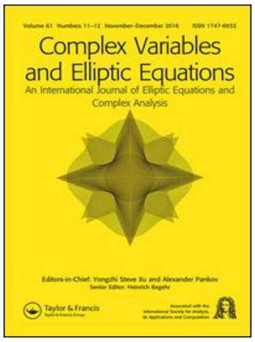

Complex Variables and Elliptic Equations
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

Holomorphic Lie group actions on Danielewski surfaces

Frank Kutzschebauch & Andreas Lind

To cite this article: Frank Kutzschebauch & Andreas Lind (2022): Holomorphic Lie group actions on Danielewski surfaces, Complex Variables and Elliptic Equations, DOI: 10.1080/17476933.2022.2076843

To link to this article: https://doi.org/10.1080/17476933.2022.2076843

Published online: 13 Jul 2022.

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
https://doi.org/10.1080/17476933.2022.2076843

Holomorphic Lie group actions on Danielewski surfaces

Frank Kutzschebauch[a] and Andreas Lind[b]

[a]Institute of Mathematics, University of Berne, Bern, Switzerland; [b]Department of Mathematics, Mid Sweden University, Sundsvall, Sweden

ABSTRACT
We prove that any Lie subgroup G (with finitely many connected components) of an infinite-dimensional topological group $\mathcal{G}$ which is an amalgamated product of two closed subgroups can be conjugated to one factor. We apply this result to classify Lie group actions on Danielewski surfaces by elements of the overshear group (up to conjugation).

ARTICLE HISTORY
Received 31 January 2022
Accepted 9 May 2022

COMMUNICATED BY
S. Krantz

KEYWORDS
Danielewski surfaces; automorphisms; overshears; free amalgamated product; Lie group actions; one-parameter subgroups

AMS SUBJECT CLASSIFICATIONS
Primary 32M17; Secondary 22E60

1. Introduction

The motivation of this paper is the study of holomorphic automorphisms of Danielewski surfaces. These are affine algebraic surfaces defined by an equation $D_p := \{xy - p(z) = 0\}$ in $\mathbb{C}^3$, where $p \in \mathbb{C}[z]$ is a polynomial with simple zeros. These surfaces are intensively studied in affine algebraic geometry, and their algebraic automorphism group has been determined by Makar-Limanov [1, 2]. More results on algebraic automorphisms of Danielewski surfaces can be found in Refs. [3–7].

From the holomorphic point of view, their study began in the paper of Kaliman and Kutzschebauch [8] who proved that they have the density and volume density property, important features of the so called Andersén–Lempert theory. For definitions and an overview over Andersén–Lempert theory, we refer to Ref. [9].

Another important study in the borderland between affine algebraic geometry and complex analysis is the classification of complete algebraic vector fields on Danielewski surfaces by Leuenberger [10]. In fact we explain in Remark 4.1 how to use his results together with our Classification Theorem 1.3 to find holomorphic automorphisms of Danielewski surfaces which are not contained in the overshear group.

In Ref. [11], we define the notion of an overshear and shear on Danielewski surfaces as follows:

CONTACT Andreas Lind Andreas.Lind@miun.se

Definition 1.1: A mapping $O_{f,g} : D_p \to D_p$ of the form

$$O_{f,g}(x, y, z) = \left(x, y + \frac{1}{x}\left(p(z\,\mathrm{e}^{xf(x)} + xg(x)) - p(z)\right), z\,\mathrm{e}^{xf(x)} + xg(x)\right)$$

(or with the role of first and second coordinates exchanged, $IO_{f,g}I$) is called an *overshear map*, where $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions (and the involution I of D_p is the map interchanging x and y). When $f \equiv 0$, we say that $S_g := O_{0,g}$ is a *shear map* on D_p.

These mappings are automorphisms of D_p. The maps of the form $O_{f,g}$ form a group, which we call O_1. It can be equivalently described as the subgroup of $\mathrm{Aut}(D_p)$, leaving the function x invariant. It is therefore a closed subgroup of $\mathrm{Aut}(D_p)$ (endowed with compact-open topology). Analogously, the maps $IO_{f,g}I$ form a group, the closed subgroup of $\mathrm{Aut}(D_p)$ leaving y invariant, which we call O_2.

The main result of Ref. [11] says that the group generated by overshears, i.e. by O_1 and O_2 (we call it the *overshear group* $\mathrm{OS}(D_p)$), is dense (with respect to the compact-open topology) in the component of the identity of the holomorphic automorphism group $\mathrm{Aut}(D_P)$ of D_p. This fact generalizes the classical results of Andersén and Lempert [12] from $\mathbb{C}^n$. It is worth to be mentioned at this point that D_p for p of degree 1 is isomorphic to $\mathbb{C}^2$.

In Ref. [13], the authors together with Andrist proved a structure result of the overshear group.

Theorem 1.2 (Theorem 5.1 in Ref. [13]): *Let D_p be a Danielewski surface and assume that* $\deg(p) \geq 4$, *then the overshear group,* $\mathrm{OS}(D_p)$, *is a free amalgamated product of* O_1 *and* O_2.

The main result of our paper is a classification result for Lie group actions on Danielewski surfaces by elements of the overshear group.

Theorem 1.3: *Let D_p be a Danielewski surface and assume that* $\deg(p) \geq 4$. *Let a real connected Lie group G act on D_p by automorphisms in* $\mathrm{OS}(D_p)$. *Then G is abelian, isomorphic to the additive group* $(\mathbb{R}^n, +)$ *and is conjugated (in* $\mathrm{OS}(D_p)$*) to a subgroup of* O_1.

The exact formulas for such actions are described in Corollary 3.3.

For the overshear group of $\mathbb{C}^2$ (instead of Danielewski surfaces), many results in the same spirit have been proven by Ahern and Rudin [14] for G finite cyclic group, by Kutzschebauch and Kraft [15] for compact G and for one-parameter subgroups in the thesis of Andersén [16] by de Fabritiis [17], Ahern and Forstnerič [18] and Ahern et al. [19]. For Danielewski surfaces, our result is the first of that kind. The proof relies on our second main result, which seems to be of independent interest.

Theorem 1.4: *Let $\mathcal{G}$ be a topological group which is a free amalgamated product $O *_{O\cap L} L$ of two closed subgroups O, L. Furthermore, let G be a Lie group with finitely many connected components and $\varphi\colon G \to \mathcal{G}$ be a continuous group homomorphisms. Then $\varphi(G)$ is conjugate to a subgroup of O or L.*

The outline of this paper is the following. In Section 2, we prove Theorem 1.4. In Section 3, we prove Theorem 1.3. In Section 4, we apply Theorem 1.2 to give new examples of holomorphic automorphisms of D_p not contained in the overshear group $\mathrm{OS}(D_p)$.

2. Lie subgroups of a free amalgamated product

The aim of this section is to prove the following theorem. For the notion of amalgamated product, we refer the reader to Ref. [20].

Theorem 2.1: *Let $\mathcal{G}$ be a topological group which is a free amalgamated product $O *_{O\cap L} L$ of two closed subgroups O, L. Furthermore, let G be a Lie group with finitely many connected components and $\varphi\colon G \to \mathcal{G}$ be a continuous group homomorphism. Then $\varphi(G)$ is conjugate to a subgroup of O or L.*

We need the following facts:

Proposition 2.2: *Every element of a free amalgamated product $O *_{O\cap L} L$ is conjugate either to an element of O or L or to a cyclically reduced element. Every cyclically reduced element is of infinite order.*

Proof: See Proposition 2 in Section 1.3 in Ref. [20]. ■

Lemma 2.3: *A subgroup H of a free amalgamated product $O *_{O\cap L} L$ is conjugate to a subgroup of O or L if and only if H is of bounded length.*

Proof: This is a direct consequence of Proposition 2.2. ■

Lemma 2.4: *Let g_1 and g_2 be two commuting elements of $O *_{O\cap L} L$ with lengths ≥ 1, then $l(g_1)$ and $l(g_2)$ are both even or both odd.*

Proof: Assume that $g_1 = a_1 \cdots a_m$ and $g_2 = b_1 \cdots b_n$ are two commuting elements. Assume, for a contradiction, that $l(g_1)$ is even and $l(g_2)$ is odd. Since g_1 has even length, the first and last element of the chain $a_1, \ldots, a_m$ have to alter between O and L. Similarly, the first and last element of the chain g_2s has to be contained in either O or L.

Assume first that $a_1 \in O$ and $a_m \in L$ and that $b_1, b_n \in O$. Then, since a_m and b_1 alter between L and O, $l(g_1g_2) = m + n$. The assumption that g_1 and g_2 are commuting yields that the corresponding length of $g_2 \cdot g_1$ has to be the same as the length of $g_1 \cdot g_2$. Clearly,

$$b_1 \cdots b_n \cdot a_1 \cdots a_m = b_1 \cdots b_{n-1} \cdot c \cdot a_2 \cdots a_m,$$

where $c = b_n \cdot a_1 \in O$. Hence, $l(g_2g_1) = m + n - 1 < m + n = l(g_1g_2)$, which contradicts our assumption.

If we assume that $a_1 \in O$ and $a_m \in L$ and that $b_1, b_n \in L$, a similar contradiction is obtained. In fact, $l(g_1g_2) = m + n - 1 < m + n = l(g_2g_1)$.

Similar calculations are obtained if $a_1 \in L$ and $a_m \in O$, where we have to consider both of the cases $b_1, b_n \in L$ and $b_1, b_n \in O$. ■

Lemma 2.5: *If an element g of a free amalgamated product $O *_{O\cap L} L$ has roots of arbitrary order, then it is conjugate to an element in O or to an element in L.*

Proof: Assume that g is not conjugate to an element in O or to an element in L. Then, by Proposition 2.2, g is conjugate to a cyclically reduced element, say $h^{-1}gh$, which has even length ≥ 2 by definition of a cyclically reduced element. For each $n > 0$, we have that $h^{-1}gh = h^{-1}(g^{1/n})^n h$, since g as roots of arbitrary order. Hence, $h^{-1}g^{1/n}h$ is not an element of O or L, since it equals $h^{-1}gh$. Furthermore,

$$h^{-1}(g^{1/n})^n h \cdot h^{-1}gh = h^{-1}(g^{1/n})^n gh = h^{-1}ggh =$$
$$= h^{-1}g(g^{1/n})^n h = h^{-1}gh \cdot h^{-1}(g^{1/n})^n h.$$

We conclude that $h^{-1}gh$ and $h^{-1}g^{1/n}h$ commute. Whence, Lemma 2.4 implies that $h^{-1}g^{1/n}h$ has even length (since $h^{-1}gh$ has even length) and is thus cyclically reduced. Hence,

$$l(h^{-1}gh) = l(h^{-1}(g^{1/n})^n h) = |n| l(h^{-1}g^{1/n}h) \geq |n|,$$

for all $n > 0$, contradicting the fact that all elements of $O *_{O\cap L} L$ have finite length. ■

First let us establish Theorem 2.1 in the case of a one-parameter subgroup:

Proposition 2.6: *Let $\mathcal{G}$ be a topological group which is a free amalgamated product $O *_{O\cap L} L$ of two closed subgroups O and L. Let $\varphi\colon \mathbb{R} \to \mathcal{G}$ be a continuous one-parameter subgroup. Then, $\varphi(\mathbb{R})$ is conjugate to a subgroup of O or L.*

Proof: Since φ is a group homomorphism, we know that $\varphi(1)$ and $\varphi(\sqrt{2})$ have roots of all orders. Hence, we can use Lemma 2.5 to conjugate both elements to O or L. Consider the dense subgroup $H = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ of $\mathbb{R}$. Since

$$l(\varphi(m + n\sqrt{2})) = l(\varphi(m)\varphi(n\sqrt{2})) \leq l(\varphi(1)^m \varphi(\sqrt{2})^n),$$

we conclude that $\varphi(H)$ have bounded length. Therefore, Lemma 2.3 implies that $\varphi(H)$ is conjugate to O or L. Let $c \in O *_{O\cap L} L$ be an element such that $c\varphi(H)c^{-1}$ is contained in O or L. Finally, as O and L are closed, we get that

$$c\varphi(\overline{H})c^{-1} = c\varphi(\mathbb{R})c^{-1} \subseteq \overline{c\varphi(H)c^{-1}}$$

is contained in O or L. ■

The key ingredient in the Proof of Theorem 1.4 will rely on the following result which seems to be of independent interest. In the language of Ref. [21], this means that every Lie group G is uniformly finitely generated by one-parameter subgroups.

Proposition 2.7: *For any connected real Lie group G, there are finitely many elements $V_i \in$ Lie(G), $i = 1, 2, \ldots, N$, for which the product map of the one-parameter subgroups*

$$\Phi_{V_1, V_2, \ldots, V_N} : \mathbb{R}^N \to G$$

defined by

$$(t_1, t_2, \ldots, t_N) \mapsto \exp(t_1 V_1)\exp(t_2 V_2) \cdots \exp(t_N V_N)$$

is surjective.

Proof: By Levi-Malcev decomposition [22] and Iwasawa decomposition [23], we can write

$$G = S \cdot R = K \cdot A \cdot N \cdot R,$$

where S is semisimple, R is solvable, A is abelian, N is nilpotent and K is compact.

If we can prove the claim of the proposition for each of the factors in the above decomposition, we will be done.

For abelian groups, the fact holds trivially.

Case 1: K a compact connected Lie group: Take any basis $(k_1, \dots, k_n)$ of the Lie algebra Lie(K). Then the product map $\Phi_{k_1,k_2,\dots,k_n} : \mathbb{R}^n \to K$ is a submersion at the unit element. Thus its image contains an open neighborhood U of the unit element. Since the powers of a neighborhood U of the unit element in any connected Lie group cover the whole group, for a compact Lie group K there is a finite number m such that $U^m = K$. This means that for our purpose $\Phi^m_{k_1,k_2,\dots,k_n} : \mathbb{R}^{nm} \to K$ is surjective.

Case 2: Consider N, a nilpotent connected Lie group. Then $N \cong \tilde{N}/\Gamma$ for the universal covering $\tilde{N}$ and Γ a normal discrete subgroup of $\tilde{N}$. Since the exponential map for $\tilde{N}$ factors over $\pi : \tilde{N} \to N$, it is enough to prove the claim for simply connected N.

Then, the following fact (due to Malcev [24]) is true: If N is simply connected then for a certain (Malcev) basis $(V_1, \dots, V_n)$ of Lie(N), the map $(t_1, t_2, \dots, t_n) \mapsto \exp t_1 V_1 + t_2 V_2 + \cdots + t_n V_n$ is a diffeomorphism. We will now prove the claim by induction of the length of the lower central series of Lie(N). For length 1, the group is abelian and the fact holds trivially. Let $g = \exp(t_1 V_1 + t_2 V_2 + \cdots t_n V_n)$. By repeated use of Lemma 2.8, we write

$$\begin{aligned} g &= \exp(t_1 V_1) \exp(t_2 V_2 + \cdots + t_n V_n) \exp K_1 \\ &= \exp(t_1 V_1) \exp(t_2 V_2) \exp(t_3 V_3 + \cdots + t_n V_n) \exp K_2 \exp K_1 \\ &= \exp(t_1 V_1) \exp(t_2 V_2) \dots \exp(t_n V_n) \exp K_n \dots \exp K_2 \exp K_1 \end{aligned} \tag{1}$$

with $K_i \in [\mathrm{Lie}(N), \mathrm{Lie}(N)]$.

Since $[\mathrm{Lie}(N), \mathrm{Lie}(N)]$ has shorter length of lower central series, by the induction hypothesis, each of the factors $\exp K_i$ is a product of one-parameter subgroup. This proves the claim.

Case 3: R is solvable: Let R' denote the commutator subgroup of R. Then R' is nilpotent and $A := R/R'$ is abelian. If $x \in R$ is any element, we can per definition write its image $\bar{x}$ in A as $\bar{x} = \exp(t_1 A_1) \cdots \exp(t_n A_n)$ for some A_i:s in Lie(A) which form a basis. Let π : Lie(R) $\to$ Lie(A) denote the quotient map and let $\tilde{A}_i \in$ Lie(R) be elements with $\pi(\tilde{A}_i) = A_i$. Thus we get $x = \exp(t_1 \tilde{A}_1) \cdots \exp(t_n \tilde{A}_n) g$ for some $g \in R'$. Since R' is nilpotent this reduces our problem to case 2. ■

Lemma 2.8: *For a nilpotent Lie group G with Lie algebra $\mathfrak{g} = Lie(G)$ and $x, y \in \mathfrak{g}$, there is $K(x, y) \in [\mathfrak{g}, \mathfrak{g}]$ with*

$$\exp(x + y) = \exp(x) \exp(y) \exp(K(x, y)).$$

Proof: The key fact is the Baker–Campbell–Hausdorff formula proven by Dynkin [25]. In the nilpotent case, it says that it is a finite sum of iterated Lie brackets $Z(x, y)$ (number of iterations of brackets bounded by the lower central series of $\mathfrak{g}$) such that for all $x, y \in \mathfrak{g}$

$$\exp(x)\exp(y) = \exp Z(x, y).$$

Moreover, $Z(x, y) = x + y + [x, y] +$ higher brackets. Now

$$\begin{aligned} \exp(x + y) &= \exp(x)\exp(y)\exp(-Z(x, y)\exp(x + y)) \\ &= \exp(x)\exp(y)\exp(Z(-Z(x, y), x + y)). \end{aligned} \quad (2)$$

Setting $K(x, y) := Z(-Z(x, y), x + y))$ finishes the proof, since the terms without bracket cancel, i.e. $K(x, y) \in [\mathfrak{g}, \mathfrak{g}]$. ■

Now we are ready to prove the main result of this section.

Proof of Theorem 1.4.: Let G_0 denote a connected component of G containing the identity. By Proposition 2.7, there are finitely many one-parameter subgroups $\mathbb{R}_i$ such that the product map $\mathbb{R}_1 \times \mathbb{R}_2 \times \cdots \times \mathbb{R}_N \to G_0$ is surjective. By Proposition 2.6 and Lemma 2.3, the elements of each of the $\varphi(\mathbb{R}_i)$ have bounded length, say $a(i)$. Thus the length of the elements in $\varphi(G_0)$ is bounded by $\sum_{i=1}^{N} a(i)$. As G has only finitely many connected components, the lengths of elements of $\varphi(G)$ are bounded. The assertion now follows from Lemma 2.3. ■

3. Classification of Lie group actions by overshears

In this section, we prove Theorem 1.3 from the introduction. We assume $\deg(p) \geq 4$ and use Theorem 1.2 from the introduction stating that $\mathrm{OS}(D_p)$ is a free amalgamated product $O_1 * O_2$, where O_1 is generated by $O^x_{f,g}$ and O_2 is generated by $IO^x_{f,g}I$. By Theorem 2.1, we can conjugate any Lie group G with finitely many components acting continuously on D_p by elements of $\mathrm{OS}(D_p)$ into O_1 or O_2. Without loss of generality, we can assume that we can conjugate any connected Lie subgroup G of $\mathrm{OS}(D_p)$, in particular any one-parameter subgroup, to O_1. Now we have reduced our problem to classify Lie subgroups of O_1. We start with one-parameter subgroups.

We recall the definitions of overshear fields and shear fields from Ref. [11].

(V1) $OF^x_{f,g} := p'(z)(zf(x) + g(x))\frac{\partial}{\partial y} + x(zf(x) + g(x))\frac{\partial}{\partial z}$
(V2) $SF^x_f := p'(z)f(x)\frac{\partial}{\partial y} + xf(x)\frac{\partial}{\partial z}$

where f, g are entire functions on $\mathbb{C}$. In the special case, $f \equiv 0$ then $OF^x_{f,g}$ is the shear field SF^x_g.

The set of overshear fields is a Lie algebra which consists of complete vector fields only. The formula for the bracket is given by Equation (4).

Any one-parameter subgroup of $\mathrm{Aut}(D_p)$ which is contained in the overshear group O_1 is the flow of an overshear field. Let us prove this. The connection between a vector field $V(x, y, z)$ and the flow $\varphi(x, y, z, t)$ is given by the ODE

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \varphi(x,y,z,t) = V(\varphi(x,y,z,t_0)), \quad \varphi(x,y,z,0) = (x,y,z). \tag{3}$$

Since any action of a real Lie group on a complex space by holomorphic automorphisms is real analytic [26, 1.6], we can write the flow $\varphi(x,y,z,t) = (x,\dots,z\exp(xf(t,x)) + xg(t,x))$ contained in O_1 as

$$\left(x,\dots,z\exp\left(x\sum_{i=0}^{\infty} f_i(x)t^i\right) + x\sum_{i=0}^{\infty} g_i(x)t^i\right)$$

for entire functions f_i and g_i. Using Equation (3) for $t_0 = 0$ leads to $V(x,y,z,t) = p'(z)(zf_0(x) + g_1(x))\frac{\partial}{\partial y} + \{xf_1(x)\exp(xf_0(x))z + xg_1(x)\}\partial/\partial z$, an overshear field.

Calculating the commutator, we find that for any f, g, h and k, entire functions on $\mathbb{C}$, we have

$$[OF^x_{f,g}, OF^x_{h,k}] = x \cdot SF_{gh-kf}. \tag{4}$$

In particular, shear fields commute and

$$[SF^x_h, OF^x_{f,g}] = x \cdot SF^x_{fh} = xf(x) \cdot SF^x_h. \tag{5}$$

Proposition 3.1: *Let f, g and h be fixed holomorphic functions with $f,h \not\equiv 0$. Then the Lie algebra $\mathrm{Lie}(OF^x_{f,g}, SF^x_h)$ generated by $OF^x_{f,g}$ and SF^x_h is of infinite dimension.*

Proof: By expression (5) and the fact that shear fields commute, we get that

$$\mathrm{Lie}(OF^x_{f,g}, SF^x_h) = \mathrm{span}\{OF^x_{f,g}, SF_{x^n f^n h}; n = 0,1,2\dots\}.$$

Assume that the Lie algebra is of finite dimension. This means that there is an n and there are constants $a_0,\dots,a_n, b$ such that

$$bOF^x_{f,g} + \sum_{j=0}^{n} a_j x^j f^j(x) SF^x_h = x^{n+1} f^{n+1}(x) SF^x_h.$$

It follows that $b = 0$, whence we get a functional equation of the form

$$\sum_{j=0}^{n} a_j y^j(x) = y^{n+1}(x),$$

where y is holomorphic and has a zero at $x = 0$. This is impossible for non-zero functions y, since the right-hand side has a higher order of vanishing at $x = 0$ than the left-hand side. ■

Proposition 3.2: *Let $\mathfrak{g}$ be a Lie algebra contained in OS_1 and suppose that $\dim(\mathfrak{g}) < +\infty$. Then $\mathfrak{g}$ is abelian.*

Proof: Assume that $\mathfrak{g}$ is not abelian. Let $\Theta_1, \Theta_2 \in \mathfrak{g}$ be two non-commuting vector fields. As explained above, they are overshear fields and since they do not commute, their bracket $[\Theta_1, \Theta_2]$ is by Equation (4) a nontrivial shear field. Now the result follows from Proposition 3.1. ■

Proof of Theorem 1.3.: As explained in the beginning of the section, the action of G on D_p by overshears can be conjugated into O_1. The action of G by elements of O_1 gives rise to a Lie algebra homomorphism of Lie(G) into the Lie algebra of vector fields on D_p fixing the variable x. This Lie algebra is exactly the set of overshear vector fields $OF^x_{f,g}$ (which consists of complete fields only). By Proposition 3.2, the finite dimensional Lie algebra Lie(G) has to be abelian. Since all one-parameter subgroups of G give rise to an overshear vector field, they are isomorphic to $(\mathbb{R}, +)$ (not S^1). Thus G is isomorphic to the additive group $\mathbb{R}^n$ generated by the flows of n linear independent commuting overshear vector fields $OF^x_{f_i,g_i}$, $i = 1, 2, \ldots, n$ which commute. By formula (4), this is equivalent to $f_i g_j - f_j g_i = 0$ $\forall\ i, j$. An equivalent way of expressing this is that the meromorphic functions $h_i := g_i/f_i$ are the same for all i or that all f_i are identically zero. ■

Corollary 3.3: *Suppose* $\deg(p) \geq 4$. *Every one-parameter subgroup of* $\mathrm{OS}(D_p)$ *is conjugate by elements of* $\mathrm{OS}(D_p)$ *to the flow of an overshear field* $OF^x_{f,g}$ *which in turn is given by the formula*

$$(x, y, z, t) \mapsto \left(x, y + \frac{p\left(\mathrm{e}^{xf(x)t} z + (g(x)/f(x))(\mathrm{e}^{xf(x)t} - 1)\right) - p(z)}{x}, \mathrm{e}^{xf(x)t} z \right.$$
$$\left. + \frac{g(x)}{f(x)} (\mathrm{e}^{xf(x)t} - 1) \right).$$

Here the expression $(\mathrm{e}^{ab} - 1)/a$ *for* $a = 0$ *is interpreted as the limit of this expression for* $a \to 0$, *i.e. as* b.

Remark 3.1: It is directly seen from Theorem 1.3 that any action of a real Lie group G on D_p extends to a holomorphic action of the universal complexification $G^{\mathbb{C}}$, which in our case has just the additive group $\mathbb{C}^n$ as connected component. This is a general fact proven by the first author in Ref. [27].

4. Examples of automorphisms of D_p not contained in OS(D_p)

In Ref. [13], it is shown that the overshear group is a proper subset of the automorphism group. In fact, using Nevanlinna theory, it is shown that the hyperbolic mapping

$$(x, y, z) \mapsto (x\,\mathrm{e}^z, y\,\mathrm{e}^{-z}, z)$$

is not contained in the overshear group. This is analogous to the result by Andersén [28], who showed that the automorphism of $\mathbb{C}^2$ defined by

$$(x, y) \mapsto (x\,\mathrm{e}^{xy}, y\,\mathrm{e}^{-xy})$$

is not finite compositions of shears. Hence, the shear group is a proper subgroup of the group of volume-preserving automorphisms. For another proof of this fact, see also Ref.

[15]. Note that our Classification Theorem 1.3 immediately implies that the elements of the $\mathbb{C}^*$-action $\lambda \mapsto (\lambda x, \lambda^{-1} y, z)$ cannot all be contained in $\mathrm{OS}(D_p)$, since there are no S^1-actions in $\mathrm{OS}(D_p)$.

We will present yet another way of finding an automorphism of a Danielewski surface which is not a composition of overshears.

Theorem 4.1: *Assume that* $\deg(p) \geq 4$. *Then, the overshear group* $\mathrm{OS}(D_p)$ *is a proper subset of the component of the identity of* $\mathrm{Aut}_{\mathrm{hol}}(D_p)$.

Proof: We look at complete algebraic vector fields on Danielewski surfaces. These are algebraic vectorfields which are globally integrable, however their flow maps are merely holomorphic maps. As shown in Ref. [29], there is always a $\mathbb{C}$- or a $\mathbb{C}^*$-fibration adapted to these vector fields. That is, there is a map $\pi : D_p \to \mathbb{C}$ such that the flow of the complete field θ maps fibers of π to fibers of π. These maps π have general fiber $\mathbb{C}$ or $\mathbb{C}^*$. In case of at least two exceptional fibers, the vector field θ has to preserve each fiber, i.e. it is tangential to the fibers of π. For example, the overshear fields in OS_1 have adapted fibration $\pi_0 : (x, y, z) \mapsto x$. They are tangential to this $\mathbb{C}$-fibration, the fibers outside $x = 0$ are parametrized by $z \in \mathbb{C}$ via $z \mapsto (x, p(z)/x, z)$. The exceptional fiber is $\pi_0^{-1}(0)$ consisting of $\deg(p)$ copies of $\mathbb{C}$, one for each zero z_i of the polynomial p and parametrized by $y \in \mathbb{C}$ via $y \mapsto (0, y, z_i)$. A typical example of a field with adapted $\mathbb{C}^*$-fibration is the hyperbolic field $x(\partial/\partial x) - y(\partial/\partial y)$ with adapted fibration $(x, y, z) \mapsto z$. There are $\deg(p)$ exceptional fibers at the zeros of the polynomial p, each of them isomorphic to the cross of axis $xy = 0$. The same $\mathbb{C}^*$-fibration is adapted to the field $f(z)(x(\partial/\partial x) - y(\partial/\partial y))$ for a nontrivial polynomial f.

Now take any complete algebraic vector field θ with an adapted $\mathbb{C}^*$-fibration (and thus generic orbits $\mathbb{C}^*$). Assume that the flow maps (or time-t maps) $\varphi_t \in \mathrm{Aut}_{hol}(D_p)$ of θ are all contained in the overshear group $\mathrm{OS}(D_p)$. Then by Theorem 1.3, this one-parameter subgroup $t \mapsto \varphi_t$ can be conjugated into O_1. This would mean that the one-parameter subgroup would be conjugate to a one-parameter subgroup of an overshear field $OF^x_{f,g}$ (since these are all complete fields respecting the fibration x). This would imply that the generic orbit of the overshear field is $\mathbb{C}^*$, which is equivalent to $f \neq 0$. However, the generic orbits of these fields $OF_{f,g}$ (isomorphic to $\mathbb{C}^*$) are not closed in D_p, they contain a fixed point in their closure. Thus our assumption that all φ_t are contained in $\mathrm{OS}(D_p)$ leads to a contradiction. In particular, we have shown that for any non-zero entire function f, there is a $t \in \mathbb{R}$ such that the time t-map of the hyperbolic field given by

$$(x, y, z) \mapsto (x\, \mathrm{e}^{f(z)t}, y\, \mathrm{e}^{-f(z)t}, z)$$

is not contained in $\mathrm{OS}(D_p)$. ■

Remark 4.1: More examples of complete algebraic vector fields on D_p with adapted $\mathbb{C}^*$-fibration can be found in the work of Leuenberger [10] who up to automorphism classifies all complete algebraic vector fields on Danielewski surfaces. Interesting examples (whose flow maps are not algebraic) are fields whose adapted $\mathbb{C}^*$-fibration is given by $(x, y, z) \mapsto x^m(x^l(z + a) + Q(x))^n$ for coprime numbers $m, n \in \mathbb{N}$, $a \in \mathbb{C}$ and $0 \leq l < \deg(Q)$. The exact formula for these fields can be found in the Main Theorem of loc.cit.

Remark 4.2: Without specifying a concrete automorphism which is not in the group generated by overshears, Andersén and Lempert use an abstract Baire category argument in Ref. [12] to show that the group generated by overshears in $\mathbb{C}^n$ is a proper subgroup of the group of holomorphic automorphisms $\mathrm{Aut}_{hol}(\mathbb{C}^n)$ of $\mathbb{C}^n$. We do believe that such a proof could work in the case of Danielewski surfaces as well.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Makar-Limanov L. On groups of automorphisms of a class of surfaces. Israel J Math. 1990;69(2):250–256.
[2] Makar-Limanov L. On the group of automorphisms of a surface $x^n y = P(z)$. Israel J Math. 2001;121:113–123.
[3] Crachiola AJ. On automorphisms of Danielewski surfaces. J Algebraic Geom. 2006;15(1): 111–132.
[4] Daigle D. Locally nilpotent derivations and Danielewski surfaces. Osaka J Math. 2004;41(1): 37–80.
[5] Daigle D. On locally nilpotent derivations of $kX_1, X_2, Y/(\phi(Y) - X_1X_2)$. J Pure Appl Algebra. 2003;181(2–3):181–208.
[6] Dubouloz A. Additive group actions on Danielewski varieties and the cancellation problem. Math Z. 2007;255(1):77–93.
[7] Dubouloz A, Poloni PM. On a class of Danielewski surfaces in affine 3-space. J Algebra. 2009;321(7):1797–1812.
[8] Kaliman S, Kutzschebauch F. Density property for hypersurfaces $UV = P(\overline{X})$. Math Z. 2008;258(1):115–131.
[9] Forstnerič F, Kutzschebauch F. The first thirty years of Andersen–Lempert theory. arXiv:2111.08802.
[10] Leuenberger M. Complete algebraic vector fields on Danielewski surfaces. Ann Inst Fourier (Grenoble). 2016;66(2):433–454.
[11] Kutzschebauch F, Lind A. Holomorphic automorphisms of Danielewski surfaces I – density of the group of overshears. Proc Am Math Soc. 2011;139(11):3915–3927.
[12] Andersén E, Lempert L. On the group of holomorphic automorphisms of $\mathbb{C}^n$. Invent Math. 1992;110(2):371–388.
[13] Andrist R, Kutzschebauch F, Lind A. Holomorphic automorphisms of Danielewski surfaces II – structure of the overshear group. Journal of Geometric Analysis, 25(3):1859–1889.
[14] Ahern P, Rudin W. Periodic automorphisms of $\mathbb{C}^n$. Indiana Univ Math J. 1995;44(1): 287–303.
[15] Kutzschebauch F, Kraft HP. Equivariant affine line bundles and linearization. Math Res Lett. 1996;3(5):619–627.
[16] Andersén E. Algebraic and analytic properties of groups of holomorphic automorphisms of $\mathbb{C}^n$ [doctorial thesis]. Lund: Lunds tekniska högskola; 1994. ISSN:0347-8475.
[17] De Fabritiis C. One-parameter groups of volume-preserving automorphisms of $\mathbb{C}^2$. Rend Istit Mat Univ Trieste. 1994;26(1–2):21–47.
[18] Ahern P, Forstnerič F. One parameter automorphism groups on $\mathbb{C}^2$. Complex Var Theory Appl. 1995;27(3):245–268.
[19] Ahern P, Forstnerič F, Varolin D. Flows on $\mathbb{C}^2$ with polynomial time one map. Complex Var Theory Appl. 1996;29(4):363–366.
[20] Serre JP. Trees. Berlin: Springer-Verlag; 1980.

[21] Silva Leite F. Bounds on the order of generation of $SO(n, R)$ by one-parameter subgroups, current directions in nonlinear partial differential equations (Provo, UT, 1987). Rocky Mountain J Math. 1991;21(2):879–911.
[22] Levi EE. Sulla struttura dei gruppi finiti e continui. In: Atti della Reale Accademia delle Scienze di Torino. B XL. 1905;551–565.
[23] Iwasawa K. On some types of topological groups. Ann Math. 1949;50(3):507–558.
[24] Malcev AI. On a class of homogeneous spaces. Izv Akad Nauk SSSR Ser Mat. 1949;13:9–32 [Russian].
[25] Dynkin EB. Calculation of the coefficients in the Campbell–Hausdorff formula. Dokl Akad Nauk SSSR (NS). 1947;57:323–326 [Russian].
[26] Akhiezer DN. Lie group actions in complex analysis. Braunschweig: Friedr. Vieweg Sohn; 1995. (Aspects of mathematics; E27).
[27] Kutzschebauch F. Compact and reductive subgroups of the group of holomorphic automorphisms of $\mathbb{C}^n$, singularities and complex analytic geometry (Japanese) (Kyoto, 1997). Surikaisekikenkyusho Kokyuroku. 1998;1033:81–93.
[28] Andersén E. Volume-preserving automorphisms of $\mathbb{C}^n$. Complex Var Theory Appl. 1990;14(1–4):223–235.
[29] Kaliman S, Kutzschebauch F, Leuenberger M. Complete algebraic vector fields on affine surfaces. Int J Math. 2020;31(3):Article ID 2050018, 50 pp.