



Some new inequalities for beta distributions

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ABSTRACT

This note provides some new tail inequalities and exponential inequalities of Hoeffding and Bernstein type for beta distributions.

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1. Introduction

Beta distributions play an important role in statistics and probability theory (Gupta and Nadarajah, 2004), and they occur in various scientific fields (Skorski, 2021). A frequent obstacle in problems involving beta distributions is the lack of analytic expressions for their distribution function, the normalized incomplete beta function. Therefore one often resorts to inequalities and approximations, as, for example, in the proofs of Dimitriadis et al. (2022, Theorem 4.1) and Dümbgen and Wellner (2023, Lemma S.8).

This paper provides some new inequalities for the beta distribution $\text{Beta}(a, b)$ with parameters $a, b > 0$, its distribution function $B_{a,b}$, survival function $\bar{B}_{a,b} = 1 - B_{a,b}$ and density function $\beta_{a,b}$ on $[0, 1]$. The latter is given by

$$\beta_{a,b}(x) := B(a, b)^{-1} x^{a-1} (1-x)^{b-1}, \quad x \in (0, 1),$$

where $B(a, b) := \int_0^1 x^{a-1} (1-x)^{b-1} dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, and $\Gamma(\cdot)$ denotes the gamma function. In Section 2, we refine the lower and upper bounds for $B_{a,b}$ and $\bar{B}_{a,b}$ by Segura (2016) which are particularly accurate in the tails of $\text{Beta}(a, b)$. As a by-product we obtain refinements of bounds for the gamma distribution and survival functions by Segura (2014). In Section 3 we present new exponential inequalities which are stronger than previously known inequalities of Dümbgen (1998), Marchal and Arbel (2017) and Skorski (2021). Most proofs are deferred to Section 4. The arXiv version of this article, available at <https://arxiv.org/abs/2202.06718>, additionally provides Gaussian tail inequalities, discusses the approximation of the symmetric distribution $\beta_{a,a}$ by Gaussian densities with mean 1/2 in the spirit of Dümbgen et al. (2021), and contains the proof of Corollary 9.

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2. Sharp tail inequalities

In what follows, let $p := a/(a+b)$, the mean of $\text{Beta}(a, b)$. In a general setting including noncentral beta distributions, Segura (2016, inequalities (27), (29), (30)) uses extensions of l'Hopital's rule to derive inequalities for $B_{a,b}$ and $\bar{B}_{a,b}$. For symmetry reasons, we only consider $B_{a,b}$, because $\bar{B}_{a,b}(x) = B_{b,a}(1-x)$. We rephrase Segura's inequalities in terms of the ratio

$$Q_{a,b}(x) := \frac{B_{a,b}(x)}{x^a/[aB(a, b)]}.$$

This is motivated by the fact that $\beta_{a,b}(x) = B(a, b)^{-1}x^{a-1}(1+O(x))$ and thus $B_{a,b}(x) = x^a/[aB(a, b)](1+O(x))$ as $x \rightarrow 0$. The goal is to find bounds for $Q_{a,b}(x)$ tending to 1 as $x \rightarrow 0$. Now, for $x \in (0, 1)$,

$$(1-x)^b(1+c_{a,b}x) \leq Q_{a,b}(x) \leq \frac{(1-x)^b}{(1-x/p)^+}, \quad (1)$$

where $c_{a,b} := (a+b)/(a+1)$. Numerical examples reveal that these inequalities are rather accurate unless x is close to or larger than p . Our first contribution is an improvement of Segura's bounds. In particular, the upper bound remains valid if x/p is replaced with the strictly smaller term $\max\{c_{a,b}, 1\}x$. The results are stated in terms of the following auxiliary functions:

$$\begin{aligned} q_{a,b}^{(1)}(x) &:= \left(1 - \frac{ax}{a+1}\right)^{b-1}, \\ q_{a,b}^{(2)}(x) &:= \frac{a(1-x)^{b-1} + 1}{a+1} - \frac{a(b-1)(b-2)x^2(1-x)^{(b-3)^+}}{2(a+1)(a+2)}, \\ q_{a,b}^{(3)}(x) &:= \frac{(1-x)^b}{(1-c_{a,b}x)^+}. \end{aligned}$$

Theorem 1. For $x \in (0, 1)$,

$$(1-x)^b(1+c_{a,b}x) < Q_{a,b}^L(x) \leq Q_{a,b}(x) \leq Q_{a,b}^U(x) \leq \frac{(1-x)^b}{(1-\max\{c_{a,b}, 1\}x)^+},$$

where

$$\begin{aligned} Q_{a,b}^L(x) &:= \begin{cases} q_{a,b}^{(1)}(x) & \text{if } b \notin \{1, 2\}, \\ q_{a,b}^{(2)}(x) & \text{if } b \in [1, 2], \end{cases} \\ Q_{a,b}^U(x) &:= \begin{cases} q_{a,b}^{(2)}(x) & \text{if } b \leq 1, \\ q_{a,b}^{(1)}(x) & \text{if } b \in [1, 2], \\ \min\{q_{a,b}^{(2)}(x), q_{a,b}^{(3)}(x)\} & \text{if } b > 2. \end{cases} \end{aligned}$$

Fig. 1 illustrates the bounds for $B_{a,b}$ resulting from (1) and Theorem 1 in case of $(a, b) = (4, 8), (2, 0.5)$.

Remark 2. The new inequalities for $Q_{a,b}$ are equalities in the case of $b \in \{1, 2\}$, because $q_{a,1}^{(1)}(x) = q_{a,1}^{(2)}(x) = Q_{a,1}(x) = 1$ and $q_{a,2}^{(1)}(x) = q_{a,2}^{(2)}(x) = Q_{a,2}(x) = 1 - ax/(a+1)$. Moreover, the upper bound is exact for $b = 3$, because $q_{a,3}^{(2)}(x) = Q_{a,3}(x) = 1 - 2ax/(a+1) + ax^2/(a+2)$.

Remark 3. Note that the ratio $Q_{a,b}$ as well as the bounds $Q_{a,b}^L, Q_{a,b}^U$ are equal to $1 - d_{a,b}x + O(x^2)$ as $x \rightarrow 0$, where $d_{a,b} = (b-1)a/(a+1)$. The lower bound in (1) has the same property, but the upper bound does not.

Gamma distributions. There is a rich literature about inequalities for gamma distribution and survival functions, see, for instance, Qi and Mei (1999), Neuman (2013), Segura (2016) and Pinelis (2020). We just illustrate that our bounds in Theorem 1 yield a connection to that literature. It is well-known that for a random variable $X_{a,b} \sim \text{Beta}(a, b)$, the rescaled variable $bX_{a,b}$ converges in distribution to a gamma random variable with shape parameter a and scale parameter 1 as $b \rightarrow \infty$. Denoting the corresponding distribution and survival function with G_a and $\bar{G}_a = 1 - G_a$, respectively, we have $G_a(x) = \lim_{b \rightarrow \infty} B_{a,b}(x/b)$, and one can deduce from Theorem 1 the following bounds.

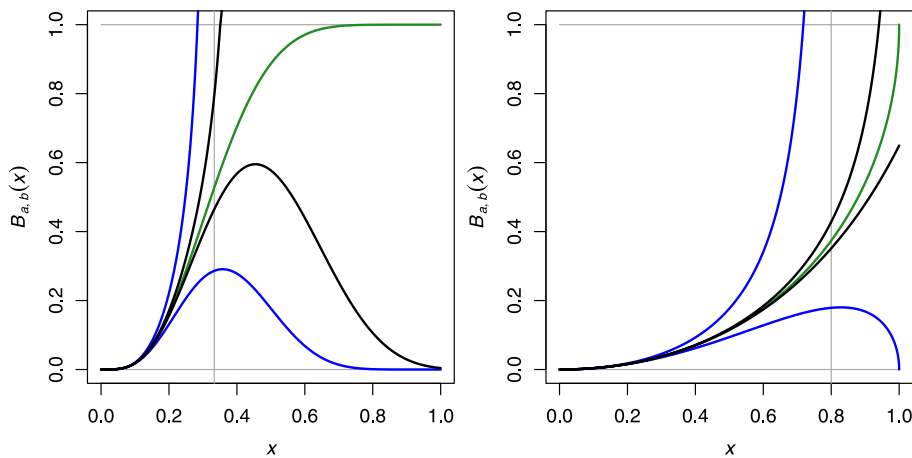


Fig. 1. Inequalities for $B_{a,b}$ when $(a,b) = (4,8)$ (left panel) and $(a,b) = (2,0.5)$ (right panel). The green line shows $B_{a,b}$, the blue lines are Segura (2016) bounds resulting from (1), and the black lines are the bounds via Theorem 1. The vertical line indicates the mean p . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Corollary 4. For $a, x > 0$,

$$\frac{x^a e^{-ax/(a+1)}}{a\Gamma(a)} \leq G_a(x) \leq \frac{x^a}{a\Gamma(a)} \cdot \min \left\{ \frac{ae^{-x} + 1}{a+1} - \frac{ax^2 e^{-x}}{2(a+1)(a+2)}, \frac{e^{-x}}{(1-x/(a+1))^+} \right\},$$

$$\frac{(x + 1_{[a \notin \{1,2\}]})^{a-1} e^{-x}}{\Gamma(a)} \leq \bar{G}_a(x) \leq \begin{cases} \frac{(x + 1_{[a > 1]})^{a-1} e^{-x}}{\Gamma(a)} & \text{if } a \leq 2, \\ \frac{x^a e^{-x}}{\Gamma(a)(x-a+1)^+} & \text{if } a > 2, \\ e^{-x}(x^2/2 + x + 1) & \text{if } a = 3. \end{cases}$$

The lower bound for $G_a(x)$ is already known from Neuman (2013, Theorem 4.1), and the upper bounds for $G_a(x)$ are a combination of Segura (2014, Theorem 10, part 3) and a slight improvement of Neuman (2013, Theorem 4.1). The lower bounds for $\bar{G}_a(x)$ are equalities if $a \in \{1, 2\}$, and the upper bounds if $a \in \{1, 2, 3\}$. Our lower bound for $\bar{G}_a(x)$ extends the lower bound of Segura (2014, Theorem 10, part 4) to $a < 1$, and it is stronger than the latter for $a > 2$. Our upper bound for $\bar{G}_a(x)$ extends the upper bound of Segura (2014, Theorem 10, part 6) to $a < 1$, and it is stronger than the latter if $1 < a \leq 2$.

3. Exponential inequalities

Although the upper bounds in Theorem 1 are numerically rather accurate in the tails, they can diverge to ∞ at $x = p$ as $a, b \rightarrow \infty$. Moreover, it is sometimes desirable to have bounds for $\log B_{a,b}(x)$ and $\log \bar{B}_{a,b}(x)$ in terms of simple, maybe rational, functions of x . Numerous exponential tail inequalities for $B_{a,b}$ and $\bar{B}_{a,b}$ have been derived already. We start with one particular result of Dümbgen (1998, Proposition 2.1). For $x \in [0, 1]$ let

$$K(p, x) := p \log\left(\frac{p}{x}\right) + (1-p) \log\left(\frac{1-p}{1-x}\right) \in [0, \infty].$$

This function $K(p, \cdot)$ is strictly convex with minimum $K(p, p) = 0$. For $x \in [0, 1]$,

$$\frac{x^a(1-x)^b}{p^a(1-p)^b} = \exp(-(a+b)K(p, x)) \geq \begin{cases} B_{a,b}(x) & \text{if } x \leq p, \\ \bar{B}_{a,b}(x) & \text{if } x \geq p. \end{cases} \quad (2)$$

In case of $a \geq 1$ or $b \geq 1$, these inequalities can be improved as follows.

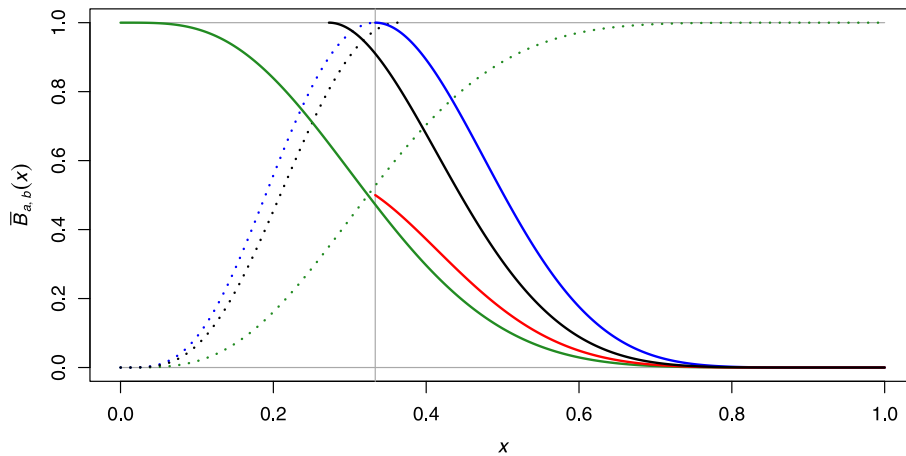


Fig. 2. Exponential tail inequalities for Beta(a, b) when $(a, b) = (4, 8)$. The green line shows $\bar{B}_{a,b}$, the black line is its upper bound from Theorem 5, and the blue line is its upper bound from (2). One also sees the distribution function $B_{a,b}$ and its bounds as dotted lines. The additional red line is the upper bound (3) from Remark 7. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Theorem 5. Suppose that $a \geq 1$. Then for $p_r := (a - 1)/(a + b - 1) < p$ and $x \in [p_r, 1]$,

$$\bar{B}_{a,b}(x) \begin{cases} \leq \frac{x^{a-1}(1-x)^b}{p_r^{a-1}(1-p_r)^b} = \exp(-(a+b-1)K(p_r, x)), \\ \geq \frac{x^{a-1}(1-x)^b}{bB(a, b)}. \end{cases}$$

Suppose that $b \geq 1$. Then for $p_\ell := a/(a + b - 1) > p$ and $x \in [0, p_\ell]$,

$$B_{a,b}(x) \begin{cases} \leq \frac{x^a(1-x)^{b-1}}{p_\ell^a(1-p_\ell)^{b-1}} = \exp(-(a+b-1)K(p_\ell, x)), \\ \geq \frac{x^a(1-x)^{b-1}}{aB(a, b)}. \end{cases}$$

The lower bound for $B_{a,b}(x)$ is less accurate than the lower bound from Theorem 1 if $b > 2$. The upper bounds are typically more accurate if x is close to p .

Remark 6. At first glance, the upper bounds in Theorem 5 seem to be weaker than the ones in (2), at least in the tail regions, because the factor $a + b - 1$ is strictly smaller than $a + b$. But elementary algebra reveals that in case of $a \geq 1$,

$$\begin{aligned} & (a + b - 1)K(p_r, x) - (a + b)K(p, x) \\ &= \log\left(\frac{x}{p}\right) + (a + b - 1)\log\left(1 + \frac{1}{a + b - 1}\right) - (a - 1)\log\left(1 + \frac{1}{a - 1}\right) \\ &> 0 \quad \text{for } x \in [p, 1], \end{aligned}$$

because $h(y) := y \log(1 + 1/y)$ (with $h(0) := 0$) is strictly increasing in $y \geq 0$. Analogously, if $b \geq 1$, then

$$\begin{aligned} & (a + b - 1)K(p_\ell, x) - (a + b)K(p, x) \\ &= \log\left(\frac{1-x}{1-p}\right) + (a + b - 1)\log\left(1 + \frac{1}{a + b - 1}\right) - (b - 1)\log\left(1 + \frac{1}{b - 1}\right) \\ &> 0 \quad \text{for } x \in (0, p]. \end{aligned}$$

Thus the bounds in Theorem 5 are strictly smaller than the bounds in (2). This is illustrated in Fig. 2 for $(a, b) = (4, 8)$.

Remark 7. The upper bound for $\bar{B}_{a,b}$ in Theorem 5 can be improved substantially if $1 \leq a \leq b$. Indeed, the proof of Theorem 5 shows that for arbitrary $0 < x_0 \leq x \leq 1$,

$$\bar{B}_{a,b}(x) \leq \frac{\bar{B}_{a,b}(x_0)}{x_0^{a-1}(1-x_0)^b} x^{a-1}(1-x)^b.$$

Specifically, it is well-known that $\text{Median}(\text{Beta}(a, b)) \leq p$, see [Groeneveld and Meeden \(1977\)](#), so $\bar{B}_{a,b}(p) \leq 1/2$ and for $x \in [p, 1]$,

$$\bar{B}_{a,b}(x) \leq \frac{x^{a-1}(1-x)^b}{2p^{a-1}(1-p)^b}. \quad (3)$$

The latter bound is strictly smaller than the upper bound of [Theorem 5](#) (restricted to $x \in [p, 1]$), provided that $2p^{a-1}(1-p)^b > p_r^{a-1}(1-p_r)^b$, and this is equivalent to $h(a-1) > h(a+b-1) - \log(2)$ with the increasing function $h(y) = y \log(1 + 1/y)$, $y > 0$. Since $h(a+b-1) < \lim_{y \rightarrow \infty} h(y) = 1$, a sufficient condition is that $h(a-1) \geq 1 - \log(2)$, which is fulfilled for $a \geq 1.152$.

The inequalities in [Theorem 5](#) imply Bernstein and Hoeffding type exponential inequalities. It follows from [Dümbgen and Wellner \(2023, Lemma S.12\)](#) and the well-known inequality $z(1-z) \leq 1/4$ for $z \in \mathbb{R}$, that

$$K(q, x) \geq \frac{(x-q)^2}{2(2x/3 + q/3)(1-2x/3 - q/3)} \geq 2(x-q)^2 \quad (4)$$

for $q, x \in [0, 1]$, where $K(0, x) := -\log(1-x)$ and $K(1, x) := -\log(x)$. This leads to the following inequalities:

Corollary 8. *If $a \geq 1$, then for $x \in [p_r, 1]$,*

$$\begin{aligned} \bar{B}_{a,b}(x) &\leq \exp\left(-\frac{(a+b-1)(x-p_r)^2}{2(2x/3 + p_r/3)(1-2x/3 - p_r/3)}\right) \\ &\leq \exp(-2(a+b-1)(x-p_r)^2). \end{aligned}$$

If $b \geq 1$, then for $x \in [0, p_\ell]$,

$$\begin{aligned} B_{a,b}(x) &\leq \exp\left(-(a+b-1)\frac{(x-p_\ell)^2}{2(2x/3 + p_\ell/3)(1-2x/3 - p_\ell/3)}\right) \\ &\leq \exp(-2(a+b-1)(x-p_\ell)^2). \end{aligned}$$

Further tail and concentration inequalities for the Beta distribution have been derived by [Marchal and Arbel \(2017\)](#) and [Skorski \(2021\)](#). [Marchal and Arbel \(2017\)](#) prove that $\text{Beta}(a, b)$ is subgaussian with a variance parameter that is the solution of an equation involving hypergeometric functions. An upper bound for the variance parameter is $(4(a+b+1))^{-1}$, which implies the inequalities

$$\exp(-2(a+b+1)(x-p)^2) \geq \begin{cases} B_{a,b}(x) & \text{if } x \leq p, \\ \bar{B}_{a,b}(x) & \text{if } x \geq p. \end{cases}$$

These bounds are weaker than the one-sided bounds in [Corollary 8](#). Indeed, for the right tails, the difference

$$(a+b-1)(x-p_r)^2 - (a+b+1)(x-p)^2$$

is strictly concave in x with value $b^2/[(a+b)^2(a+b-1)] > 0$ for $x \in \{p, 1\}$. Analogously, for the left tails, the difference

$$(a+b-1)(x-p_\ell)^2 - (a+b+1)(x-p)^2$$

is strictly concave in x with value $a^2/[(a+b)^2(a+b-1)] > 0$ for $x \in \{0, p\}$. [Skorski \(2021\)](#) derives a Bernstein type inequality. With the parameters

$$v^2 := \frac{p(1-p)}{a+b+1}, \quad c := \max\left(\frac{|1-2p|}{a+b+2}, \sqrt{\frac{p(1-p)}{a+b+2}}\right),$$

he shows that for $X \sim \text{Beta}(a, b)$ and $\varepsilon \geq 0$,

$$P(\pm(X-p) \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2(v^2 + c\varepsilon)}\right).$$

The next result shows that our bounds imply a stronger version of these inequalities if $a, b \geq 1$.

Corollary 9. *Let $a, b \geq 1$. Then for $x \in [p, 1]$,*

$$\bar{B}_{a,b}(x) \leq \exp\left(-\frac{(a+b+1)(x-p)^2}{2p(1-p) + (4/3)(1-2p)(x-p)}\right),$$

and for $x \in [0, p]$,

$$B_{a,b}(x) \leq \exp\left(-\frac{(a+b+1)(x-p)^2}{2p(1-p) + (4/3)(2p-1)(p-x)}\right).$$

With the notation of Skorski (2021), our upper bounds read

$$P(\pm(X - p) \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2(v^2 \pm \tilde{c}\varepsilon)}\right)$$

with v^2 as before and $\tilde{c} = (2/3)(1 - 2p)/(a + b + 1)$. In particular,

$$|\tilde{c}| = \frac{2(a + b + 2)}{3(a + b + 1)} \frac{|1 - 2p|}{a + b + 2} \leq \frac{2(a + b + 2)}{3(a + b + 1)} C.$$

Since $a, b \geq 1$, the factor $2(a + b + 2)/[3(a + b + 1)]$ is at most $8/9$ and converges to $2/3$ as $a + b \rightarrow \infty$.

4. Proofs

Proof of Theorem 1. Note that

$$Q_{a,b}(x) = \frac{a}{x^a} \int_0^x u^{a-1}(1-u)^{b-1} du = a \int_0^1 w^{a-1}(1-xw)^{b-1} dw.$$

Since $d^2(1-xw)^{b-1}/dw^2 = (b-1)(b-2)x^2(1-xw)^{b-3}$, the function $w \mapsto (1-xw)^{b-1}$ is convex if $b \notin (1, 2)$ and concave if $b \in [1, 2]$. Since $a \int_0^1 w^{a-1} dw = 1$, it follows from Jensen's inequality that

$$\left(a \int_0^1 w^{a-1}(1-xw) dw\right)^{b-1} = q_{a,b}^{(1)}(x)$$

is a lower bound for $Q_{a,b}$ if $b \notin (1, 2)$ and an upper bound if $b \in [1, 2]$. To compare $Q_{a,b}$ with $q_{a,b}^{(2)}$, we use a well-known formula for linear interpolation of the function $w \mapsto (1-xw)^{b-1}$ with second derivative $(b-1)(b-2)x^2(1-xw)^{b-3}$ on $[0, 1]$, namely,

$$(1-xw)^{b-1} = 1 - w + w(1-x)^{b-1} - w(1-w)(b-1)(b-2)x^2(1-x\tilde{w})^{b-3}/2$$

for some $\tilde{w} = \tilde{w}(x, w) \in (0, 1)$. Note that

$$(b-1)(b-2)(1-x\tilde{w})^{b-3} \begin{cases} \geq (b-1)(b-2) & \text{if } b \leq 1, \\ \leq (b-1)(b-2) & \text{if } b \in [1, 2], \\ \geq (b-1)(b-2)(1-x)^{(b-3)^+} & \text{if } b \geq 2. \end{cases}$$

Hence, with $h_b(w) := 1 - w + w(1-x)^{b-1} - w(1-w)(b-1)(b-2)x^2(1-x)^{(b-3)^+}/2$ we may conclude that

$$a \int_0^1 w^{a-1} h_b(w) dw = q_{a,b}^{(2)}(x)$$

is an upper bound for $Q_{a,b}(x)$ if $b \notin (1, 2)$ and a lower bound if $b \in [1, 2]$.

Concerning alternative bounds for $Q_{a,b}$, let $Q : [0, x_0] \rightarrow (0, \infty]$ be a continuous function for some $x_0 \in (0, 1]$. Viewing Q as a bound of $Q_{a,b}$, $H(x) = x^a Q(x)/[aB(a, b)]$ is a bound for $B_{a,b}(x)$. If Q is differentiable on $(0, x_0)$, then elementary calculus reveals that

$$H'(x) = \beta_{a,b}(x)J(x) \quad \text{with} \quad J(x) := \frac{Q(x) + Q'(x)x/a}{(1-x)^{b-1}}.$$

If we can show that $J \geq 1$ (or $J \leq 1$) on $(0, x_0)$, we may conclude that $Q_{a,b} \leq Q$ (or $Q_{a,b} \geq Q$) on $[0, x_0]$. For instance, let $Q(x) := (1-x)^b(1+cx)$ for some $c > 0$ and $x \in [0, 1]$. Then one can show that $J \leq 1$ on $[0, 1]$, provided that $c \geq c_{a,b} = (a+b)/(a+1)$. This yields the lower bound for $Q_{a,b}$ in (1). Now, let $Q(x) := (1-x)^b/(1-cx)$ for some $c > 0$ and $0 \leq x \leq x_0 := \min\{c^{-1}, 1\}$. For $0 < x < x_0$,

$$J(x) = 1 + \frac{x}{a(1-cx)} \left(c(a+1) - (a+b) + (c-1) \frac{cx}{1-cx} \right).$$

If $c < 1$, then the infimum of $c(a+1) - (a+b) + (c-1)cx/(1-cx)$ over all $x \in (0, x_0)$ equals $ca - (a+b) < 0$. If $c \geq 1$, that infimum equals $c(a+1) - (a+b) \geq 0$, provided that $c \geq (a+b)/(a+1)$. Consequently, $J \geq 1$ on $(0, x_0)$ if $c \geq \max\{c_\ell, 1\}$, and this yields the upper bound in (1) as well as the upper bound $q_{a,b}^{(3)}(x)$ for $Q_{a,b}(x)$ in case of $b \geq 1$.

It remains to verify the additional inequalities for $Q_{a,b}^L, Q_{a,b}^U$. Concerning the lower bound for $Q_{a,b}^L$, the inequality $q_{a,b}^{(1)}(x) > (1-x)^b(1+c_{a,b}x)$ is equivalent to

$$(1-x/(a+1-ax))^{-b} > 1 + bx/(a+1) - a(a+b)x^2/(a+1)^2.$$

Indeed, by convexity of $(1 - \cdot)^{-b}$, the left-hand side is larger than $1 + bx/(a + 1 - ax) > 1 + bx/(a + 1)$. Now let $b \geq 1$. For $b \in [1, 2]$, $q_{a,b}^{(2)}(x) \geq (a(1-x)^{b-1} + 1)/(a + 1)$, and the latter term is strictly larger than $(1-x)^b/(1 + c_{a,b}x)$ if and only if

$$(1-x)^{-(b-1)} > 1 + (b-1)x - (a+b)x^2.$$

Indeed, since $b-1 \geq 0$, the left-hand side is not smaller than $1 + (b-1)x$.

Concerning the upper bound for $Q_{a,b}^U$, if $b \leq 1$, then $c_{a,b} \leq 1$ and $(1-x)^{b-1} \geq 1$, so $q_{a,b}^{(2)}(x) \leq (a(1-x)^{b-1} + 1)/(a + 1) \leq (1-x)^{b-1} = (1-x)^b/(1 - \max\{c_{a,b}, 1\}x)^+$. If $1 < b \leq 2$, then $c_{a,b} > 1$, and the inequality $q_{a,b}^{(1)}(x) < (1-x)^b/(1 - c_{a,b}x)^+$ is equivalent to

$$(1 - (b-1)y)^+(1+y)^{b-1} < 1$$

with $y := (a+1)^{-1}x/(1-x) \in (0, \infty)$. By concavity of $(1+\cdot)^{b-1}$, $(1+y)^{b-1} \leq 1 + (b-1)y$, whence $(1 - (b-1)y)^+(1+y)^{b-1} \leq (1 - (b-1)^2y^2)^+ < 1$.

Proof of Corollary 4. Recall Stirling's approximation $\Gamma(c) = \sqrt{2\pi}c^{c-1/2}e^{-c}(1+o(1))$ as $c \rightarrow \infty$. This implies the following asymptotic expansions as $b \rightarrow \infty$:

$$\frac{1}{B(a, b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{b^a(1+a/b)^{a+b-1/2}e^{-a}(1+o(1))}{\Gamma(a)} = \frac{b^a(1+o(1))}{\Gamma(a)}.$$

Consequently,

$$G_a(x) = \lim_{b \rightarrow \infty} B_{a,b}(x/b) = \lim_{b \rightarrow \infty} \frac{(x/b)^a}{aB(a, b)} Q_{a,b}(x/b) = \frac{x^a}{a\Gamma(a)} \lim_{b \rightarrow \infty} Q_{a,b}(x/b).$$

Bounding $Q_{a,b}(x/b)$ in terms of $q_{a,b}^{(\ell)}(x/b)$, $1 \leq \ell \leq 3$, as in Theorem 1, the asserted bounds for $G_a(x)$ follow immediately from the following limits:

$$\begin{aligned} q_{a,b}^{(1)}(x/b) &= \left(1 - \frac{ax}{(a+1)b}\right)^{b-1} \rightarrow e^{-ax/(a+1)}, \\ q_{a,b}^{(2)}(x/b) &= \frac{a(1-x/b)^{b-1} + 1}{a+1} - \frac{a(b-1)(b-2)x^2(1-x/b)^{(b-3)^+}}{2b^2(a+1)(a+2)} \\ &\rightarrow \frac{ae^{-x} + 1}{a+1} - \frac{ax^2e^{-x}}{2(a+1)(a+2)}, \\ q_{a,b}^{(3)}(x/b) &= \frac{(1-x/b)^b}{(1 - [(a+b)/b]x/(a+1))^+} \rightarrow \frac{e^{-x}}{(1-x/(a+1))^+}. \end{aligned}$$

As to $\bar{G}_a(x)$, we write $\bar{G}_a(x) = \lim_{b \rightarrow \infty} \bar{B}_{a,b}(x/b) = \lim_{b \rightarrow \infty} B_{b,a}(1-x/b)$ and

$$B_{b,a}(1-x/b) = \frac{(1-x/b)^b}{bB(a, b)} Q_{b,a}(1-x/b) = \frac{e^{-x}b^{a-1}(1+o(1))}{\Gamma(a)} Q_{b,a}(1-x/b),$$

so $\bar{G}_a(x)$ is $e^{-x}/\Gamma(a)$ times $\lim_{b \rightarrow \infty} b^{a-1}Q_{b,a}(1-x/b)$. Bounding $Q_{b,1}(1-x/b)$ in terms of $q_{b,1}^{(\ell)}(1-x/b)$, $1 \leq \ell \leq 3$, as in Theorem 1, the asserted bounds for $\bar{G}_a(x)$ follow immediately from the following limits:

$$\begin{aligned} b^{a-1}q_{b,a}^{(1)}(1-x/b) &= b^{a-1}\left(\frac{x+1}{b+1}\right)^{a-1} \rightarrow (x+1)^{a-1}, \\ b^{a-1}q_{b,a}^{(2)}(1-x/b) &= \frac{bx^{a-1} + b^{a-1}}{b+1} - \frac{b^a(a-1)(a-2)(1-x/b)^2(x/b)^{(a-3)^+}}{2(b+1)(b+2)} \\ &\rightarrow \begin{cases} x^{a-1} & \text{if } a < 2, \\ x+1 & \text{if } a = 2, \\ \infty & \text{if } a > 2, a \neq 3, \\ x^2 + 2x + 2 & \text{if } a = 3, \end{cases} \\ b^{a-1}q_{b,a}^{(3)}(1-x/b) &= x^a / \left(\frac{(a+b)x - b(a-1)}{b+1}\right)^+ \rightarrow \frac{x^a}{(x-a+1)^+}. \end{aligned}$$

Proof of Theorem 5. Since $B_{a,b}(\cdot) = \bar{B}_{b,a}(1 - \cdot)$ and $K(q, x) = K(1-q, 1-x)$ for $q \in (0, 1)$ and $x \in [0, 1]$, it suffices to prove the result for $\bar{B}_{a,b}(x)$, $x \in [p_r, 1]$.

In case of $a = 1$, the asserted bounds are sharp, because $B(a, b) = 1/b$, $p_r = 0$ and $\bar{B}_{a,b}(x) = (1-x)^b$. In case of $a > 1$, the ratio

$$Q(x) := \frac{\bar{B}_{a,b}(x)}{x^{a-1}(1-x)^b} = \frac{B(a, b)^{-1}}{1-x} \int_x^1 \left(\frac{u}{x}\right)^{a-1} \left(\frac{1-u}{1-x}\right)^{b-1} du$$

is strictly decreasing in $x \in (0, 1)$. Indeed, with $w(u) := (1-u)/(1-x) \in (0, 1)$ for $u \in (x, 1)$, we have $dw(u)/du = -1/(1-x)$, and $u = 1 - (1-x)w(u)$, so

$$Q(x) = B(a, b)^{-1} \int_0^1 \left(w + \frac{1-w}{x}\right)^{a-1} w^{b-1} dw,$$

which is strictly decreasing in $x \in (0, 1)$ with limit $Q(1) = 1/[bB(a, b)]$. Consequently, we obtain that $Q(1) \leq Q(x) \leq Q(x_0)$ for $0 < x_0 \leq x \leq 1$. Multiplying these inequalities with $x^{a-1}(1-x)^b$ and setting $x_0 = p_r$ yields the asserted bounds for $\bar{B}_{a,b}(x)$.

Data availability

No data was used for the research described in the article.

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