

Massive Ray-Singer torsion and path integrals

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ABSTRACT: Zero modes are an essential part of topological field theories, but they are frequently also an obstacle to the explicit evaluation of the associated path integrals. In order to address this issue in the case of Ray-Singer Torsion, which appears in various topological gauge theories, we introduce a massive variant of the Ray-Singer Torsion which involves determinants of the twisted Laplacian with mass but without zero modes. This has the advantage of allowing one to explicitly keep track of the zero mode dependence of the theory. We establish a number of general properties of this massive Ray-Singer Torsion. For product manifolds $M = N \times S^1$ and mapping tori one is able to interpret the mass term as a flat \mathbb{R}_+ connection and one can represent the massive Ray-Singer Torsion as the path integral of a Schwarz type topological gauge theory. Using path integral techniques, with a judicious choice of an algebraic gauge fixing condition and a change of variables which leaves one with a free action, we can evaluate the torsion in closed form. We discuss a number of applications, including an explicit calculation of the Ray-Singer Torsion on S^1 for $G = \text{PSL}(2, R)$ and a path integral derivation of a generalisation of a formula of Fried for the torsion of finite order mapping tori.

KEYWORDS: Topological Field Theories, Differential and Algebraic Geometry

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1 Introduction

Many field theories suffer from issues related to zero modes which, in particular, complicate the explicit evaluation of path integrals. This is particularly true in topological field theories where the zero mode structure is all important. This obstacle appears even in the simplest topological field theories such as BF theories or Schwarz type theories in general. In this paper we suggest a method for dealing with these issues by introducing a massive version of Ray-Singer Torsion where all zero modes are lifted.

BF theories are a very simple class of topological gauge theories, and they have been with us for over 30 years [1–4]. They derive their name from the fact that their classical action is

$$S_{BF} = \int_M \text{Tr } B \wedge F_A$$

with F_A the curvature 2-form of a connection A and B a Lie algebra valued $(\dim M - 2)$ -form. Despite their (deceptive) simplicity, there have been few exact evaluations of the associated partition function on compact closed manifolds.

One case which is completely understood is that of two-dimensional BF theory with a compact gauge group, for which the partition function is the zero-coupling limit of that of two-dimensional Yang-Mills theory, which can be evaluated in closed form in a variety of ways [5–9].

In dimension 3, BF -theories with gauge group G can equivalently be regarded as Chern-Simons gauge theories with the non-compact gauge group $TG \simeq G \times \mathfrak{g}$, and in dimension greater than 3, BF theories have an intricate non-compact and moreover reducible symmetry structure. As a consequence there are different types of zero modes and various sources of potential divergences. Thus, even though the construction and properties of the quantum action for these theories are well understood [4, 10, 11], even in the case of 3-dimensional manifolds there is a dearth of examples of exact evaluations.

Nevertheless, these theories are of interest from a number of points of view. For example, from a mathematical perspective in the 3-dimensional case Witten [3] has, using semi-classical arguments, shown how they are related to the Johnson invariants [12], and it would certainly be desirable to develop tools to explicitly evaluate the partition function in that situation. From a physics perspective, BF theories are e.g. related to formulations of theories of gravity in various dimensions (see for example [13, 14] for reviews), and again it would be of interest to gain a better understanding of the corresponding partition functions in those situations that goes beyond purely formal considerations. While we will not be dealing explicitly with BF -theories in this paper, the techniques that we develop here (in a simpler context to be described next) are also applicable to BF -theories.

Some of the problems that beset BF -theories also arise in an even simpler class of theories, which have been with us even longer. There are the prototypical ‘Schwarz-type’ [15–17] topological field theories with action

$$S_{\text{RST}} = \int_M \text{Tr } \mathcal{B} \wedge d_A \mathcal{C}$$

Here A is a flat connection on a vector bundle \mathbb{E} , and the fields $(\mathcal{B}, \mathcal{C})$ are \mathbb{E} -valued differential p - and $(\dim M - p - 1)$ -forms respectively. Using covariant gauge fixing conditions for all the classical and quantum fields that appear (i.e. $d_A * \mathcal{B} = d_A * \mathcal{C} = 0$ etc.) with respect to some metric g_M on M , these actions can be seen to provide a field-theoretic realisation of the Ray-Singer Torsion [18] of (A, \mathbb{E}) , given by

$$\tau_M(A, \mathbb{E}, g_M) = \prod_{i=0}^n \left(\text{Det}_{\Omega^i(M, \mathbb{E})}(\Delta_A) \right)^{(-1)^{i+1}i/2}$$

where Δ_A is the twisted Laplacian with respect to the metric g_M acting on \mathbb{E} -valued differential forms (for a more detailed description and definition of τ_M see the next section). These $\mathcal{B} - \mathcal{C}$ models are a kind of ‘Abelian’ version of BF theories depending only on a fixed flat background connection A , and they are Gaussian (quadratic, free) theories, so one might expect the evaluation of the corresponding partition functions to be reasonably straightforward.

Surprisingly enough, even in this case there are essentially no intrinsically path integral determinations of the partition functions. The Ray-Singer Torsion can of course be calculated either by explicit calculations of the determinants appearing in its definition (as in the classic calculation of Ray for Lens spaces [19]), or by actually calculating the Reidemeister Torsion and appealing to the equivalence of the Ray-Singer Torsion and the Reidemeister Torsion conjectured by Ray and Singer [18] and established in [20–22] (see e.g. [5, 23] for some sample calculations along these lines). Nevertheless, as mentioned, there are essentially no intrinsically path integral derivations which make use of the latitude afforded by that formalism, with the exception of the Ray-Singer Torsion on S^1 , which was explicitly calculated in [24] using purely path integral methods.

One of the factors contributing to this is the fact that, here too, in general there are zero mode issues that need to be addressed. Another one is the absence of a suitable alternative computationally efficient gauge condition that would allow one to derive an alternative representation of the Ray-Singer Torsion from the path integral. We will address these (and some related) issues in this paper.

At a mathematical level, the zero mode issue is reflected in the fact that in the presence of harmonic modes the correct definition of the Ray-Singer Torsion (as a metric independent ‘topological’ quantity) requires a (metric dependent) cohomological correction term on the cohomology, first described by Ray and Singer in [25].

At the field theory level, these harmonic modes correspond to zero modes of the action and make the partition function ill-defined (zero or infinity and multiple products thereof). In [4] a BRST-invariant method to project out these harmonic modes was proposed and outlined. As an aside (although an interesting aside, we believe), we shall show here that the result of this prescription agrees precisely with that obtained from the procedure advocated by Ray and Singer (section 2.1.4). Even so, while it is pleasing to see this agreement at the formal level, this is still not a particularly useful prescription in realistic cases, in particular for the purposes of explicitly evaluating the partition function.

Therefore, in this paper we propose an alternative method of dealing with the zero modes which turns out to be computationally efficient (and also extends to BF -theories).

Namely, we pursue the seemingly naive idea that, instead of projecting out the harmonic modes, we simply ‘lift’ them by adding mass terms (section 2.2). This is of course not a totally original idea. In particular, it is closely related to the prescription of using a mass to regularise the determinants that two of us advocated on several occasions in the past [24, 26, 27] in order to lift certain degeneracies in Chern-Simons theory (however, here we assign a more fundamental role to this procedure and analyse its consequences). On 3-manifolds of the form $\Sigma \times S^1$, it is also related to the elegant equivariant prescription in complex Chern-Simons theory of Gukov and Pei [28] which was motivated by other geometric considerations.

In the case at hand, at the mathematical level this procedure simply amounts to replacing the Laplacians Δ_A that enter the definition of the Ray-Singer Torsion by their ‘massive’ versions $\Delta_A + m^2$ (which have no zero modes). We refer to the object that one obtains in this way, not very originally, as the Massive Ray-Singer Torsion. Up to a (judiciously chosen, see (2.61)) constant prefactor, the Massive Ray-Singer Torsion is thus defined by

$$\tau_M(A, \mathbb{E}, g_M, m) \propto \prod_{i=0}^n \left(\text{Det}_{\Omega^i(M, \mathbb{E})} (\Delta_A + m^2) \right)^{(-1)^{i+1}i/2}$$

If all the cohomology groups are trivial one can take the $m^2 \rightarrow 0$ limit and reproduce the usual Ray-Singer Torsion. The interest thus lies in the case when the cohomology groups are non-trivial. We will show that, perhaps somewhat unexpectedly, this massive Ray-Singer Torsion has a number of attractive properties in common with its ‘massless’ counterpart (including the triviality of the torsion in even dimensions, and a simple product formula).

At the field theory level, there are a number of issues that need to be addressed. First of all, in general one cannot expect to have a massive deformation of the above $\mathcal{B} - \mathcal{C}$ action that preserves the underlying gauge invariance (and hence gives rise to the ratio of determinants entering the definition of the massive torsion upon gauge fixing these symmetries, say). Secondly, even if we had such an action, a priori we would not have a way to evaluate the corresponding partition function on a general manifold any more than we did with the original definition.

However, as we will show, this situation brightens up considerably for certain classes of manifolds, including for S^1 , product manifolds of the form $M = N \times S^1$ as well as, more generally, mapping tori of N which are fibrations $N \rightarrow M \rightarrow S^1$ over S^1 with fibre N . Also in the latter cases, a central role is played by the Ray-Singer Torsion on S^1 itself, which we therefore look at in detail (section 2.3).

In particular, we will show that on S^1 the mass term can be interpreted as the coupling to a flat \mathbb{R}_+ connection $A^{\mathbb{R}}$ (section 2.3.1), and the massive Ray-Singer Torsion for a flat gauge field A can equivalently be regarded as the standard (metric independent) Ray-Singer Torsion for a flat gauge field $A + A^{\mathbb{R}}$ without zero modes. We determine the Ray-Singer Torsion for $G \times \mathbb{R}_+$ connections (section 2.3.3), and show how it succinctly encodes the Ray-Singer Torsions for G with different numbers of zero modes (section 2.3.4). As an example, we calculate the Ray-Singer Torsion for $G = \text{PSL}(2, \mathbb{R})$ from its massive counterpart, and show complete agreement with the calculation of the Reidemeister torsion by Stanford and

Witten in [29]. We also introduce a path integral representation of the Ray-Singer Torsion on S^1 , and evaluate it in closed form by a variant of the method originally used in [24], namely by discretisation and a suitable change of variables that trivialises the action.

Turning to higher-dimensional manifolds, we show that the gauge-theoretic realisation and interpretation of the massive Ray-Singer Torsion extends to manifolds of the form $M = N \times S^1$ (section 3) and to mapping tori (section 4). In particular, we will show that the absence of zero modes allows one to make a computationally efficient gauge choice adapted to the geometry of the situation (section 3.3). This choice of gauge is a generalisation of the familiar ‘temporal’ gauge $A_0 = 0$. On a manifold of the type $N \rightarrow M \rightarrow S^1$ described above, the p -form field \mathcal{B} has a component $B^{p-1}d\theta$ along the S^1 , and the generalised temporal gauge is the algebraic gauge condition $B^{p-1} = 0$. With this gauge choice, the path integral calculation of the partition function simplifies significantly and boils down to knowing the torsion on the circle S^1 (and that one is known and can be determined in a variety of ways from first principles) and the application of various local index theorems (section 3.4). Both ingredients have appeared previously in our calculation of Chern-Simons theory partition functions on Seifert 3-manifolds [24, 26, 27, 30].

This will also allow us to give a path integral derivation of a generalisation of a formula due to Fried [31] for the torsion of a finite order mapping torus without needing to appeal to the equivalence of Ray-Singer Torsion with Reidemeister torsion [20–22] (section 4.3). We should note here that a first attempt in this direction [32] did not get very far computationally, among other things because it attempted to deal with all, and not just finite order, diffeomorphisms.

In addition to its calculability (on mapping tori), there are a number of other advantages to using the massive Ray-Singer Torsion. Of particular importance is the ability to simultaneously keep track of zero modes of differing orders and their corresponding torsions (as seen concretely in the calculations of section 2.3.3 and 2.3.4). Moreover, as we will see, the algebraic generalised temporal gauge allows one to introduce the ghost system without the need of picking a metric on N at all. The topological nature of the theory is then assured from the outset (section 3.5.3).

Yet another benefit of using the massive Ray-Singer Torsion is that it provides an explanation of a rather puzzling fact that arises within the Abelianisation / Diagonalisation programme [9, 24, 33]. In particular, in the evaluation of the Chern-Simons partition function on various classes of 3-manifolds in [24, 26, 27, 30], one finds a localisation of the path integral onto what appear to be non-flat connections (and Ray-Singer torsions associated to them). Using the insight gained from the calculations that we have performed, we can now explain how to understand and interpret this (section 3.5.2).

2 Ray-Singer torsion revisited

We begin this section by briefly reviewing the definitions of the Analytic Torsion and the Ray-Singer Torsion [18, 25] including the treatment of harmonic modes in both cases. The standard Schwarz type path integral representation of the Ray-Singer Torsion will also be reviewed. We expand on our suggestion in [4] for dealing with non-trivial cohomology

and establish that the outcome of this procedure is equivalent to the prescription of Ray and Singer.

This is followed by introducing ‘massive Ray-Singer Torsion’ where a mass is added to the Laplacian appearing in the definition of the Ray-Singer Torsion. The addition of the mass lifts the zero modes and provides one with a different method of calculation. We establish a number of general properties of the massive Ray-Singer Torsion which parallel those of the standard torsion.

These ideas are applied in the case that the base manifold is the circle. On S^1 both the Analytic Torsion as well as the Ray-Singer Torsion with mass are determined. We show that the addition of a mass can be understood equivalently as the introduction of a flat \mathbb{R}_+ connection. One may view the results either as a regularisation procedure for a flat G connection or as a bona fide evaluation of the torsion for a flat $G \times \mathbb{R}_+$ connection (there are no zero modes for this connection).

In [24] we gave a path integral derivation of the Ray-Singer Torsion of a flat connection on S^1 in the adjoint representation, by projecting out the non-zero cohomology. Here we introduce the \mathbb{R}_+ connection directly in the Schwarz type path integral. This path integral is easily evaluated by a field redefinition that exchanges the connections for twisted boundary conditions, leaving us with otherwise an essentially free action.

2.1 Analytic torsion and Ray-Singer torsion

2.1.1 Basic definitions

We quickly recall some basic definitions. Let M be a compact oriented n -dimensional manifold equipped with a Riemannian metric g_M , and \mathbb{E} a (real) vector bundle over M with fibre E , and equipped with a flat connection ∇ (corresponding to a gauge field A). We also assume that ∇ is compatible with a fixed positive definite fibre metric $\langle \cdot, \cdot \rangle$ on \mathbb{E} (but we will keep this fibre metric fixed throughout the discussion and suppress it from the notation). These data define

- the nilpotent exterior covariant derivative d_A on \mathbb{E} -valued differential forms,

$$d_A : \Omega^k(M, \mathbb{E}) \rightarrow \Omega^{k+1}(M, \mathbb{E}) \tag{2.1}$$

with $(d_A)^2 = 0$, and the corresponding cohomology groups $H_A^k(M, \mathbb{E})$;

- the adjoint $\delta_A = d_A^*$ of d_A with respect to the Hodge scalar product

$$\langle a, b \rangle = \int_M \langle a \wedge *b \rangle, \tag{2.2}$$

on $\Omega^k(M, \mathbb{E})$;

- the Laplace operators $\Delta_A = \Delta_A^{(k)}$ on \mathbb{E} -valued k -forms,

$$\Delta_A = (d_A + \delta_A)^2 = d_A \delta_A + \delta_A d_A : \Omega^k(M, \mathbb{E}) \rightarrow \Omega^k(M, \mathbb{E}) \tag{2.3}$$

The spectrum of Δ_A is positive semi-definite, and by an analog of the usual Hodge decomposition theorem one has an isomorphism between the harmonic modes (zero modes) of the Laplace operator and the cohomology groups,

$$\text{Ker}\Delta_A^{(k)} \simeq H_A^k(M, \mathbb{E}). \tag{2.4}$$

In this setting, the *Analytic Torsion* $\tau_M(A, \mathbb{E}, g_M)$ is defined in terms of suitable ratios of determinants of these Laplace operators. In order to make sense of such infinite-dimensional determinants, Ray and Singer made use of a ζ -function regularisation of these determinants [18].

When there are no zero modes (i.e. the cohomology groups are trivial — one also says that the complex defined by d_A is acyclic), the determinant of the Laplace operator Δ_A can be defined via analytic continuation of the spectral ζ -function

$$\zeta_A(s) = \sum_{\text{Spec}(\Delta_A)} \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\exp(-t\Delta_A)) dt \tag{2.5}$$

by setting

$$\text{Det}(\Delta_A) = e^{-\zeta_A'(0)}. \tag{2.6}$$

We will also use the notation $\zeta_A^{(k)}(s)$ to denote the spectral ζ -function of the operator Δ_A acting on the space $\Omega^k(M, \mathbb{E})$ of k -forms.

When there are zero modes, Ray and Singer proposed to use a ζ -function regularisation that explicitly projects out the harmonic modes, i.e.

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\exp(-t\Delta_A) - P) dt \tag{2.7}$$

where P is a projector onto the spaces of zero modes, say $\lim_{t \rightarrow \infty} \exp(-t\Delta_A)$.

In either case, the *Analytic Torsion* $\tau_M(A, \mathbb{E}, g_M)$ is now defined as

$$\tau_M(A, \mathbb{E}, g_M) = \prod_{i=0}^n \left(\text{Det}_{\Omega^i(M, \mathbb{E})}(\Delta_A) \right)^{(-1)^{i+1}i/2} = \prod_{i=1}^n \left(\text{Det}_{\Omega^i(M, \mathbb{E})}(\Delta_A) \right)^{(-1)^{i+1}i/2} \tag{2.8}$$

For later convenience, we also write this definition in logarithmic form as

$$\log \tau_M(A, \mathbb{E}, g_M) = \sum_{k=0}^n (-1)^k (k/2) \zeta_A^{(k)'}(0). \tag{2.9}$$

E.g. in the 1-dimensional case $n = 1$ one has (by Hodge duality)

$$n = 1 : \quad \tau_M(A, \mathbb{E}, g_M) = \left(\text{Det}_{\Omega^1(M, \mathbb{E})}(\Delta_A) \right)^{1/2} = \left(\text{Det}_{\Omega^0(M, \mathbb{E})}(\Delta_A) \right)^{1/2}. \tag{2.10}$$

A key property of the Analytic Torsion is that, in the acyclic case, it is independent of the metric g_M on M [18],

$$d_A \text{ acyclic} \quad \Rightarrow \quad \tau_M(A, \mathbb{E}, g_M) = \widehat{\tau}_M(A, \mathbb{E}). \tag{2.11}$$

In the non-acyclic case, the presence of harmonic modes introduces a metric dependence, which can be cancelled [25] by introducing a suitable metric-dependent volume factor $\rho_H(g)$ on the space of harmonic modes (identified with the cohomology groups). We will recall the definition of $\rho_H(g_M)$ below. One then has

$$\hat{\tau}_M(A, \mathbb{E}) = \rho_H(g_M)\tau_M(A, \mathbb{E}, g_M). \tag{2.12}$$

In either case, the metric independent quantity $\hat{\tau}_M$ is also usually referred to as the *Ray-Singer Torsion*.

Notation: in case the vector bundle is trivial $\mathbb{E} = M \times E$ we may simplify the notation and write $\tau_M(A, E, g_M)$ for $\tau_M(A, \mathbb{E}, g_M)$ and $\hat{\tau}_M(A, E)$ for $\hat{\tau}_M(A, \mathbb{E})$. In order not to unduly burden the notation, we will occasionally suppress the vector bundle \mathbb{E} from the notation (when it is clear from the context, or simply from specifying A itself), and we will frequently also not indicate explicitly the dependence on the metric g_M (whenever this is not our main concern at that time).

2.1.2 Treatment of harmonic modes

Here we first explain how to define the cohomological correction term $\rho_H(g_M)$ following the prescription given originally by Ray and Singer [25]. We do this in some detail in order to simplify the comparison with the BRST-based path integral treatment of harmonic modes to be discussed in section 2.1.4. We then briefly indicate how to translate this into the modern perspective viewing the Ray-Singer Torsion as an element of a suitable determinant line on the cohomology.

We assume that the cohomology groups $H_A^i(M, \mathbb{E}) \equiv H_A^i$ are not all trivial, and at the outset we choose some reference basis $h_a^{(i)}$ of H_A^i , with $a = 1, \dots, \dim H_A^i$. Given a metric g_M on M , any such class $h_a^{(i)}$ has a unique harmonic representative $f_a^{(i)}$, i.e.

$$\Delta_A f_a^{(i)} = 0, \quad [f_a^{(i)}] = h_a^{(i)}. \tag{2.13}$$

In the following it will be convenient to consider a smooth 1-parameter family of metrics $g_M(u)$ on M , and to parameterise and indicate the dependence of any quantity on the metric $g_M(u)$ simply by its dependence on the parameter u . Thus for each u we have the space of u -harmonic forms $\mathcal{H}_A^i(u)$, with the $f_a^{(i)}(u) \in \mathcal{H}_A^i(u)$ furnishing a basis of $\mathcal{H}_A^i(u)$.

Moreover, for each u we let $\gamma_a^{(i)}(u)$ be an orthonormal basis of $\mathcal{H}_A^i(u)$ with respect to the Hodge metric (2.2), i.e. we require

$$\langle \gamma_a^{(i)}, \gamma_b^{(i)} \rangle (u) = \int_M \langle \gamma_a^{(i)}(u) \wedge *_u \gamma_b^{(i)}(u) \rangle = \delta_{ab}. \tag{2.14}$$

For each u we now have two sets of basis vectors for the finite-dimensional vector space $\mathcal{H}_A^i(u)$, namely the $\{f_a^{(i)}(u)\}$ and the $\{\gamma_a^{(i)}(u)\}$. There must therefore exist some invertible matrix $S_{ab}^{(i)}(u)$ such that

$$\gamma_a^{(i)}(u) = S_{ab}^{(i)}(u) f_b^{(i)}(u). \tag{2.15}$$

Then $|\det S^{(i)}(u)| \neq 0$ is independent of the choice of orthonormal basis $\gamma_a^{(i)}(u)$, and Ray and Singer show that

$$\begin{aligned} \log \widehat{\tau}_M(A, \mathbb{E}) &\equiv \sum_{k=0}^n (-1)^k \left((k/2) \zeta_A^{(k)'}(0) + \log |\det S^{(k)}(u)| \right) \\ &= \log \tau_M(A, \mathbb{E}, u) + \sum_{k=0}^n (-1)^k \log |\det S^{(k)}(u)| \end{aligned} \tag{2.16}$$

is indeed independent of u , and thus independent of the metric. Thus the multiplicative correction factor $\rho_H(u)$ is

$$\rho_H(u) = \prod_{k=0}^n |\det S^{(k)}(u)|^{(-1)^k}. \tag{2.17}$$

As already mentioned, this factor does not depend on the choice of orthonormal basis of $\mathcal{H}_A^i(u)$. It does however depend (weakly) on the initial (u -independent) choice of basis $h_a^{(i)}$ of H_A^i : under a change of basis $h^{(i)} \rightarrow L^{(i)} h^{(i)}$ for some linear (and u -independent) transformation matrix $L^{(i)}$ one has $f^{(i)} \rightarrow L^{(i)} f^{(i)}$ and therefore $S^{(i)} \rightarrow S^{(i)} L^{(i)}$ and

$$\rho_H(u) \rightarrow \rho_H(u) \prod_{k=0}^n |\det L^{(k)}|^{(-1)^k}. \tag{2.18}$$

It is now easy to see that one can redefine the Ray-Singer Torsion in such a way that it is independent of this choice of basis, at the expense of defining it to be an element of a suitable determinant line (rather than as a real number).

First of all recall that, for any vector space V its determinant line $\text{Det } V$ is defined by

$$\text{Det } V = \bigwedge^{\dim V} V \tag{2.19}$$

and that any basis $\{e_i\}$ of V defines an element

$$e_1 \wedge \cdots \wedge e_{\dim V} \in \text{Det } V. \tag{2.20}$$

Given a metric (or scalar product) on V , up to a sign (choice of orientation) an orthonormal basis with respect to this metric defines a preferred element of $\text{Det } V$. Given some reference basis v_i , with $e_i = S v_i$ for some matrix S , by the definition of the determinant one has

$$e_1 \wedge \cdots \wedge e_{\dim V} = (\det S) v_1 \wedge \cdots \wedge v_{\dim V}. \tag{2.21}$$

Thus, with respect to the fixed reference basis v_i , this element of $\text{Det } V$ can be identified with the real number $\det S$.

In particular, an orthonormal basis $\gamma_a^{(i)}(u)$ of \mathcal{H}_A^i defines an element of $\text{Det } H_A^i$. In this way, we can lift $\rho_H(g_M)$ to a well-defined element

$$\widehat{\rho}_H(g_M) \in \bigwedge^{\text{even}} \text{Det } H_A^\bullet \bigwedge^{\text{odd}} (\text{Det } H_A^\bullet)^* \tag{2.22}$$

and define the Ray-Singer Torsion as

$$\widehat{\tau}_M(A, \mathbb{E}) = \widehat{\rho}_H(g_M) \tau_M(A, \mathbb{E}, g_M) \in \bigwedge^{\text{even}} \text{Det } H_A^\bullet \bigwedge^{\text{odd}} (\text{Det } H_A^\bullet)^* \tag{2.23}$$

(see e.g. [34] Theorem 2.11 or (79) in [35]). This is independent of a choice of reference basis, but given some reference basis $h_a^{(i)}$, this reduces to the expressions for $\rho_H(g_M)$ and $\hat{\tau}_M$ given in (2.17) and (2.16).

While this may sound rather abstract, as an example, for the untwisted case the correctly normalised element of $H^0(M, \mathbb{R})$ (the constant functions) would be not the function 1 but rather

$$\frac{1}{\sqrt{\text{Vol}(M, g_M)}} \in \text{Det } H^0 \tag{2.24}$$

with $\text{Vol}(M, g_M)$ the volume of M given by the chosen metric.

Example 2.1. Consider the untwisted Laplacian $\Delta = -R^{-2}d^2/d\theta^2$ on the circle S^1 with metric $g_{S^1} = R^2d\theta \otimes d\theta$ (with $0 \leq \theta < 2\pi$ and $*1 = Rd\theta$). It has eigenmodes $e^{in\theta}$ corresponding to the eigenvalues $R^{-2}n^2$ with $n \in \mathbb{Z}$. In particular, there is a zero mode (the constant function) for $n = 0$. A standard ζ -function calculation (excluding the zero mode $n = 0$) yields

$$\zeta_\Delta(s) = \sum_{n \neq 0} \lambda_n^{-s} = 2R^{2s} \sum_{n > 0} n^{-2s} \equiv 2R^{2s} \zeta(2s). \tag{2.25}$$

Thus, using $\zeta(0) = -1/2, \zeta'(0) = -(1/2) \log(2\pi)$,

$$\zeta'_\Delta(0) = 2 \log R^2 \zeta(0) + 4\zeta'(0) = -\log R^2 - 2 \log 2\pi = -\log(2\pi R)^2. \tag{2.26}$$

Therefore the ζ -function regularised determinant of the Laplace operator is

$$\text{Det } \Delta = e^{-\zeta'_\Delta(0)} = (2\pi R)^2. \tag{2.27}$$

The Analytic Torsion is the square root of the determinant of this operator (with the zero mode removed), and thus

$$\tau_{S^1}(A = 0 d\theta, g_{S^1}) \equiv \tau_{S^1}(R) = 2\pi R. \tag{2.28}$$

Notice that this is metric dependent, as expected (because we have removed the zero mode). To determine the correction factor $\rho_H(R)$ (thinking of R as playing the role of the parameter u in this example), we note that the harmonic 0- and 1-forms on the circle are (up to normalisation) just the constant function 1 and the 1-form $d\theta$, with Hodge norm

$$\int 1 * 1 = 2\pi R, \quad \int d\theta * d\theta = 2\pi R^{-1} \tag{2.29}$$

Therefore, an orthonormal basis of the harmonic modes is provided by $(2\pi R)^{-1/2}$ (in agreement with (2.24) above) and $(2\pi/R)^{-1/2}d\theta$. This provides precisely the overall factor $\sim R^{-1}$ required to cancel the metric dependence $\sim R$ of the Analytic Torsion, but there are metric-independent factors that depend on the choice of reference basis for H^0 and H^1 :

$$\begin{aligned} \left\{ h^{(0)} = [1], h^{(1)} = [d\theta] \right\} &\Rightarrow \rho_H(R) = (2\pi R)^{-1/2} (2\pi/R)^{+1/2} = R^{-1} \\ &\Rightarrow \hat{\tau}_{S^1}(A = 0d\theta) = \rho_H(R) \tau_{S^1}(R) = 2\pi \\ \left\{ h^{(0)} = [1], h^{(1)} = [d\theta/2\pi] \right\} &\Rightarrow \rho_H(R) = (2\pi R)^{-1/2} (2\pi R)^{-1/2} = (2\pi R)^{-1} \\ &\Rightarrow \hat{\tau}_{S^1}(A = 0d\theta) = \rho_H(R) \tau_{S^1}(R) = 1 \end{aligned} \tag{2.30}$$

Of course, by construction both choices lead to (equally valid) metric independent results. To be specific, in the following, whenever dealing with the Ray-Singer Torsion on S^1 , we make the second choice, i.e. we normalise the Ray-Singer Torsion such that $\widehat{\tau}_{S^1} = 1$ for the trivial connection,

$$S^1 : \quad \rho_H(R) = (2\pi R)^{-1} \quad \Rightarrow \quad \widehat{\tau}_{S^1}(A = 0d\theta) = 1. \quad (2.31)$$

2.1.3 Actions for the Ray-Singer torsion in any dimension

Here we first quickly review the standard field theory actions for the Ray-Singer Torsion, in order to fix notation and as background for subsequent discussions. We then discuss in some detail the BRST-based path integral approach to dealing with harmonic modes suggested in [4], and establish the precise equivalence with the mathematical approach described in section 2.1.2. The traditional field theory approach to Ray-Singer Torsion [4, 15–17] uses first and second order actions and is based on the fact that $\Delta_A = TT^\dagger$ with $T = *d_A + d_A*$, in this case $T = \pm T^\dagger$, and both are understood to act on direct sums of spaces of forms. The starting action in such a path integral presentation is

$$S_M(A) = \int_M \langle \mathcal{B}, d_A \mathcal{C} \rangle \quad (2.32)$$

for A a flat connection on a bundle \mathbb{E} while \mathcal{B} and \mathcal{C} are sections of associated bundles. For real vector bundles this is enough. If \mathbb{E} is a complex vector bundle we add the complex conjugate of (2.32) to obtain a real action. In the following we suppress, unless needed explicitly, the complex conjugate fields, though one should be aware that they are there.

The symmetry is

$$\delta \mathcal{B} = d_A \Sigma, \quad \delta \mathcal{C} = d_A \Lambda \quad (2.33)$$

for Λ and Σ forms of appropriate type. The δ_A component of T comes from the requisite (covariant) gauge fixing of (2.33). Indeed, on any manifold M one can gauge fix the shift symmetries (2.33) of the action by imposing the usual covariant gauge conditions

$$d_A * \mathcal{B} = 0, \quad d_A * \mathcal{C} = 0 \quad (2.34)$$

on the fields \mathcal{B} and \mathcal{C} , and this leads to the operator $\Delta_A = TT^\dagger$. As we will briefly recall later on, the complete quantisation of such a system involves the gauge fixing of a hierarchy of symmetries and gauge fixing and ghosts for ghosts, arising from the reducibility of the gauge parameters, as in $\Sigma \rightarrow \Sigma + d_A \Sigma'$ etc. Including all the analogous covariant gauge fixing and measure or ghost terms also for the two towers of gauge symmetries of gauge symmetries, it was shown in detail in [16] in terms of resolvents and in [4, 10] in terms of BRST-symmetry and the Batalin-Vilkovisky procedure [36] that in the end the partition function

$$Z_M[A] = \int D\Phi \exp \left(i \int_M \langle \mathcal{B}, d_A \mathcal{C} \rangle + \text{gauge fixing and ghost terms} \right) \quad (2.35)$$

(where Φ represents all the classical and quantum fields to be integrated over) of these theories reduces precisely to the ratio of determinants of Laplace operators that defines the

Ray-Singer Torsion; more precisely one has

$$Z_M[A] = \tau_M(A, \mathbb{E}, g_M)^{(-1)^{p-1}}. \quad (2.36)$$

The partition function also has a classical gauge invariance

$$A \rightarrow g^{-1}Ag + g^{-1}dg, \quad \Phi \rightarrow \rho(g)\Phi, \quad (2.37)$$

acting on both the dynamical fields $\{\Phi\} = \{\mathcal{B}, \mathcal{C}, \dots\}$ and the background gauge field A , provided that the gauge fixing conditions of the shift gauge symmetry (2.33) are chosen to respect this symmetry (and the covariant gauge fixing conditions do). Here $\rho(g)$ is the transformation in the relevant representation for the quantum field in question and under which the measure $D\Phi$ is invariant.

2.1.4 Treatment of harmonic modes in the path integral

When there are harmonic modes, the action has more symmetries than those given in (2.33) above. Indeed, the symmetries are then shifts by d_A -closed rather than just by d_A -exact forms,

$$\delta\mathcal{B} = \Gamma_A, \quad d_A\Gamma_A = 0, \quad \delta\mathcal{C} = \Psi_A, \quad d_A\Psi_A = 0 \quad (2.38)$$

Here each Γ_A can be split non-uniquely into a d_A -exact piece and a form representing a non-trivial cohomology class,

$$\Gamma_A = d_A\phi + \beta_A \quad \text{with} \quad [\beta_A] \in H_A^p \equiv H_A^p(M, \mathbb{E}). \quad (2.39)$$

As in section 2.1.2, we consider a 1-parameter family of metrics $g_M(u)$. For each u , gauge fixing of the shift by the d_A -exact piece $d_A\phi$ of Γ_A can, as usual, be accomplished e.g. by the covariant gauge fixing conditions (2.34) with respect to the metric $g_M(u)$. As far as the shift symmetries by cohomologically non-trivial β_A are concerned, in line with the definition of the Analytic Torsion or Ray-Singer Torsion, for each u and for every field that appears in the classical (or quantum) action we would like to gauge fix these by projecting out the u -harmonic piece of the field. Moreover, we would like to do so in such a way that the BRST-operator itself is metric independent so that (formally at least) BRST-invariance of the path integral implies metric independence.

We will now show how this can be accomplished by following the prescription suggested in [4]. We will then show that this leads precisely to the cohomological correction term $\rho_H(g_M)$ in the Ray-Singer Torsion discussed in detail in section 2.1.2, i.e. we will show that (2.36) gets modified precisely to

$$\text{BRST gauge fixing of harmonic modes} \quad \Rightarrow \quad Z_M[A] = \hat{\tau}_M(A, \mathbb{E}, g_M)^{(-1)^{p-1}}. \quad (2.40)$$

In the path integral the symmetries (2.38) go over to BRST symmetries together with appropriate transformations for a whole tower of ghost, anti-ghost and multiplier fields. The antighosts and the multiplier fields are naturally paired but with opposite statistics so that their measures are naturally given. This means that we do not have to be explicit about zero modes of these as they can be ‘canonically’ excised in pairs (we will briefly come

back to this below). The ghosts and ghosts for ghosts associated with the hierarchy of reducible symmetries

$$\delta\mathcal{B} = d_A\Sigma^{(p-1)}, \quad \delta\Sigma^{(p-1)} = d_A\Sigma^{(p-2)}, \quad \dots \quad (2.41)$$

will each have a symmetry of the type (2.38), (2.39). Denoting \mathcal{B} and its hierarchy of ghosts collectively by Ω^i (of form-degree i , and with $\Omega^p = \mathcal{B}$ and $i = 0, \dots, p$), we write the corresponding BRST symmetry as

$$\begin{aligned} Q\Omega^i &= d_A\Omega^{i-1} + c^a\beta_a^{(i)} & Qc^a &= 0 \\ Q\bar{c}^a &= \tau^a & Q\tau^a &= 0 \end{aligned} \quad (2.42)$$

Here the $\beta_a^{(i)}$ with $a = 1, \dots, \dim H_A^i$ are a fixed choice of forms such that the

$$h_a^{(i)} = [\beta_a^{(i)}] \quad (2.43)$$

are a fixed basis of H_A^i . c^a, \bar{c}^a and τ^a are constant (zero-form) ghosts and multipliers respectively. The ghosts have ghost number $p - i + 1$. Let us note the following:

- The choice of basis of H_A^i is precisely the same ingredient as the one that enters the Ray-Singer prescription described in section 2.1.2.
- Moreover, nothing depends on the actual choice of the $\beta_a^{(i)}$ in their cohomology class, since any change $\beta_a^{(i)} \rightarrow \beta_a^{(i)} + d_A\gamma_a^{(i-1)}$ can be absorbed into a redefinition of the Ω^{i-1} .
- The BRST operator defined above is independent of the metric $g_M(u)$ (that will enter only in the gauge-fixing conditions).
- There is a similar system for \mathcal{C} which ranges from $i = 0, \dots, \dim M - p - 1$ (or dually $i = p + 1, \dots, \dim M$).

Apart from the usual covariant gauge fixing terms, we add to the action terms of the form

$$\begin{aligned} Q \int_M \langle \Omega^i, *_u \bar{c}^a \gamma_a^{(i)}(u) \rangle &= \int_M \left((-1)^{p-i} \langle \Omega^i, *_u \tau^a \gamma_a^i(u) \rangle \right. \\ &\quad \left. + \langle d_A \Omega^{i-1} + c^b \beta_b^{(i)}, *_u \bar{c}^a \gamma_a^{(i)}(u) \rangle \right) \end{aligned} \quad (2.44)$$

Here, precisely as in section 2.1.2, the $\gamma_a^i(u)$ form an orthonormal basis of u -harmonic forms, i.e. (2.14)

$$\Delta_A(u)\gamma_a^{(i)}(u) = 0, \quad \int_M \langle \gamma_a^{(i)}(u) \wedge *_u \gamma_b^{(i)}(u) \rangle = \delta_{ab}. \quad (2.45)$$

Let us take a look at the two terms in turn:

1. The effect of the first term is to precisely project out the u -harmonic terms of Ω^i . Indeed, decomposing the field Ω^i as

$$\Omega^i = \Omega_a^i(u)\gamma_a^{(i)}(u) + \text{non-harmonic terms}, \quad (2.46)$$

where the $\Omega_a^i(u)$ are constant on M , one sees that (due to the orthogonality of harmonic and non-harmonic terms in the scalar product) the latter do not contribute, and the first term is simply

$$\int_M \langle \Omega^i, *_u \tau^a \gamma_a^{(i)}(u) \rangle = \Omega_b^i(u) \tau^a \int_M \langle \gamma_b^{(i)}(u), *_u \gamma_a^{(i)}(u) \rangle = \Omega_a^i(u) \tau^a. \quad (2.47)$$

Thus the integral over the multipliers τ_a will simply impose the requirement that the $\Omega_a^i(u) = 0$, i.e. that the u -harmonic part of Ω^i is equal to 0. It thus has the same effect as the projector P has in the definition of the ζ -function regularised determinants appearing in the Analytic Torsion.

However, this part all by itself would not lead to a metric independent answer, just like projecting out the harmonic modes in the definition of the Analytic Torsion all by itself does not accomplish this. As we will see, metric independence will be restored by including the second (ghost) term, as would have been expected on the basis of BRST-invariance.

2. In the second (ghost) term, the d_A -exact piece $d_A \Omega^{i-1}$ does not contribute, because it is orthogonal to the harmonic term it is paired with. In order to evaluate the remaining term $\int_M \langle c^b \beta_b^{(i)}, *_u \bar{c}^a \gamma_a^{(i)}(u) \rangle$, we split $\beta_a^{(i)}$ as

$$\beta_a^{(i)} = f_a^{(i)}(u) + d_A g_a^{(i-1)}(u) \quad (2.48)$$

where $f_a^{(i)}(u)$ is u -harmonic. This split is unique, because by Hodge theory every cohomology class $h_a^i = [\beta_a^{(i)}]$ contains a unique u -harmonic representative. Note that this $f_a^{(i)}(u)$ is exactly the same as the one that appeared in (2.13). By the same reasoning as just above, the d_A -exact piece does not contribute, and thus one is left with the task of evaluating $c^b \bar{c}^a \int_M \langle f_b^{(i)}(u), *_u \gamma_a^{(i)}(u) \rangle$. By (2.15), the bases $\{f_a^{(i)}(u)\}$ and $\{\gamma_a^{(i)}(u)\}$ of u -harmonic forms are related by the matrix $S^{(i)}(u)$, and therefore

$$c^b \bar{c}^a \int_M \langle f_b^{(i)}, *_u \gamma_a^{(i)}(u) \rangle = (S^{(i)})_{ab}^{-1} c^b \bar{c}^a. \quad (2.49)$$

In this term, the ghost c has ghost number $p - i + 1$, and therefore from the ghost integral one obtains a contribution proportional to

$$(S^{(i)})_{ab}^{-1} c^b \bar{c}^a \Rightarrow |\det S^{(i)}|^{(-1)^{p-i+1}}. \quad (2.50)$$

Collecting all the ghost terms, including those coming from the field \mathcal{E} , one thus finds that the ghost determinants contribute

$$\begin{aligned} \log Z_M &= (-1)^{p-1} \log \tau_M + \sum_{i=0}^n (-1)^{p-i+1} \log |\det S^{(i)}| \\ &= (-1)^{p-1} \left(\log \tau_M + \sum_{i=0}^n (-1)^i \log |\det S^{(i)}| \right) \\ &= (-1)^{p-1} \log \widehat{\tau}_M \end{aligned} \quad (2.51)$$

as claimed, because the 2nd term in the 2nd line is precisely equal to the Ray-Singer correction term in (2.16).

It remains to explain what to do with all the other (anti-ghost and multiplier) fields of various form degrees that appear upon gauge fixing the theory, but these are also easily dealt with [4]: they always come in pairs of k -forms $(\bar{\sigma}, \pi)$ with $Q\bar{\sigma} = \pi, Q\pi = 0$. In order to gauge fix their harmonic modes to zero, we again introduce a pair (\bar{c}^a, τ^a) of zero-forms, and simply add to the action the term

$$Q \int_M \langle \bar{\sigma} *_u \bar{c}^a \gamma_a^{(k)} \rangle = \int_M \langle \pi *_u \bar{c}^a \gamma_a^{(k)} \rangle \pm \int_M \langle \bar{\sigma} *_u \tau^a \gamma_a^{(k)} \rangle \quad (2.52)$$

which simultaneously projects out the u -harmonic modes of π and $\bar{\sigma}$ in a manifestly BRST-invariant manner.

Thus the path integral is now well-defined, and we have reproduced on the nose the cohomological (or determinant line) correction factor to the Ray-Singer Torsion, required to restore metric independence in the presence of harmonic modes, from a BRST-invariant gauge fixing of the harmonic modes in the path integral.

The prescription we have just described is, therefore, perfect for all formal purposes; unfortunately, for practical calculations it is somewhat cumbersome, as one would need to know the harmonic modes and a more convenient way of handling them. This is one of the main reasons for introducing the Massive Ray-Singer Torsion as it allows us to by-pass that need.

2.2 Massive Ray-Singer torsion: definition and general properties

Instead of projecting out the zero modes right away, we are able to follow them around (and project them out later if so required). To lift the zero modes we add a small mass term to the Laplacian,

$$\Delta_A \rightarrow \Delta_A + m^2. \quad (2.53)$$

Since the spectrum of the twisted Laplacian Δ_A is positive semi-definite, adding the small mass lifts the zero-modes so that the spectrum of the massive Laplacian, $\Delta_A + m^2$ is positive definite. Its spectral zeta function

$$\zeta_A(s; m) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \exp(-t(\Delta_A + m^2)) dt \quad (2.54)$$

then defines the determinant

$$\text{Det}(\Delta_A + m^2) = \exp(-\zeta'_A(0; m)). \quad (2.55)$$

Formally one has

$$\text{Det}(\Delta_A + m^2) = |m|^{2 \dim H_A} \prod_{\lambda \in \text{Spec}'} \lambda \cdot \left(1 + \frac{m^2}{\lambda}\right) \quad (2.56)$$

where Spec' is the non-zero spectrum of Δ_A . Rather more explicitly from the definition (2.54) we have

$$\zeta_A(s; m) = \zeta_m(s) + \widehat{\zeta}_A(s; m) \quad (2.57)$$

with

$$\zeta_m(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \left(P \exp \left(-tm^2 \right) \right) dt \quad (2.58)$$

and

$$\widehat{\zeta}_A(s; m) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \left((1 - P) \exp \left(-t \left(\Delta_A + m^2 \right) \right) \right) dt \quad (2.59)$$

The projection onto the non-zero modes of Δ_A can then be accomplished by considering

$$\begin{aligned} \lim_{m \rightarrow 0} \text{Det} \left(\Delta_A + m^2 \right) \Big|_{\Omega_A^\bullet \ominus \text{H}_A^\bullet} &\equiv \lim_{m \rightarrow 0} \frac{\text{Det} \left(\Delta_A + m^2 \right) \Big|_{\Omega_A^\bullet}}{\text{Det} \left(m^2 \right) \Big|_{\text{H}_A^\bullet}} \\ &= \lim_{m \rightarrow 0} |m|^{-2 \dim \text{H}_A^\bullet} \text{Det} \left(\Delta_A + m^2 \right) \Big|_{\Omega_A^\bullet} \end{aligned} \quad (2.60)$$

Here $\text{Det} \left(m^2 \right) \Big|_{\text{H}_A^\bullet}$ is the determinant that arises on using the ζ -function $\zeta_m(s)$ (2.58).

From this point of view the mass regularised or massive Ray-Singer (or Analytic) Torsion is defined to be

$$\tau_M(A, \mathbb{E}, g_M, m) = |m|^{e(M) \cdot \text{rk}(\mathbb{E}) \cdot \dim M/2} \prod_{i=0}^n \left(\text{Det}_{\Omega^i(M, \mathbb{E})} \left(\Delta_A + m^2 \right) \right)^{(-1)^{i+1} i/2} \quad (2.61)$$

where $e(M)$ is the Euler characteristic of M . The ratio of determinants is exactly the same as that in the definition of the Analytic Torsion in (2.8). Thus it is only the prefactor (which is only non-trivial in even dimensions) which may require some justification. As we will explain below, this factor is chosen such that the massive regularised Ray-Singer Torsion, just as the standard Analytic Torsion itself, is trivial (equal to 1) for even dimensional manifolds.

While the original Ray-Singer Torsion $\tau_M(A, \mathbb{E}, g_M) \cdot \rho_H(g_M)$ is independent of the choice of Riemannian metric, in general $\tau_M(A, \mathbb{E}, g_M, m)$ will not be (even though there are no zero modes of the operator $\Delta_A + m^2$), and one should also not expect there to be a simple correction factor analogous to the cohomological correction term $\rho_H(g_M)$ which would render the massive torsion metric independent in general. We will discuss in much more detail in subsequent sections, under which circumstances the massive Ray-Singer Torsion can be understood or interpreted as defining a metric-independent quantity. For the time being, however, we note that in general the massive torsion has a number of useful and interesting properties (some of which are identical to those of the standard (massless) Ray-Singer or Analytic Torsion):

- By removing the prefactor and projecting out the harmonic modes of Δ_A , as in (2.60), the standard Analytic Torsion can be recovered from the massive torsion as

$$\tau_M(A, g_M) = \lim_{m \rightarrow 0} |m|^{\left(\sum_{i=1}^n (-1)^i i \dim \text{H}_A^i - e(M) \cdot \text{rk}(\mathbb{E}) \cdot \dim M/2 \right)} \cdot \tau_M(A, g_M, m) \quad (2.62)$$

- The alternating sum of weighted dimensions of the cohomology groups that appears here is given by the derivative of the Poincaré polynomial

$$P_A(t) = \sum_{i=1}^n t^i \dim \text{H}_A^i \quad (2.63)$$

at $t = -1$,

$$\left. \frac{dP_A(t)}{dt} \right|_{t=-1} = - \sum_{i=1}^n (-1)^i i \dim H_A^i. \tag{2.64}$$

Since the limit $m \rightarrow 0$ in (2.62) is well-defined, one has an expansion of the massive torsion in non-negative powers of m as

$$\tau_M(A, \mathbb{E}, g_M, m) = |m|^{(e(M) \cdot \text{rk}(\mathbb{E}) \cdot \dim M/2 + \dot{P}_A(-1))} \sum_{n=0}^{\infty} a_n(M, A) m^n \tag{2.65}$$

In particular, for M even-dimensional one has

$$\dim M = 2k \quad \Rightarrow \quad \dot{P}_A(-1) = -e(M) \cdot \text{rk}(\mathbb{E}) \dim M/2 \tag{2.66}$$

and therefore

$$\dim M = 2k \quad \Rightarrow \quad \tau_M(A, \mathbb{E}, g_M, m) = \sum_{n=0}^{\infty} a_n(M, A) m^n. \tag{2.67}$$

As an aside we note that the quantity $\dot{P}_A(-1)$ naturally appears in a variety of contexts, and in some sense is an odd-dimensional analogue of the Euler characteristic — see for example [37].

- If M is even dimensional the massive Ray-Singer Torsion is actually trivial, that is it satisfies

$$\tau_M(A, \mathbb{E}, g_M, m) = 1 \tag{2.68}$$

for all m . In particular the coefficients in (2.67) are $a_0(M, A) = 1$ and $a_i(M, A) = 0$, $i \geq 1$.

This is just as for the standard (massless) Analytic Torsion or Ray-Singer Torsion, which is trivial (and =1) in even dimensions in the acyclic case. The proof relies on the fact that both d_A and the Hodge operator $*$ commute with the Laplacian Δ_A , leading to a cancellation of non-zero eigenvalues between k -forms and $(n - k - 1)$ -forms in even dimensions in the expression for the Analytic Torsion. This argument extends verbatim to the massive case, as also the operator $\Delta_A + m^2$ commutes with both d_A and $*$. An alternative argument, using a result of [4], is given in appendix A.

- There is another property of the Ray-Singer Torsion that is also shared by its massive counterpart, namely a product formula. Suppose M and N are compact, closed and oriented manifolds and equip $M \times N$ with the product metric. Let E be a vector bundle over $M \times N$ which allows for flat connections with local form $A \in \Omega^1(M, \mathfrak{g}) \otimes \Omega^0(N)$, up to gauge equivalence. Then

$$\tau_{M \times N}(A, \mathbb{E}, g_{M \times N}, m) = \tau_M(A, \mathbb{E}, g_M, m)^{e(N)} \cdot \tau_N(g_N, m)^{e(M) \text{rk} \mathbb{E}} \tag{2.69}$$

The proof (appendix A) follows the original proof of Theorem 2.5 in Ray and Singer [18] (with a slight generalisation we gave in [24]).

In view of the triviality of the massive torsion for even-dimensional manifolds, and that of the Euler characteristic for odd-dimensional manifolds, this product formula is only of interest if $\dim M$ is odd and $\dim N$ is even. In this case the product formula (2.69) reduces to

$$\tau_{M \times N}(A, \mathbb{E}, g_{M \times N}, m) = \tau_M(A, \mathbb{E}, g_M, m)^{e(N)} \quad (2.70)$$

- Lastly, there are a number of attractive results, due to Fried [31], for the Reidemeister Torsion (and hence the Ray-Singer Torsion). The one of interest to us here is the following:

Let M be an odd dimensional mapping torus $N \rightarrow M \xrightarrow{\pi} S^1$ with f a finite order diffeomorphism acting on N . For E a bundle over the base S^1 with connection $\phi d\theta$ where ϕ is constant and $f_i^* : H^i(N, \mathbb{R}) \rightarrow H^i(N, \mathbb{R})$. Then

$$\tau_M(\phi, \pi^* \mathbb{E}) = \prod_{i=0}^{\dim N} \left| \det \left(1 - \rho \left(e^{-2\pi\phi} \right) \otimes f_i^* \right) \right|^{(-1)^i} \quad (2.71)$$

We will establish a somewhat stronger result (including the mass term) in section 4 using field theory techniques.

Example 2.2. For the untwisted Laplacian with mass $m > 0$ on S^1 one has

$$\tau_{S^1}(R, m) = \frac{1}{m} \prod_{n \geq 0} R^{-2} (n^2 + R^2 m^2) = \frac{1}{m} \prod_{n \geq 0} R^{-1} (n + iRm) \prod_{n \geq 0} R^{-1} (n - iRm) \quad (2.72)$$

which suggests making use of the Hurwitz ζ -function $\zeta(s; a) = \sum_{n \geq 0} (n + a)^{-s}$ ($a \in \mathbb{C}$ but not 0 or a negative integer). We therefore define

$$\tau_{S^1}(R, m) = \lim_{s \rightarrow 0} \frac{1}{m} \cdot \exp \left(-d/ds \cdot [R^s \zeta(s, iRm) + R^s \zeta(s, -iRm)] \right) \quad (2.73)$$

Note that $\zeta(0, a) = 1/2 - a$ and $\zeta'(0, a) = \ln \Gamma(a) - \ln(2\pi)/2$. Hence

$$\tau_{S^1}(R, m) = \frac{2\pi}{mR \Gamma(imR) \Gamma(-imR)} = 2i \sin(\pi(1 + imR)) = 2 \sinh(\pi mR), \quad (2.74)$$

the second last equality arising from Euler's reflection formula. As $m \rightarrow 0$ this vanishes, reflecting the presence of the zero mode. Indeed we have

$$\lim_{m \rightarrow 0} \tau_{S^1}(R, m) = 2\pi Rm + \dots \quad (2.75)$$

where the exact same metric dependence is maintained, at this order, as in the massless case.

2.3 Massive Ray-Singer torsion on S^1

Note that the massive spectrum determined in Example 2.2, with $n \rightarrow n \pm iRm$, looks like it could also be produced by the coupling to a gauge field $\sim iRm$, in which case the issue of metric (in-) dependence presents itself in a slightly different way. We shall now reconsider the above example from this point of view and then generalise it in a variety of ways.

2.3.1 Mass and \mathbb{R}_+ connections

Thus consider once again the circle S^1 with local coordinate $0 \leq \theta < 2\pi$ and with a metric $g_{S^1} = R^2 d\theta \otimes d\theta$ such that $\text{Vol}(S^1, g_{S^1}) = 2\pi R$, $*1 = Rd\theta$ and $*d\theta = R^{-1}$. However, rather than looking at the untwisted Laplacian (as in the above Example), consider now a vector bundle \mathbb{E} over S^1 , equipped with a (necessarily flat) connection $A = \phi d\theta$, with associated exterior covariant derivative

$$d_A = d\theta D_\phi, \quad D_\phi = \partial_\theta + \phi \tag{2.76}$$

Take ϕ to be anti-Hermitian (with respect to the fibre metric $\langle \cdot, \cdot \rangle$). The Ray-Singer Torsion on S^1 of the flat connection $A = \phi d\theta$ is formally the positive root

$$\tau_{S^1}(\phi, \mathbb{E}) = \left(\text{Det}_{\Omega^1(S^1, \mathbb{E})} \Delta_A \right)^{1/2} \tag{2.77}$$

of the twisted Laplacian Δ_A on $\Omega^k(S^1, \mathbb{E})$, i.e. $\Delta_A = -(*d_A)^2$ on $\Omega^0(S^1, \mathbb{E})$ and $\Delta_A = -(d_A *)^2$ on $\Omega^1(S^1, \mathbb{E})$. The spectra of the two Laplacians are the same (by Hodge duality, and $\Delta_A * = * \Delta_A$). The positive square root can then be taken to be either $|\text{Det}(id_A *)|$ or $|\text{Det}(i * d_A)|$ on the appropriate space of forms,

$$\tau_{S^1}(\phi, \mathbb{E}) = \left| \text{Det}_{\Omega^1(S^1, \mathbb{E})}(id_A *) \right| = \left| \text{Det}_{\Omega^0(S^1, \mathbb{E})}(i * d_A) \right| \tag{2.78}$$

Adding a mass term to the twisted Laplacian amounts to the substitution

$$\Delta_A \rightarrow \Delta_A + m^2 = R^{-2}(iD_\phi + imR)(iD_\phi - imR). \tag{2.79}$$

Note that, since $i\phi \pm imR$ is the sum of a Hermitian and an anti-Hermitian part, the individual factors on the right-hand side are neither Hermitian nor anti-Hermitian.

However, at this point we can regard the mass term mR as an \mathbb{R}_+ connection (or as the imaginary component of a $U(1)_\mathbb{C}$ connection). This means we take $\mathbb{E} \otimes \mathbb{L}^{\pm 1}$ to be a $G \times \mathbb{R}_+$ bundle over S^1 with a flat connection $A^\pm = (\phi \pm \mu)d\theta$. The relationship between the connection μ and the mass term m is

$$\mu = Rm \quad \Leftrightarrow \quad \mu d\theta = *m. \tag{2.80}$$

In this case the massive Ray-Singer Torsion takes the more symmetric form

$$\tau_{S^1}(\phi, \mathbb{E}, m) = \sqrt{\tau_{S^1}(\phi + \mu, \mathbb{E} \otimes \mathbb{L}) \tau_{S^1}(\phi - \mu, \mathbb{E} \otimes \mathbb{L}^{-1})} \tag{2.81}$$

where we have set

$$\tau_{S^1}(\phi \pm \mu, \mathbb{E} \otimes \mathbb{L}^\pm) = \text{Det}_{\Omega^1(S^1, \mathbb{E})} \left(R^{-1}(iD_\phi \pm i\mu) \right) \tag{2.82}$$

The factorised form for the Ray-Singer Torsion (2.81) agrees with what was found in [38] for complex Chern-Simons theory. In the following we fix on one sign and consider for concreteness $\text{Det}_{\Omega^1(S^1, \mathbb{E} \otimes \mathbb{L})} (R^{-1}(iD_\phi + i\mu))$, the other case is easily obtained.

2.3.2 Metric independence of the $U(1)_{\mathbb{C}}$ analytic torsion

In this subsection, as a warm-up, we consider the Abelian case, i.e. $G = U(1)$, extended by the above \mathbb{R}_+ gauge field to a $U(1)_{\mathbb{C}}$ -connection on a complex line bundle \mathcal{L} , i.e. we study the determinant

$$\text{Det}_{\Omega^1(S^1, \mathcal{L})} \left(R^{-1} \left(i \frac{d}{d\theta} + \nu \right) \right) \tag{2.83}$$

of the operator

$$R^{-1} \left(i \frac{d}{d\theta} + \nu \right) = R^{-1} \left(i \frac{d}{d\theta} + (\alpha + i\mu) \right) \tag{2.84}$$

with $\nu = \alpha + i\mu \in \mathbb{C}$ representing a constant $U(1)_{\mathbb{C}}$ gauge field, the metric dependence being encoded in the “einbein” R^{-1} corresponding to the metric $g_{S^1} = R^2 d\theta \otimes d\theta$.

The spectrum of this operator is simply $\{R^{-1}(n + \nu)\}$ with $n \in \mathbb{Z}$. In particular, note that for $\mu \neq 0$ this operator has no zero modes for any value of (the $U(1)$ -part of the connection) α , including $\alpha = 0$. As a consequence, we expect the Analytic Torsion to be metric independent for $\mu \neq 0$. On the other hand, we can think of μ as encoding the mass parameter, and from that point of view we would expect the Analytic Torsion to be metric dependent.

To see how this works out, note that the spectral ζ -function $\widehat{\zeta}(s, \nu)$ for this Dirac type operator can, somewhat as in Example 2.2 above, be written in terms of the Hurwitz ζ -function $\zeta(s; a) = \sum_{n \geq 0} (n + a)^{-s}$ as

$$\widehat{\zeta}(s, \nu) = (R)^s \left[\zeta(s, \nu) + (-1)^{-s} \zeta(s, 1 - \nu) \right] \tag{2.85}$$

The R -dependent term after taking the derivative evaluated at $s = 0$ gives (with $\zeta(0; a) = 1/2 - a$)

$$\ln(R) [\widehat{\zeta}(0, \nu) + \widehat{\zeta}(0, 1 - \nu)] = \ln(R) \left[\frac{1}{2} - \nu - \frac{1}{2} + \nu \right] = 0 \tag{2.86}$$

and so we see explicitly that, as anticipated, there is no metric dependence with ν , and hence μ , regarded as a connection. Then, using again $\zeta'(0, a) = \ln \Gamma(a) - \ln(2\pi)/2$ and $\Gamma(a)\Gamma(1 - a) = \pi/\sin(\pi a)$, the determinant evaluates to

$$\text{Det}_{\Omega^1(S^1, \mathcal{L})} \left(R^{-1} \left(i \frac{d}{d\theta} + \nu \right) \right) = e^{-i\pi/2} e^{i\pi\nu} 2 \sin \pi\nu. \tag{2.87}$$

If, however, μ is simply used to encode the (metric independent) mass term m , as in

$$R^{-1} i \frac{d}{d\theta} + im = R^{-1} \left(i \frac{d}{d\theta} + i\mu \right) \tag{2.88}$$

with $\mu = mR$, then the situation is different. As explained previously one may consider $\mu d\theta$ to be a connection on a real line bundle \mathbb{L} . While we have shown that the overall R^{-1} prefactor does not enter into the final answer, the result depends on R via the dependence of μ on R . Indeed, then one has

$$\text{Det}_{\Omega^1(S^1, \mathbb{L})} \left(R^{-1} (id/d\theta + i\mu) \right) = e^{-i\pi/2} e^{(-\pi\mu)} 2 \sin(i\pi\mu) \tag{2.89}$$

Clearly, for μ a metric independent connection, this result does not depend on the metric. On the other hand, for $\mu = mR$, equally clearly the result *is* metric dependent, and for small mass we get the characteristic dependence

$$\text{Det}_{\Omega^1(S^1)}(id/d\theta + imR) \longrightarrow 2\pi mR \tag{2.90}$$

on the metric. This resolves the issue raised at the beginning of this section regarding the metric (in-) dependence of this quantity in a satisfactory (all is as it should be) way.

2.3.3 Ray-Singer torsion for $G \times \mathbb{R}_+$ connections

We now return to the general problem of determining (2.82). Without loss of generality, on S^1 we can assume that ϕ is constant (we will later on re-derive the result from the path integral point of view without this assumption). We expand in terms of Fourier modes to obtain

$$\begin{aligned} \tau_{S^1}(\phi + \mu, \mathbb{E} \otimes \mathbb{L}) &= \text{Det}_{\Omega^1(S^1, \mathbb{E} \otimes \mathbb{L})} \left(R^{-1}(iD_\phi + i\mu) \right) \\ &= \prod_{\lambda} \prod_{n \in \mathbb{Z}} R^{-1}(n + \lambda(\phi) + i\mu) \end{aligned} \tag{2.91}$$

where the $\lambda(\phi)$ are the real eigenvalues of the action of $i\phi$ in the representation defined by \mathbb{E} . Note that, by a reordering of the integers n one can take $0 \leq \lambda(\phi) < 1$ (as long as $\mu \neq 0$). As we need to make sense of this first order operator we make use of the Hurwitz ζ -function regularisation, which amounts to setting

$$\text{Det}_{\Omega^1(S^1, \mathbb{E} \otimes \mathbb{L})}(iD_\phi + i\mu) = \prod_{\lambda} \exp \left(-\widehat{\zeta}'(0, \lambda(\phi) + i\mu) \right) \tag{2.92}$$

where

$$\begin{aligned} \widehat{\zeta}(s, \lambda(\phi) + i\mu) &= \sum_{n \in \mathbb{Z}} \frac{R^s}{(n + \lambda(\phi) + i\mu)^s} \\ &= \sum_{n \geq 0} \frac{R^s}{(n + \lambda(\phi) + i\mu)^s} + (-1)^{-s} \sum_{n \geq 0} \frac{R^s}{(n + 1 - \lambda(\phi) - i\mu)^s} \\ &= R^s \left[\zeta(s, \lambda(\phi) + i\mu) + e^{-i\pi s} \zeta(s, 1 - \lambda(\phi) - i\mu) \right] \end{aligned} \tag{2.93}$$

with the ζ -functions on the right hand side being the usual Hurwitz ζ -function and we have taken $-1 = \exp(i\pi)$. Note that none of the derivatives of the Hurwitz ζ -functions are singular at $s = 0$.

As in the previous Abelian example, with this regularisation there is no R dependence as the only source for such a dependence comes from differentiating the overall R^s term which gives, at $s = 0$,

$$\ln R \cdot [\zeta(0, \lambda(\phi) + i\mu) + \zeta(0, 1 - \lambda(\phi) - i\mu)] = 0 \tag{2.94}$$

on using $\zeta(0, a) = 1/2 - a$. Consequently, with $z = \lambda(\phi) + i\mu$ and $\zeta'(0, a) = \ln \Gamma(a) - \ln(2\pi)/2$, one finds

$$\begin{aligned} \widehat{\zeta}'(0, z) &= \zeta'(0, z) + \zeta'(0, 1 - z) - i\pi \zeta(0, 1 - z) \\ &= \ln \Gamma(z) \Gamma(1 - z) - \ln 2\pi + i\frac{\pi}{2} - i\pi z \end{aligned} \tag{2.95}$$

Hence

$$\begin{aligned} \text{Det}_{\Omega^1(S^1, \mathbb{E} \otimes \mathbb{L})} (iD_\phi + i\mu) &= \prod_\lambda -2i \sin \pi(\lambda(\phi) + i\mu) \cdot e^{i\pi(\lambda(\phi) + i\mu)} \\ &= \text{Det}_{E \otimes L} \left(1 - e^{-2\pi\mu} \cdot \rho \left(e^{2\pi\phi} \right)^{-1} \right) \end{aligned} \quad (2.96)$$

where $\rho(e^{2\pi\phi})$ and $e^{2\pi\mu} = h$ are the holonomies of the connections ϕ and μ along the circle respectively. More generally, for $\phi = \phi(s)$ and $\mu = \mu(s) \neq 0$ not necessarily constant we will set

$$g = \text{Pe} \int_0^{2\pi} \phi(s) ds, \quad h = e^{\int_0^{2\pi} \mu(s) ds}. \quad (2.97)$$

We have therefore established that the Ray-Singer Torsion is

$$\tau_{S^1}(\phi + \mu, \mathbb{E} \otimes \mathbb{L}) = \text{Det}_{E \otimes L} \left(1 - e^{-2\pi\mu} \cdot \rho(e^{2\pi\phi})^{-1} \right) \quad (2.98)$$

One should note that with all our assumptions the determinant that appears here has no zeros for $\mu > 0$ and as there are no zero modes one has that the Analytic Torsion and Ray-Singer Torsion agree

$$\widehat{\tau}_{S^1}(\phi + \mu, \mathbb{E} \otimes \mathbb{L}) = \tau_{S^1}(\phi + \mu, \mathbb{E} \otimes \mathbb{L}) \quad (2.99)$$

Example 2.3. Let $g \in \text{SU}(2)$ and ρ be the fundamental representation. There is just one type of conjugacy class and one has

$$g = k^{-1} \cdot \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \cdot k$$

so that the Ray-Singer Torsion for $\text{SU}(2) \times \mathbb{R}_+$ is

$$\widehat{\tau}_{S^1}(\phi + \mu, \mathbb{C}^2 \otimes L) = 1 - 2h^{-1} \cos \alpha + h^{-2} = (1 - h^{-1})^2 + 4h^{-1} \sin^2 \alpha / 2$$

Clearly, this does not vanish for any positive $h \neq 1$ (i.e. for any real non-zero $\mu(x)$), reflecting the fact that the zero-modes have been lifted by the mass term.

In the previous example h was introduced to lift the zero modes and it did just that. Surprisingly even for non-compact groups, for which not all of our assumptions (like the compatibility of the connection with a positive-definite scalar product on the fibres) need to be satisfied, it may happen that the introduction of a mass term eliminates the zero modes, though the situation may be considerably more complicated. An example of interest is a calculation arising in the context of JT gravity in 2 dimensions [29]. Amongst other things the authors determine the Reidemeister Torsion on the circle for the group $\text{SL}(2, \mathbb{R})$. In the following example we determine the Ray-Singer Torsion and establish equivalence with the result in [29].

Example 2.4. Let $g \in \text{PSL}(2, \mathbb{R})$ then there are three distinct types of conjugacy classes depending on whether the trace is less than 2 (elliptic), equal to 2 (parabolic) or greater than 2 (hyperbolic),

$$g = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \begin{pmatrix} 1 \pm 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

with $\theta \in [0, \pi)$ and $b \geq 1$. The corresponding Ray-Singer Torsion in the adjoint representation $\mathfrak{psl}(2, \mathbb{R}) \simeq \mathbb{R}^3$ is

$$\tau_{S^1}(\phi + \mu, \mathbb{R}^3 \otimes L) = \begin{cases} (1 - h^{-1})^3 + (1 - h^{-1}) h^{-1} 4 \sin^2 \theta \\ (1 - h^{-1})^3 \\ (1 - h^{-1})^3 - (1 - h^{-1}) h^{-1} (b - b^{-1})^2 \end{cases}$$

respectively. In all three cases there are zero modes for $\mu = 0$, i.e. for $h = 1$. However, for $h \neq 1$, the torsion is clearly regularised (that is it is not zero due to zero modes) in the elliptic and parabolic case. In the hyperbolic case, there are zero modes for $h = b^{\pm 2}$, but nevertheless the torsion is regularised for all other values, in particular for $h \in (b^{-2}, b^2)$, and one can then look at the limit $h \rightarrow 1$ with impunity.

At the identity matrix ($\theta = 0$ or $b = 1$) and in the parabolic case, the above expressions have a zero of order 3 in μ . In all other cases, one has zeros of order one. We can now readily determine the Ray-Singer Torsion in each case. To that end one first passes to the massive Ray-Singer Torsion in factorised form (2.81) and then takes the limit as in (2.60) with the appropriate power of μ . One finds

$$\hat{\tau}_{S^1}(\phi, \mathbb{R}^3) = \begin{cases} 4 \sin^2 \theta, & 1 \text{ if } \theta = 0 \\ 1 \\ (b - b^{-1})^2, & 1 \text{ if } b = 1 \end{cases}$$

This result agrees (for the hyperbolic conjugacy class) with the Reidemeister Torsion calculation in [29] (cf. their equation (3.42) with $b = \exp a/2$). The ‘accidental’ zero modes occur in every finite dimensional representation of $\text{PSL}(2, \mathbb{R})$ as then the holonomy for the hyperbolic elements times the \mathbb{R}_+ holonomy will be

$$h \begin{pmatrix} b^r & 0 & \dots & 0 \\ 0 & b^{r-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & b^{2-r} & 0 \\ 0 & \dots & 0 & b^{-r} \end{pmatrix}$$

which has eigenvalue 1 whenever $h = b^r, b^{(r-2)}, \dots, b^{(2-r)}, b^{-r}$.

2.3.4 Passing from $G \times \mathbb{R}_+$ to G Ray-Singer torsion

We briefly review the relationship between zero modes and reducibility of the holonomy g (the unit eigenvalues of $\rho(g)$). Let $\psi \in \Gamma(S^1, \mathbb{E})$ be an eigenvector with zero eigenvalue then

$$D_\phi \psi = 0 \tag{2.100}$$

The solution is

$$\psi(\theta) = \text{P exp} \left(\int_0^\theta \rho(\phi(s)) ds \right) \cdot \psi(0) \tag{2.101}$$

However, periodicity requires that $\psi(2\pi) = \psi(0)$ or that

$$(1 - \rho(g))\psi(0) = 0 \tag{2.102}$$

Clearly with $\mu = 0$ the zero modes of the Laplacian translate into the zeros of the determinant on the right hand side of (2.98).

So in order to isolate the contribution of the zero modes in general, split the representation space E as $E = E_{\parallel} \oplus E_{\perp}$ where the vectors in E_{\parallel} are invariant under the action of $g = e^{2\pi\phi}$, that is they satisfy (2.102). Then according to (2.60) the Analytic Torsion is obtained as the limit

$$\begin{aligned} \tau_{S^1}(\phi, \mathbb{E}) &= \lim_{m \rightarrow 0} m^{-\dim E_{\parallel}} \cdot \text{Det}_E \left(1 - h^{-1} \rho(g)^{-1} \right) \\ &= (2\pi R)^{\dim E_{\parallel}} \text{Det}_{E_{\perp}} \left(1 - \rho(g)^{-1} \right) \end{aligned} \tag{2.103}$$

This is not quite the Ray-Singer Torsion as we still need to multiply by the metric factor $\rho_H(R)$ which we will do below.

Example 2.5. *The situation we most often come across in gauge theory is where \mathbb{E} is the $\text{ad } P$ bundle with fibre \mathfrak{g} the Lie algebra of the compact gauge group G . Split the complexified Lie algebra as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$ where \mathfrak{t} is the Cartan subalgebra. Providing ϕ is a regular element of \mathfrak{g} so that its centraliser can be taken to be \mathfrak{t} , write (somewhat incorrectly) $E = \mathfrak{t} \oplus \mathfrak{k}$ where $\mathfrak{t} = E_{\parallel}$ and $\mathfrak{k} = E_{\perp}$ so that in this case from (2.103) we have*

$$\begin{aligned} \tau_{S^1}(\phi, \mathfrak{g}) &= \lim_{m \rightarrow 0} (m)^{-\dim \mathfrak{t}} \cdot \text{Det}_{\mathfrak{g}} \left(1 - h^{-1} \text{Ad}(g)^{-1} \right) \\ &= (2\pi R)^{\dim \mathfrak{t}} \text{Det}_{\mathfrak{k}} (1 - \text{Ad}(g)) \end{aligned}$$

This will go over to the Ray-Singer Torsion that we found in [24] or as Reidemeister torsion as found by Freed [23] or Witten [5] once we have included the $\rho_H(R)$ contribution (and we do this in section 2.3.4 below). Note that if ϕ is not regular, its centraliser (and thus E_{\parallel}) is larger but we may still apply (2.103) to get a sensible answer. For example, if $\phi = 0$, then $E_{\parallel} = \mathfrak{g}$, thus E_{\perp} is simply absent, and

$$\tau_{S^1}(\phi = 0) = (2\pi R)^{\dim \mathfrak{g}}$$

In summary, we have seen that the introduction of a mass has not only lifted the zero modes: it has also allowed us to follow the non-trivial cohomology in case the connection does not lead to acyclic cohomology and to pick out precisely the terms where that cohomology has been projected out. One may, if one wishes, retain and follow the μ (or h) dependence and we will do so further along.

Within the context of the massive Ray-Singer Torsion we must reconsider the issue of metric dependence given the twin role played by μ . If one considers μ to be an \mathbb{R}_+ flat connection then there is nothing to do. Indeed by (2.98) the Analytic Torsion for the $G \times \mathbb{R}_+$ connection is the same as the Ray-Singer Torsion and is respectably metric independent.

If, on the other hand, $\mu = mR$ is simply a mass regulator then to arrive at the Ray-Singer Torsion for a G connection there are two passes that we need to make. Firstly one

should take the $m \rightarrow 0$ limit as in (2.60) of the Analytic Torsion with mass and then one must multiply the result by ρ_H to arrive at the metric-independent Ray-Singer Torsion. We will now see that the only possible natural combination of m and R arises in this way, namely the combination $\mu = mR$, in terms of which (as we have just seen) there is indeed no metric dependence.

By the normalisation convention (2.31), we find a factor of $(2\pi R)^{-1}$ for each orthonormal basis vector e_i of the vector space E_{\parallel} . Therefore,

$$\rho_H(R) = (2\pi R)^{-\dim E_{\parallel}} \tag{2.104}$$

while from (2.103) we are asked to divide by $m^{\dim E_{\parallel}}$. The combination then is to take

$$\begin{aligned} \widehat{\tau}_{S^1}(\phi, \mathbb{E}) &= \lim_{m \rightarrow 0} \rho_H(mR) \tau_{S^1}(\phi + \mu, \mathbb{E}) = \lim_{\mu \rightarrow 0} \rho_H(\mu) \tau_{S^1}(\phi + \mu, \mathbb{E}) \\ &= \text{Det}_{E_{\perp}} \left(1 - \rho(g)^{-1} \right) \end{aligned} \tag{2.105}$$

From this perspective, dividing by a suitable power of the constant \mathbb{R}_+ connection $\mu = mR$ rather than of the mass ensures that at no point in the calculation any metric dependence arises.

In particular the Ray-Singer Torsion for the adjoint representation Example 2.5 is (for ϕ regular)

$$\widehat{\tau}_{S^1}(\phi, \mathfrak{g}) = \text{Det}_{\mathfrak{k}} (1 - \text{Ad}(g)) \tag{2.106}$$

agreeing with the results obtained in [24] and the Reidemeister form in [5, 23].

Before concluding this section we observe that as there are no zero modes on \mathbb{E}_{\perp} that

$$\widehat{\tau}_{S^1}(\phi, \mathbb{E}_{\perp}) = \tau_{S^1}(\phi, \mathbb{E}_{\perp}) \tag{2.107}$$

and that essentially the prescription for dealing with the zero modes says that the Ray-Singer Torsion on \mathbb{E} is the Analytic Torsion on \mathbb{E}_{\perp} so we end up with a cycle of equivalences

$$\widehat{\tau}_{S^1}(\phi, \mathbb{E}) = \widehat{\tau}_{S^1}(\phi, \mathbb{E}_{\perp}) = \tau_{S^1}(\phi, \mathbb{E}_{\perp}) \tag{2.108}$$

This seems to imply that the prescription of Ray and Singer for calculating the Ray-Singer Torsion on a bundle \mathbb{E} is essentially to calculate the Analytic Torsion on the bundle \mathbb{E}_{\perp} .

2.3.5 Path integral representation

It is difficult to see, in general, how to construct a first order action with mass which has a quantum gauge symmetry without introducing more fields and such that it reproduces the action (2.32) as the mass goes to zero. Consequently, it is not clear how to create the Schwarz type topological theory corresponding to the Massive Ray-Singer Torsion that we have defined.

However, as we have already seen, on S^1 a mass term can be converted into a gauge field, and moreover on S^1 the gauge shift symmetry is not an issue. As we will see later on, these favourable features are inherited by Schwarz-type gauge theories on certain fibrations over S^1 , where the torsion on S^1 will continue to play a central role.

Given its importance, we therefore now turn to the torsion on S^1 from a path integral perspective. In particular we will evaluate the path integral for the torsion of S^1 by a small but useful variation of the method used in [24], as this new method turns out to generalise in a rather straightforward way to more general manifolds (such as mapping tori — see section 4).

The action that we use for the path integral to represent the determinants that we have already studied in section 2.3.3 is (see for example (3.9) in [24])

$$iS_F = \int_{S^1} R^{-1} \langle \bar{\eta}, (iD_\phi \pm i\mu)\eta \rangle d\theta \tag{2.109}$$

where η and $\bar{\eta}$ are Grassmann odd sections of \mathbb{E} and \mathbb{E}^* respectively. Because of its metric dependence, encoded in the “einbein” R^{-1} , this is not strictly speaking at first sight just a Grassmann-odd one-dimensional version of the Schwarz-type actions (2.32), which are written in terms of differential forms and are thus explicitly metric-independent. Rather, written in this way the action is more like a 1-dimensional Dirac action. Nevertheless, once we have established the R -independence of the result (and we will do this in detail below), we may just set $R = 1$ and view the above action as nothing other than a Grassmann-odd Schwarz action (cf. also the comment in section 3.5.3 below).

Now let us turn to the path integral. If one uses a Fourier mode expansion the measure is taken to be

$$D\eta D\bar{\eta} = \prod_n d\eta_n d\bar{\eta}_n \tag{2.110}$$

so that the integral over the exponential of (2.109) yields $\text{Det}_{\Omega^1(S^1, \mathbb{E} \otimes \mathbb{L})}(iD_\phi \pm i\mu)$ and one could evaluate it with the ζ -functions of the previous section.

Our aim here is to give an evaluation of the corresponding partition function

$$Z[\phi \pm \mu, R] = \int D\eta D\bar{\eta} \exp\left(\int_{S^1} R^{-1} \langle \bar{\eta}, (iD_\phi \pm i\mu)\eta \rangle d\theta\right) \tag{2.111}$$

where the Grassmann fields are periodic on S^1 . The usual normalisation of the path integral measure is so that the one loop contribution has no coupling constant pre-factor. In this case it would mean that we have the measure

$$D(\eta/\sqrt{R})D(\bar{\eta}'/\sqrt{R}) \tag{2.112}$$

and a scaling $(\eta, \bar{\eta}) \rightarrow (\sqrt{R}\eta', \sqrt{R}\bar{\eta}')$ would eliminate R from both the action as well as the measure. Rather more generally, if we start with the measure which does not involve R , then any R -dependence that arises can be eliminated by a renormalisation, depending on the regularisation, of the log of the volume of the space. So we could always multiply the partition function by a term of the form

$$\exp(a + b \ln R) \tag{2.113}$$

This is quite analogous to the “standard renormalisation” ambiguities encountered e.g. in Yang-Mills theory on a Riemann surface [5, 8]. With the ζ -function regularisation we see

that there is no need to renormalise with the volume $2\pi R$. For any other regularisation the counter terms would need to be chosen so that the metric independence is maintained. At that point the definition of what one means by the Ray-Singer Torsion in terms of path integrals would include the renormalisation prescription. With this understood we may set $R = 1$ for now and in so doing we have that the action (2.109) is of the standard metric-independent Schwarz type (albeit with Grassmann-odd fields).

In the evaluation of this path integral we do not presume that ϕ or μ are time independent. We will evaluate this partition function by using a slight, but useful, variant of the trick used in [24]. In order to do so we will have to explicitly impose the conditions that the variables are periodic. To do this one incorporates the Grassmann delta functions

$$\delta(\eta(2\pi) - \eta(0)) \delta(\bar{\eta}(2\pi) - \bar{\eta}(0)) \tag{2.114}$$

in the measure.

In order to simplify the evaluation of the path integral, we now perform the change of variables

$$\eta = e^{\mp \int_0^\theta \mu(s) ds} \rho(g(\theta))^{-1} \cdot \eta', \quad \bar{\eta} \rightarrow e^{\pm \int_0^\theta \mu(s) ds} \bar{\eta}' \cdot \rho(g(\theta)) \tag{2.115}$$

where $g(\theta)$ solves the equation

$$g(\theta) D_\phi g(\theta)^{-1} = 0 \quad g(0) = 1, \tag{2.116}$$

and ρ indicates the representation defined by \mathbb{E} . Note that by (2.97) we have

$$g(\theta = 2\pi) \equiv g = \text{Pe} \int_0^{2\pi} \phi(s) ds \tag{2.117}$$

The periodicity constraints (2.114) now go over to

$$\delta\left(\eta'(0) - h^{\mp 1} \rho(g)^{-1} \cdot \eta'(2\pi)\right) \delta\left(\bar{\eta}'(0) - h^{\pm 1} \bar{\eta}'(2\pi) \rho(g)\right) \tag{2.118}$$

with $h = \exp\left(\int_0^{2\pi} \mu(s) ds\right)$ (2.97).

The point of these transformations is that in terms of the new fields the action is free

$$iS_F = \int_{S^1} d\theta \langle \bar{\eta}', i \frac{\partial}{\partial \theta} \eta' \rangle \tag{2.119}$$

while the measure goes over to $D\eta' D\bar{\eta}'$, however, with the new boundary conditions (2.118). Next we discretise the circle as in [39]. Replace the continuous variable with the distinct points $i \in [0, N]$ with separating distance between consecutive points being $2\pi/N$. Because of the delta function ‘periodicity’ conditions the path integral can be taken to be over all η'_i and $\bar{\eta}'_i$ for $i = 0, \dots, N$. The idea is to evaluate the finite dimensional discretised path integral and then take the $N \rightarrow \infty$ limit.

The choice for the discretised version of (2.119), as explained in [39], is

$$iS_F \rightarrow i \sum_{j=1}^N \langle \bar{\eta}'_j, (\eta'_j - \eta'_{j-1}) \rangle \tag{2.120}$$

Notice that $\bar{\eta}'_0$ makes no appearance in the action so that one can use the periodicity condition to remove it completely from the path integral. η'_0 does appear in the action but is determined by the boundary conditions. The integral over the $\bar{\eta}'_j$ gives a set of delta functions. Integrating over $\bar{\eta}'_1$ means we can do the η'_1 integral which sets it to η'_0 . The $\bar{\eta}'_2$ integral sets η'_2 to η'_1 which is now η'_0 and so on until we get to the last integral over $\bar{\eta}'_N$ which sets η'_N to $\eta'_{N-1} = \eta'_0$. The finally we end up with

$$i^{N \dim \mathbb{E}} \int d\eta'_0 \delta\left(\left(1 - h^{\mp 1} \rho(g)^{-1}\right) \eta'_0\right) = i^{N \dim \mathbb{E}} \text{Det}_{\mathbb{E}}\left(1 - h^{\mp 1} \rho(g)^{-1}\right) \quad (2.121)$$

We should take the $N \rightarrow \infty$ limit to arrive at the continuum. Only the prefactor changes and that can be eliminated by a finite renormalisation on appropriate choice of a in (2.113) (e.g. the regularisation used in [24] does not give rise to the $i^{N \dim \mathbb{E}}$ prefactor). That the answer depends only weakly on how one treats the circle (and is essentially independent of the discretisation) is a manifestation of the topological nature of the theory. Then we are formally left with

$$Z[\phi \pm \mu, R] = \text{Det}_{E \otimes L^{\pm 1}}\left(1 - h^{\mp 1} \rho(g)^{-1}\right) \quad (2.122)$$

as the partition function. We have thus rederived the result (2.96) directly from the path integral.

3 Massive Ray-Singer torsion on $N \times S^1$

Here we draw together several of the ingredients that have, separately, appeared in the previous section: the field theoretic realisation of the Ray-Singer Torsion, the massive deformation of the Ray-Singer Torsion, and the interpretation of the mass term (on S^1) as a flat background gauge field. We combine these by coupling the standard Schwarz-type action (2.32) for the Ray-Singer Torsion to a flat \mathbb{R}_+ gauge field, and then studying situations in which this gauge theory can indeed be regarded as a massive deformation (and regularisation) of the original theory. Manifolds on which this is the case include $M = N \times S^1$ (studied in this section), and mapping tori $M = N_f$ (which we will look at in section 4). We show that on such manifolds a purely algebraic gauge condition is available which greatly simplifies the evaluation of the partition function and (essentially) reduces it to the calculation of the massive Ray-Singer Torsion on S^1 , already discussed in detail in the previous section.

The field theory approach allows us to quantise in a way that from the outset does not depend on the details of the product metric that can be put on the manifold and allow us to obtain the product formula (that is also derived through traditional means in the appendix). As a side benefit we also obtain a deeper understanding of why in the Abelianisation programme a connection that does not appear to be flat appears as the argument of the Ray-Singer Torsion. We include a derivation of the Schwarz type field theories directly from a path integral that involves only Laplacians (no Dirac type operators) that obviously represent the torsion.

3.1 The classical action and its symmetries

As just explained, the action of interest is now the action (2.32) for the fields \mathcal{B} and \mathcal{C} coupled to a flat background gauge field A , but with A now extended to include an \mathbb{R}_+ connection. Thus the action is

$$S_M(A) = \int_M \langle \mathcal{B}, d_A \mathcal{C} \rangle \tag{3.1}$$

for the fields with values in $\mathbb{E} \otimes \mathbb{L}^{\pm 1}$ with \mathbb{E} a vector bundle and \mathbb{L} a real line bundle over M . Specifically we choose $\mathcal{B} \in \Omega^p(M, \mathbb{E} \otimes \mathbb{L})$, $\mathcal{C} \in \Omega^{n-p-1}(M, \mathbb{E} \otimes \mathbb{L}^{-1})$, for $0 \leq p \leq n-1$ where $\dim_{\mathbb{R}} M = n$. Moreover, A is now a flat $G \times \mathbb{R}_+$ connection for some (compact) gauge group G , which splits into a $\mathfrak{g} \oplus \mathbb{R}$ part as

$$A = A^{\mathfrak{g}} + A^{\mathbb{R}}, \tag{3.2}$$

and the action of \mathbb{R}_+ on the fields is

$$A \rightarrow A + d\alpha, \quad \mathcal{B} \rightarrow e^{\alpha} \cdot \mathcal{B}, \quad \mathcal{C} \rightarrow e^{-\alpha} \cdot \mathcal{C}, \quad \alpha : M \rightarrow \mathbb{R} \tag{3.3}$$

Flatness of the extended connection A implies that the action then still has the shift symmetry (2.33),

$$\delta \mathcal{B} = d_A \Sigma, \quad \delta \mathcal{C} = d_A \Lambda \tag{3.4}$$

Thus, provided that one has a non-trivial flat connection $A = A^{\mathfrak{g}} + A^{\mathbb{R}}$ (both parts need to be flat separately), the action has the required shift gauge invariance. Moreover, by construction, the (covariantly) gauge fixed partition function will formally calculate the Ray-Singer Torsion $\tau_{S^1}(M, A^{\mathfrak{g}} + A^{\mathbb{R}}, \mathbb{E} \otimes \mathbb{L})$.

However, while on the circle S^1 , the non-trivial flat connection $A^{\mathbb{R}} = \mu d\theta$ was enough to regularise the theory by providing a mass term that lifts the zero modes of the kinetic term, on more general manifolds a non-trivial flat connection $A^{\mathbb{R}}$ will in general not be sufficient to regularise the theory.

We will not try to give a complete answer to the question, under which conditions the coupling to a flat \mathbb{R}_+ -connection is sufficient to ensure that all the zero modes of the theory are lifted. However, we can easily identify two necessary conditions, and we can then also identify situations in which we see (by explicit calculation) that these conditions are also sufficient:

1. First of all, there are no global topological issues associated with an \mathbb{R}_+ -bundle (with its contractible structure group). Thus we can globally identify the gauge field $A^{\mathbb{R}}$ with a 1-form on M . Flatness of $A^{\mathbb{R}}$ is then simply the statement that $A^{\mathbb{R}}$ is closed, $dA^{\mathbb{R}} = 0$. In order for such an $A^{\mathbb{R}}$ to not be globally gauge equivalent to $A^{\mathbb{R}} = 0$ (which would not be helpful), $A^{\mathbb{R}}$ must be closed but not exact. Thus our first necessary condition is that

$$H^1(M, \mathbb{R}) \neq 0. \tag{3.5}$$

2. Secondly, we would only expect $A^{\mathbb{R}}$ to act like a mass term that can lift all the zero modes if it is nowhere vanishing. By duality (using any metric g_M on M), this requires M to admit a nowhere vanishing vector field, and a necessary condition for this is that the Euler characteristic of M vanishes,

$$e(M) = 0. \tag{3.6}$$

In the following we will look at two classes of manifolds where these two conditions are satisfied, namely product manifolds $M = N \times S^1$ (in this section) and finite order mapping tori $M = N_f$, which are non-trivial fibrations over S^1 (in section 4 below). In both cases, one can readily identify a non-vanishing vector field (∂_θ or ∂_t respectively) or appeal to the multiplicativity of the Euler characteristic for fibrations to conclude

$$M = N \times S^1 \quad \text{or} \quad M = N_f \quad \Rightarrow \quad e(M) = 0. \tag{3.7}$$

Moreover, in both cases, $d\theta$ or dt respectively represents a non-trivial element of $H^1(M, \mathbb{R})$.

3.2 Decomposition of fields on $N \times S^1$

We now consider the $G \times \mathbb{R}_+$ theory described by the classical action (3.1) on a manifold of the form $M = N \times S^1$. Given the product structure of the manifold, it will be convenient to decompose all the fields and gauge parameters accordingly.

- The Connection and the Covariant Derivative

On $M = N \times S^1$ we can further split the connection $A = A^{\mathfrak{g}} + A^{\mathbb{R}}$ as

$$A^{\mathfrak{g}} = A_N^{\mathfrak{g}} + \phi d\theta, \quad A^{\mathbb{R}} = A_N^{\mathbb{R}} + \mu d\theta, \quad A = A_N + (\phi + \mu)d\theta \tag{3.8}$$

where $\phi \in \mathfrak{g}$ is anti-Hermitian and μ is real. Correspondingly, the covariant derivative splits as

$$d_A = d_N + A_N + d\theta(\partial_\theta + \phi + \mu) \equiv d_{A_N} + d\theta D_{\phi+\mu}. \tag{3.9}$$

The operator $D_{\phi+\mu}$ will appear repeatedly in the following, and we will therefore abbreviate it to

$$D \equiv D_{\phi+\mu}. \tag{3.10}$$

- The Laplace Operator

Also the massive Laplacian with respect to the product metric $g_M = g_N + R^2 d\theta \otimes d\theta$ splits neatly as

$$\Delta_A^M + m^2 = \Delta_A^N + R^{-2}(iD_{\phi+\mu})(iD_{\phi-\mu}) \tag{3.11}$$

where we have once again used the identification $\mu = mR$ (2.80).

- The Fields \mathcal{B} and \mathcal{C}

We can also decompose the fields \mathcal{B} and \mathcal{C} on $M = N \times S^1$ as

$$\mathcal{B}^p = B^p + B^{p-1}d\theta, \quad \mathcal{C}^{n-p-1} = C^{n-p-1} + C^{n-p-2}d\theta \tag{3.12}$$

The fields that appear on the right hand side of these equations are all horizontal

$$\iota_\xi B^p = 0, \quad \iota_\xi B^{p-1} = 0, \quad \iota_\xi C^{n-p-1} = 0, \quad \iota_\xi C^{n-p-2} = 0 \quad (3.13)$$

with respect to the S^1 vector field ξ (normalised so that $\iota_\xi d\theta = 1$). This means that they are forms of type $\Omega^\bullet(N, \mathbb{E} \otimes \mathbb{L}^{\pm 1}) \otimes \Omega^0(S^1)$.

- The Action

With respect to this decomposition, the action (3.1) takes the form

$$S_M(A) = (-1)^p \int_M d\theta \langle B^p D C^q - B^{p-1} d_{A_N} C^q + (-1)^q B^p d_{A_N} C^{q-1} \rangle \quad (3.14)$$

with $q = n - p - 1$.

- Decomposition of all other Fields

All fields Φ other than \mathcal{B} and \mathcal{C} will be decomposed as follows

$$\Phi^k = \varphi^k + \Phi_{k-1} d\theta \quad (3.15)$$

Capital letter for the starting field Φ^k of degree k (exhibited as a superscript), small letter for the component φ^k which is a k -form on N (exhibited as a superscript) and a capital letter for the component of the form of mixed degree Φ_{k-1} which is a $k - 1$ -form on N (exhibited as a subscript).

- The Shift Symmetry Parameters

With the previous notation in hand, the parameters $\Lambda \in \Omega^{n-p-2}(M, \mathbb{E} \otimes \mathbb{L}^{-1})$ and $\Sigma \in \Omega^{p-1}(M, \mathbb{E} \otimes \mathbb{L})$ appearing in the shift symmetry (3.4) can also be decomposed

$$\Sigma^{p-1} = \sigma^{p-1} + \Sigma_{p-2} d\theta, \quad \Lambda^{q-1} = \lambda^{q-1} + \Lambda_{q-2} d\theta \quad (3.16)$$

so that the symmetry reads

$$\begin{aligned} \delta B^p &= d_A^N \sigma^{p-1}, & \delta B^{p-1} &= d_A^N \Sigma_{p-2} + (-1)^{p-1} D \sigma^{p-1} \\ \delta C^q &= d_A^N \lambda^{q-1}, & \delta C^{q-1} &= d_A^N \Lambda_{q-2} + (-1)^{q-1} D \lambda^{q-1} \end{aligned} \quad (3.17)$$

In general there are also ghost for ghosts coming from the fact that the symmetry (3.4) is ‘reducible’ namely, as $d_A^2 = 0$ it does not change under

$$\Sigma^{p-1} \rightarrow \Sigma^{p-1} + d_A \Sigma^{p-2}, \quad \Lambda^{q-1} \rightarrow \Lambda^{q-1} + d_A \Lambda^{q-2} \quad (3.18)$$

and so on. Under the decomposition

$$\Sigma^{p-2} = \sigma^{p-2} + \Sigma_{p-3} d\theta, \quad \Lambda^{q-2} = \lambda^{q-2} + \Lambda_{q-3} d\theta \quad (3.19)$$

we have

$$\sigma^{p-1} \rightarrow \sigma^{p-1} + d_A^N \sigma^{p-2}, \quad \Sigma_{p-2} \rightarrow \Sigma_{p-2} + d_A^N \Sigma_{p-3} + (-1)^p D \sigma^{p-2} \quad (3.20)$$

and a similar expression for Λ . One keeps getting gauge symmetries of gauge symmetries until the last variation is of the form $d_A \Sigma^0$ and, as it is a zero-form, in the above notation one has $\Sigma^0 = \sigma^0$. For B^p there is a total nesting of p symmetries (and hence p parameters) while for C^q there is a total nesting of q symmetries.

3.3 Algebraic gauge fixing conditions

As mentioned before, on any manifold M one can gauge fix the shift symmetries (2.33) or (3.4) of the action by imposing the usual covariant gauge conditions (2.34),

$$d_A * \mathcal{B} = 0, \quad d_A * \mathcal{C} = 0 \tag{3.21}$$

on the fields \mathcal{B} and \mathcal{C} .

However, it turns out that with the extra structure afforded to us by having a preferred direction and an \mathbb{R}_+ connection (mass term) on $M = N \times S^1$, an algebraic gauge condition is available: namely one may simply set the S^1 -components B^{p-1} and C^{q-1} of the fields \mathcal{B} and \mathcal{C} to zero,

$$B^{p-1} = 0, \quad \text{and} \quad C^{q-1} = 0 \tag{3.22}$$

Before turning to a proof of this statement, let us comment on some implications of this result:

- The great advantage of such a gauge choice is that the action (3.14) simplifies immensely; indeed evidently it reduces to the simple action

$$S_M(A) = (-1)^p \int_M d\theta \langle B^p, DC^q \rangle \tag{3.23}$$

- The same argument that establishes (3.22) can also be used to set the S^1 -components of the gauge (and gauge for gauge) parameters, namely all the Σ_k and Λ_k , to zero. This also leads to a tremendous simplification of the full quantum action (see the examples and the discussion below), and of the calculation of the partition function.
- Moreover, the gauge condition (3.22) is manifestly metric independent, something that should not be underrated in the context of topological field theories.

Let us now turn to the gauge conditions (3.22) themselves. These conditions are of course in some sense the higher degree form analogues of the temporal gauge $A_0 = 0$ in QED or QCD. However, it is important to also keep in mind the differences: for a standard gauge field A with gauge transformation $A \rightarrow A^g = g^{-1} A g + g^{-1} dg$, this temporal gauge is available on any manifold of the form $M = N \times \mathbb{R}$. However, in general, on a manifold of the form $M = N \times S^1$, this gauge condition cannot be imposed (the obstruction being the possible non-trivial holonomy of A along the S^1), and the “time”-dependent gauge transformations can only be used to impose the weaker condition $\partial_0 A_0 = 0$.

What we will now show is that for a field \mathcal{B} transforming with what we have called the shift symmetry $\delta \mathcal{B} = d_A \Sigma$ (rather than like a connection), we *can* impose such a temporal gauge even on $M = N \times S^1$, provided that the S^1 -component D of the operator d_A is invertible. In the case at hand this is accomplished by the mass term or \mathbb{R}_+ -connection — after all, this was the purpose of introducing the mass term in the first place (to lift the zero modes).

Recall that under the shift symmetry, the S^1 -component B^{p-1} of \mathcal{B} transforms as (3.17)

$$\delta B^{p-1} = d_A^N \Sigma_{p-2} + (-1)^{p-1} D \sigma^{p-1} \tag{3.24}$$

We will now show that we can use just the second term (Σ_{p-2} is not required) to achieve $\delta B^{p-1} = -B^{p-1}$. Then for the transformed field one has $B^{p-1} + \delta B^{p-1} = 0$, as desired. This means that we have to solve the equation

$$(-1)^p B^{p-1} = D\sigma^{p-1}. \tag{3.25}$$

We will construct the explicit solution for σ^{p-1} below, but we can see immediately that a necessary (and sufficient) condition for a solution to exist is that the operator D is invertible, i.e. has no zero modes (with the appropriate periodic boundary conditions). Recalling that the determinant of iD is the Ray-Singer Torsion $\tau_{S^1}(\phi + \mu, \mathbb{E})$, we see that this gauge choice is available precisely when the massive Ray-Singer Torsion is non-zero.

In order to solve (3.25), the first step is to write the operator D as

$$D = \partial_\theta + \phi + \mu = k(\theta)^{-1} \partial_\theta k(\theta) \tag{3.26}$$

with

$$k(\theta) = \text{P exp} \left(\int_0^\theta (\phi + \mu)(s) ds \right), \quad k(2\pi) \equiv k = hg \tag{3.27}$$

the path ordered exponential of $\phi + \mu$, with h and g the previously introduced holonomies of μ and ϕ respectively (2.97). We shall also write

$$\rho(k) = h\rho(g). \tag{3.28}$$

Then one finds by straightforward integration that the solution is

$$\sigma^{p-1} = \rho(k(\theta))^{-1} \left[\int_0^\theta \rho(k(s)) (-1)^p B^{p-1}(s) ds + \gamma \right] \tag{3.29}$$

where the integration constant matrix γ is determined by the requirement of periodicity of σ^{p-1} on S^1 to be

$$\gamma = (-1)^p (h\rho(g) - 1)^{-1} \int_0^{2\pi} \rho(k(s)) B^{p-1}(s) ds \tag{3.30}$$

Note that it is here that the requirement that D be invertible enters, through the equivalent condition that $\text{Det}_{E \otimes L} (1 - \rho(k)^{-1}) \neq 0$, which is, as anticipated above, precisely the condition that the massive Ray-Singer Torsion on the circle does not vanish,

$$\text{Det}_{E \otimes L} (1 - \rho(k)^{-1}) \neq 0 \quad \Leftrightarrow \quad \tau_{S^1}(\phi + \mu, \mathbb{E}) \neq 0. \tag{3.31}$$

In any case, as announced, we have now established that we can then achieve the gauge fixing conditions (3.22).

As mentioned above, we have not used $\Sigma_{p-2} d\theta$ thus far; however, by the second symmetry (3.20) we can choose σ^{p-2} to set $\Sigma_{p-2} = 0$ (as we now know that for such a shift symmetry this gauge choice is achievable). If $p = 2$ we have finished but if $p \geq 3$ then we have not made use of $\Sigma_{p-3} d\theta$ but fortunately we have a next level of gauge symmetry and we can use the parameter of the next level down with degree $(p - 3)$ to set $\Sigma_{p-3} = 0$ and so

on all the way until the gauge parameter has degree 0 — which means that there is no $d\theta$ component and so nothing to gauge fix to zero.

We can summarise this by saying that we can use (and only need) the shift gauge symmetry parameters σ^k with $k = 0, \dots, p - 1$, to set to zero the $d\theta$ -component B^{p-1} of \mathcal{B}^p and the $d\theta$ -components Σ_k of the shift gauge symmetry parameters themselves,

$$\{\sigma^k, k = 0, \dots, p - 1\} \Rightarrow B^{p-1} = 0, \quad \Sigma_k = 0 \quad k = 0, \dots, p - 2. \quad (3.32)$$

3.4 Path integral derivation of the massive Ray-Singer torsion

In this section we will explicitly evaluate the partition function of the field theory models for the massive Ray-Singer Torsion introduced above in the algebraic gauge. As a warm-up, and as it is already quite instructive in its own right, we first discuss two 3-dimensional examples, in order to illustrate how

- the same ratio of determinants (of Laplacians) emerges from the usual covariant gauge (involving derivatives of the fields) and the algebraic gauge;
- in the algebraic gauge one can work with a significantly reduced field content compared with the covariant gauge (which requires the full field content of ghosts, anti-ghosts and ghosts for ghosts and their multipliers, as specified by the Batalin-Vilkovisky (BV) triangle [36]).

3.4.1 3-dimensional examples: covariant vs algebraic gauge

We consider the action (3.1) on a 3-manifold of the form $M = N \times S^1 \equiv \Sigma \times S^1$, with the form degree (p, q) of $(\mathcal{B}, \mathcal{C})$ chosen to be $(p, q) = (1, 1)$ and $(p, q) = (2, 0)$ respectively. The first case is irreducible and in a sense the square of Abelian (or 1-loop) Chern-Simons theory [40], while the second case, with \mathcal{B} a 2-form, is reducible.

Example 3.1. *The classical action is*

$$S_M(A) = \int_M \langle \mathcal{B}^1, d_A \mathcal{C}^1 \rangle$$

The shift gauge transformation $\delta \mathcal{B}^1 = d_A \Sigma^0$ (3.4) translates into the BRST-symmetry $Q \mathcal{B}^1 = d_A \Omega^0$ (and analogously for the 1-form \mathcal{C}^1). The complete BRST algebra for the fields \mathcal{B}^1 and \mathcal{C}^1 is then (regardless of the gauge fixing conditions)

$$\begin{aligned} Q \mathcal{B}^1 &= d_A \Omega^0, & Q \Omega^0 &= 0, & Q \bar{\Omega}^0 &= \Pi^0, & Q \Pi^0 &= 0, & Q^2 &= 0 \\ Q \mathcal{C}^1 &= d_A \Psi^0, & Q \Psi^0 &= 0, & Q \bar{\Psi}^0 &= \Phi^0, & Q \Phi^0 &= 0, & Q^2 &= 0 \end{aligned}$$

The usual covariant gauge fixing condition (3.21) is imposed by adding to the classical action $Q \int_M \langle \bar{\Omega}^0, d_A * \mathcal{B}^1 \rangle + Q \int_M \langle \bar{\Psi}^0, d_A * \mathcal{C}^1 \rangle$, so that the complete quantum action is

$$\int_M \left(\langle \mathcal{B}^1, d_A \mathcal{C}^1 \rangle + \langle \Pi^0, d_A * \mathcal{B}^1 \rangle + \langle \bar{\Omega}^0, d_A * d_A \Omega^0 \rangle + \langle \Phi^0, d_A * \mathcal{C}^1 \rangle + \langle \bar{\Psi}^0, d_A * d_A \Psi^0 \rangle \right)$$

The integral over the Grassmann-odd ghosts and anti-ghosts gives a $(\text{Det}_{\Omega^0(M, \mathbb{E} \otimes \mathbb{L})} \Delta_A)^2$, while the integral over the Grassmann-even fields $\mathcal{B}, \mathcal{C}, \Pi^0, \Phi^0$ gives the determinant of the operator $L_A = *d_A + d_{A*}$ on odd forms (say). By squaring this operator and taking the square root (unlike in [40] there is no phase to worry about because of the doubled field content and on including the complex conjugate field set if there is one), and using Hodge duality, one finally finds that the partition function is precisely equal to the Analytic Torsion,

$$Z_M[A] = \left(\text{Det}_{\Omega^0(M, \mathbb{E} \otimes \mathbb{L})} \Delta_A \right)^{3/2} \left(\text{Det}_{\Omega^1(M, \mathbb{E} \otimes \mathbb{L})} \Delta_A \right)^{-1/2} = \tau_M(A, \mathbb{E} \otimes \mathbb{L}),$$

in agreement with (2.36). On the product manifold $M = \Sigma \times S^1$, one could now use the product formula (2.70) and the triviality of the Analytic Torsion on Σ to deduce that

$$Z_{\Sigma \times S^1}[A] = \tau_{S^1}(\phi + \mu, \mathbb{E})^{e(\Sigma)}.$$

We will now show how to obtain this formula directly by evaluating the path integral on $\Sigma \times S^1$ in the algebraic gauge $B^0 = C^0 = 0$ (3.22), where $\mathcal{B}^1 = B^1 + B^0 d\theta$, $\mathcal{C}^1 = C^1 + C^0 d\theta$ (and we recall that for 0-form fields on $\Sigma \times S^1$ we have $\Omega^0 = \omega^0$ etc.). These algebraic conditions are thus imposed by adding to the classical action $Q \int_M *(B^0, \bar{\omega}^0) + Q \int_M *(C^0, \bar{\psi}^0)$, leading to the complete action

$$S_M(A) + \int_M * \left(\langle B^0, \pi^0 \rangle + \langle D\omega^0, \bar{\omega}^0 \rangle + \langle C^0, \phi^0 \rangle + \langle D\psi^0, \bar{\psi}^0 \rangle \right)$$

This clearly sets $B^0 = C^0 = 0$ and integrating out the ghosts gives rise to the square of the determinant $\text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L}) \otimes \Omega^0(\Sigma)}(iD)$. The classical action, on the other hand, reduces to $\int_M d\theta B^1 DC^1$ (3.23), and thus gives rise to the inverse of the determinant $\text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L}) \otimes \Omega^1(\Sigma)}(iD)$. Putting all the pieces together we get

$$Z_M[A] = \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})}(iD)^{2 \dim \Omega^0(\Sigma) - \dim \Omega^1(\Sigma)}$$

At a heuristic level, one has (from the Hodge decomposition)

$$2 \dim \Omega^0(\Sigma) - \dim \Omega^1(\Sigma) = 2 \dim H^0(\Sigma) - \dim H^1(\Sigma) = e(\Sigma),$$

so that

$$Z_{\Sigma \times S^1}[A] = \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})}(iD)^{e(\Sigma)} = \tau_{S^1}(\phi + \mu, \mathbb{E})^{e(\Sigma)}$$

A correct account requires a regularisation of the objects involved as given in [9, 24]. We will come back to this in section 3.4.2 below. In any case, we have seen that (modulo this proviso) in the algebraic gauge we have reproduced, on the nose, the result obtained before from combining the calculation in the covariant gauge with the product theorem for the Analytic Torsion.

We now look at an example that illustrates the issues that arise when the theory is reducible.

Example 3.2. Once more let $M = \Sigma \times S^1$ but with and $(p, q) = (2, 0)$ in which case \mathcal{B} is a 2-form while \mathcal{C} is a 0-form. The BRST algebra for the field \mathcal{B} is

$$Q\mathcal{B}^2 = d_A \Omega^1, \quad Q\Omega^1 = d_A \Omega^0, \quad Q\bar{\Omega}^1 = \Pi^1, \quad Q\bar{\Omega}^0 = \Pi^0, \quad Q\gamma^0 = \tau^0 \quad Q^2 = 0$$

and which holds once more irrespective of the gauge fixing condition. There are no transformations for \mathcal{C}^0 . Here Ω^0 is the ghost for ghost, required due to the reducibility of the transformation $Q\mathcal{B}^2 = d_A\Omega^1$, and $\bar{\Omega}^0$ is its anti-ghost. The fields (τ^0, γ^0) are the extra ghost and its multiplier, required in general in the presence of this reducible symmetry, with τ^0 completing the BV triangle:

$$\begin{array}{ccc} & \mathcal{B}^2 & \\ & \bar{\Omega}^1 & \Omega^1 \\ \tau^0 & \bar{\Omega}^0 & \Omega^0 \end{array}$$

We will see now that (and why) τ^0 is indeed required in the covariant gauge, while it is not required in the algebraic gauge. This generalises to the higher-dimensional theories and higher rank forms and will significantly simplify the calculation of the partition function in section 3.4.2 below.

In the covariant gauge, in order to gauge fix the shift gauge symmetry of the field \mathcal{B}^2 and its ghost Ω^1 , one can add to the action the terms $Q \int_M \langle \bar{\Omega}^1, d_A * \mathcal{B}^2 \rangle + Q \int_M \langle \bar{\Omega}^0, d_A * \Omega^1 \rangle$. However, one now sees that the multiplier Π^1 , which only appears in the combination $\int_M \Pi^1 d_A * \mathcal{B}^2$, has its own gauge invariance (as the d_A -exact part of Π^1 does not appear in the action). This is the rationale for the necessity of the extra ghost τ^0 and its multiplier γ^0 in the covariant gauge, as these allow one to gauge fix Π^1 (and its multiplier $\bar{\Omega}^1$), by adding a third covariant gauge fixing term, namely

$$Q \int_M \langle \gamma^0, d_A * \bar{\Omega}^1 \rangle = \int_M \langle \tau^0, d_A * \bar{\Omega}^1 \rangle + \int_M \langle \gamma^0, d_A * \Pi^1 \rangle.$$

The resulting partition function is now well defined and can be evaluated much like in the previous example, with the result

$$Z_M[A] = (\text{Det}_{\Omega^0(M, \mathbb{E} \otimes \mathbb{L})} \Delta_A)^{-3/2} (\text{Det}_{\Omega^1(M, \mathbb{E} \otimes \mathbb{L})} \Delta_A)^{+1/2} = \tau_M(A, \mathbb{E} \otimes \mathbb{L})^{-1},$$

once again in agreement with (2.36). Using again the product formula, on $M = \Sigma \times S^1$ this is equal to

$$Z_{\Sigma \times S^1}[A] = \tau_{S^1}(\phi + \mu, \mathbb{E})^{-e(\Sigma)}.$$

Working now directly on $\Sigma \times S^1$ instead, recall that we have the decompositions $\mathcal{B}^2 = B^2 + B^1 d\theta$, $\Omega^1 = \omega^1 + \Omega_0 d\theta$, $\Pi^1 = \pi^1 + \Pi_0 d\theta$ etc., as well as $\Omega^0 = \omega^0$ etc. for 0-form fields. The algebraic gauge conditions on the fields \mathcal{B}^2 and its ghost Ω^1 are the conditions $B^1 = 0, \Omega_0 = 0$, which can be imposed by adding $Q \int_M (\langle B^1, *\bar{\Omega}^1 \rangle + \langle \Omega_0, *\bar{\Omega}^0 \rangle)$ to the action. However, because B^1 is a 1-form on Σ , $\langle B^1, *\bar{\Omega}^1 \rangle = \langle B^1, *\bar{\omega}^1 \rangle$, and therefore at this stage neither $\bar{\omega}_0$ nor its multiplier π_0 will appear anywhere in the action. If one insists on using the full BV field content, then one can once again use the ghost for ghost τ^0 and its multiplier to set these fields to zero (algebraically), by adding $Q \int_M \gamma^0 * \bar{\Omega}_0$. Using also that

$$Q\mathcal{B}^2 = d_A\Omega^1 \quad \Rightarrow \quad QB^1 = d_A^\Sigma \Omega_0 + D\omega^1,$$

one then finds that the terms that need to be added to the classical action to arrive at a well-defined partition function are

$$\begin{aligned} & \int_M Q \left(\langle B^1, *\bar{\omega}^1 \rangle + \langle \gamma^0, *\bar{\Omega}_0 \rangle + \langle \Omega_0, *\bar{\omega}^0 \rangle \right) \\ &= \int_M \left(\langle B^1, *\pi^1 \rangle + \langle (d_A^\Sigma \Omega_0 + D\omega^1), *\bar{\omega}^1 \rangle + \langle \tau^0, *\bar{\Omega}_0 \rangle + \langle \gamma^0, *\Pi_0 \rangle + \langle \Omega_0, *\pi^0 \rangle + \langle D\omega^0, *\bar{\omega}^0 \rangle \right) \end{aligned}$$

After integrating out all the multiplier fields imposing the algebraic gauge conditions, one is then (including the classical action) left with

$$\int_M d\theta \langle B^2, DC^0 \rangle + \int_M \langle D\omega^1, *\bar{\omega}^1 \rangle + \int_M \langle D\omega^0, *\bar{\omega}^0 \rangle$$

This immediately leads to

$$Z_{\Sigma \times S^1}[A] = \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})} (D)^{-\dim \Omega^2(\Sigma) + \dim \Omega^1(\Sigma) - \dim \Omega^0(\Sigma)} = \tau_{S^1}(\phi + \mu, \mathbb{E})^{-e(\Sigma)}$$

(using the same heuristic reasoning as before), in agreement with the result found in the covariant gauge.

While this is somewhat satisfactory, it is also clear from the derivation that the procedure we have followed here in the second example (with its full field content as dictated by the BV triangle) is overly baroque and complicated. Indeed, if only the field π^1 in the decomposition $\Pi^1 = \pi^1 + \pi_0 d\theta$ of the multiplier field is required to impose the algebraic gauge condition $B^1 = 0$, then there is no reason to introduce π_0 in the first place. And if one does not introduce π_0 , then one does also not need a multiplier γ^0 (and its extra ghost) to set it (and its anti-ghost) to zero. The upshot is that if one simply does not introduce those fields, one does not need the extra ghost.

3.4.2 General calculation in the algebraic gauge

We will now show that this kind of reasoning can be extended to arbitrary $(p, q = n - p - 1)$, where it leads to a rather more significant simplification of the calculation of the partition function than in the above (after all barely non-trivial) 3-dimensional example.

For reducible gauge systems (displaying the ghost for ghost and extra ghost phenomena), the Batalin-Vilkovisky triangle [36] has a right edge which includes the classical field to be quantised as well as the ghost terms. All the other fields in the triangle are the anti-ghosts and extra ghosts. Each of the anti-ghosts and extra ghosts have matching multiplier fields. In the case at hand, the $\mathcal{B}^p - \mathcal{C}^q$ system, we would have BV triangles of height $p + 1$ and $q + 1$ respectively, with a corresponding explosion in the number of fields.

However, for an algebraic gauge condition the only place from which a non-trivial determinant (involving derivatives of the fields) can arise is from the classical action and from the terms involving the BRST variations of the classical fields and the ghosts on the right edge of the triangle. These only couple to the anti-ghosts and their multiplier fields. Thus only the fields that appear on the first and second right edges of the triangle (and

their multipliers) are required, as indicated here for the field \mathcal{B}^p :

$$\begin{array}{ccc}
 & \mathcal{B}^p & \\
 (\bar{\Omega}^{p-1}, \Pi^{p-1}) & & \Omega^{p-1} \\
 & (\bar{\Omega}^{p-2}, \Pi^{p-2}) & \Omega^{p-2} \\
 & \dots & \dots
 \end{array}$$

Concretely, since we just need the π^i -components of the multiplier fields $\Pi^i = \pi^i + \Pi_{i-1}$ to impose the algebraic gauge conditions $B^{p-1} = 0, \Omega_i = 0$, the reduced set of fields is the minimal one required for the gauge fixing as indicated in (3.32). The BRST transformations on this reduced set of fields in the \mathcal{B}^p -sector are simply

$$Q\Omega_i = D\omega^i, \quad Q\omega^i = 0, \quad Q\bar{\omega}^i = \pi^i, \quad Q\pi^i = 0, \quad Q^2 = 0 \quad (3.33)$$

with $\Omega_{p-1} = B^{p-1}$ and $i = 0, \dots, p-1$. The simplified set of gauge fixing conditions are

$$\sum_{i=0}^{p-1} \int_M Q \langle \Omega_i, * \bar{\omega}^i \rangle = \sum_{i=0}^{p-1} \int_M \left(\langle \Omega_i, * \pi^i \rangle + \langle D\omega^i, * \bar{\omega}^i \rangle \right) \quad (3.34)$$

From the various fields we get the following determinants:

$$\begin{aligned}
 (B^p, C^{n-p-1}) &\Rightarrow \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})} (D)^{-\dim \Omega^p(N)} \\
 \mathcal{B} \text{ ghosts} &\Rightarrow \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})} (D)^{\sum_{i=0}^{p-1} (-1)^{p+1+i} \dim \Omega^i(N)} \\
 \mathcal{C} \text{ ghosts} &\Rightarrow \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})} (D)^{\sum_{i=0}^{n-p-2} (-1)^{n-p+i} \dim \Omega^i(N)}
 \end{aligned} \quad (3.35)$$

(we have assumed that both \mathcal{B} and \mathcal{C} are Grassmann even — otherwise one would have to simply invert this result). One can use duality to set $\sum_{i=0}^{n-p-2} (-1)^{n-p+i} \dim \Omega^i(N) = \sum_{i=p+1}^{n-1} (-1)^{p+1+i} \dim \Omega^i(N)$, so that the product of the determinants is the partition function

$$Z_{\Sigma \times S^1}[A] = \text{Det}_{\Omega^0(S^1, \mathbb{E} \otimes \mathbb{L})} (D)^{(-1)^{p+1} \sum_{i=0}^{n-1} (-1)^i \dim \Omega^i(N)} \quad (3.36)$$

As before, formally the exponent is (up to a sign) the Euler characteristic $e(N)$ of N .

We now explain how meaning is given to these formulae. Clearly one must regularise the fact that one is dealing with all forms on N . The McKean-Singer formula [41] uses a Heat Kernel regularisation for these objects. Namely one writes [24]

$$\ln \left(\tau_{S^1}(\phi + \mu, \mathbb{E})^{\sum_{i=0}^{n-1} (-1)^i \dim \Omega^i(N)} \right) = \lim_{\epsilon \rightarrow 0} \sum_i \text{Tr}_{\Omega^i(N)} (-1)^i \ln \tau_{S^1}(\phi + \mu, \mathbb{E}) e^{-\epsilon \Delta_A}$$

We use the obvious (position space) notation for the functional trace,

$$\text{Tr} \ln \tau_{S^1}(\phi + \mu, \mathbb{E}) e^{-\epsilon \Delta_A} = \int_N \ln \tau_{S^1}(\phi + \mu, \mathbb{E})(x) \langle x | e^{-\epsilon \Delta_A} | x \rangle \quad (3.37)$$

The coefficients arising in the expansion of the heat kernel are known to be local and gauge invariant, and since $F_A = 0$ we can replace $\Delta_A \rightarrow \Delta$ in the above calculation. We can then

make use of the fact that the index density of the deRham operator is the Pfaffian of the Riemann curvature tensor (see e.g. equation (1) in [42]),

$$\lim_{\epsilon \rightarrow 0} \sum_i (-1)^i \langle x | e^{-\epsilon \Delta} | x \rangle = \text{Pf}(R) \tag{3.38}$$

Thinking of $\text{Pf}(R)$ as the Pfaffian of the $\mathfrak{so}(n-1)$ -valued Riemann curvature 2-form of N (thought of as a top-form on N), we can write

$$\lim_{\epsilon \rightarrow 0} \sum_i \text{Tr}_{\Omega^i(N)} (-1)^i \ln \tau_{S^1}(\phi + \mu, \mathbb{E}) e^{-\epsilon \Delta A} = \int_N \ln \tau_{S^1}(\phi + \mu, \mathbb{E}) \text{Pf}(R) \tag{3.39}$$

The partition function is, therefore,

$$Z_{\Sigma \times S^1}[A] = \exp \left((-1)^{p+1} \int_N \ln \tau_{S^1}(\phi + \mu, \mathbb{E}) \text{Pf}(R) \right) \tag{3.40}$$

In case that $\phi + \mu$ are constant on N we have

$$Z_{\Sigma \times S^1}[A] = \tau_{S^1}(\phi + \mu, \mathbb{E})^{(-1)^{p+1}e(N)} \tag{3.41}$$

justifying our rather cavalier formulae in the examples above.

3.5 Remarks and observations

3.5.1 Independence of the connections on N

Notice that the result for the Analytic Torsion does not depend explicitly on the component of the flat connection A_N on N . This rather surprising result also is a generalisation of the results of Ray and Singer. In order to see that it is not obvious that the component of the connection on N does not appear we take a quick look at the flat connections on $N \times I$ and those that are actually flat on $N \times S^1$ to show that there is no a priori reason to ignore A_N .

The flat connections on $N \times I$, with the interval $I = [0, 2\pi]$, satisfy

$$F_A^N = 0, \quad \dot{A}_N = d_A^N \phi \tag{3.42}$$

where the superscript N indicates that the exterior derivative and connection are those on N . The first equation says that A_N must be a flat connection on N for all ‘time’ θ while the second says that the time evolution of A_N is given by a gauge transformation parameterised by ϕ . The evolution equation is solved by

$$A_N(x, \theta) = g^{-1}(x, \theta) A_N(x, 0) g(x, \theta) + g^{-1}(x, \theta) d^N g(x, \theta) \tag{3.43}$$

where

$$g(x, \theta) = \text{P exp} \left(\int_0^\theta \phi(x, s) ds \right) \tag{3.44}$$

and to satisfy the first equation $A_N(x, 0)$ is a flat connection on N .

One passes to flat connections on $N \times S^1$ by imposing the periodicity conditions

$$A_N(x, 2\pi) = A_N(x, 0), \quad \phi(x, 2\pi) = \phi(x, 0) \tag{3.45}$$

This gives us, from (3.43), that $A_N(x, 0)$ satisfies

$$A_N(x, 0) = g^{-1}(x, 2\pi)A_N(x, 0)g(x, 2\pi) + g^{-1}(x, 2\pi)d^N g(x, 2\pi) \quad (3.46)$$

We have two extremes, the first is with $\phi = 0$, for which this equation gives no condition and $A_N(x, 0)$ any flat connection on N while the second is with $A_N(x, 0) = 0$ and then ϕ is constant on N and hence a flat connection on S^1 . For inbetween values we have that $A_N(x, 0)$ is reducible and the reducibility is determined by the holonomy on S^1 of ϕ .

Consequently we see that there are, in principle, many connections that require both A_N and ϕ to be non-zero and so it is a non-empty result that the torsion does not depend on A_N .

3.5.2 Application to Abelianisation

Quite surprisingly the fact that A_N does not make an appearance in the formulae for the Ray-Singer Torsion also puts into context a perplexing aspect of the Abelianisation programme initiated in [9, 24, 33], and applied to the evaluation of the Chern-Simons partition function on various classes of 3-manifolds of increasing complexity in [24, 26, 27, 30].

Concretely, in Chern-Simons theory (or its relatives such as BF theory and the G/G gauged sigma model) one is able to choose a gauge condition that Abelianises certain fields so that, within the evaluation of the associated path integral, one encounters a Ray-Singer Torsion that is a function only of that Abelianised field, even though that Abelian field is typically, by itself, not a flat connection. However, here we see that ϕ (and μ) while they may not be flat by themselves, can be regarded as components of a flat connection on M which also includes a non-trivial A_N , while the result does not depend on the explicit form of A_N at all.

In order to fix ideas, consider the Chern-Simons path integral on $M = \Sigma \times S^1$ with Σ a compact closed Riemann surface [24]. Let ϕ be the component of the connection along the S^1 direction. After integrating over the other components of the connection as well as the ghost terms one finds that the absolute value of the partition function is proportional to the Ray-Singer Torsion (there is still an integral over ϕ to perform)

$$\tau_{S^1}(\phi, \mathfrak{k})^{e(\Sigma)/2}. \quad (3.47)$$

Since we have already integrated over all the components of the connection A_Σ , it is not at all obvious that ϕ is a component of some flat connection on M . However, as we have seen, ϕ can be thought of as a component of a flat connection (\tilde{A}_Σ, ϕ) as the Ray-Singer Torsion will not, in any case, depend on the rest of the flat connection. This in turn means that one may therefore compare with the traditional approach of determining the Chern-Simons path integral which involves an expansion about the flat connections. The crucial advantage here is that we do not actually need to know the flat connections completely, but only their possible ϕ components.

3.5.3 Metric independence

The pairing that one uses for the ghost terms in the gauge fixing procedure makes use of the Hodge star, so of the metric on the underlying Riemannian manifold. One need not do

this. For example one could in QED consider the ghost ω to be a 0-form while taking both the anti-ghost $\bar{\omega}$ and multiplier field π to be n -forms. In this way the covariant gauge fixing and ghost terms would be

$$\int_M (\langle \delta A, \pi \rangle + \langle \omega, \Delta \bar{\omega} \rangle) \tag{3.48}$$

Usually, nothing is gained by doing this as the Hodge star resides in the Laplacian as well.

However, with the algebraic conditions (3.34) if one opts to use the Hodge duals of the anti-ghosts and multiplier fields then one sees quite straightforwardly that there is no metric at all in the gauge fixed action. But what of the path integral measure? We note that the anti-ghosts and the multiplier fields are always evenly matched and always of opposite statistics so that they form a type of supersymmetric pair. Consequently, the easiest way to see the metric independence of the theory is to use the fields adapted so there is no explicit metric dependence in the action.

We should also make a comparison with the path integrals of section 2.3.5. So consider N to be a point and suppose that $p = 0$. Then the action that we should consider to arrive at the torsion is the Grassmann odd version of the action (3.23) with $(B, C) \rightarrow (\bar{\eta}, \eta)$, i.e.

$$(-1)^p \int_M d\theta \langle B^p, DC^q \rangle \rightarrow \int_{S^1} d\theta \langle \bar{\eta}, D\eta \rangle, \tag{3.49}$$

with $\bar{\eta}$ and η being Grassmann zero-forms on S^1 . There is no gauge fixing involved and so the theory is metric independent from the start and coincides with that considered in section 2.3.5 (after we set $R = 1$).

3.5.4 From 2nd to 1st order actions on $N \times S^1$

We will give yet another derivation of the formulae for the torsion on $N \times S^1$ with metric $g_M = g_N + R^2 d\theta \otimes d\theta$ and a background \mathbb{R}_+ connection switched on. The standard Schwarz path integral for the Ray-Singer Torsion requires an interpretation of the determinants of first order operators T as being formally given as $\text{Det}(TT^\dagger)$ with the adjoint operator T^\dagger being a metric adjoint. Our aim here is to show that the ratio of determinants of Laplacians, that goes into the definition of the Ray-Singer Torsion, devolves to the ratios of the determinants of the first order operators of the previous sections, completely by-passing the need to give a separate definition for the determinants.

This derivation requires that the overall path integral is metric independent and that we may take the singular limit $R \rightarrow 0$. The simplest way to ensure this is to consider the massive theory as there are no zero modes to contend with. We will make use of the knowledge that we have gained previously in that rather than the mass we should use μ and on the circle there is no metric dependence on using this variable, in particular it is R independent.

We begin with the square of the product of determinants that go into the definition (2.61) with mass

$$\widehat{\tau}_{N \times S^1}^2(A, m) = \prod_{j=0}^n \left(\text{Det}_{\Omega^j(M, \mathbb{E})} (\Delta_A + m^2) \right)^{(-1)^{j+1}j} \tag{3.50}$$

Now use a standard representation of the determinant of the Laplacian as a Gaussian field theory so that we may express (3.50) in terms of path integrals as

$$\widehat{\tau}_M^2(A, m) = \prod_i \int DBDC \exp\left(i \int_M \langle B, (\Delta_A + m^2)C \rangle\right) \quad (3.51)$$

where the product is over i copies of each set of $(i, n - i)$ forms (B^i, C^{n-i}) and the pairs are Grassmann even if B^i is of even degree while they are Grassmann odd if B^i is of odd degree. This product, representing the Ray-Singer Torsion, will be metric independent.

For every Gaussian path integral factor on $M = N \times S^1$ we split the Laplacian as follows

$$\exp\left(i \int_M \langle B, (\Delta_A^M + m^2)C \rangle\right) = \exp\left(i \int_M \langle B, \Delta_A^N C \rangle\right) \exp\left(i \int_M \langle B, (\Delta_A^{S^1} + m^2)C \rangle\right)$$

and introduce a second path integral

$$\begin{aligned} & \exp\left(i \int_M \langle B, (\Delta_A^{S^1} + m^2)C \rangle\right) \\ &= \int D\widehat{B}D\widehat{C} \exp\left(i \int_M \left(\langle B, D_{\phi+\mu}\widehat{C} \rangle + \langle \widehat{B}, D_{\phi-\mu}C \rangle + R^2 \langle \widehat{B}, \widehat{C} \rangle\right)\right) \end{aligned} \quad (3.52)$$

Notice that in this formula we have once more set $\mu = Rm$. We substitute back into (3.51) to arrive at

$$\begin{aligned} \widehat{\tau}_M^2(\phi, \mu) &= \prod_i \int DBDCD\widehat{B}D\widehat{C} \\ & \exp\left(i \int_M \langle B, \Delta_A^N C \rangle + \langle B, D_{\phi+\mu}\widehat{C} \rangle + \langle \widehat{B}, D_{\phi-\mu}C \rangle + R^2 \langle \widehat{B}, \widehat{C} \rangle\right) \end{aligned} \quad (3.53)$$

Now we take the $R \rightarrow 0$ limit while keeping μ fixed. As $D_{\phi\pm\mu}$ are invertible we may ignore the term involving the Laplacian on N altogether as the integrals over $(\widehat{B}, \widehat{C})$ essentially set (B, C) to zero. We thus arrive at

$$\widehat{\tau}_M^2(\phi, \mu) = \prod_i \int DBDCD\widehat{B}D\widehat{C} \exp\left(i \int_M \langle B, D_{\phi+\mu}\widehat{C} \rangle + \langle \widehat{B}, D_{\phi-\mu}C \rangle\right) \quad (3.54)$$

We have therefore derived the representation of the Ray-Singer Torsion in terms of path integrals of 1st order actions from its definition in terms of products of determinants of Laplace operators (path integrals of 2nd order actions), thus in a sense reversing the usual derivation of the Ray-Singer Torsion from the 1st order Schwarz actions.

As an aside we note that it might seem strange that we are taking the $R \rightarrow 0$ limit while keeping μ fixed which amounts to simultaneously taking $m \rightarrow \infty$, while it may be the intention to eventually take the $m \rightarrow 0$ limit. However, we saw in section 2.3.4 that the limit is correctly reformulated in terms of μ as then no metric dependence arises.

To complete the derivation we note that the forms in question are on M and that we should break them down to their form degree on N ,

$$\Omega^i(M, \mathbb{E}) = \Omega^i(N, \mathbb{E}) \otimes \Omega^0(S^1) \oplus \Omega^{i-1}(N, \mathbb{E}) \otimes \Omega^1(S^1) \quad (3.55)$$

Under this decomposition we have

$$\left(B^i, \widehat{C}^{n-i}\right) \rightarrow \left(D^i, \overline{E}^{n-i-1}\right) \oplus \left(\overline{D}^{i-1}, E^{n-i}\right) \quad (3.56)$$

(at the same time leading to an explicit $d\theta$ in the action) and a similar decomposition for the (\widehat{B}^i, C^{n-i}) pair.

The Grassmann count for $(D^i, \overline{E}^{n-i-1})$ is $(-1)^i$ while that for the pair $(\overline{D}^i, E^{n-i-1})$ the Grassmann degree is $(-1)^{i+1}$. From the multiplicity of determinants in (3.50) the determinants of $D_{\phi+\mu}$ acting on i forms are i of the type that arise with $(D^i, \overline{E}^{n-i-1})$ and $i+1$ of the type $(\overline{D}^i, E^{n-i-1})$ and these two types have opposite statistics. Happily enough all the path integrals now have the form of the previous section leading to an overall determinant

$$\begin{aligned} \widehat{\tau}_{N \times S^1}^2(\phi, \mu) &= \prod_{i=0}^{n-1} \text{Det}_{\mathbb{E} \otimes \Omega^i(N)}(D_{\phi+\mu})^{(-1)^i} \cdot \text{Det}_{\mathbb{E} \otimes \Omega^i(N)}(D_{\phi-\mu})^{(-1)^i} \\ &= \widehat{\tau}_{N \times S^1}(\phi + \mu) \cdot \widehat{\tau}_{N \times S^1}(\phi - \mu) \end{aligned} \quad (3.57)$$

This formula generalises that derived when N is a point (2.81).

For $\phi \pm \mu$ constant and following the regularisation of previous sections we arrive at

$$\widehat{\tau}_{N \times S^1}^2(A, m) = \widehat{\tau}_{S^1}(\phi + \mu, \mathbb{E})^{e(N)} \cdot \widehat{\tau}_{S^1}(\phi - \mu, \mathbb{E})^{e(N)} \quad (3.58)$$

We end this section with one last comment on metric dependence. Had we used the ‘physics’ path integral measure then its conventional normalisation would have been

$$D(BR^{-1}) D(CR^{-1}) \quad (3.59)$$

However, the correct measure for the $(\widehat{B}, \widehat{C})$ systems is

$$D(\widehat{B}R) D(\widehat{C}R) \quad (3.60)$$

and the product measure is

$$D(BR^{-1}) DC(R^{-1}) D(\widehat{B}R) D(\widehat{C}R) = DBDCD\widehat{B}D\widehat{C} \quad (3.61)$$

which is R independent. The complete R dependence is then in the action and so taking the $R \rightarrow 0$ limit amounts to just dropping the $R^2 \langle \widehat{B}, \widehat{C} \rangle$ term.

4 Massive Ray-Singer torsion on mapping tori

We now apply the techniques introduced previously to the case when M is a finite order mapping torus. The approach that we use is to twist the fields by the finite order diffeomorphism along the circle. We will simultaneously perform a change of variables that pushes the mass dependence out of the action and into the ‘boundary conditions’ along the S^1 , as we did in previous sections. In particular we will establish a generalisation of Fried’s formula (2.71) [31].

At a technical level we will need to establish that the equivalent of the temporal gauge still applies on these manifolds. We will also need to use a Lefschetz version of the index theorem.

4.1 Fields on mapping tori

Let N be a smooth (compact, closed) oriented manifold and let $f \in \text{Diff}(N)$ be a diffeomorphism of N . Then the *mapping torus* $M = N_f$ of f can be concretely described as the manifold

$$M = N_f \equiv \frac{N \times I}{(x, 0) \sim (f(x), 1)} \tag{4.1}$$

where $I = [0, 1]$ is the unit interval. This shows that N_f is a fibration $N \rightarrow N_f \xrightarrow{\pi} S^1$ over S^1 , with $\pi(x, t) = t$ and typical fibre N .

Fields $\Phi(x, t)$ on N_f can be regarded

- either as functions on $N \times I$ satisfying the twisted boundary condition

$$\Phi(x, 0) = f^* \Phi(x, 1) \tag{4.2}$$

- or as functions on $N \times \mathbb{R}$ satisfying the twisted periodicity conditions

$$\Phi(x, t) = f^* \Phi(x, t + 1) \tag{4.3}$$

Likewise, connections and gauge transformations can be chosen to satisfy the same periodicity rules

$$A(x, t) = f^* A(x, t + 1), \quad g(x, t) = f^* g(x, t + 1) \tag{4.4}$$

In the following, we will consider the case that f has finite order n_f , i.e. $f^{n_f} = \text{Id}$ with Id the identity map (note that $f^{n_f} = f \circ \dots \circ f$ (n_f times) refers to the group structure in $\text{Diff}(N)$). In this case, $N \times S^1$ (with the circle coordinate $t \in [0, n_f)$) is an n_f -fold cover of N_f . This is reflected in the fact that the n_f -fold iteration of the twisted periodicity conditions gives

$$\Phi(x, t) = (f^*)^{n_f} \Phi(x, t + n_f) = \Phi(x, t + n_f) \tag{4.5}$$

(and likewise for A and g).

As on $N \times S^1$, one may unambiguously decompose the connection as

$$A(x, t) = A_N(x, t) + \phi dt \tag{4.6}$$

with $\iota_\xi A_N(x, t) = 0$ with ξ the vector field $\partial/\partial t$.

4.2 The partition function and gauge fixing

The partition function of a theory on the mapping torus, as will be considered by us below, is over the class of fields on $N \times I$ that satisfy (4.2). We will write the partition function of the theory yielding the Analytic Torsion on M as

$$Z_{N_f} = \int_{\Phi(0)=f^*\Phi(1)} D\Phi \exp(S_{N \times I}(\Phi)) \tag{4.7}$$

for

$$S_{N \times I}(\Phi) = \int_{N \times I} \langle \mathcal{B}, d_A \mathcal{C} \rangle \tag{4.8}$$

The shift gauge symmetry

$$\delta\mathcal{B} = d_A\Sigma, \quad \delta\mathcal{C} = d_A\Lambda \tag{4.9}$$

now requires that the symmetry parameters, as every other field, obey the twisted boundary conditions

$$\Sigma(0, x) = f^*\Sigma(x, 1), \quad \Lambda(0, x) = f^*\Lambda(x, 1) \tag{4.10}$$

We must check to see if the algebraic gauge fixing conditions that were so effective on the product $N \times S^1$ can also be used on the mapping torus $N \rightarrow M \rightarrow S^1$. This amounts to showing that a solution to (3.25) exists within the set of forms with twisted boundary conditions. The solution, without boundary conditions on $N \times I$ remains (3.29) and our task is determine the analogue of the integration constant (3.30) for the twisted boundary conditions. As the general case is a bit messy, we only provide the solution for $\phi + \mu$ which are constant on N , and do so to convince the reader that this is indeed possible. It is

$$\gamma = (-1)^p (h^{n_f} \rho(g)^{n_f} - 1)^{-1} \sum_{i=0}^{n_f-1} (h\rho(g)f^*)^i \int_0^1 h^s \rho(g)^s B^{p-1}(s) ds \tag{4.11}$$

Now that we know that we have the ability to gauge fix as before we can go to the BRST structure. The ghost for ghost system that one has in the mapping torus case is the same as that for the product manifold. In any case this means that we have exactly the same system as before (3.34) with the only difference being that the fields $(\omega^i, \bar{\omega}^i, \pi^i)$ now also satisfy the twisted boundary conditions.

4.3 Path integral derivation of a generalisation of Fried’s formula

After gauge fixing we are confronted with products of ‘independent’ partition functions of the form

$$Z_i = \int D\alpha D\beta e^{(i \int_{N \times I} dt \langle \alpha(t), D\beta(t) \rangle)} \delta(\alpha(0) - f^*\alpha(1)) \delta(\beta(0) - f^*\beta(1)) \tag{4.12}$$

with $\alpha \in \Omega^i(N, \mathbb{E}) \otimes \Omega^0(I)$, the appropriate boundary conditions on the mapping torus being specified and implemented by the explicit δ -function insertions in the path integral. By using (3.26), one can now write the action as

$$\int_{N \times I} \langle \alpha(t), k^{-1}(t) \frac{\partial}{\partial t} k(t) \beta(t) \rangle \tag{4.13}$$

Changing variables to $\tilde{\alpha}(t) = \alpha(t)k^{-1}(t)$ and $\tilde{\beta}(t) = k(t)\beta(t)$, the action simplifies to

$$\int_{N \times I} \langle \tilde{\alpha}, \frac{\partial}{\partial t} \tilde{\beta} \rangle \tag{4.14}$$

while the boundary conditions go over to

$$\begin{aligned} \tilde{\alpha}(0) &= f^*(\tilde{\alpha}(1)\rho(k)) \equiv f^* \cdot \tilde{\alpha}(1)\rho(k) \\ \tilde{\beta}(0) &= f^*(\rho(k)^{-1}\tilde{\beta}(1)) \equiv f^* \cdot \rho(k)^{-1}\tilde{\beta}(1) \end{aligned} \tag{4.15}$$

where here now $k \equiv k(t = 1)$. The path integral can now be evaluated by discretising the interval as we did for the $\eta\text{-}\bar{\eta}$ system in section 2.3.5. In this way we arrive at

$$\int D\tilde{\beta}(0)\delta\left((1 - f^* \cdot \rho(k)^{-1})\tilde{\beta}(0)\right) = \text{Det}_{\Omega^i(N, \mathbb{E})} \left(1 - f^* \cdot \rho(k)^{-1}\right)^{\pm 1} \quad (4.16)$$

the power depending on the statistics of α and β . This only differs from the situation on $N \times S^1$ by the pull back with respect to f which acts both on the holonomies as well as the space of forms $\Omega^i(N, \mathbb{E})$. Consequently, we may make use of the work done in section 3.4 to arrive at

$$Z_{N_f} = \prod_{i=0}^{n-1} Z_i = \prod_{i=0}^{n-1} \text{Det}_{\Omega^i(N, \mathbb{E})} \left(1 - f^* \cdot \rho(k)^{-1}\right)^{(-1)^{p+1+i}}. \quad (4.17)$$

The appropriate regularised form of the determinant can be obtained by writing (as usual) $\log \text{Det} = \text{Tr} \log$, the heat kernel regularised logarithm being

$$\lim_{\epsilon \rightarrow 0} \sum_{i=0}^{n-1} (-1)^i \text{Tr}_{\Omega^i(N, \mathbb{E})} \left(\ln \left(1 - f^* \cdot \rho(k)^{-1}\right) e^{-\epsilon \Delta_A} \right) \quad (4.18)$$

Compared with the analogous calculation on $N \times S^1$ performed in detail in the previous section, here we now also have a ‘twist’ coming from taking into account the action of the finite order diffeomorphism on the spaces of forms. It is also convenient to use an f -invariant metric (since f is of finite order, this can always be achieved for any seed metric g_N by ‘averaging’) and we do so.

When ϕ and μ are constant on N , we may separate the action on the space of forms as

$$\lim_{\epsilon \rightarrow 0} \sum_{i=0}^{n-1} (-1)^i \text{Tr}_{\Omega^i(N, \mathbb{E})} \left(\ln \left(1 - \rho(k)^{-1} f^*\right) e^{-\epsilon \Delta_A} \right) \quad (4.19)$$

Note that since $(f^*)^r$ is an isometry, one has

$$\lim_{\epsilon \rightarrow 0} \sum_{i=0}^{n-1} (-1)^i \text{Tr}_{\Omega^i(N, \mathbb{E})} \left(e^{-\epsilon \Delta_A} (f^*)^r \right) = \sum_{i=0}^{n-1} (-1)^i \text{Tr}_{H_A^i(N, \mathbb{E})} \left((f^*)^r \right) \quad (4.20)$$

Thus, on expanding the logarithm we understand that we may replace the signed sum over the spaces of forms with the signed sum of the cohomology groups so that on putting these pieces together we find

$$\tau_M(\phi + \mu, \mathbb{E}) = \prod_{i=0}^{n-1} \text{Det}_{H^i(N, \mathbb{E})} \left(1 - \rho(k)^{-1} \otimes f_i^*\right)^{(-1)^i} \quad (4.21)$$

This is already a generalisation of Fried’s formula (2.71) as the connection does not have to be the pullback of a connection on the base (but could include an arbitrary flat component A_N along N) and includes the \mathbb{R}_+ connection (or a mass, as one prefers).

In general, when $\phi + \mu$ is not constant on N evaluating (4.17) may be laborious. One way is to split the $\Omega^i(N, \mathbb{E})$ in terms of eigenspaces of f (with eigenvalues characters of the finite cyclic group generated by f) as

$$\Omega^i(N, \mathbb{E}) = \bigoplus_{s=0}^{n-1} \Omega_{(s)}^i(N, \mathbb{E}), \quad f^* \omega_{(s)}^i = \chi_s(f) \omega_{(s)}^i \quad \text{for } \omega_{(s)}^i \in \Omega_{(s)}^i(N, \mathbb{E}) \quad (4.22)$$

Also with the notation at hand we must understand that there is a corresponding decomposition of ρ , with $\rho_{(s)}$ acting as

$$\rho_{(s)} : \Omega_{(r)}^i(N, \mathbb{E}) \rightarrow \Omega_{(r+s)}^i(N, \mathbb{E}) \tag{4.23}$$

Then the determinants in (4.17) may be expressed in terms of products of determinants over the spaces $\Omega_{(s)}^i(N, \mathbb{E})$. One would then find a result of the general (but enlightening) form

$$Z = \prod_{s=0}^{n_f-1} \left(\prod_{i=0}^{n-1} \text{Det}_{\Omega_{(s)}^i(N, \mathbb{E})} (T_s(f, \rho(k)))^{(-1)^{p+1+i}} \right) \tag{4.24}$$

At this point a McKean-Singer type argument could be used for the alternating signed sum over the $\Omega_{(s)}^\bullet$ for each eigenspace separately.

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A Properties of the massive Ray-Singer torsion

Here we establish two properties of the massive Ray-Singer Torsion mentioned in section 2.2, namely

1. the triviality of the massive Ray-Singer Torsion on even-dimensional manifolds M (2.68)

$$\dim M = 2\ell \quad \Rightarrow \quad \tau_M(A, \mathbb{E}, g_M, m) = 1 \tag{A.1}$$

2. a product formula for the massive Ray-Singer Torsion on product manifolds $M \times N$, when the flat connection A is of the form $A \in \Omega^1(M, \mathfrak{g}) \otimes \Omega^0(N)$, namely (2.69)

$$\tau_{M \times N}(A, \mathbb{E}, m) = \tau_M(A, \mathbb{E}, m)^{e(N)} \cdot \tau_N(m)^{e(M) \cdot \text{rk } \mathbb{E}} \tag{A.2}$$

Both of these properties are strict analogues of those for the traditional Ray-Singer Torsion and, indeed, the proofs of these properties can also be modelled on the traditional proofs.

A.1 Triviality of the massive Ray-Singer torsion in even dimensions

We already outlined one proof of this assertion (based on the generalisation of the original argument of Ray and Singer to the massive case) in section 2.2. Here we will generalise an alternative argument to the massive case, namely one originally given in [4] and inspired by the path integral representation of the Ray-Singer Torsion.

First of all, we recall the definition of the massive Ray-Singer Torsion, namely

$$\tau_M(A, \mathbb{E}, g_M, m) = |m|^{e(M) \cdot \text{rk}(\mathbb{E}) \cdot \dim M/2} \prod_{i=0}^{\dim M} \left(\text{Det}_{\Omega^i(M, \mathbb{E})} (\Delta_A + m^2) \right)^{(-1)^{i+1} i/2} \quad (\text{A.3})$$

Using Hodge duality, in the even-dimensional case $\dim M = 2\ell$ this can be rewritten as (suppressing the arguments of τ_M , and abbreviating $\Omega^i(M, \mathbb{E}) \equiv \Omega^i$)

$$\tau_M = |m|^{e(M) \text{rk}(\mathbb{E})\ell} \left(\text{Det}_{\Omega^\ell} (\Delta_A + m^2) \right)^{(-1)^{\ell+1} \ell/2} \prod_{i=0}^{\ell-1} \left(\text{Det}_{\Omega^i} (\Delta_A + m^2) \right)^{(-1)^{i+1} \ell} \quad (\text{A.4})$$

On the other hand, in [4] a simple scaling argument based on the path integral was used to establish the identity (Proposition 4 in [4], in present notation)

$$\text{Det}'_{\Omega^\ell} f(\Delta_A) = \prod_{i=0}^{\ell-1} \left(\text{Det}'_{\Omega^{\ell-i-1}} f(\Delta_A) \right)^{2(-1)^i} \quad (\text{A.5})$$

where $f(\Delta_A)$ is some function of the Laplacian, and the notation Det' indicates that the harmonic modes of Δ_A were projected out. This formula can be applied to the massive Laplacian, i.e. to the simple function

$$f(\Delta_A) = \Delta_A + m^2, \quad (\text{A.6})$$

but in order to conform with our definition of the determinant of $\Delta_A + m^2$, which also included the harmonic modes of Δ_A , each determinant appearing in the above identity needs to be divided explicitly by the contribution of the harmonic modes to the determinant, i.e.

$$\text{Det}'_{\Omega^i} (\Delta_A + m^2) = m^{-2 \dim H_A^i(M, \mathbb{E})} \text{Det}_{\Omega^i} (\Delta_A + m^2). \quad (\text{A.7})$$

Collecting all the powers of the mass m , one then arrives at the identity

$$\text{Det}_{\Omega^\ell} (\Delta_A + m^2) = m^{2(-1)^\ell e(M) \text{rk}(\mathbb{E})} \prod_{i=0}^{\ell-1} \left(\text{Det}_{\Omega^{\ell-i-1}} (\Delta_A + m^2) \right)^{2(-1)^i} \quad (\text{A.8})$$

This is now the ‘massive’ generalisation of Proposition 4 of [4]. Comparing this with the expression (A.4) for the massive Ray-Singer Torsion obtained above, we conclude that $\tau_M = 1$.

A.2 Product formula for the massive Ray-Singer torsion on $M \times N$

We now turn to a proof of the product formula (A.2) for the massive Ray-Singer Torsion. In [24] we gave a slight generalisation of Theorem 2.5 in [18] which we extend here to the massive Ray-Singer Torsion $\tau_M(A, \mathbb{E}, m)$. Suppose M and N are compact, closed and oriented manifolds and equip $M \times N$ with the product metric. Let E be a vector bundle over $M \times N$ which allows for connections with local form $A \in \Omega^1(M, \mathfrak{g}) \otimes \Omega^0(N)$, up to gauge equivalence. (Thus in contrast to Ray and Singer, we do not require $\pi_1(N)$ to be trivial; and we also do not need \mathbb{E} and A to be pullbacks from M to $M \times N$.) Then one has

$$\tau_{M \times N}(A, \mathbb{E}, m) = \tau_M(A, \mathbb{E}, m)^{e(N)} \cdot \tau_N(m)^{e(M) \cdot \text{rk} \mathbb{E}} \quad (\text{A.9})$$

Note that here $\tau_N(m)$ simply refers to the massive Ray-Singer Torsion of the untwisted Laplacian $\Delta + m^2$ on differential forms on N .

By the previous result, the only interesting case is when $\dim M$ is even and $\dim N$ is odd (or perhaps vice-versa), but we will give the proof in general. The proof is essentially identical to that of Ray and Singer, the difference being that we allow for zero eigenvalues of the Laplacian. With the product metric one has that the twisted Laplacian acts ‘diagonally’,

$$\Delta_A(\omega_M \otimes \omega_N) = (\Delta_A \omega_M) \otimes \omega_N + \omega_M \otimes (\Delta \omega_N), \quad \omega_M \in \Omega(M, \mathbb{E}), \quad \omega_N \in \Omega(N)$$

The ζ function for the massive Laplacian acting on $\Omega^r(M \times N, \mathbb{E})$ is thus

$$\zeta_r(M \times N, \mathbb{E})(s) = \sum_{\lambda, \nu} \sum_{p+q=r} (\lambda + \nu + m^2)^{-s} M_p(\lambda, M) M_q(\nu, N)$$

with $M_p(\lambda, M)$ and $M_q(\nu, N)$ being the multiplicity of the eigenvalues λ and ν of the Laplacian on the spaces of forms $\Omega^p(M, \mathbb{E})$ and $\Omega^q(N)$ respectively. The alternating sum which goes into the definition of the torsion is

$$\begin{aligned} & \sum_{r=0}^{\dim(M \times N)} (-1)^r r \zeta_r(M \times N, \mathbb{E})(s) \\ &= \sum_{\lambda, \nu} (\lambda + \nu + m^2)^{-s} \sum_{p=0}^{\dim M} \sum_{q=0}^{\dim N} (-1)^{p+q} (p+q) M_p(\lambda, M) M_q(\nu, N) \\ &= \sum_{\lambda, \nu} (\lambda + \nu + m^2)^{-s} \sum_{p=0}^{\dim M} (-1)^p p M_p(\lambda, M) \sum_{q=0}^{\dim N} (-1)^q M_q(\nu, N) \\ &+ \sum_{\lambda, \nu} (\lambda + \nu + m^2)^{-s} \sum_{p=0}^{\dim M} (-1)^p M_p(\lambda, M) \sum_{q=0}^{\dim N} (-1)^q q M_q(\nu, N) \end{aligned} \tag{A.10}$$

Ray and Singer show that (we have allowed $\lambda = 0$)

$$\sum_{p=0}^{\dim M} (-1)^p M_p(\lambda, M) = e(M) \cdot \text{rk}(\mathbb{E}) \delta_{\lambda 0}, \quad \sum_{q=0}^{\dim N} (-1)^q M_q(\nu, N) = e(N) \delta_{\nu 0}$$

(non-zero eigenvalues of the Laplacian do not contribute to the Euler characteristic and so their total signed sum must be zero) whence

$$\begin{aligned} & \sum_{r=0}^{\dim(M \times N)} (-1)^r r \zeta_r(M \times N, \mathbb{E})(s) \\ &= e(N) \sum_{\lambda} (\lambda + m^2)^{-s} \sum_{p=0}^{\dim M} (-1)^p p M_p(\lambda, M) \\ &+ e(M) \cdot \text{rk}(\mathbb{E}) \sum_{\nu} (\nu + m^2)^{-s} \sum_{q=0}^{\dim N} (-1)^q q M_q(\nu, N) \\ &= e(N) \sum_{r=0}^{\dim M} (-1)^r r \zeta_r(M, \mathbb{E})(s) + e(M) \cdot \text{rk}(\mathbb{E}) \sum_{r=0}^{\dim N} (-1)^r r \zeta_r(N)(s) \end{aligned} \tag{A.11}$$

Taking the derivative at $s = 0$ now yields the desired product formula (A.9).

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