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Almost positive links are strongly quasipositive

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Abstract

We prove that any link admitting a diagram with a single negative crossing is strongly quasipositive. This answers a question of Stoimenow's in the (strong) positive. As a second main result, we give a simple and complete characterization of link diagrams with quasipositive canonical surface (the surface produced by Seifert's algorithm). As applications, we determine which prime knots up to 13 crossings are strongly quasipositive, and we confirm the following conjecture for knots that have a canonical surface realizing their genus: a knot is strongly quasipositive if and only if the Bennequin inequality is an equality.

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Introduction

Notions of quasipositivity for links and surfaces were introduced and explored by Rudolph in a series of papers (cited in the text). Their study is motivated, for example,

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by connections with complex algebraic plane curves [6,20] and relationships to contact geometry [3,12].

Quasipositive links, strongly quasipositive links, and quasipositive Seifert surfaces are usually defined in terms of braids. In this paper, however, our focus lies more on geometry and less on braids, and so we omit the original definitions in favor of the following characterizations: a Seifert surface is called *quasipositive* if it is an incompressible subsurface of the fiber surface of a positive torus link (*incompressible* meaning that the map induced by inclusion on the fundamental group is injective) and *strongly quasipositive links* are precisely those links that arise as the boundary of a quasipositive Seifert surface. That these characterizations are equivalent to the original definitions is due to Rudolph [21]. We are not concerned with (non-strongly) quasipositive links in this text.

Quasipositive Seifert surfaces are of maximal Euler characteristic; not just among Seifert surfaces of the given link, but even among smooth slice surfaces [13,22].

Main results

Links that admit a positive diagram, in other words a diagram without negative crossings, are known as *positive links*. Positive links are strongly quasipositive [18,24]. Our first main result generalizes this to *almost positive links*—links admitting an almost positive diagram, in other words a diagram with a single negative crossing. This gives a positive answer to a question of Stoimenow [27, Question 4].

Theorem A *Almost positive links are strongly quasipositive.*

Note that the hypothesis cannot be weakened further (at least in the most obvious way), since links admitting diagrams with two negative crossings need not even be quasipositive (for example, the figure eight knot).

Almost positive links have been studied before they were given this name, and their similarity in many respects to positive links has been observed. For example, Cromwell showed that almost positive links have Conway polynomials with non-negative coefficients [7]; Przytycki and Taniyama proved they have negative signature [19]; Stoimenow showed that non-trivial almost positive links are chiral and non-slice [26]; and Tagami proved that the 3-genus, 4-genus, and $s/2$ (for s the Rasmussen invariant) of almost positive knots agree [31]. Theorem A can be seen in this context. In particular, Theorem A recovers the last result: the slice-Bennequin inequality implies that for any strongly quasipositive knot (and thus for any almost positive knot) the 3-genus, 4-genus, and all slice-torus invariants agree [14,15,22]. This also proves chirality and non-sliceness for non-trivial almost positive knots. Here, a *slice-torus invariant* [14,15] is a homomorphism y from the smooth concordance group to $\mathbb{R}$ such that for all knots K, $y(K)$ is a lower bound for the 4-genus of K, and for positive torus knots K, $y(K)$ is equal to the 4-genus of K. Examples of such y include τ from knot Floer homology, and $s/2$.

To prove Theorem A, we explicitly exhibit quasipositive Seifert surfaces for all almost positive links. For a certain type of almost positive diagram (which will later be referred to as type I), we prove that in fact the *canonical surface* is quasipositive (the canonical surface is that produced from the diagram by Seifert's algorithm).

Canonical surfaces have been studied extensively; it is for example a classical result that the canonical surfaces of alternating diagrams are genus-minimizing [8,17], which generalizes to homogeneous diagrams [7], and has recently been scrutinized further [28,29]. In this light, our proof of Theorem A naturally begs the question: *which canonical surfaces are quasipositive?* The complete answer to this question forms our second main result: namely a criterion in terms of the Seifert graph, which is combinatorial and algorithmic.

Theorem B *A canonical surface is quasipositive if and only if all cycles of its Seifert graph have strictly positive total weight.*

Here, the *Seifert graph* $\Gamma(D)$ of a diagram D has the Seifert circles as vertex set, and one edge between k and k' for each crossing connecting the Seifert circles k and k'. It is a bipartite graph, possibly with multiple edges between two vertices. Its edges carry a *weight* of ± 1 corresponding to the sign of the crossing. The *total weight* of a cycle is understood as the sum of the weights of the cycle's edges. The reader will find more details on Seifert graphs at the beginning of Sect. 1.

Applications of Theorem B

Theorem B implies a purely geometric criterion for quasipositivity of a canonical Seifert surface, which we state as the following corollary.

Corollary C *Let Σ be a Seifert surface that is isotopic to a canonical surface. Then Σ is quasipositive if and only if every unknot contained in Σ bounds a disk in Σ or has negative induced framing by Σ.*

Corollary C does not generalize to non-canonical Seifert surfaces; in fact, there exist non-quasipositive Seifert surfaces Σ such that all incompressible annuli of Σ are quasipositive Seifert surfaces [4].

Next we observe the following criterion for quasipositivity. This allows, see the example below, to determine the strong-quasipositivity status of all prime knots up to 13 crossings, in particular recovering the recently completed calculation [10] of the strong-quasipositivity status of prime knots up to 12 crossings. Throughout this subsection, let y denote a slice-torus invariant.

Theorem D *If K is a knot with a canonical surface Σ such that $y(K) = \operatorname{genus}(\Sigma)$, then Σ is a quasipositive Seifert surface; in particular, K is a strongly quasipositive knot.*

Recall that the Bennequin inequality states $\frac{\mathrm{sl}(K)+1}{2} \leq g(K)$, where $\mathrm{sl}(K)$ is defined in either of the following two equivalent ways; see [5]:

$$\mathrm{sl}(K) := \max\{\mathrm{sl}(T) \mid T \text{ is a transverse representative of } K\}$$
$$\mathrm{sl}(K) := \max\{\operatorname{writhe}(\beta) - n \mid \beta \text{ is an } n\text{-braid with closure } K\}.$$

It is a conjecture (popularized by Hedden, Etnyre and Van Horn-Morris amongst others) that the Bennequin inequality is an equality if and only if K is strongly quasipositive; compare also [10]. As a consequence of Theorem D, we confirm this conjecture for knots with canonical genus $\tilde{g}$ (the minimum genus of a canonical surface) equal to the genus.

Corollary E *Let K be a knot with $\tilde{g}(K) = g(K)$, i.e. a knot for which the genus $g(K)$ is realized by a canonical surface Σ. The following are equivalent:*

(1) *Σ is quasipositive,*
(2) *K is strongly quasipositive,*
(3) *for K the Bennequin-inequality is an equality, and*
(4) *$y(K) = g(K)$.*

Note that for K a fibered knot with fiber surface Σ, the conditions of (1)–(4) in Corollary E are also equivalent [12].

Applied to the canonical surface $\Sigma(p_1, \ldots, p_{2n+1})$ of genus n of the $P(p_1, \ldots, p_{2n+1})$ pretzel knot with all $p_1, \ldots, p_n \in \mathbb{Z}$ odd, Theorem B immediately yields that this surface is quasipositive if and only if $p_i + p_j < 0$ for all $1 \leq i < j \leq n$, recovering a result of Rudolph's [22,25]. Moreover, since $\Sigma(p_1, \ldots, p_{2n+1})$ is genus-minimizing [9, Theorem 3.2], Corollary E now implies the following.

Corollary F *The $P(p_1, \ldots, p_{2n+1})$ pretzel knot with all $p_1, \ldots, p_n$ odd is strongly quasipositive if and only if $p_i + p_j < 0$ for all $1 \leq i < j \leq n$.* □

Example We claim that if K is a prime knot with crossing number $c(K) \leq 13$, then K is strongly quasipositive if and only if $y(K) = g(K)$. The 'only if' direction holds for all knots. To show the 'if' direction, we rely on Stoimenow's calculation [30] that for all prime knots K with $c(K) \leq 13$, it holds that $\tilde{g}(K) = m(K)$, where m denotes Morton's lower bound [16] for $\tilde{g}$ coming from the Homflypt-polynomial. So, using Theorem D, it is enough to show that all prime knots with $c(K) \leq 13$ satisfy $\tau(K) < g(K)$ or $\tau(K) = m(K)$. This is readily verified by a computer calculation.

Note that this criterion is algorithmic, and can be used in practice to determine the strong quasipositivity status of a given prime knot K with $c(K) \leq 13$: simply calculate $\tau(K)$ and $m(K)$; K is strongly quasipositive if and only if $\tau(K) = m(K)$.

We do not know whether $y(K) = g(K)$ also implies strong quasipositivity for prime knots K with $c(K) = 14$. For $c(K) = 15$, we know it does not: the 2-twisted positive Whitehead double of the right-handed trefoil knot is a prime 15-crossing knot (15n115646 in the table) with $\tau = g = 1$, which is not quasipositive since its Rasmussen invariant s is 0 [11].

Remark Theorem B also provides a new proof for Baader's theorem [1] that a knot K is positive if and only if it is strongly quasipositive and homogeneous.

Here, following [7], a knot is called *homogeneous* if it admits a homogeneous diagram D; and D is called homogeneous if all edges within each block of the Seifert graph $\Gamma(D)$ carry the same weight; where a block B of a graph G is either an isolated vertex of G, or a maximal subgraph of G with the property that B is connected and $B \setminus v$ is connected for all vertices v of B.

We leave it as an exercise in graph theory to show that a knot diagram D is homogeneous if and only if all edges within each cycle of $\Gamma(D)$ carry the same weight.

The 'only if' direction of Baader's theorem is clear. For the 'if' direction, let a strongly quasipositive and homogeneous knot K be given. Let D be a homogeneous diagram of K that is also reduced (has no nugatory crossings). For Σ the canonical surface of D, we have $g(\Sigma) = g(K)$ [7]. So, by Corollary E, Σ is quasipositive. We are going to show that D is a positive diagram. Let an edge e of $\Gamma(D)$ be given. Since the crossing corresponding to e is not nugatory, e is contained in a cycle C of $\Gamma(D)$. Because Σ is quasipositive, by Theorem B, the cycle C has positive total weight. Therefore, C must contain at least one edge with weight $+1$. But as discussed above, the homogeneity of D implies that all edges within C carry the same weight, so in particular, e has weight $+1$. This concludes the proof.

Approach to proofs

In this subsection we give more details regarding the proofs of the main theorems. We distinguish two types of diagrams.

Definition We say that D is of *type I* if D is positive, or if D is almost positive and there is no positive crossing *parallel* to the unique negative crossing (in other words connecting the same pair of Seifert circles).

Definition We say that D is of *type II* if it is almost positive, and there is a positive crossing parallel to the unique negative crossing.

We opted to include positive diagrams in type I because they behave similarly in our constructions as almost positive diagrams of type I; for example our proof of Theorem A in Sect. 1 for links with a diagram of type I recovers Rudolph's result that positive links are strongly quasipositive.

The distinction between type I and II is rather natural and has been made previously [27,29,31]. Stoimenow shows that each of the two types is realized by knots that do not admit diagrams of the other type; he further shows that an almost positive diagram has a minimal genus canonical surface if and only if it is of type I. We strengthen this result from 'minimal genus' to 'quasipositive'.

Indeed, we show that if D is of type I, then the canonical surface $\Sigma(D)$ is quasipositive. If D is of type II, we construct a quasipositive Seifert surface $\Sigma'(D)$ with $\partial\Sigma'(D) = \partial\Sigma(D)$ of genus one less than $\Sigma(D)$. The alert reader has spotted that the quasipositivity of $\Sigma(D)$ for D of type I also follows from Theorem B. Nevertheless, we are going to supply an independent proof, which also serves as a warm-up for the other proofs.

The quasipositivity of links admitting diagrams of type II can be shown using a method due to Baader [2] as remarked by Tagami [31]. It does not seem clear, however, how this approach could be strengthened to give a proof of strong quasipositivity. For links admitting only diagrams of type I, even quasipositivity has not hitherto been established.

The proof strategy is similar for both theorems. The quasipositivity of the surface $\Sigma(D)$ or $\Sigma'(D)$ is established by induction over some measure of the complexity of the Seifert graph $\Gamma(D)$. For the induction step, the following two facts about quasipositive Seifert surfaces are crucial:

(M) Murasugi sums of quasipositive Seifert surfaces are quasipositive [23],

(S) Incompressible subsurfaces of quasipositive Seifert surfaces are quasipositive.

Note that (S) is an immediate consequence of the characterization of quasipositive Seifert surfaces that we use.

Outline of the paper

The remainder of the paper contains the proofs of the main results. The proof of Theorem A is split into Proposition 1.1 for type I in Sect. 1, and Proposition 2.1 for type II in Sect. 2. Theorem B is proven in Sect. 3. Corollary C, Theorem D and Corollary E are proven in Sect. 4. Sections 1, 2, 3 and 4 can essentially be read independently.

1 Almost positive links of type I

The goal of this section is to prove the following.

Proposition 1.1 *The canonical surface $\Sigma(D)$ of a diagram D of type I is quasipositive.*

Let us start by providing details regarding the Seifert graph $\Gamma(D)$ of a diagram D, which was briefly defined in the introduction.

The set of edges adjacent to a vertex k of $\Gamma(D)$ carries a cyclic ordering, which comes from the ordering of crossings around the Seifert circle k. Moreover, k separates $\mathbb{R}^2$ into an interior and an exterior. So each edge adjacent to k carries the additional information of *on which side* of k it lies. We say two Seifert circles k and k' are *nested* if one lies in the interior of the other.

If D has no nested Seifert circles, that is the interior of every Seifert circle is empty, then shrinking every Seifert circle to a point provides a canonical embedding of $\Gamma(D)$ into $\mathbb{R}^2$. Thus, if D has no nested Seifert circles, we shall treat $\Gamma(D)$ as a plane graph (i.e. a graph with a fixed embedding into $\mathbb{R}^2$).

Next, we will need two lemmas giving sufficient diagrammatic conditions for the canonical surface being a Murasugi sum and a Hopf plumbing, respectively.

Lemma 1.2 (cf. [7]) *Let D be a non-split link diagram* (*i.e. D is not a disjoint union of link diagrams*) *and let k be a Seifert circle of D. Let D_i and D_e be the link diagrams forming the closure of the interior and the exterior of k, respectively* (*so that $D_i \cap D_e = k$*)*. Then $\Sigma(D)$ is a Murasugi sum of $\Sigma(D_i)$ and $\Sigma(D_e)$.* □

Lemma 1.3 *Let D be a link diagram with a positive crossing c between two Seifert circles k and k' that are not nested. Let D' be the diagram obtained from D by inserting another positive crossing c' that is parallel to c and such that c is the next crossing after c' with respect to both the cyclic orderings of crossings around k and around k'. Then $\Sigma(D')$ is the plumbing of $\Sigma(D)$ and a positive Hopf band.*

Proof See Fig. 1. □

Proof of Proposition 1.1 We prove that $\Sigma(D)$ is quasipositive by induction over the sum of the number of Seifert circles of D and the number of crossings of D.

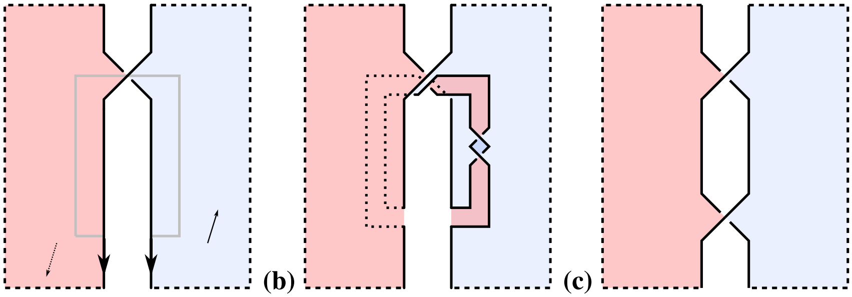
(a) (b) (c)


Fig. 1 Inserting a positive crossing next to another one by positive Hopf plumbing. Red and blue indicate the two sides of oriented surfaces. Dotted lines are hidden below a surface. **a** Two Seifert circles connected by a positive crossing. The small arrows indicate positive normal vectors of the surfaces. **b** Surface obtained from **a** by plumbing a positive Hopf band along the gray curve on the positive side of the surface. **c** This surface is isotopic to **b** (pull the Hopf band away from the crossing). Note that the central white region could contain infinity

Suppose that D is a diagram with a Seifert circle that has empty exterior and non-empty interior. By 'moving infinity' we may change this to a diagram D' with no such circle and such that $\Sigma(D') = \Sigma(D)$. So without loss of generality we may assume that D has no Seifert circle with empty exterior and non-empty interior.

Consider the following cases; we will prove below that they are exhaustive.

(1) If D consists of a single Seifert circle:
The canonical surface $\Sigma(D)$ is a disk, which is quasipositive.
(2) If D is split, i.e. $D = D_1 \sqcup D_2$ for link diagrams D_1 and D_2:
The surfaces $\Sigma(D_i)$ are quasipositive by induction, and, thus, so is $\Sigma(D) = \Sigma(D_1) \sqcup \Sigma(D_2)$.
(3) If there is a nugatory crossing (i.e. an edge removing which would disconnect $\Gamma(D)$):
Let D' denote the diagram obtained by untwisting. Then the surfaces $\Sigma(D)$ and $\Sigma(D')$ are isotopic and, by induction, $\Sigma(D')$ is quasipositive.
(4) If D has a Seifert circle with non-empty interior and non-empty exterior:
If D is a split diagram, we proceed as in case (2). Otherwise, the canonical Seifert surface $\Sigma(D)$ is the Murasugi sum of two canonical surfaces by Lemma 1.2. Since these two summands are quasipositive by induction, $\Sigma(D)$ is quasipositive by (M).
(5) If there is a Seifert circle with empty interior that is adjacent to exactly two crossings, one of which is a positive crossing and the other is a negative crossing:
A Reidemeister-II-move removes that circle and the crossings adjacent to it, producing a diagram D' such that $\Sigma(D)$ and $\Sigma(D')$ are isotopic and $\Sigma(D')$ is quasipositive by induction.
(6) If a pair of non-nested Seifert circles are connected by two positive crossings that are next to each other (i.e. as in the hypothesis of Lemma 1.3):
Denote by D' the diagram obtained by deleting one of these crossings. Then $\Sigma(D')$ is quasipositive by induction, and $\Sigma(D)$ is a plumbing of $\Sigma(D')$ and a positive Hopf band by Lemma 1.3. Thus $\Sigma(D)$ is quasipositive by (M).

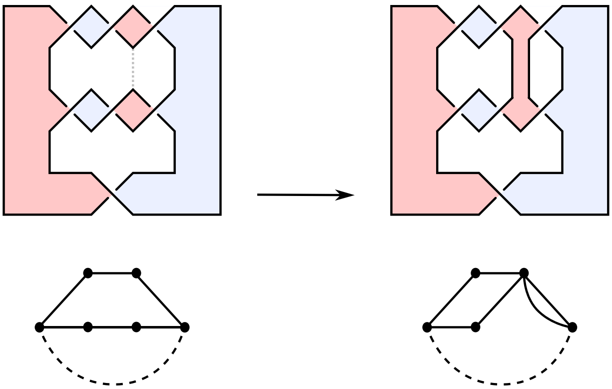


Fig. 2 Top and left: a type I diagram D (the $(-3,-3,1)$-pretzel diagram) and $\Sigma(D)$. Below: its Seifert graph, with the unique edge of weight -1 drawn dashed. On the right: a diagram D' obtained from D by applying (7) to the closed interval drawn gray and dotted, the surface $\Sigma(D')$, and the graph $\Gamma(D')$

(7) If there is a closed interval embedded in the plane such that

(a) its interior is disjoint from D,
(b) its endpoints lie on two distinct Seifert circles k_1, k_2,
(c) k_1, k_2 are oriented coherently,
(d) if k_1 and k_2 are both connected to a third circle, then both of the connecting edges are positive:

Denote by D' the diagram obtained by adding a 1–handle along that closed interval. Because D' has one fewer Seifert circle than D and D' is of type I by (7d), $\Sigma(D')$ is quasipositive by induction. Since $\Sigma(D')$ contains $\Sigma(D)$ as an incompressible subsurface, $\Sigma(D)$ is quasipositive by (S). See Fig. 2 for an example of this case.

□

Let us prove that the above cases are exhaustive. For this, let us assume that (1)–(6) are not satisfied, and deduce that (7) is. Note that the exclusion of (2) and (4) implies that no Seifert circles in D are nested, so its Seifert graph $\Gamma(D)$ can be seen canonically as a plane graph. Furthermore, the exclusion of (2) implies that $\Gamma(D)$ is a connected graph, the exclusion of (1) and (3) imply that all vertices of $\Gamma(D)$ have degree at least 2, and by the exclusion of (5) vertices adjacent to a negative edge have degree at least 3.

We distinguish two cases based on whether $\Gamma(D)$ contains a negative edge or not. In both cases, we succeed in finding an interval as in (7). See Fig. 3 for an illustration.

Case 1. Suppose there is no negative edge. Pick any edge c connecting circles k and k_1. Since k has degree at least 2, one may walk from the edge c in clockwise direction around k until the next edge c', which connects k to some Seifert circle k_2.

If $k_1 \neq k_2$ then there is a closed interval as in (7) which connects k_1 and k_2 by following c and c'. Note that because there is no negative edge, (7d) is vacuously true.

If $k_1 = k_2$, then walk clockwise around k_2 from where c' meets k_2 until we meet the next edge c''. If $c'' = c$, then we are in case (6). If $c'' \neq c$, note that topologically c'' cannot connect to k since we would then have met it when walking clockwise along

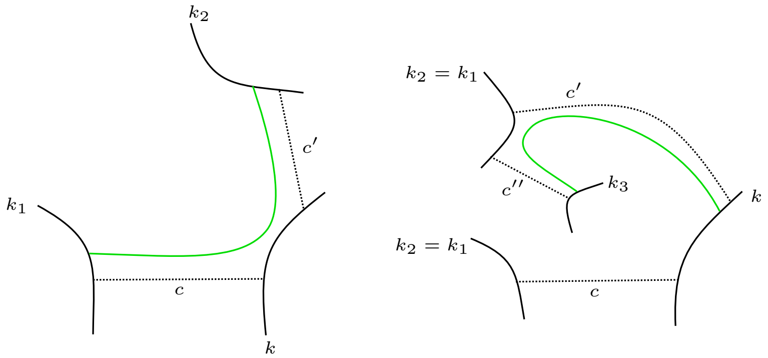


Fig. 3 How to find an interval as in (7) (drawn green)

k from c to c'. So c'' connects to a Seifert circle $k_3 \neq k$. There is then an interval connecting k to k_3 by following c' and c''.

Case 2. Now suppose there is a negative edge connecting circles k and k_0. Because k has degree at least 3, one may walk from that edge in clockwise direction around k until an edge c connecting k and some Seifert circle k_1, and still further until an edge c' connecting k and some Seifert circle k_2. Because the diagram is type I, one has $k_1 \neq k_0$ and $k_2 \neq k_0$. Now one may proceed exactly as in the previous case to find an interval as needed for (7). We note that the intervals as constructed above also satisfy (7d) because none of k_1, k_2 and k_3 are adjacent to the negative edge. □

2 Almost positive links of type II

In this section, we prove the second half of Theorem A.

Proposition 2.1 *If D is a link diagram of type II, then D represents a strongly quasipositive link.*

We first describe how to associate a Seifert surface $\Sigma'(D)$ to such a diagram D, which is similar to the canonical surface but with smaller first Betti number. Afterwards we will show that $\Sigma'(D)$ is a quasipositive Seifert surface.

Construction 1 (Generalized Seifert algorithm) Let D be a diagram with exactly one negative crossing c_-. Further suppose c_- is parallel to a positive crossing c_+. If there is more than one positive crossing parallel to c_-, we fix a choice of c_+. We describe a version of Seifert's algorithm adapted to this setting that associates a Seifert surface $\Sigma'(D)$ with the diagram D as follows.

Resolve all crossings except for c_- and for c_+ in the oriented manner (as in Seifert's algorithm). This produces a two-crossing diagram D_0 of an unlink L_0. The diagram D_0 consists of two twice transversely intersecting curves, which we call s_1 and s_2, and simple closed curves that are pairwise disjoint and disjoint from s_1 and s_2, which we refer to as *Seifert circles*. We take s_1 to be the curve that goes over. We refer to the

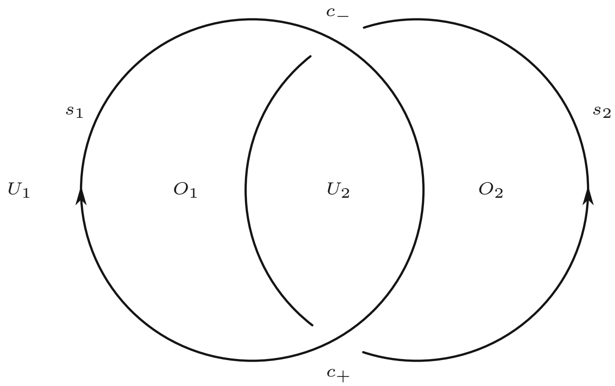


Fig. 4 s_1 and s_2 cut the plane into four regions

union of the Seifert circles and $\{s_1, s_2\}$ as *generalized Seifert circles*. For the rest of this section, we only consider D such that s_1 is oriented clockwise and s_2 is oriented counterclockwise; which, if not the case, can be achieved by 'moving infinity' without changing the associated link. See Fig. 4.

As in Seifert's algorithm, pick a disjoint union of oriented disks d_i in $\mathbb{R}^3$ with constant z-coordinate, one for each generalized Seifert circle k_i, such that the boundary of d_i projects to k_i preserving orientation, and glue in a twisted ribbon for each crossing to obtain $\Sigma'(D)$. We choose the z-coordinates for the disks as follows.

(1) The disk corresponding to s_1 has to lie above the disk corresponding to s_2. In other words (using the convention that s_1 is oriented clockwise and goes over s_2), the positive sides of the disks face each other.
(2) Let k_1 be a generalized Seifert circle lying wholly inside a generalized Seifert circle k_2. The disk d_1 corresponding to k_1 lies to the positive side of the disk d_2 corresponding to k_2. In other words, a positive normal to d_2 points in the direction of d_1.

Any such choice of z-coordinates assures that glueing in the twisted ribbons provides an embedded surface.

The choice of z-coordinate for disks corresponding to Seifert circles nested in s_1 and s_2 is crucial. For other disks, other choices work equally well in what is done below (save small changes in the details of the proof of Lemma 2.7).

Remark 2.2 The above generalized Seifert algorithm produces a Seifert surface out of any link diagram D with two marked crossings that are parallel and of opposite sign.

When no c_- and c_+ are specified, the above algorithm produces a Seifert surface from a link diagram (simply ignore (1)). However, that Seifert surface is in general not isotopic to the canonical surface, since z-coordinates are usually chosen differently in the usual Seifert algorithm (nested implies higher, independent of the orientations of the disks). In this section, we write $\Sigma(\cdot)$ for the Seifert surface constructed by the Seifert algorithm using the height order given in (2).

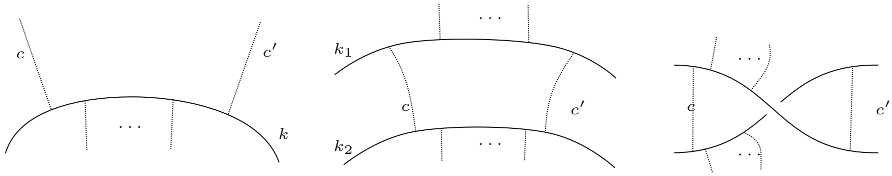


Fig. 5 Left: crossings c and c' next to each other on k. Middle and right: crossings c and c' next to each other

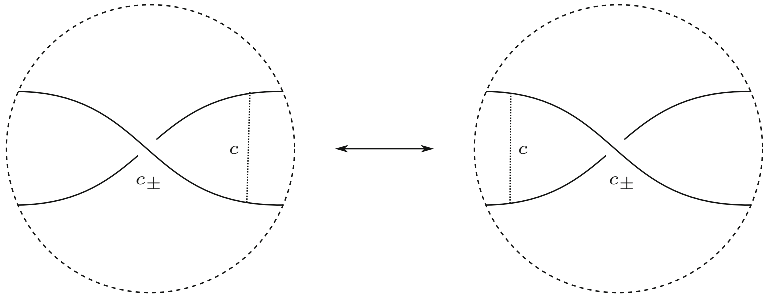


Fig. 6 Swapping the crossing c, which is adjacent to s_1 and s_2. Note that no other Seifert circles or crossings are present in the disk where the modification occurs

We note that the proof of Proposition 1.1 holds *verbatim* for the z-coordinate conventions in this section. Hence if D is a positive diagram we already know that $\Sigma(D)$ is a quasipositive Seifert surface.

In the usual way, we specify each crossing (different from c_- and c_+) by giving an embedded closed interval c (which we shall also call a *crossing*) in $\mathbb{R}^2$. The boundary points of each such crossing will lie on two different generalized Seifert circles, and each such crossing will be disjoint from c_- and c_+, with its interior disjoint from all generalized Seifert circles. See Fig. 7 for examples.

Definition 2.3 We give a few notions which we shall refer to throughout the remainder of this section.

- From here on, unless otherwise stated, a *crossing* refers to a crossing of D that is neither c_+ nor c_-.
- A crossing is said to be *adjacent* to the generalized Seifert circles on which its endpoints lie.
- Two crossings c and c' are said to be *next to each other on a generalized Seifert circle* k, if they are both adjacent to k, they both lie to the same side of k, and there is a closed subinterval I of k with endpoints on c and c' such that I does not contain c_- or c_+ and there are no crossings with endpoints on I that lie to the same side of k as c and c'. See Fig. 5(left).
- Two crossings c and c' are said to be *next to each other*, if they are both adjacent to the same two generalized Seifert circles k_1 and k_2, they are next to each other on both k_i, witnessed by intervals I_i, such that the union $S = c \cup c' \cup I_1 \cup I_2$ has

the property that one of the two components of $\mathbb{R}^2 \setminus S$ contains no generalized Seifert circles and no crossings. See Fig. 5(middle).

- We also say c and c' are *next to each other* if they are next to each other after swapping one of them over c_- or c_+; see Fig. 5(right).
- Here *swapping a crossing c over c_-* or *over c_+* is the operation on diagrams defined by a modification of a diagram in a disk as described in Fig. 6.
- The union of s_1 and s_2 separates $\mathbb{R}^2$ into four regions. Two of these have inconsistently oriented boundaries induced from the orientations of s_1 and s_2. We denote the unbounded region U_1 and the other U_2. We denote the remaining two regions by O_1 and O_2, where O_i is the region contained inside s_i. See Fig. 4.

We note that if the diagram D' arises from D by swapping a crossing, then $\Sigma'(D')$ and $\Sigma'(D)$ are isotopic Seifert surfaces.

Proof of Proposition 2.1 As in the proof of Proposition 1.1, we shall proceed by induction on the sum of the number of Seifert circles and the number of crossings and consider a list of cases. That these cases are exhaustive is the content of Lemma 2.5 below. We shall refer back to the proof of Proposition 1.1 for how to proceed with some of these cases.

(1') If D is one of the 8 diagrams indicated in Fig. 7 or a diagram obtained from one of them by deleting Seifert circles or crossings:
The Seifert surface $\Sigma'(D)$ is a quasipositive Seifert surface, as demonstrated in Lemma 2.6.

(2') If D is split:
Proceed as in (2) (using $\Sigma'(\cdot)$ rather than $\Sigma(\cdot)$ for the part of the diagram that contains the s_i). Explicitly, writing the diagram D as $D_- \sqcup D_+$, where D_- contains c_- (and thus also c_+), we have that $\Sigma'(D) = \Sigma'(D_-) \sqcup \Sigma(D_+)$ is quasipositive, since $\Sigma'(D_-)$ is quasipositive by induction, and $\Sigma(D_+)$ is quasipositive because D_+ is positive.

(3') If there is a nugatory crossing:
Proceed as in (3).

(4') If a Seifert circle has non-empty interior and exterior:
Proceed as in (4).

(5') If two generalized Seifert circles are connected by two crossings that are next to each other:
By Lemma 2.7 (analog of Lemma 1.3 provided at the end of this section), we obtain a diagram D' by removing one of the two crossings such that $\Sigma'(D)$ is quasipositive if and only if $\Sigma'(D')$ is quasipositive. However, by induction, $\Sigma'(D')$ is a quasipositive Seifert surface.

(6') If there is a closed interval embedded in $\mathbb{R}^2$ such that

  (a) its interior is disjoint from D,
  (b) one of its endpoints lies on a Seifert circle k_1 and the other endpoint lies on a generalized Seifert circle k_2, and
  (c) k_1, k_2 are oriented coherently:

Denote by D' the diagram obtained by adding a 1-handle along that closed interval. Because D' has one fewer Seifert circle than D, $\Sigma'(D')$ is quasipositive by

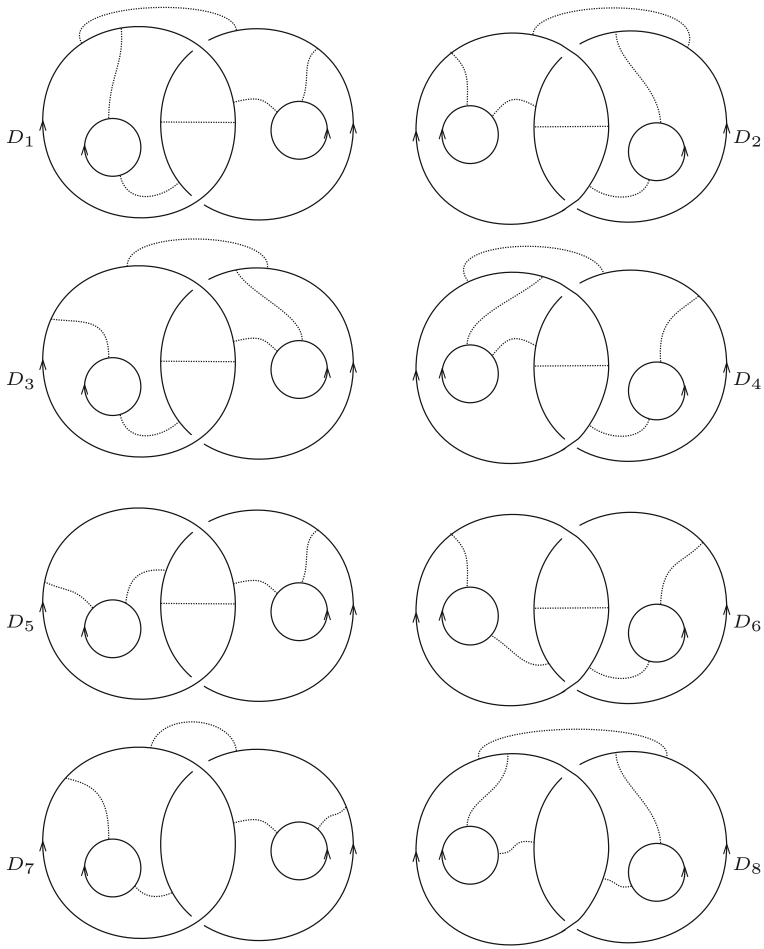


Fig. 7 Diagrams to which (2')–(6') do not apply

induction. And since $\Sigma'(D')$ contains $\Sigma'(D)$ as an incompressible subsurface, $\Sigma'(D)$ is quasipositive by (S).

Let us first establish that the above cases are exhaustive.

Lemma 2.4 *If the conditions of none of (2')–(6') are satisfied, then*

(i) *the regions U_1 and U_2 contain no Seifert circles,*
(ii) *O_1 and O_2 each contain at most one Seifert circle,*
(iii) *each of the Seifert circles has exactly 2 positive crossings adjacent to it,*
(iv) *in each U_i there is at most one crossing between s_1 and s_2.*

We postpone the proof of Lemma 2.4 and apply it to prove the following.

Lemma 2.5 *If D is a diagram such that (2')–(6') do not apply, then (1') applies to D.*

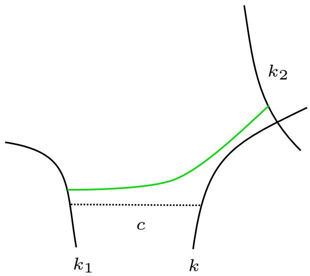


Fig. 8 k is s_1 or s_2 and c is next to one of the two intersection points of s_1 and s_2 (bottom)

Proof By Lemma 2.4, it suffices to consider diagrams satisfying i)–iv).

First we consider the case where D has a crossing in both U_1 and U_2 and O_1 and O_2 each contain a Seifert circle. Once one has fixed the endpoints of the crossings in U_1 and U_2, there are four possibilities for how the two crossings adjacent to the unique Seifert circle in O_1 can lie without being next to each other (as otherwise (5') applies to D). Similarly, there are four possibilities for how the two crossings adjacent to the Seifert circle in O_2 can lie without being next to each other.

Thus in this case there are 16 diagrams satisfying i)–iv). However, in 12 of these diagrams there is a crossings in U_1 that is next to a crossing of U_2 (using the notion of next to each other that uses swapping). Hence for these 12 diagrams (5') applies. The four remaining diagrams, which we denote by D_1, D_2, D_3, and D_4, are indicated in Fig. 7.

Next we consider the case where D has a crossing in exactly one of U_1 or U_2, and two Seifert circles. There are eight such diagrams satisfying i)–iv). Four of these arise by deleting a crossing in one of the diagrams D_1, D_2, D_3, and D_4. The other four, denoted by D_5, D_6, D_7, and D_8, are indicated in Fig. 7.

Finally, it is easy to see that any case not yet considered is obtained from at least one of the D_i by deleting Seifert circles or crossings. □

Proof of Lemma 2.4 **i)** Assume towards a contradiction that there is at least one Seifert circle, say k_1, in U_i. There is a crossing c connecting k_1 to a different generalized Seifert circle k. We distinguish two cases.

Case 1: Assume that on k there is a crossing c' next to c. Let k_2 be the generalized Seifert circle adjacent to c' that is not k; see Fig. 3.

If $k_1 \neq k_2$, then (6') applies (see green interval in Fig. 3(left)), hence we obtain a contradiction. Thus we have that $k_1 = k_2$. The crossings c' and c (compare Fig. 3) are next to each other on k, which implies that there must be another crossing c'' adjacent to k_2 that is between c and c', since otherwise the crossings c and c' are next to each other (this uses that D is not split). Let k_3 be the other Seifert circle adjacent to c''. Now (6') applies (see green interval in Fig. 3 (right)), hence we obtain a contradiction.

Case 2: Assume that on k there is no crossing next to c. This implies that k is s_1 or s_2 (if k were a Seifert circle, having no crossing next to c on k would imply that c is the only crossing on one side of k, thus c would be nugatory).

Thus, (6') applies (see green interval in Fig. 8), contradiction.

ii) Assume towards a contradiction that there are at least two Seifert circles in O_1 (without loss of generality).
Case 1: Assume that inside O_1 there are two distinct crossings c, c' adjacent to s_i for some i. Calling the Seifert circle k_1 that is also adjacent to c, we can now argue verbatim as in Case 1 of i) above.
Case 2: Assume that inside O_1 there is at most one crossing adjacent to s_1 and at most one crossing adjacent to s_2. Then, to avoid nugatory crossings, we know that there is exactly one crossing c_1 adjacent to s_1 and exactly one crossing c_2 adjacent to s_2.
Call the Seifert circle k_1 that is also adjacent to c_1. If k_1 is adjacent to no other crossings then c_1 is nugatory. If k_1 is adjacent to c_1, to c_2, and to no other crossing then either the diagram is disconnected, k_1 has non-empty interior, or there is no other Seifert circle in O_1. If k_1 is adjacent to some crossing different from c_1 and c_2, then pick $c \neq c_2$ to be a crossing next to c_1 on k_1. Then there is an arc connecting s_1 to the other Seifert circle adjacent to c.

iii) Let $i \in \{1, 2\}$. Let k be a Seifert circle in O_i. All crossings adjacent to k are adjacent to s_1 or s_2 since there are no other Seifert circles in O_i by ii). If there are at least three crossings, then two of them are adjacent to the same s_j and are next to each other since there are no other Seifert circles in O_i by ii).

iv) Let $i \in \{1, 2\}$. If there are two or more crossings in U_i, then two of them are next to each other since there are no Seifert circles in U_i by i).

It remains to show that for the diagrams D_i for $i \in \{1, \dots, 8\}$ given in Fig. 7, $\Sigma'(D_i)$ is a quasipositive surface. This implies that $\Sigma'(D)$ is a quasipositive Seifert surface for any diagram D obtained from some D_i by deleting crossings or Seifert circles. (Because in this case $\Sigma'(D)$ is an incompressible subsurface of $\Sigma'(D_i)$ and thus a quasipositive Seifert surface by (S).)

Lemma 2.6 *The Seifert surface $\Sigma'(D_i)$ is quasipositive for all $i \in \{1, \dots, 8\}$.*

Proof We discuss each of the surfaces $\Sigma'(D_i)$ in turn.

- $\mathbf{6'(D_1)}$. We write L_1 for the boundary of $\Sigma'(D_1)$. Note that L_1 has two components and further note that $\Sigma'(D_1)$ has Euler characteristic $\chi(\Sigma'(D_1)) = -2$. Hence $\Sigma'(D_1)$ is a twice punctured surface of genus 1.
By inspection, L_1 is the two component link consisting of the positive trefoil and a meridian positively linking the trefoil.
Note that the link L_1 is the boundary of the surface F given as the connected sum (a special case of a Murasugi sum) of the fiber surface of the positive trefoil $F_{2,3}$ and the positive Hopf band $F_{2,2}$. In particular, F is a fiber surface (since Murasugi sum preserves fiberedness) for L with Euler characteristic -2. Since F is a fiber surface, it is the unique Euler characteristic maximizing Seifert surface for L_1; therefore $\Sigma'(D_1)$ is isotopic to F. However, F is a quasipositive Seifert surface by (M) since it is the Murasugi sum of the two quasipositive Seifert surfaces $F_{2,3}$ and $F_{2,2}$.
- $\mathbf{6'(D_2)}$. This is seen to be isotopic to $\Sigma'(D_1)$, for example via rotation about the vertical axis in the plane.

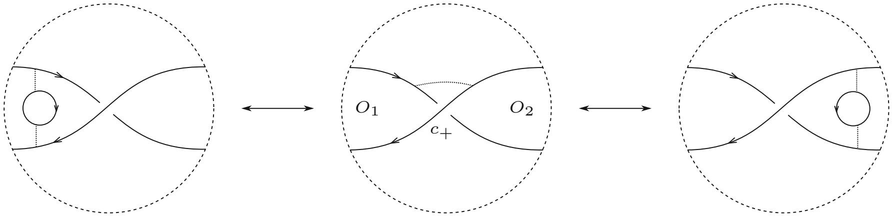


Fig. 9 Left-to-middle and right-to-middle: swapping a Seifert circle over c_+ into a crossing. Left-to-right: swapping a Seifert circle from O_1 over c_+ into a Seifert circle in O_2

- $\mathbf{6'(D_3)}$. We consider a diagram move (similar to swapping a crossing) indicated in Fig. 9 that swaps a Seifert circle with two adjacent crossings into a crossing and vice versa.
  Note that $\Sigma'(D) = \Sigma'(D')$ for diagrams D and D' that are related by this move. We apply this move (Fig. 9(left-to-middle)) to D_3: swap the Seifert circle in O_1 over c_+ into a crossing in U_2 to get a diagram D_3' with two crossings in U_2 and one crossing in U_1. One of the crossings in U_2 is next to the other crossing in U_2 and also next to the crossing in U_1. By Lemma 2.7, $\Sigma'(D_3')$ is quasipositive if and only if $\Sigma'(D_3'')$ is quasipositive, where D_3'' is the diagram obtained from D_3' by deleting the crossing in U_1 and one of the crossings in U_2.
  Finally, we observe that $\Sigma'(D_3'')$ is a quasipositive surface. This follows since D_3'' can be obtained from D_1 by deleting crossings and Seifert circles establishing that $\Sigma'(D_3'')$ is an incompressible subsurface of the quasipositive surface $\Sigma'(D_1)$.
- $\mathbf{6'(D_4)}$. This is seen to be isotopic to $\Sigma'(D_3)$, for example via rotation about the vertical axis in the plane.
- $\mathbf{6'(D_5)}$. First use the move depicted in Fig. 9(middle-to-left) to swap the crossing in U_2 over c_+ to result in a diagram with two Seifert circles in O_1. Then swap the Seifert circle in O_2 over c_+ to O_1 using the move depicted in Fig. 9(right-to-left) resulting in a diagram D_5' with three Seifert circles in O_1.
  The Seifert surface $\Sigma'(D_5')$ is isotopic to $\Sigma'(D_5)$ and is easily seen to be the Seifert surface of a positive diagram of the $(-2, -2, -2)$–pretzel link, and hence quasipositive.
- $\mathbf{6'(D_6)}$. The surface $\Sigma'(D_6)$ is a thrice-punctured sphere since it has Euler characteristic -1 and its boundary is a three component link L_6.
  By inspection, the link L_6 is the three-component link given as an unknot with two parallel positively linked meridians. Thus, we note that L_6 is the boundary of the Seifert surface S of Euler characteristic -1 given as the connected sum of two positive Hopf bands. As for $\Sigma'(D_1)$, we conclude that S is a quasipositive Seifert surface that is isotopic to $\Sigma'(D_6)$. Thus, $\Sigma'(D_6)$ is a quasipositive Seifert surface.
- $\mathbf{6'(D_7)}$. The surfaces $\Sigma'(D_7)$ and $\Sigma'(D_6)$ are isotopic since D_6 can be turned into D_7 by swapping a crossing.
- $\mathbf{6'(D_8)}$. The surfaces $\Sigma'(D_8)$ and $\Sigma'(D_5)$ are isotopic since swapping a crossing in D_5 (and then moving the crossing over infinity) turns D_5 into D_8.

We end the section with the generalization of Lemma 1.3 used above.

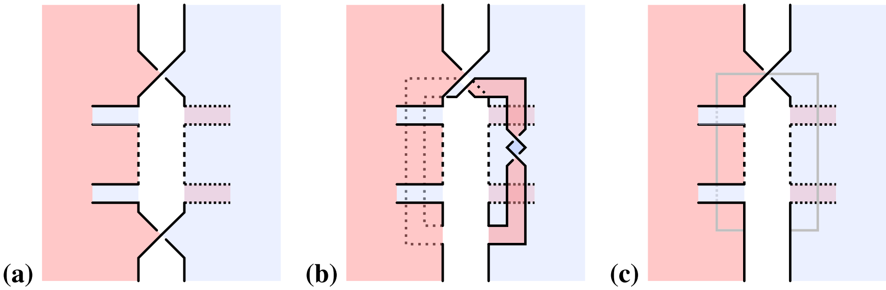


Fig. 10 Inserting a positive crossing next to another one by positive Hopf plumbing (generalizing Fig. 1). **a** Local picture of $\Sigma'(D)$ containing the two ribbons corresponding to two crossings c_1 and c_2 that are next to each other. Note that the central white region could contain infinity. **b** The result of plumbing a positive Hopf band in **c**. **c** A closed interval (gray) in $\Sigma'(D')$ along which a positive Hopf band gets plumbed to the blue side

Lemma 2.7 *Let D be a diagram with two marked crossings of opposite sign c_- and c_+. If two positive crossings c_1 and c_2 are next to each other, then for some $i \in \{1, 2\}$, the diagram D' obtained by deleting c_i satisfies: $\Sigma'(D)$ is a quasipositive Seifert surface if and only $\Sigma'(D')$ is a quasipositive Seifert surface.*

Proof of Lemma 2.7 One direction of the Lemma follows immediately since $\Sigma'(D')$ is an incompressible subsurface of $\Sigma'(D)$. The other direction shall be proven in a similar way as Lemma 1.3. We may and do assume that c_1 and c_2 are next to each other without swapping needed (otherwise swap a crossing first and consider the resulting diagram as D). Writing k and k' for the generalized Seifert circles adjacent to c_1 and c_2, we wish to prove that the local situation is isotopic to Fig. 10a. Then, while there may be twisted ribbons attached to k and to k' between c_1 and c_2, either all of those ribbons lie above the disk corresponding to k, and below the disk corresponding to k', or vice versa. So, the Hopf band in Fig. 10b may be plumbed to Fig. 10c such that it does not interfere with the ribbons.

To make this plan work, we distinguish two cases depending on whether k and k' are nested or not; in each of the cases we pay attention to the possibility that k and k' may be s_1 or s_2.

Case when k and k' are not nested. It turns out that D' can be chosen such that $\Sigma'(D)$ arises from plumbing a positive Hopf band to $\Sigma'(D')$, which implies that $\Sigma'(D)$ is a quasipositive Seifert surface if and only if $\Sigma'(D')$ is. The argument is more involved version of the proof of Lemma 1.3; in particular, we will have to be careful which of the two crossings c_1 and c_2 to eliminate in D to obtain D'.

We first consider the case that at least one of the generalized Seifert circles k or k' is a Seifert circle (i.e. not an s_i). The situation is as depicted in Fig. 10a (we note that picture only depicts a closed range of z-coordinate height—far above or below there could be further disks corresponding to generalized Seifert circles that contain both k and k').

This is due to the z-coordinate convention of nested disks assuring that all the ribbons corresponding to crossings on k and k' to the other side than the c_i are to the positive (red) side of the disks corresponding to k and k'. Then, the surface can be isotoped to be the result (see Fig. 10b) of plumbing a positive Hopf band to the negative (blue) side of $\Sigma'(D')$ (see Fig. 10c), where D' is the diagram obtained by deleting the crossing c_i in D that is depicted at the bottom of Fig. 10a.

If instead k and k' are s_1 and s_2, then the crossings c_1 and c_2 lie in U_1 or U_2. If they lie in U_1, the situation is again exactly as depicted in Fig. 10a. If instead the crossings c_1 and c_2 lie in U_2, then $\Sigma'(D)$ can be isotoped to look like Fig. 10a by locally pulling the disks corresponding to s_1 and s_2 apart, so the first is no longer above the second. Note that, different from the previous cases, the positive side of $\Sigma'(D)$ is depicted as blue and the plumbing of the positive Hopf band happens to the positive (blue) side of $\Sigma'(D')$. Again, here D' is the appropriate diagram obtained from D by deleting either c_1 or c_2.

Case when k and k' are nested.

First remark that k or k' is a Seifert circle since s_1 and s_2 are not nested. Second, note that either the two c_i and the two s_i all lie to the same side of k, or they all lie to the same side of k'. All in all, we suppose w.l.o.g. that k is a Seifert circle and the s_i and the c_i all lie to the same side of k.

We now split $\Sigma'(D)$ as a Murasugi sum along k. Let D_- and D_+ be the link diagrams so that $D = D_- \cup D_+$ and $D_- \cap D_+ = k$ (as in Lemma 1.2), where we let D_- be the link diagram that contains the s_i and the c_i. The simple case that D_+ consists only of k and $D_- = D$ is possible. The Seifert surface $\Sigma'(D)$ is a Murasugi sum of $\Sigma(D_+)$ and $\Sigma'(D_-)$. Since D_+ has no negative crossings, $\Sigma(D_+)$ is a quasipositive Seifert surface and, thus, $\Sigma'(D)$ is a quasipositive Seifert surface if and only if $\Sigma'(D_-)$ is by (M).

We now argue that $\Sigma'(D_-)$ arises by positive Hopf plumbing on $\Sigma'(D'_-)$, where D'_- is a diagram obtained from deleting one of the c_i in D_-. For this we note that the Seifert surface $\Sigma'(D_-)$ can be isotoped (by folding the disk corresponding to either k or k', which ever contains the other, along the part of its boundary connecting the two ribbons corresponding to c_1 and c_2) to look like Fig. 10a.

In fact, the situation is necessarily simpler as depicted in Fig. 10a: on the disk corresponding to k there will be no ribbons leaving between c_1 and c_2 on either side (red or blue). In other words, the situation is as depicted in Fig. 10a to one side and as depicted in Fig. 1c on the other side.

So then, as before, $\Sigma'(D_-)$ is the result (see Fig. 10b) of plumbing a positive Hopf band to $\Sigma'(D'_-)$ (see Fig. 10c), where D'_- is the appropriate diagram obtained from D_- by deleting one of the c_i. We note that both plumbing to the positive side and plumbing to the negative side can occur.

Finally, we set D' to be the union of D'_- and D_+. The Seifert surface $\Sigma'(D')$ is a Murasugi sum of $\Sigma(D_+)$ and $\Sigma'(D'_-)$ and, thus, $\Sigma'(D')$ is a quasipositive Seifert surface if and only if $\Sigma'(D'_-)$ is (by (M)). Therefore, we conclude that $\Sigma'(D)$ is a quasipositive Seifert surface if and only if $\Sigma'(D')$ is, as desired. □

3 Canonical quasipositive surfaces

Let us start with some graph theoretic concepts.

Definition 3.1

- A *path* P is a sequence $e_1, \ldots, e_n$ of distinct edges in which e_i has vertices v_i^1 and v_i^2 such that $v_i^2 = v_{i+1}^1$ and such that every vertex appears at most twice as endpoint of an edge of P.
- The *length* of such a path P is denoted by $\ell(P) = n$.
- A *cycle* C is a path as above with $v_n^2 = v_1^1$.
- A *region* of a plane graph G is a connected component of $\mathbb{R}^2 \setminus G$.
- A graph G is *2–connected* if it has at least three vertices, is connected, and the result of removing any vertex is again connected.
- A *weighted graph* is a graph in which each edge carries either the weight $+1$ or the weight -1. For a collection E of edges of a weighted graph, we denote by $w(E) \in \mathbb{Z}$ the *total weight* of E, i.e. the sum of the weights of the edges in E.

Our main theorem of this section is the following.

Theorem B. *A canonical surface is quasipositive if and only if all cycles of its Seifert graph have strictly positive total weight.*

Proof A cycle C of $\Gamma(D)$ lifts to a non-null-homologous unknot in $\Sigma(D)$ with framing $w(C)$. A tubular neighborhood of that unknot in $\Sigma(D)$ is an annulus with $w(C)$ full twists, and an incompressible subsurface of $\Sigma(D)$. So if $\Sigma(D)$ is quasipositive, then $w(C) > 0$ follows from (S). This establishes the necessity of the cycle condition for quasipositivity.

To see that that the cycle condition for quasipositivity is sufficient, suppose some diagram D has $\Gamma(D)$ satisfying the hypothesis of Theorem B.

If D is a split diagram, then $\Gamma(D)$ is not connected and the cycle condition can be checked on each connected component individually. Therefore we may and do assume that D is non-split.

If there is a Seifert circle k in D that has non-empty interior and exterior, then $\Sigma(D)$ may be expressed as the Murasugi sum of some $\Sigma(D_i)$ and $\Sigma(D_e)$ (see Lemma 1.2). Since $\Gamma(D_i)$ and $\Gamma(D_e)$ are both subgraphs of $\Gamma(D)$, we conclude that Theorem B follows from considering diagrams where each Seifert circle either has empty interior or empty exterior.

By moving infinity we move to a different diagram but with an isotopic canonical surface. So, by possibly moving infinity, we may and do assume that every Seifert circle of D has empty interior.

Hence we have reduced the proof to Proposition 3.2. □

Proposition 3.2 *If D is a non-split link diagram such that every Seifert circle of D has empty interior, and such that all cycles of $\Gamma(D)$ have positive total weight, then $\Sigma(D)$ is quasipositive.*

Recall from the paragraph after Theorem B in Sect. 1 that if D is a link diagram with all Seifert circles having empty interior, then its Seifert graph $\Gamma(D)$ is naturally a plane graph, while also being bipartite and weighted. All graphs considered in this section will be bipartite weighted plane graphs.

Proof of Proposition 3.2 Let D be a link diagram satisfying the hypothesis of the proposition. Since D is non-split, $\Gamma(D)$ is connected.

Suppose that D is a connected sum of diagrams D_1 and D_2, each with at least one crossing. Then $\Sigma(D)$ is a connected sum (a special case of a Murasugi sum) of $\Sigma(D_1)$ and $\Sigma(D_2)$ and it suffices to consider the summands by (M). Therefore we only consider diagrams that are not such connected sums.

If D has only one Seifert circle, $\Sigma(D)$ is a disk, which is quasipositive. Similarly if D has only two Seifert circles and only one crossing.

If D has two Seifert circles and $n \geq 2$ crossings, then each crossing is positive since otherwise the cycle positivity condition of $\Gamma(D)$ would be violated. Then we see that $\Sigma(D)$ is the fiber surface of the positive $(2, n)$–torus link, which is quasipositive. This case will be the root case of a proof by induction.

We assume now that D has at least three Seifert circles. Since D is not a non-trivial connected sum, $\Gamma(D)$ is 2–connected (see Definition 3.1). Then it is a straightforward graph theoretic result about 2–connected plane graphs that the boundary of each region of $\mathbb{R}^2 \setminus \Gamma(D)$ is a cycle. We will call such a cycle *boundary cycle*.

Our strategy is a proof by induction over the following measure of complexity of $\Gamma(D)$. Suppose $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ are two infinite sequences of integers with only finitely many non-zero integers. Define $x > y$ iff the rightmost non-zero entry of $x - y$ is positive. Let f_i be the number of boundary cycles of length $2i$. We define the infinite sequence

$$f(\Gamma(D)) = (f_1, f_2, f_3, \ldots).$$

Given a link diagram D satisfying the hypothesis of Proposition 3.2, our idea is to produce a new link diagram D' also satisfying the hypothesis. Furthermore we aim to do this so that $f(\Gamma(D')) < f(\Gamma(D))$ and so that $\Sigma(D)$ is a quasipositive surface if $\Sigma(D')$ is a quasipositive surface. Having already verified the root case that all boundary cycles have length 2 (which implies having two Seifert circles), the induction will give us the result.

If $\Gamma(D)$ contains a vertex of degree 2 adjacent to a positive and a negative edge, then we have $\Sigma(D) = \Sigma(D')$ where D' is obtained from D by removing two crossings via a Reidemeister II move. Furthermore D' satisfies the hypothesis of Proposition 3.2 and has $f(\Gamma(D')) < f(\Gamma(D))$. So we may and do assume that $\Gamma(D)$ contains no degree 2 vertices adjacent to both a positive and negative edge.

Let us now introduce a new move, which generalizes (7) from the proof of Theorem A. Suppose v, w are vertices of $\Gamma(D)$ on the boundary C of a region of $\mathbb{R}^2 \setminus \Gamma(D)$. Let d be the distance (lengthwise, not weighted) between v and w along C and suppose $d \geq 2$. We now describe a diagram D' obtained from D. It is enough to describe $\Gamma(D')$, which is obtained from $\Gamma(D)$ by adding a chord consisting of a path of $(d-2)$ positive edges between v and w inside of the region. In the special case of $d = 2$, adding a chord of length 0 is understood as merging v and w. The two crucial observations are:

- We have that $f(\Gamma(D')) < f(\Gamma(D))$ because the regions of $\mathbb{R}^2 \setminus \Gamma(D')$ correspond to those of $\mathbb{R}^2 \setminus \Gamma(D)$, except for the region with boundary C, which is split into two regions, each of them with strictly fewer edges than C.

- The surface $\Sigma(D)$ is an incompressible subsurface of $\Sigma(D')$.

So to conclude the proof of the proposition, it suffices to show that this move is always possible in such a way that D' still satisfies the hypothesis of the proposition. Note that the only cycles of $\Gamma(D')$ not occurring as cycles of $\Gamma(D)$ are those that pass through the new chord. So it suffices to pick v, w and C such that any path Q in $\Gamma(D)$ between v and w satisfies $w(Q) + d - 2 \geq 2$. That this is always possible is the contents of Propositions 3.5 and 3.6. □

Definition 3.3 We say that a link diagram D has property (*) if it satisfies the following.

- All Seifert circles of D have empty interior.
- All cycles of $\Gamma(D)$ have positive total weight.
- $\Gamma(D)$ is 2-connected (see Definition 3.1).
- $\Gamma(D)$ contains no degree 2 vertices adjacent to both a positive and a negative edge.

Definition 3.4 Suppose that D has property (*) and C is the boundary cycle of a region of $\mathbb{R}^2 \setminus \Gamma(D)$. We say that C is *splittable* if there exist vertices v and w of C, distance $d \geq 2$ apart on C, such that every path Q in $\Gamma(D)$ connecting v to w satisfies $w(Q) \geq 4 - d$.

Proposition 3.5 *Suppose that D has property (*). Then there is a region of $\mathbb{R}^2 \setminus \Gamma(D)$ whose boundary cycle C has $w(C) \geq 4$.*

We postpone the proof of this proposition to the end of the section.

Proposition 3.6 *Suppose that D has property (*) and C is the boundary cycle of a region of $\mathbb{R}^2 \setminus \Gamma(D)$ with $w(C) \geq 4$. Then C is splittable.*

Proof The proof is divided into two cases given as Lemmas 3.7 and 3.8. □

Lemma 3.7 *Suppose that D has property (*), and that C is the boundary cycle of a region of $\mathbb{R}^2 \setminus \Gamma(D)$ with $w(C) \geq 4$ and at least one edge of C negative. Then C is splittable.*

Lemma 3.8 *Suppose that D has property (*), and that C is the boundary cycle of a region of $\mathbb{R}^2 \setminus \Gamma(D)$ with $w(C) \geq 4$ and every edge of C positive. Then C is splittable.*

For the proof of Lemmas 3.7 and 3.8 we first collect some straightforward facts, without proof, in the following lemma.

Lemma 3.9 *We have the following.*

(1) *Suppose that H is a graph and v and w are distinct vertices of H. Then paths from v to w are exactly the minimal subgraphs of H in which v and w are the only vertices with odd degree.*
(2) *If H is a graph with vertices of only even degree then its set of edges can be written as a disjoint union of cycles.*

(3) *Suppose that H is a graph with exactly two vertices v and w of odd degree and that P is any path from v to w (such a P exists by the first part of this lemma). Then removing the set of edges of P from H leaves a graph whose set of edges is a disjoint union of cycles.* □

Proof of Lemma 3.7 Let us pick v and w as follows. Walking around C, pick v such that the next edge is positive and the one immediately after is negative, and continue walking until the next positive edge, and call its farther vertex w. Then one has walked along a path P with $w(P) = 4 - \ell(P)$. The other path P' in C between v and w satisfies $w(P') + w(P) \geq 4$, and so $\ell(P') \geq w(P') \geq 4 - w(P) = \ell(P)$. Hence P is not the longer path between v and w around C, and $d = \ell(P)$.

For every path Q in $\Gamma(D)$ between v and w, we must show that $w(Q) \geq 4 - d$, which is equivalent to showing $w(Q) \geq w(P)$ since $4 - d = 4 - \ell(P) = w(P)$.

So let Q be such a path. Note that $(P \cup Q) \setminus (P \cap Q)$ is a union of cycles (see Lemma 3.9(2)), say $Z_1, \ldots, Z_n$. Let us write $Z_i = P_i \sqcup Q_i$ for a decomposition of each Z_i into sets of edges $P_i \subset P$ and $Q_i \subset Q$. Then we have that

$$w(P) = w(P_1) + \cdots + w(P_n) + w(P \cap Q),$$
$$w(Q) = w(Q_1) + \cdots + w(Q_n) + w(P \cap Q).$$

So we shall be done if we can show that $w(Q_i) \geq w(P_i)$ for all i.

Since P contains only two positive edges, we must have $w(P_i) \leq 2$, with equality only when P_i consists of exactly the two positive edges of P (in other words the first and last edge of P). Since the path Q_i of course cannot enter the region bounded by C the only way that $w(P_i) = 2$ can happen is if Q_i consists of a path connecting v to w and another path connecting the other two endpoints of the first and last edge of P. But Q is a path connecting v to w and $Q_i \subset Q$, so we have a contradiction by Lemma 3.9(1). Therefore we must have $w(P_i) \leq 1$ for all i. Further note that $w(Q_i) + w(P_i) = w(Z_i) \geq 2$ since Z_i is a cycle. Hence we can conclude that $w(Q_i) \geq w(P_i)$ for all i. □

A heuristic important for our proof of Lemma 3.8 is that if two paths of a weighted graph intersect, one can resolve them and obtain two new paths which have the same total weight. This heuristic turns out, when formalized in the following lemma, only to yield an inequality rather than an equality.

Lemma 3.10 *Let D be a diagram which has property (*) and consider the boundary cycle C of a region of $\mathbb{R}^2 \setminus \Gamma(D)$. Suppose that $\ell(C) \geq 4$, and let v_i be distinct vertices of C for $i \in \mathbb{Z}/4$, occurring in the cyclic ordering around C. Suppose that P_{02} and P_{13} are paths from v_0 to v_2 and from v_1 to v_3 respectively.*

Then for some $i \in \{0, 1\}$ there exists a path $P_{i,i+1}$ from v_i to v_{i+1} and a path $P_{i+2,i+3}$ from v_{i+2} to v_{i+3} such that

$$w(P_{i,i+1}) + w(P_{i+2,i+3}) \leq w(P_{02}) + w(P_{13}).$$

Proof Consider the subgraph H of $\Gamma(D)$

$$H = (P_{02} \cup P_{13}) \setminus (P_{02} \cap P_{13}).$$

The vertices in H of odd degree are exactly the vertices v_i for $i \in \mathbb{Z}/4$. Let $\widetilde{H}$ be the subgraph of H obtained by removing those connected components of H not containing any of the vertices v_i. Again the vertices of $\widetilde{H}$ of odd degree are exactly the vertices v_i. Therefore $\widetilde{H}$ is either connected or has exactly two components each containing two of the vertices P_i. Any path connecting v_0 and v_2 must intersect any path connecting v_1 and v_3. Hence, it cannot be the case that there are two components of which one contains v_0 and v_2 and while the other contains v_1 and v_3. Therefore v_0 is in the same component as v_1 or as v_3. Let us assume the former for now.

Let Q be a path in $\widetilde{H}$ connecting v_0 to v_1. Note that Q is a subgraph of H and so, by construction, each edge of Q occurs either in P_{02} or in P_{13} but not in both. Therefore we have

$$w(P_{02}) + w(P_{13}) = w((P_{02} \cup Q) \setminus (P_{02} \cap Q)) + w((P_{13} \cup Q) \setminus (P_{13} \cap Q)).$$

Now v_1 and v_2 are exactly the vertices of $(P_{02} \cup Q) \setminus (P_{02} \cap Q)$ of odd degree. Therefore $(P_{02} \cup Q) \setminus (P_{02} \cap Q)$ can be written as the disjoint union of a path P_{12} from v_1 to v_2 and some cycles (by Lemma 3.9). Since by assumption each cycle has weight ≥ 2, we must have that

$$w(P_{12}) \leq w((P_{02} \cup Q) \setminus (P_{02} \cap Q)).$$

Similarly there is a path P_{30} from v_3 to v_0 satisfying

$$w(P_{30}) \leq w((P_{13} \cup Q) \setminus (P_{13} \cap Q)).$$

Hence we have

$$\begin{aligned} w(P_{12}) + w(P_{30}) &\leq w((P_{02} \cup Q) \setminus (P_{02} \cap Q)) + w((P_{13} \cup Q) \setminus (P_{13} \cap Q)) \\ &= w(P_{02}) + w(P_{13}). \end{aligned}$$

The other case follows from making the assumption that v_0 is in the same component of $\widetilde{H}$ as v_3. □

Lemma 3.11 *Suppose that D has property (*) and that C is the boundary of a region of* $\mathbb{R}^2 \setminus \Gamma(D)$ *such that all edges of C have weight* $+1$. *Let v and w be vertices of C such that the shortest path in C between v and w has length* $d \geq 1$. *Then there is no path between v and w of weight less than* $2 - d$.

Proof Let P be a path between v and w. Let us write Q for a path of length d contained in C connecting v to w. Consider $P \cap Q$. Each connected component of $P \cap Q$, since it is a subgraph of Q, is a path in C. Furthermore the set of edges of $(P \cup Q) \setminus (P \cap Q)$

can be written as a disjoint union of cycles. Let us write c for the number of cycles in such a decomposition of $(P \cup Q) \setminus (P \cap Q)$.

In the case that $c = 0$ then $P = Q$ and $w(Q) = w(P) = d \geq 2 - d$.

In the case that $c \geq 1$ then

$$\begin{aligned} w(P) &= w((P \cup Q) \setminus (P \cap Q)) + 2w(P \cap Q) - w(Q) \\ &\geq 2c + 2w(P \cap Q) - d \geq 2c - d \geq 2 - d. \end{aligned}$$

Now we are in a position to give the proof of Lemma 3.8, thus establishing Proposition 3.6.

Proof of Lemma 3.8 Let us proceed by contradiction. So assume that C is a boundary cycle with no negative edges, of total weight $w(C) = \ell(C) = 2n \geq 4$, and assume C is not splittable. That is to say that for any pair of vertices v, w on C of distance d along C, there is a path Q in $\Gamma(D)$ with $w(Q) < 4 - d$, thus $2n \leq 2 - d$.

Pick v_i for $i \in \mathbb{Z}/4$ on C in the cyclic ordering, such that the distances between v_i and v_{i+1} are $1, 1, n-1$ and $n-1$ for $i = 0, 1, 2, 3$. Thus there are paths P_{02} and P_{13} from v_0 to v_2 and from v_1 to v_3, respectively, such that $w(P_{02}) \leq 0$ and $w(P_{13}) \leq 2-n$. By Lemma 3.11 it follows that in fact $w(P_{02}) = 0$ and $w(P_{13}) = 2-n$. By Lemma 3.10, there are paths $P_{i,i+1}$ and $P_{i+2,i+3}$ for some $i \in \{0, 1\}$ such that $w(P_{i,i+1}) + w(P_{i+2,i+3}) \leq 2 - n$. This contradicts the fact that by positivity of cycle weights, $w(P_{i,i+1}) > -1$ and $w(P_{i+2,i+3}) > 1 - d$ and thus $w(P_{i,i+1}) \geq 1$ and $w(P_{i+2,i+3}) \geq 3 - d$. □

Finally, we turn to Proposition 3.5, to whose proof we devote the remainder of this section.

Proof of Proposition 3.5 We divide the proof of this proposition into Lemmas 3.14 and 3.15. □

Let us first prove the following.

Lemma 3.12 *Suppose that D is a diagram which has property (*). Suppose further that the boundaries of all regions of $\mathbb{R}^2 \setminus \Gamma(D)$ have total weight 2. Then there is a* positive *vertex of $\Gamma(D)$, in other words a vertex not adjacent to a negative edge.*

Proof As before, let f_i be the number of boundary cycles of length $2i$. Let f be the total number of regions, e the number of edges, e_- the number of negative edges and v the number of vertices. Then we have

$$f = \sum_{i=2}^{\infty} f_{2i} \quad \text{and} \quad e = \sum_{i=2}^{\infty} i f_{2i} \quad \text{so that} \quad v = 2 + \sum_{i=2}^{\infty} (i-1) f_{2i}.$$

Then, since every region of $\mathbb{R}^2 \setminus \Gamma(D)$ has two more positive edges in its boundary than negative edges, we have

$$e_- = \sum_{i=2}^{\infty}(i-1)f_{2i}/2 \quad \text{and so} \quad v > 2e_-.$$

Definition 3.13 Let D be a diagram which has property (*) and let v be a positive vertex of $\Gamma(D)$. Let $C_1, \ldots, C_n$ be the boundaries of those regions of $\mathbb{R}^2 \setminus \Gamma(D)$ adjacent to v. We say that v is a *wicked* positive vertex if $C_i \cap C_j$ is connected for all i, j.

Lemma 3.14 *Suppose that D is a diagram which has property (*) and which contains a wicked positive vertex. Then $\mathbb{R}^2 \setminus \Gamma(D)$ contains a region whose boundary cycle has weight at least* 4.

Proof For a contradiction, suppose that D is a diagram which has property (*) and in which there is no region of $\mathbb{R}^2 \setminus \Gamma(D)$ whose boundary has weight 4 or greater, and suppose that v is a wicked positive vertex of $\Gamma(D)$.

Let $C_1, \ldots, C_n$ be the boundaries of those regions of $\mathbb{R}^2 \setminus \Gamma(D)$ adjacent to v, in counterclockwise order around v, where the subscripts are considered modulo n. If all of the C_i had length 2, then v would have n edges adjacent to a vertex w, and no further edges, which would contradict property (*). Since $C_i \cap C_{i+1}$ is connected, it is a path starting at v (or containing v in case $n = 2$). Since D has property (*), all these paths contain only positive edges, and thus have positive total weight. Now, using that not all C_i have length 2, the edges of

$$(C_1 \cup \cdots \cup C_n) \setminus \bigcup_i (C_i \cap C_{i+1})$$

form a cycle C with

$$\begin{aligned} w(C) &= \sum_i w(C_i) - 2\sum_i w(C_i \cap C_{i+1}) \\ &= 2n - 2\sum_i w(C_i \cap C_{i+1}) \\ &\leq 2n - 2n = 0. \end{aligned}$$

But this contradicts the property (*). □

Lemma 3.15 *If D is diagram which has property (*) then either $\Gamma(D)$ has a wicked positive vertex, or $\mathbb{R}^2 \setminus \Gamma(D)$ contains a region whose boundary cycle has weight at least* 4.

Proof of Lemma 3.15 Suppose for a contradiction that there exists at least one diagram with property (*) whose Seifert graph contains no positive wicked vertices and no boundaries of weight at least 4. Consider such diagrams with the minimal number of positive vertices, and let D be one of these with the minimal number of crossings.

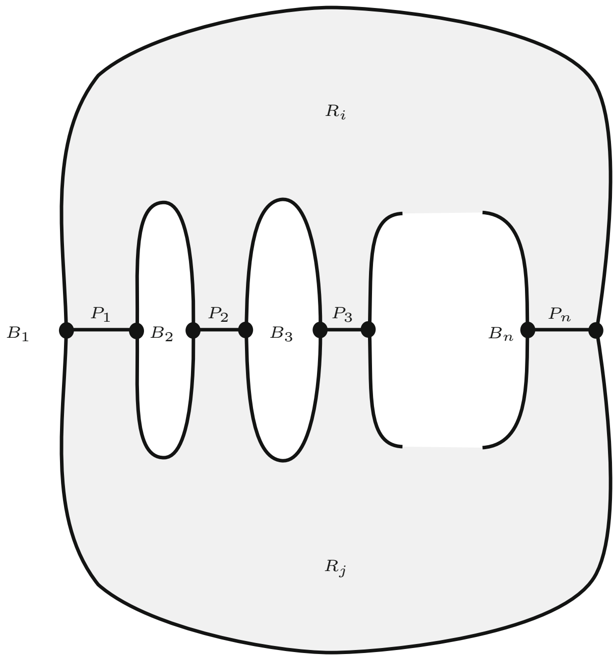


Fig. 11 A diagram of the situation of Lemma 3.15

We know by Lemma 3.12 that D has a positive vertex; let us call it v. Let $C_1, \ldots, C_n$ be the boundaries of those regions of $\mathbb{R}^2 \setminus \Gamma(D)$ adjacent to v, in counterclockwise order around v, where the subscripts are considered modulo n. Then, since v is not wicked by assumption, for some $i \neq j$ we have that $C_i \cap C_j$ has $m \geq 2$ components (note that one of these components contains the positive vertex v). We write R_i and R_j for the closed bounded regions of $\mathbb{R}^2$ whose boundaries are C_i and C_j respectively. Then $\mathbb{R}^2 \setminus (R_i \cup R_j)$ has m components $B_1, \ldots, B_n$ and the boundary of each $\overline{B_k}$ is a cycle in $\Gamma(D)$. Let us write Z_k for the boundary of $\overline{B_k}$, and $P_1, P_2, \ldots, P_n$ for the paths which are the components of $R_i \cap R_j$.

The situation is illustrated in Fig. 11. Note that the interiors of the regions R_i and R_j do not contain any vertices or edges of $\Gamma(D)$. Note also that some paths P_k could consist of single vertices. Note further that the vertices of each P_k which are not endpoints of P_k have degree 2 in $\Gamma(D)$. Since $\Gamma(D)$ has property (*) it follows that no P_k has both positive and negative edges.

Now for any k consider the subgraph of $\Gamma(D)$ that lies within $\overline{B_k}$. This is the Seifert graph of a diagram D' that also satisfies property (*), apart from possibly containing a degree 2 vertex adjacent to both a positive and a negative edge. By Reidemeister II moves D' may be converted to a diagram D'' with property (*) such that D'' has the same number of Seifert circles as D' and possibly fewer crossings. Hence by induction $\mathbb{R}^2 \setminus \Gamma(D'')$ contains a region whose boundary has weight 4 or more. However the regions of $\mathbb{R}^2 \setminus \Gamma(D'')$ are in an obvious correspondence with those of $\mathbb{R}^2 \setminus \Gamma(D')$ under which the weights the boundary cycles are invariant. Hence $\mathbb{R}^2 \setminus \Gamma(D')$ contains a region whose boundary has weight 4 or more. We conclude that either $\mathbb{R}^2 \setminus \Gamma(D)$ does as well (and we are done) or that $w(Z_k) \geq 4$.

We assume for a contradiction that we have $w(C_k) = 2$ for all C_k. Then we have

$$\begin{aligned} 2(w(P_1) + \cdots + w(P_n)) &= 2w(C_i \cap C_j) \\ &= w(C_i) + w(C_j) - (w(Z_1) + \cdots + w(Z_n)) \\ &= 4 - (w(Z_1) + \cdots + w(Z_n)) \\ &\leq 4 - 4n \leq -2n, \end{aligned}$$

where the last inequality is because $n \geq 2$.

Since no P_k has both positive and negative edges, and at least one of the P_k, say P_α, (that containing the vertex v) contains no positive edge, it follows that at least one of the P_i contains two consecutive negative edges. Let w be the midpoint of these two consecutive edges. Note that there is an embedded circle in $\mathbb{R}^2$ which meets $\Gamma(D)$ only at v and w. Flyping along this circle to move one of the negative edges to lie adjacent to v creates a new graph which is the Seifert graph $\Gamma(D')$ of a diagram D'.

Now, in the case that the path P_α contains at least one edge, D' admits simplification via a Reidemeister II move to a new diagram D'' such that D'' has property (*). Furthermore $\Gamma(D'')$ has either the same number or one fewer positive vertices than $\Gamma(D)$. Hence by assumption, $\Gamma(D'')$ must contain a wicked positive vertex and so by Lemma 3.14 contains a region whose boundary cycle has weight 4.

But there is an obvious correspondence between the regions of $\mathbb{R}^2 \setminus \Gamma(D'')$ and those of $\mathbb{R}^2 \setminus \Gamma(D)$, and the weight of the boundary cycle of each region is preserved under this correspondence. Hence we get a contradiction in this case.

Finally consider the remaining case that the path P_α contains no edges. In this case we have that $\Gamma(D')$ has property (*), has no wicked positive vertices, and has one fewer positive vertices than does $\Gamma(D)$. Hence by assumption $\mathbb{R}^2 \setminus \Gamma(D')$ must contain a region whose boundary cycle has weight at least 4. But again, there is a weight-preserving correspondence between the regions of $\mathbb{R}^2 \setminus \Gamma(D')$ and those of $\mathbb{R}^2 \setminus \Gamma(D)$. Hence we have a contradiction. □

4 Applications

This section contains the proofs of the applications of Theorem B mentioned in the introduction.

Corollary C. *Let Σ be a Seifert surface that is isotopic to a canonical surface. Then Σ is quasipositive if and only if every unknot contained in Σ bounds a disk in Σ or has negative induced framing by Σ.*

Proof For the 'only if' direction (for which the assumption that Σ is isotopic to a canonical surface is not necessary), assume Σ is quasipositive and let $\gamma \subset \Sigma$ be an unknot that does not bound a disk in Σ, with framing k induced by Σ. Let B be an annular neighborhood of γ in Σ. As an incompressible subsurface of Σ, B is itself quasipositive. But as an unknotted band, B is quasipositive only if its core curve γ has negative induced framing by B (which equals the framing induced by Σ). It follows that $k < 0$, concluding the proof of the 'only if' direction.

For the 'if' direction, let Σ be a canonical Seifert surface. By Theorem B, it suffices to check that all cycles in the Seifert graph of Σ have strictly positive total weight. For a cycle c in the Seifert graph, we let A_c be the embedded annulus in Σ given by the union of Seifert circles and half-bands that correspond to the vertices and edges that make up c. Note that A_c is an unknotted annulus and incompressible in Σ. So, by assumption, A_c has strictly negative framing. Twice the framing of A_c equals minus the total weight of c. To see this, note that the framing is calculated as the linking of the two boundary components of A_c where the orientation is reversed on one component, hence every positive crossing traversed by A_c contributes $-\frac{1}{2}$ to the framing, while it contributes $+1$ to the total weight of c (and analogously for negative crossings). With this we established that all cycles have strictly positive total weight, as required. □

Recall that by y we denote a slice-torus invariant, as defined in the introduction.

Theorem D. *If K is a knot with a canonical surface Σ such that $y(K) = \mathrm{genus}(\Sigma)$, then Σ is a quasipositive Seifert surface; in particular, K is a strongly quasipositive knot.*

Proof Let g denote the genus of Σ. Assume toward a contradiction that Σ is not strongly quasipositive. Then, by Corollary C, Σ contains a homologically non-trivial unknot U whose framing induced by Σ is some non-negative integer k. Choose a closed disk $D \subset S^3$ such that D intersects Σ transversely in a proper arc $I \subset \Sigma$ that lies in the interior of D and, in Σ, I intersects U transversely in exactly one point. Let $\Sigma' \subset S^3$ be the surface obtained from Σ by a $\pm 1/k$ surgery along ∂D, where the sign is chosen such that $U \subset \Sigma'$ has framing 0 induced by Σ'. Note that one gets from the knot $K = \partial\Sigma$ to the knot $J := \partial\Sigma'$ by k crossing changes from negative to positive, and so $y(K) \leq y(J)$. Hence ambient surgery in B^4 of Σ' along U (i.e. replacing an annular neighborhood of U in Σ' by two discs properly embedded in B^4) produces a slice surface F of J with $\mathrm{genus}(F) = \mathrm{genus}(\Sigma') - 1 = g - 1$. It follows that $y(K) \leq y(J) \leq g - 1$, contradicting the assumption $y(K) = g$. □

Corollary E. *Let K be a knot with $\tilde{g}(K) = g(K)$, i.e. a knot for which the genus $g(K)$ is realized by a canonical surface Σ. The following are* equivalent:

(1) *Σ is quasipositive,*
(2) *K is strongly quasipositive,*
(3) *for K the Bennequin-inequality is an equality, and*
(4) *$y(K) = g(K)$.*

Proof (1) ⇒ (2) holds by definition; (2) ⇒ (3) is implied by $\mathrm{writhe}(\beta) - n + 1 = 2g(K)$ for a strongly quasipositive braid β on n strands whose closure is K; (3) ⇒ (4) follows since $\frac{\mathrm{sl}(K)+1}{2} \leq y(K) \leq g(K)$ holds for all knots K; and (4) ⇒ (1) is immediate from Theorem D. □

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