\right)^{Q-1} \int_{\sigma B} F_{\phi}^{Q,1}(\widetilde{g}(x))\, dx, \tag{6.11}$$

whenever the right-hand side is finite (see Subsect. 3.2 for the relevant definitions).

Theorem 3.6 states that there is a gauge $\phi \in \mathcal{A}^{Q,1}$ such that $\phi(|\widetilde{g}(x)|) > 0$ for almost every $x \in X$ with $|\widetilde{g}(x)| > 0$, and such that $F_{\phi}^{Q,1} \circ \widetilde{g} \in L^1(X)$. The gauge ϕ depends only on the data and $\widetilde{g}$.

Define $\Theta\colon X\to\mathbb{R}$ by

$$\Theta(x)=C\left(\int_0^\infty \phi^{\frac{1}{Q}}(t)\,dt\right)^{Q-1}F_\phi^{Q,1}(g(x)).$$

Then $\Theta\in L^1(X)$, and it depends only on the data and $\widetilde{g}$. Combining (6.10) and (6.11), we see that

$$(\operatorname{diam} f(B\backslash N))^Q\le C\int_{\sigma B}\Theta\,d\mu.$$

Since Θ does not depend on B, it follows that f is uniformly continuous on $X\backslash N$. Since N has measure zero, it has empty interior, and hence we may extend $f|_{X\backslash N}$ to a continuous function $\widetilde{f}$ on X. By the continuity of $\widetilde{f}$,

$$(\operatorname{diam}\widetilde{f}(B))^Q=\left(\operatorname{diam}\widetilde{f}(B\backslash N)\right)^Q=(\operatorname{diam} f(B\backslash N))^Q\le C\int_{\sigma B}\Theta\,d\mu,$$

as desired.

Now suppose that $Q=1$. For any ball $B=B(x,r)$ in X, we may find points y and z in B such that

$$\operatorname{diam} f(B)\le 2|f(x)-f(y)|.$$

Since X is doubling and supports a 1-Poincaré inequality, it is quasiconvex with constant depending only on the data. (see e.g., [10] or [16]). Hence there is a path $\gamma\colon[0,1]\to X$ such that $\gamma(0)=x,\ \gamma(1)=y$, and length $\gamma\le Cd(x,y)$. Hence $\gamma([0,1])\subseteq CB$. The definition of an upper gradient implies that

$$\operatorname{diam} f(B)\le 2\int_\gamma g\,ds.$$

We claim that

$$\int_\gamma g\,ds\le C\int_{CB}g\,d\mu.$$

Since $L^{1,1}(X)=L^1(X)$, it suffices to show the claim. As in [9, Exercise 8.11], we may assume without loss of generality that μ is the one-dimensional Hausdorff measure $\mathcal{H}^1$. Proposition 15.1 of [23] implies that we may also assume that γ is injective and parameterized so that

$$\int_\gamma g\,ds=C\int_{\gamma([0,1])}g\,d\mathcal{H}^1.$$

The claim follows. □

A similar result for metric space valued mappings now follows easily.

Corollary 6.7 *Let $Q\ge 1$, and assume that X is complete, doubling, Q-regular on small scales, and supports a Q-Poincaré inequality. Let Y be a separable metric space, and let $f\colon X\to Y$ be a continuous mapping with an upper gradient g in the Lorentz space $L^{Q,1}(X)$. Then f satisfies the Rado-Reichelderfer condition on small scales with a weight Θ depending only on g and the constants associated to the assumptions on X.*

Proof Recall that as Y is separable, there is an isometric embedding $\iota\colon Y \hookrightarrow \ell^\infty$. For each $k \in \mathbb{N}$, let $T_k\colon \ell^\infty \to \mathbb{R}$ denote the 1-Lipschitz projection defined by

$$T_k(\{a_n\}_{n\in\mathbb{N}}) = a_k.$$

Then g is again an upper gradient of the continuous real-valued mapping $T_k \circ \iota \circ f \in L^1_{\text{loc}}(X)$. Hence, by Theorem 6.6, each mapping $T_k \circ \iota \circ f$ satisfies the Q-Rado-Reichelderfer condition with the same weight Θ and scaling factor σ, which depend only on the data and on g. Thus, if B is a sufficiently small ball, the definition of the metric on ℓ^∞ implies that

$$\begin{aligned}(\operatorname{diam} f(B))^Q = (\operatorname{diam} \iota \circ f(B))^Q &= \left(\sup_{x,y\in B}\sup_{k\in\mathbb{N}} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|\right)^Q \\ &= \sup_{k\in\mathbb{N}}\left(\sup_{x,y\in B} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|\right)^Q \\ &\le \int_{\sigma B} \Theta \, d\mu,\end{aligned}$$

yielding the desired result. □

Theorem 1.4 states that certain mappings do not increase dimension. One step in the proof is to show that the mappings under consideration satisfy Lusin's condition N, implying that no set of measure zero can be mapped onto a set of higher dimension. We also need to show that large sets cannot be mapped onto a set of higher dimension. This is true even for Sobolev mappings, as the following well-known statement shows. See also [26, Theorem 1].

Lemma 6.8 *Assume that X is doubling and supports a Q-Poincaré inequality, $Q \ge 1$. Suppose that $f \in L^1_{\text{loc}}(X;Y)$ is continuous and has an upper gradient in $L^Q(X)$. Then there are subsets $E_1 \supseteq E_2 \supseteq \dots$ such that $\mu(E_i) < 1/i$ and $f|_{E_i}$ is Lipschitz for each $i \in \mathbb{N}$. In particular, the set $E = \bigcap_{i\in\mathbb{N}} E_i$ has measure zero, and the Hausdorff dimension of $f(X\backslash E)$ is no greater than the Hausdorff dimension of X.*

Proof Let $\iota\colon Y \to \ell^\infty(Y)$ be an isometric embedding. By Proposition 3.9, the mapping $\iota \circ f$ is locally Bochner integrable, and it has an upper gradient g in $L^Q(X)$. By [11, Theorem 4.3], the mapping $\iota \circ f$ satisfies a Q-Poincaré inequality. Applying [7, Theorem 3.2] to each component of $\iota \circ f$ now shows that f satisfies a pointwise inequality of the form

$$\begin{aligned}d_Y(f(x), f(y)) = ||\iota \circ f(x) - \iota \circ f(y)||_{\ell^\infty(Y)} \lesssim d_X(x,y)((M(g^Q)(x))^{1/Q} \\ +(M(g^Q)(y))^{1/Q}),\end{aligned}$$

where M denotes the Hardy-Littlewood maximal function. Since M maps $L^1(X)$ to weak-$L^1(X)$, the result follows. □

Proof of Theorem 1.4 We assume that X is complete, doubling, Q-regular on small scales, and supports a Q-Poincaré inequality, and that $f \in L^1_{\text{loc}}(X;Y)$ is a continuous surjection with an upper gradient in the space $L^{Q,1}(X)$. Since X is doubling, it is separable, and hence the Q-regularity on small scales and the countable sub-additivity of $\mathcal{H}^Q$ imply that X has Hausdorff dimension Q. By Corollary 3.5, the space $L^{Q,1}(X)$ is contained in $L^Q(X)$. Thus, by Lemma 6.8 there is a set E such that the Hausdorff dimension of $f(X\backslash E)$ is no greater than Q, and $\mu(E) = 0$. On the other hand, Theorem 6.3 and Corollary 6.7 imply that f satisfies Lusin's condition N_Q, and so $\mathcal{H}^Q(f(E)) = 0$ as well. Since f is a surjection, we see that Y has Hausdorff dimension no greater that Q. □

Remark 6.9 Theorem 1.5 shows that the conclusion of Theorem 1.4 does not hold for the Cantor diamond space X_λ, introduced in Subsect. 4.2. However, recent work of Marola and Ziemer allows us to make the following statement [19, Corollary 6.2]. Let $m \geq 3$. If $f\colon X_\lambda \to \mathbb{R}^m$ is a continuous mapping with an upper gradient in the space $L^p(X_\lambda)$ for some $p > p_\lambda$, then f satisfies Lusin's condition N_2. Thus, by Lemma 6.8, we may conclude that f is not a surjection.

Finally, we prove Theorem 4.5, giving conditions under which a point does not have zero Lorentz $(Q, 1)$-capacity. We employ a Sobolev-Lorentz embedding theorem that is valid in great generality [21, Theorem 2.1].

Proof of Theorem 4.5 The case that $Q = 1$ is handled by an argument similar to the one given in the same case of Theorem 6.6, and we leave it to the reader. Hence we let $Q > 1$, and assume that (X, d, μ) is complete, doubling, Q-regular at small scales, and supports a Q-Poincaré inequality. It follows that X is proper and quasiconvex, and hence bi-Lipschitz equivalent to a geodesic space [10], [16]. As the conclusion of the theorem is invariant under bi-Lipschitz mappings, we may assume that X is geodesic. Hence the proof of [21, Proposition 1.4] shows that the hypotheses of [21, Theorem 2.1] are met under our assumptions. As before, we let C be a number, possibly varying at each instance, that depends only on the constants associated to our assumptions.

Towards a contradiction, suppose that a point $a \in X$ has zero continuous $(Q, 1)$-Lorentz capacity. Let r_0 be the scale below which the Q-regularity condition holds, and let $0 < r < r_0$ and $\epsilon > 0$. By assumption, we may find a continuous map $\eta\colon X \to [0, 1]$ that is compactly supported in $B(a, r)$, takes the value 1 on a neighborhood of a, and has an upper gradient g satisfying $||g||_{L^{Q,1}} < \epsilon$. As in the proof of Theorem 6.6, there is a Hajłasz upper gradient $\widetilde{g}$ of η such that $||\widetilde{g}||_{L^{Q,1}} < C\epsilon$. The Sobolev-Lorentz Embedding Theorem [21, Theorem 2.1] now shows that for almost every $x, y \in B(a, r)$,

$$|\eta(x) - \eta(y)| \leq C||\widetilde{g}||_{L^{Q,1}} \leq C\epsilon.$$

Choosing $\epsilon < 1/C$ now yields a contradiction. □

Remark 6.10 Given Theorems 4.5 and 1.4, it is natural to ask what can be said about a mapping $f\colon X \to Y$ with an upper gradient in some Banach function space $L^{\mathcal{G}}(X)$, given that each point of X does not have zero continuous $\mathcal{G}$-capacity. Is there a bound on how much such mappings can increase dimension, in terms of $\mathcal{G}$?

Remark 6.11 If (X, d) is a separable space, all of the results of this section, including Theorems 1.4 and 4.5, make conclusions only involving small scales. However, as is standard in the literature, we often assume the doubling condition and a Poincaré inequality, which control behavior at large scales as well. This inconsistency can be resolved by assuming separability and small scale versions of the doubling condition and the Poincaré inequality. However, one must verify that the tools used in the proofs (such as the Lebesgue differentiation theorem, the existence of a homogeneous measure on a doubling space, and the self-improvement of the Poincaré inequality) have appropriate small scale analogues. This is very likely to be true, though detailed proofs are beyond the scope of this paper.

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