

## 3 definitions of BF theory on homology 3-spheres

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**ABSTRACT:** 3-dimensional BF theory with gauge group  $G$  (= Chern-Simons theory with non-compact gauge group  $TG$ ) is a deceptively simple yet subtle topological gauge theory. Formally, its partition function is a sum/integral over the moduli space  $\mathcal{M}$  of flat connections, weighted by the Ray-Singer torsion. In practice, however, this formal expression is almost invariably singular and ill-defined.

In order to improve upon this, we perform a direct evaluation of the path integral for certain classes of 3-manifolds (namely integral and rational Seifert homology spheres). By a suitable choice of gauge, we sidestep the issue of having to integrate over  $\mathcal{M}$  and reduce the partition function to a finite-dimensional Abelian matrix integral which, however, itself requires a definition. We offer 3 definitions of this integral, firstly via residues, and then via a large  $k$  limit of the corresponding  $G \times G$  or  $G_{\mathbb{C}}$  Chern-Simons matrix integrals (obtained previously). We then check and discuss to which extent the results capture the expected sum/integral over all flat connections.

**KEYWORDS:** Chern-Simons Theories, Topological Field Theories

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## 1 Introduction

Gauge theories with non-compact gauge groups are notoriously difficult to make sense of, issues including questions of convergence and propagators with the wrong signs which make unitarity and hence the physical meaning of the theory far from clear. One set of theories for which one may make some headway are topological field theories where a particle

interpretation is not required. Our first investigation in that direction where the partition function can be evaluated exactly [1] was to consider the Schwarz type theories [2–4] formally representing the Ray-Singer Torsion [5, 6] which have non-compact symmetries. These symmetries though are essentially Abelian and we would like to consider now a class of theories with non-Abelian non-compact gauge groups. In particular we have in mind the topological  $BF$  theories [7–10] based on a compact gauge group  $G$ . In two dimensions such a theory is the zero coupling limit of Yang-Mills theory with gauge group  $G$  and as such does not have an associated non-compact symmetry. In dimension greater than or equal to 3, however,  $BF$  theories possess additional non-compact shift symmetries. We will focus on 3-dimensional  $BF$  theory here, where the combined gauge group turns out to be the non-compact group  $TG$ , the tangent bundle group of  $G$ .

Witten [11] first introduced 3-dimensional  $BF$  theories as a variant of Chern-Simons theory [12] for non-compact gauge groups (in fact, it can be regarded as a Chern-Simons theory for the non-compact gauge group  $TG$ ). The action is

$$I_{BF} = \int_M \text{Tr} (B \wedge F_A) \tag{1.1}$$

where  $A$  is a connection on a  $G$ -bundle over  $M$  and  $B$  is a Lie algebra valued 1-form. The action enjoys the usual compact  $G$  gauge symmetry, as well as a non-compact shift gauge symmetry  $B \rightarrow B + d_A \Lambda$ . This theory is deceptively simple. The path integral over  $B$  yields a delta function constraint on the curvature 2-form  $F_A$ , so that formally the path integral for the partition function of  $BF$  theory is

$$Z_{BF}[M, G] = \int DADB \exp \left( i \int_M \text{Tr} (B \wedge F_A) \right) \simeq \int DA \delta (F_A) \tag{1.2}$$

This seems to suggest that the path integral simply reduces to an integral over the moduli space of flat connections, with some measure. In general, however, this is not correct. Rather, the complete classical equations of motion are

$$F_A = 0, \quad d_A B = 0 \tag{1.3}$$

which taken together (and modulo gauge transformations) can formally be viewed as describing the tangent bundle of the moduli space  $\mathfrak{M}[M, G]$  of flat  $G$  connections on the 3-manifold  $M$ ,  $T\mathfrak{M}[M, G]$ . The partition function is then an integral over this space (and ghost zero modes etc.), again with some measure to be determined. However, in general this space can be very singular and the finite dimensional integral is ill-defined. The various types of singularities that lurk inside the path integral associated with the action (1.1) include issues with reducible connections (manifested as ghost zero modes) as well as the non-compactness of the moduli space itself (due to  $B$  zero modes).

One situation in which it is pretty clear what sort of contribution one should find from the path integral is when one expands the path integral around an isolated irreducible flat connection. In that case, there are no non-trivial solutions to the equations of motion for  $B$  and no ghost zero modes, and following Witten [11], the contribution of such an *acyclic* connection  $\omega$  on a 3-manifold  $M$  can be shown to be precisely the Ray-Singer analytic

torsion  $\tau_M(\omega)$  [5, 6] of that connection. Thus on manifolds on which the moduli space of flat connections includes such acyclic flat connections, one should perhaps expect the partition function to take the form

$$Z_{BF}[M, G] = \sum_{\text{acyclic}} \tau_M(\omega_{(\alpha)}) + \dots \tag{1.4}$$

where the ellipses indicate the contributions from the non-acyclic flat connections. If  $M$  is an Integral Homology Sphere ( $\mathbb{Z}$  Homology Sphere,  $H_1(M, \mathbb{Z}) = 0$ ), then for  $G = \text{SU}(2)$  the trivial connection is the only reducible flat connection, and there are indeed  $\mathbb{Z}$  Homology Spheres for which all the flat connections are isolated. In such a situation one should expect the  $BF$  partition function to be largely captured by the above expression. One of the aims of this paper will be to check to which extent this expectation is borne out by actual explicit evaluations of the path integral of  $BF$  theory.

Moreover, by using the Massive Ray-Singer Torsion  $\tau_M(\omega, \mu)$  introduced in [1] (among other reasons precisely with its application to  $BF$  theory in mind), one can extend the considerations of Witten to the case that the flat connections are not necessarily irreducible and isolated, and thus obtain a formal expression for the regularised partition function of  $BF$  theory (section 2), which takes the form (2.20)

$$Z_{BF}[M, G; \mu] = \int \tau_M(\omega_{(\alpha)}, \mu) \tag{1.5}$$

of a sum / integral over the moduli space of flat connections. This expression can in turn be expanded in the mass-parameter  $\mu$  to reproduce different kinds of contributions to the partition function [1]. However, this approach outlined above to defining the partition function of  $BF$  theory is rather formal: one would need to determine by hand the spaces of flat connections and zero modes associated with them, calculate the (massive) Ray-Singer torsion by some means, etc. So this does not all by itself lead to an *evaluation* of the  $BF$  partition function via path integral methods.

In order to be able to verify to which extent an actual evaluation of the  $BF$  path integral reproduces the above formal expectations, one needs a class of manifolds for which explicit path integral calculations can be performed. To that end we will concentrate on Seifert fibred 3-manifolds. These spaces allow for a significant simplification in the calculations via suitable gauge choices. The first of these is that the ‘time’ (fibre) components of both  $A$  and  $B$  (the components of  $A$  and  $B$  along the fibres of the defining Seifert fibrations) can be taken to be constant along the fibre. This is a generalisation of the usual temporal gauge  $\partial_0 A_0 = 0$  of a gauge field on a circle, and allows one to partially push the calculation down a dimension, to the base of the Seifert fibration. And now further simplification comes from the approach [13–18] to evaluating the path integral through Abelianisation. For Chern-Simons theory with gauge group  $G$  the end result is a finite dimensional integral over the Cartan subalgebra  $\mathfrak{t}$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . One of the characteristic features of this approach is that the reduction to a finite-dimensional integral (over a linear space) completely bypasses the need to define (and integrate over) the moduli space of flat connections. Nevertheless, this approach can be shown to reproduce the results of perturbative or localisation calculations (when available), which in principle require an exact evaluation of the latter.

Returning to  $BF$  theory, the main technical result of this paper is to show that this calculational method can be extended to  $BF$  theory with its non-compact gauge group  $TG$ . The result is a finite-dimensional integral over the space  $\mathfrak{t} \oplus \mathfrak{t}$ , which plays the role of the Cartan subalgebra of the Lie algebra of  $TG$ . This once again bypasses the need to integrate over the moduli space of flat connections. The finite-dimensional integrals that we find are similar to those for Chern-Simons theory, but they are more singular than their Chern-Simons counterparts, having poles on the integration contour an integral over  $\lambda$  which is a distribution, and hence require a definition. For example, the partition function for  $SU(2)$  and  $M$  a  $\mathbb{Z}$  Homology Sphere devolves to

$$Z_{BF}[M, SU(2)] \simeq \sum_{n_0} \int_{\mathbb{R} \times \mathbb{R}} \tau_M(\phi) \cdot \exp(i\lambda(\phi/P + 2\pi i n_0)) \tag{1.6}$$

(see the body of the paper for all the definitions). The important point to note here is that the Ray-Singer Torsion  $\tau_M(\phi)$  has poles at  $\phi = m\pi$   $m \neq sP$  and zero when  $m$  is proportional to  $P$ . The  $\lambda$  integral sets  $\phi = Pn_0$  which formally avoids the poles. Still we can only conclude what the integral should be after it is properly regularised (defined).

In the case of Chern-Simons theory the integrals that we find there also have poles, however, there we could argue that the contributions at the poles should not be included (lying on the walls of the Weyl chamber) and we introduced a mass regulator that essentially allowed us to avoid them. Unfortunately, in the present setting of  $BF$  theory, we do not have a guiding principle and so we offer 3 different possible definitions of  $BF$  theory and discuss their advantages as well as their shortcomings:

1. Direct Definition via Residues

The first, and the one we spend most time on (section 4), is through the direct definition of the finite-dimensional integral (1.6). In order to get a handle on the poles of the Ray-Singer Torsion on  $\mathfrak{t} \times \mathfrak{t}$ , one defines the theory in such a way that it is given by the residue of all the possible poles including delta function contributions. We find that, rather remarkably, this does reproduce the expected contributions of *reducible* flat connections which we can follow by making use of the Massive Ray-Singer Torsion. In particular we show in detail how this gives the expected results, as a sum over contributions from the isolated non-trivial flat connections, for Lens spaces. Indeed, this definition appears to capture the essence of the contribution of reducible connections in general. However, this approach does not reproduce the expected result (1.4) for connections that are isolated and irreducible. We note that, suprisingly, some of the poles on the contour correspond to irreducible flat connections, while others correspond to ‘complex’ flat connections. We offer a brief attempt at an explanation (related to gauge fixing) for why the irreducible connections arise in this way.

In view of this, we approach the problem from a different point of view in section 5, based on the fact that the gauge group  $TG$  can be regarded as a contraction of either the compact group  $G \times G$  or the complex group  $G_{\mathbb{C}}$ . Correspondingly, the other 2 definitions

that we consider are to regard  $BF$  theory as arising either in the large  $k$  limit of  $G \times G$  Chern-Simons theory at levels  $k$  and  $-k$ , or in certain limits of the coupling  $t = k + is$  of  $G_{\mathbb{C}}$  Chern-Simons theory. This regularises the theory to some extent and also side-steps the gauge-fixing issue alluded to above.

## 2. Definition via a large $k$ Limit of $G \times G$ Compact Chern-Simons Theory

The advantage of using the compact  $G \times G$  theory is that the Hilbert space of states is finite dimensional and only goes over to the infinite dimensional Hilbert space of states of  $BF$  theory in the limit. This therefore acts as a natural regulator. A disadvantage is that certain conditions must be met by the connections so that the correspondence that we need exists. We offer examples of Brieskorn  $\mathbb{Z}$  Homology Spheres (Seifert manifolds with 3 exceptional fibres) [19] where the conditions are met and one obtains the partition function of  $BF$  theory in the expected form (1.4). These conditions are not met by all manifolds, however. For example, the equality between  $BF$  theory and the large  $k$  limit of the  $G \times G$  theory fails for certain Lens spaces. As the Lens spaces are  $\mathbb{Q}$  Homology Spheres one would like to conjecture that the correspondence holds for isolated flat connections of  $\mathbb{Z}$  Homology Spheres in general.

## 3. Definition via a large $k$ Limit of $G_{\mathbb{C}}$ Complex Chern-Simons Theory

As for  $G_{\mathbb{C}}$  Chern-Simons theory [20–22] with complex level  $k+is$ , there are many formal correspondences with  $BF$  theory. The most obvious is to set  $k = 0$  and let  $s \rightarrow \infty$ , which leads directly to the  $BF$  action (1.1) and its non-compact symmetries (2.2). Given the difficulties we face with this action, this is not the limit that we consider. Rather, we take  $s = 0$  and  $k \rightarrow \infty$ . An advantage of this approach is that the finite dimensional integrals that arise are slight variants of those discussed by Lawrence and Rozansky [24] for  $SU(2)$  and for general  $G$  by Mariño [25] in the context of  $G$  Chern-Simons theory. In principle then the same strategies that apply there can be used here, though we leave that for the future.

To the extent that the perturbative large  $k$  evaluation of Chern-Simons theory reduces to a sum over contributions from flat connections, this is then also true for the partition function of  $BF$  theory. Overall, the definition of  $BF$  theory via  $G_{\mathbb{C}}$  Chern-Simons theory appears to be the most complete (however, one needs to ensure that one does not count complex flat connections that cannot be conjugated into flat  $G$  connections).

One approach not followed here but which is certainly of interest is resurgence [26]. Gukov, Mariño and Putrov [27] show that for  $SU(2)$  one can start with the Abelian contribution to the Chern-Simons partition function in the large  $k$  limit and Borel resum in order to obtain contributions from non-Abelian flat connections. Once in the complex space of connections there is the possibility that  $SL(2, \mathbb{C})$  connections which are not (conjugate to)  $SU(2)$  flat connections will contribute. They show that miraculously this does not happen. The importance of this for us is that, as we will see, we appear to obtain contributions from connections in our first approach that would, from this point of view be considered to be ‘complex’.

This paper is organised as follows. Section 2 follows Witten’s [11] approach to obtaining the partition function for isolated irreducible flat connections. There we also consider what formally happens when the connections are either not isolated or are reducible. In these cases we can profitably make use of the massive Ray-Singer Torsion introduced in [1]. This allows us to express our expectations for the form of the partition function in concrete terms.

In section 3 we formulate  $BF$  theory on Seifert 3-manifolds which are either  $\mathbb{Z}$  or  $\mathbb{Q}$  homology spheres. Particular attention is paid to issues with gauge fixing of  $\phi$  and  $\lambda$ , the components of  $A$  and  $B$  along the fibre respectively. We will show that  $\phi$  and (with an assumption about  $\phi$ )  $\lambda$  can be chosen to be constant along the fibre (temporal gauge), and to simultaneously take values in a fixed Cartan subalgebra  $\mathfrak{t} \oplus \mathfrak{t}$  of  $TG$  (Abelianisation in  $TG$ ). We then give a brief outline of how to pass from the functional integral to a finite dimensional integral over  $\mathfrak{t} \oplus \mathfrak{t}$ .

As already mentioned, the integral in question has poles on the integration contour and so requires a definition. In section 4 we thus give our first definition of  $BF$  theory, as a residue of a particular function related to the integrand of the finite dimensional integral, and analyse its consequences. Given the shortcomings of that definition, the alternative Definitions 2 and 3 described above are explored in section 5.

Certain technical details are relegated to an appendix (A for information about the group  $TG$  and its Lie algebra, and B for details about a certain useful discrete symmetry of complex Chern-Simons theory on Seifert manifolds).

## 2 $BF$ -theory on a 3-manifold and path integrals

In this section we give a brief review of 3-dimensional  $BF$  theory and Witten’s formula for the path integral which holds when there are no zero modes. This is followed by a discussion of a generalisation which takes zero modes into account.

### 2.1 $BF$ theory as a $TG$ Chern-Simons theory

As recalled in the Introduction, 3-dimensional  $BF$  theory on a 3-manifold  $M$  is defined by the action (1.1)

$$I_{BF} = \int_M \text{Tr}(B \wedge F_A) \tag{2.1}$$

with  $A$  a connection on a  $G$ -bundle over  $M$  and  $B \in \Omega^1(M, \mathfrak{g})$ . The gauge symmetries of this action are the usual  $G$  gauge symmetry acting on  $A$  and  $B$ , as well as a shift gauge symmetry for the field  $B$ ,

$$(A, B) \rightarrow (g^{-1}Ag + g^{-1}dg, g^{-1}(B + d_A\Lambda)g) \tag{2.2}$$

This shift gauge symmetry is non-compact, parametrised by  $\Lambda \in \Omega^0(M, \mathfrak{g})$ , and thus, even for  $G$  a compact Lie group, 3-dimensional  $BF$  theory is an example of a gauge theory with a non-compact gauge group (and this is the main reason we are interested in this theory here). In fact,  $BF$ -theory is not just some non-compact analogue of Chern-Simons theory: in a precise sense it is a Chern-Simons theory for the non-compact gauge group  $TG$ , the tangent bundle group of  $G$ . As it will frequently be useful to have this perspective in the



back of one's mind in the following, here (and in more detail in appendix A) we quickly review the relevant facts regarding the group  $TG$ , its Lie algebra  $\mathfrak{tg}$ , and Chern-Simons theory based on it.

Thus, consider a compact semi-simple Lie group  $G$  (throughout we will also assume that  $G$  is connected and simply-connected) with Lie algebra  $\mathfrak{g}$ , a basis of generators  $j_a$  and commutation relations  $[j_a, j_b] = f_{ab}^c j_c$ . Then the Lie algebra  $\mathfrak{tg}$  of the Lie group  $TG$  has generators  $(j_a, p_a)$  and commutation relations (A.8)

$$[j_a, j_b] = f_{ab}^c j_c, \quad [j_a, p_b] = [p_a, j_b] = f_{ab}^c p_c, \quad [p_a, p_b] = 0. \quad (2.3)$$

For the considerations of section 5 it will be useful to keep in mind, that this algebra can be obtained as a contraction of the Lie algebra of  $G \times G$  or  $G_C$  (see (A.12)).

Given an invariant non-degenerate scalar product  $g_{ab} = \text{Tr } j_a j_b$  on  $\mathfrak{g}$ , an invariant and non-degenerate scalar product on  $\mathfrak{tg}$  is (A.18)

$$\ll j_a, j_b \gg = \ll p_a, p_b \gg = 0, \quad \ll j_a, p_b \gg = g_{ab}. \quad (2.4)$$

A  $TG$ -gauge field can be expanded as

$$C = A^a j_a + B^a p_a, \quad (2.5)$$

with field strength

$$F_C = dC + \frac{1}{2}[C, C] = F_A^a j_a + d_A B^a p_a. \quad (2.6)$$

The Chern-Simons action for  $C$  with respect to the above scalar product is (with a for present purposes convenient choice of normalisation, and with an integration by parts)

$$I_{CS} \equiv \frac{1}{2} \int_M \ll C, dC + \frac{1}{3}[C, C] \gg = \int_M \text{Tr } B \wedge F_A = I_{BF} \quad (2.7)$$

The equations of motion (1.3) of  $BF$ -theory are then evidently simply the flatness conditions for the connection  $C$ ,

$$F_C = 0 \quad \Leftrightarrow \quad F_A = 0, \quad d_A B = 0 \quad (2.8)$$

Moreover, the gauge symmetries (2.2) of  $BF$ -theory are precisely the  $TG$  gauge symmetries of the  $TG$ -connection  $C$ .

## 2.2 Path integral for $BF$ theory: formal considerations

Evidently (and as recalled in the Introduction), formally, the path integral for the partition function of  $BF$  theory is

$$\begin{aligned} Z_{BF}[M, G] &= \int DADB \exp \left( i \int_M \text{Tr} (B \wedge F_A) \right) \\ &\simeq \int DA \delta(F_A) \end{aligned} \quad (2.9)$$

so that we may expand about flat connections to give a more complete evaluation. Following Witten [11], we use the standard Faddeev-Popov covariant gauge-fixing procedure around



these flat connections. More details about gauge fixing Chern-Simons theory with non-compact gauge groups can be found in [28]. The action and symmetries about a flat background connection  $\omega_{(\alpha)}$  are

$$I_{BF} = \int_M \text{Tr} \left( B \wedge F_{A+\omega_{(\alpha)}} \right) \tag{2.10}$$

$$(A + \omega_{(\alpha)}, B) \rightarrow (g^{-1}(A + \omega_{(\alpha)})g + g^{-1}dg, g^{-1} \left( B + d_{A+\omega_{(\alpha)}} \Lambda \right) g) \tag{2.11}$$

while the covariant background field gauge fixing conditions

$$d_{\omega_{(\alpha)}} * A = 0. \quad d_{\omega_{(\alpha)}} * B = 0 \tag{2.12}$$

require us to make a choice of metric on  $M$ . Then the ghost and gauge fixing action is

$$I_{GF} = \int_M \text{Tr} \left( ud_{\omega_{(\alpha)}} * B + vd_{\omega_{(\alpha)}} * A + \bar{f}d_{\omega_{(\alpha)}} * d_{(\omega_{(\alpha)}+A)}f + \bar{g}d_{\omega_{(\alpha)}} * d_{(\omega_{(\alpha)}+A)}g \right) \tag{2.13}$$

If the space of flat connections is not made up of isolated points then one would need to integrate over them which one could do through the introduction of collective coordinates. One should also take into account the possible zero modes of both  $B$  and of the ghosts. Those zero modes form non-compact spaces  $H_{\omega_{\alpha}}^1(M, \mathfrak{g})$ , and  $H_{\omega_{\alpha}}^0(M, \mathfrak{g})$  respectively. Taking the above caveats into account the path integral formally becomes

$$\begin{aligned} Z_{BF}[M, G] &= \sum_{\alpha} \int DADB \exp(iI_{BF} + iI_{GF}) \\ &= \int_{T\mathfrak{M}_{\alpha}} \tau_M(\omega_{(\alpha)}) \end{aligned} \tag{2.14}$$

Here the sum and integral symbol over  $T\mathfrak{M}_{\alpha}$  is meant to indicate that one sums over distinct components of the moduli spaces of classical solutions and integrates over the moduli of continuous families of solutions. These spaces include the zero modes of  $A$  and  $B$  as well as of those of the ghosts  $f$  and  $g$  (the multiplier field and anti-ghost zero modes canonically cancel each other). The space  $T\mathfrak{M}_{\alpha}$  space can be very singular and so (2.14) as it stands is rather symbolic in general.

On the 3-manifolds of interest in this paper, the moduli space includes acyclic flat connections, i.e. flat connections which are isolated and irreducible, so that there are no zero modes at all at these solutions. The path integral around such a connection simplifies by using the following scaling argument. About such an isolated and irreducible connection send

$$A \rightarrow tA, \quad B \rightarrow t^{-1}B, \quad t \in \mathbb{R}_+ \tag{2.15}$$

(with compensating scaling for the multiplier fields). This transformation has trivial Jacobian, and the action remains well defined in the limit  $t \rightarrow 0$ . Indeed, in this limit,

$$\begin{aligned} I_{BF} &\rightarrow \int_M B d_{\omega_{(\alpha)}} A \\ I_{GF} &\rightarrow \int_M \text{Tr} \left( ud_{\omega_{(\alpha)}} * B + vd_{\omega_{(\alpha)}} * A + \bar{f} * \Delta_{\omega_{(\alpha)}} f + \bar{g} * \Delta_{\omega_{(\alpha)}} g \right) \end{aligned} \tag{2.16}$$

and the partition function around this connection is simply a standard path integral representation [3, 9] of the Ray-Singer torsion [5]  $\tau_M(\omega_{(\alpha)})$  of the flat connection  $\omega_{(\alpha)}$ ,

$$\tau_M(\omega_{(\alpha)}) = \left(\text{Det } \Delta_{\omega_{(\alpha)}}^0\right)^{3/2} \cdot \left(\text{Det } \Delta_{\omega_{(\alpha)}}^1\right)^{-1/2} \tag{2.17}$$

(the superscripts on the twisted Laplacians indicating the degrees of the forms on which they act). Thus on such 3-manifolds, the partition function now becomes

$$Z_{BF}[M, G] = \sum_{\text{acyclic}} \tau_M(\omega_{(\alpha)}) + \dots \tag{2.18}$$

where the ellipses indicate the rest of the contributions to the path integral. As we have already explained, there may be zero modes to deal with in the extra terms (2.18) which manifest themselves as zeros of the determinants in (2.17) and which invalidate the simplifications that we were allowed to make for the isolated irreducible connections.

One way to proceed is to adopt the prescription of Ray and Singer [6] by first excising the zero modes and then adding a correction term to ensure metric independence. This ‘extra’ gauge fixing of the zero modes may be achieved by a BRST procedure [9] the details of which have been explained in some detail in [1]. The advantage of this method is that then the torsion is a natural measure for the finite dimensional integral that appears in (2.14). While this defines the integrand, one is still confronted with the problem of determining and defining the space over which this is to be integrated. Thus, while formally this appears to be a good definition, at a practical level it seems somewhat intractable at the moment.

### 2.3 Path integral for *BF* theory and massive Ray-Singer torsion

An alternative method for regularising such zero modes, and for keeping track of the ellipses in the formula (2.18) was advocated in [1]. The idea is to add a mass to the Laplacians that appear in (2.17), thus lifting all the zero modes and side-stepping (or at least initially bypassing) the problem of having to deal with them directly. As a first step, what this means is that instead of using the twisted Laplacian  $\Delta_\omega$  in the definition of the Ray-Singer Torsion we instead use the massive Laplacian  $\Delta_\omega + \mu^2$  which now manifestly has a positive definite spectrum. The Massive Ray-Singer Torsion, on a 3-manifold, for a flat connection  $\omega$ , is then defined to be

$$\tau_M(\omega, \mu) = \left(\text{Det } (\Delta_\omega^0 + \mu^2)\right)^{3/2} \cdot \left(\text{Det } (\Delta_\omega^1 + \mu^2)\right)^{-1/2} \tag{2.19}$$

One can then define a regularised partition function by

$$Z_{BF}[M, G; \mu] = \sum \tau_M(\omega_{(\alpha)}, \mu) \tag{2.20}$$

where we do not necessarily attempt to integrate over the tangent space as those modes have been lifted. Here a caveat is required: in [1] we showed that for manifolds which are mapping tori it is possible to maintain complete gauge invariance even with the introduction of a mass. However, a mass term cannot be introduced in a gauge invariant way on a general 3-manifold. We accept that the gauge symmetry is broken and may be reinstated if required by some renormalisation.

The way in which one obtains the actual Ray-Singer Torsion from the massive Ray-Singer torsion is to take a limit

$$\tau_M(\omega) \equiv \lim_{\mu \rightarrow 0} \mu^{-3 \dim H_\omega^0 + \dim H_\omega^1} \tau_M(\omega, \mu) \tag{2.21}$$

In this way the  $BF$  partition function will then take the form

$$Z_{BF}[M, G, \mu] = \sum \mu^{3 \dim H_\omega^0 - \dim H_\omega^1} (\tau_M(\omega) + \dots) \tag{2.22}$$

and we will need to specify which terms we are interested in. The ellipses in this formula refer to terms higher order in  $\mu$  than the zero-th order Torsion (essentially constants and derivatives of the Ray-Singer Torsion).

At a formal level this is, perhaps, as far as one may go. It must be said that this situation is not very satisfying and has calculational drawbacks. The formula (2.20) is very implicit and requires knowledge outside of the path integral in order to be used. The flat connections need to be found, the cohomology groups about the flat connections must be determined and a formula for the Massive Ray-Singer Torsion must be given.

It is therefore of considerable interest to consider 3-manifolds for which the BF partition function can also be calculated directly and explicitly, and where the result can be compared with the formal expectations for the partition function outlined above. One such class of manifolds is Seifert manifolds.

### 2.4 Expectations for the partition function on Seifert manifolds

The Ray-Singer torsion for Seifert 3-manifolds is known (through its equivalence to Reidemeister Torsion [29]). About an acyclic flat connection, i.e. with trivial twisted cohomology, it is

$$\tau_M(\omega) = \tau_{S^1}(\phi)^{2-N} \prod_{i=1}^N \tau_{S^1}(\phi/a_i) \tag{2.23}$$

where  $\phi$  is the component of the connection in the direction of the fibre of the Seifert 3-manifold and  $N$  (the number of orbifold points) and  $a_i$  (the order of the isotropy group at the  $i$ 'th orbifold point) are integers that are part of the defining data of a Seifert 3-manifold (see section 3.1). We should also specify the representation of the group  $G$  for which we are evaluating the Torsion; however, as throughout we fix on the adjoint representation, we do not need to indicate it in the notation.

Including the contributions from the non-acyclic connections, one expects the partition function to take the form (2.18)

$$Z_{BF}[M, G] \simeq \sum_{\text{acyclic}} \tau_{S^1}(\phi)^{2-N} \prod_{i=1}^N \tau_{S^1}(\phi/a_i) + \dots \tag{2.24}$$

and, as in the general situation above, one way to keep track of the ellipses in the above formula is to use the massive Ray-Singer torsion. In the case of Seifert manifolds all we

need is the massive Ray-Singer torsion on the circle, which is well understood [1]. Thus a suitably regularised version of (2.24) is

$$Z_{BF}[M, G; \mu] = \sum \tau_{S^1}(\phi, \mu)^{2-N} \prod_{i=1}^N \tau_{S^1}(\phi/a_i, \mu) + \dots \quad (2.25)$$

One can now use a mass expansion to identify the individual contributions to the regularised partition function.

The analysis that we have presented thus far, in the covariant gauges, is an extension of that of Witten to those flat connections which are not necessarily flat and isolated. One shortcoming with this approach, as we have already explained, is that with a background field approach one must, by independent means, find the flat connections and determine the cohomology groups about them. One would prefer an approach that evaluates the path integral directly and does not require the split between the classical and quantum fields. The rest of the paper is devoted to producing just such an approach.

In particular, we will show in section 3 that the procedure developed in [13–18] to reduce the partition function of Chern-Simons theory for a compact gauge group  $G$  to a finite-dimensional integral (over the Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ ) can be extended to the case at hand, namely  $BF$  theory, or Chern-Simons theory with gauge group  $TG$ .

The finite dimensional integrals have poles on the contour of integration and so it then remains to define and give a meaning to them. As we will see in section 4, a direct calculation of this finite-dimensional integral reproduces the expected form of the partition function only partially. In particular it captures the contributions of reducible flat connections as expected but surprisingly does not give the expected results for the isolated and irreducible flat connections.

This prompts us to investigate two other possible definitions in section 5 which do lead to the expected form of the partition function discussed in this section.

### 3 $BF$ -theory on a Seifert fibred 3-manifold

In this section we specialise to Seifert fibred 3-manifolds. The extra structure afforded by the  $S^1$  principal bundle structure (over an orbifold base) allows for convenient choices of gauge as well as for regularising the Ray-Singer Torsion with the introduction of a mass term [1]. We are very brief with the background material on Seifert manifolds as it has appeared before. We spend some time on the gauge fixing as this is quite new for the  $TG$  theories, while the evaluation of the determinants is so close to that of the determinants evaluated in [17] that we borrow liberally from there.

#### 3.1 Seifert 3-manifolds briefly

From now on  $M$  denotes a Seifert 3-manifold. Then  $M$  is a 3-manifold which is a circle  $V$ -bundle over the 2 dimensional orbifold  $\Sigma_V$  of genus  $g$  with  $N$  orbifold points. A general Seifert manifold is written as  $M[\text{deg } \mathcal{L}_M, g, (a_1, b_1), \dots, (a_N, b_N)]$  where the  $a_i$  are the isotropies of the orbifold points, the  $b_i$  are the weights of the line  $V$ -bundle at the orbifold

points and  $\text{deg } \mathcal{L}_M$  is the degree of that line bundle. As the Seifert 3-manifold is the circle V-bundle of the line V-bundle  $\mathcal{L}_M$  it is also designated by  $S(\mathcal{L}_M)$ .

Let  $\mathcal{L}_0$  be a topological line bundle at some smooth point on  $\Sigma_V$  with degree 1 and  $\mathcal{L}_i$  be the line V-bundles at the  $i$ -th orbifold points with degree  $1/a_i$  then

$$\mathcal{L}_M = \mathcal{L}_0^{b_0} \otimes \mathcal{L}_1^{b_1} \otimes \dots \otimes \mathcal{L}_N^{b_N} \tag{3.1}$$

and  $M$  is the circle bundle of  $\mathcal{L}_M$  [30] (we use normalised Seifert data so that  $1 \leq b_i < a_i$ ).

The Seifert 3-manifold is smooth iff  $\text{gcd}(a_i, b_i) = 1$  for each  $i$ . It is a  $\mathbb{Z}$  Homology Sphere ( $H_1(M, \mathbb{Z}) = 0$ ) iff the line bundle  $\mathcal{L}_M$  that defines it satisfies

$$g = 0, \quad c_1(\mathcal{L}_M) = b_0 + \sum_{i=1}^N \frac{b_i}{a_i} = \pm \frac{1}{a_1 \dots a_N} \equiv \pm \frac{1}{P} \tag{3.2}$$

(here we have introduced the notation  $P = a_1 \dots a_N$ ). One consequence of these conditions is that  $\text{gcd}(a_i, a_j) = 1$  for  $i \neq j$ . If one takes a tensor power of this line V-bundle,  $\mathcal{L}_{M_d} = \mathcal{L}_M^{\otimes d}$ , then the Seifert manifold  $M_d = M/\mathbb{Z}_d$  is a  $\mathbb{Q}$  Homology Sphere ( $H_1(M_d, \mathbb{Q}) = 0$ ) with

$$g = 0, \quad c_1(\mathcal{L}_{M_d}) = c_1(\mathcal{L}_M^{\otimes d}) = \pm \frac{d}{a_1 \dots a_N} = \pm \frac{d}{P} \tag{3.3}$$

and

$$|d| = |H_1(M_d, \mathbb{Z})| \tag{3.4}$$

These are the 3-manifolds that we will exclusively concentrate on. The reason for this choice of  $M$  is two fold.

Firstly, if  $M$  is neither a  $\mathbb{Z}$  nor a  $\mathbb{Q}$  homology sphere, then it will necessarily have a non-zero dimensional moduli space of Abelian flat connections which in turn means that there will be Abelian  $B$  zero modes and hence non-compact directions to deal with.<sup>1</sup> We do not want to have to worry about such a situation here, as it is somewhat tangential to the other issues that we wish to address.

Far from all  $\mathbb{Z}$  or  $\mathbb{Q}$  homology spheres are Seifert. The reason for choosing  $M$  to be more specifically a Seifert  $\mathbb{Z}$  or  $\mathbb{Q}$  homology sphere is a more pragmatic one. Having  $M$  a Seifert manifold means that we have Fourier mode decomposition of all the fields, so their components can be ultimately viewed as living on  $\Sigma_V$ , and, as shown in [13–18], there are specific gauge choices available that allow one to significantly simplify the evaluation of the partition function.

### 3.2 BF action and gauge transformations on a Seifert manifold

As the Seifert fibred 3-manifold  $M$  is a principal  $U(1)$  bundle, it also comes equipped with the fundamental vector field  $\xi$  which generates the  $U(1)$  action. We also equip  $M$  with a (nowhere vanishing) connection 1-form  $\kappa$ , i.e.

$$\iota_\xi \kappa = 1, \quad L_\xi \kappa = \iota_\xi d\kappa = 0, \tag{3.5}$$

---

<sup>1</sup>Even if  $M$  is a  $\mathbb{Z}$  or  $\mathbb{Q}$  homology sphere, there may be non-zero dimensional components of the moduli space of flat connections, but there are many  $M$  for which the moduli space of flat connections is made up of a finite number of points.

and

$$\int_M \kappa \wedge d\kappa = c_1(\mathcal{L}_M). \quad (3.6)$$

Note that, acting on a form of any degree, one has

$$\iota_\xi \kappa + \kappa \iota_\xi = 1 \quad (3.7)$$

so that  $\iota_\xi \kappa$  and  $\kappa \iota_\xi$  are projection operators onto the space of horizontal and vertical forms of fixed degree respectively.

Given a trivial  $G$ -bundle on a Seifert 3-manifold  $M$ , we correspondingly decompose the connection  $A$  and 1-form  $B$  of  $BF$  theory as

$$A = A_\Sigma + \kappa\phi, \quad B = B_\Sigma + \kappa\lambda \quad (3.8)$$

where

$$A_\Sigma = \iota_\xi \kappa A, \quad \phi = \iota_\xi A, \quad B_\Sigma = \iota_\xi \kappa B, \quad \lambda = \iota_\xi B \quad (3.9)$$

We may also decompose the exterior derivative as

$$d = \iota_\xi \kappa d + \kappa \iota_\xi d = d^\Sigma + \kappa \iota_\xi d \quad (3.10)$$

with twisted versions

$$d_A = \iota_\xi \kappa d_A + \kappa \iota_\xi d_A \quad (3.11)$$

which acts on 0-forms as

$$d_A \beta = d_A^\Sigma \beta + \kappa D_\phi \beta \quad (3.12)$$

In terms of this decomposition, the gauge transformations (2.2) take the form, with  $t$  a local fibre coordinate

$$\begin{aligned} A_\Sigma &\rightarrow g^{-1} d_A^\Sigma g, & \phi &\rightarrow g^{-1} (\partial_t + \phi) g \\ B_\Sigma &\rightarrow g^{-1} (B_\Sigma + d_A^\Sigma \Lambda) g, & \lambda &\rightarrow g^{-1} (\lambda + D_\phi \Lambda) g \end{aligned} \quad (3.13)$$

and the action (2.1) becomes

$$I_{BF} = \int_M \text{Tr} \left( B_\Sigma \wedge \kappa D_\phi A_\Sigma + \kappa_\Sigma B \wedge d^\Sigma \phi + \lambda \kappa \wedge F_{A_\Sigma} + \kappa \wedge d\kappa \phi \lambda \right) \quad (3.14)$$

with horizontal curvature

$$F_{A_\Sigma} = d^\Sigma A_\Sigma + A_\Sigma^2 \quad (3.15)$$

### 3.3 Gauge fixing 1: temporal gauge for $A$ and $B$

One of the great benefits of considering a Seifert manifold is that there are very useful non-covariant gauges available. In particular, on a Seifert manifold we can always choose the “temporal gauge” that the fibre-component  $\phi$  of  $A$  is constant along the fibre,

$$\iota_\xi d\phi = 0. \quad (3.16)$$

It turns out that (for generic  $\phi$ ) one can also impose the analogous condition on  $\lambda$ ,

$$\iota_\xi d\lambda = 0. \tag{3.17}$$

In local coordinates with  $t$  a fibre coordinate, these conditions simply read

$$\frac{\partial}{\partial t}\phi = 0, \quad \text{and} \quad \frac{\partial}{\partial t}\lambda = 0 \tag{3.18}$$

The first of these can be achieved by having  $\phi$  gauge equivalent to some ‘time’ independent field  $\phi_0$ ; i.e. we have to solve

$$g(t)^{-1}(\partial_t + \phi)g(t) = \phi_0 \quad \Leftrightarrow \quad \partial_t g(t) = g(t)\phi_0 - \phi(t)g(t) \tag{3.19}$$

This equation has the solution

$$g(t) = P \exp\left(\int_t^0 \phi(s) ds\right) e^{t\phi_0} \tag{3.20}$$

(where we have without loss of generality chosen  $g(0) = 1$ ). Periodicity  $g(1) = g(0)$  of the gauge parameter now determines  $\phi_0$  to be the average value of  $\phi$  in the sense that

$$e^{\phi_0} = P \exp\left(\int_0^1 \phi(s) ds\right) \tag{3.21}$$

The second gauge condition,  $\partial_t \lambda = 0$ , is perhaps not familiar and as there is a hidden subtlety we will deal with it in some detail. Suppose we have already implemented the first gauge choice (thus in the following  $\phi = \phi_0$  is taken to satisfy  $\partial_t \phi = 0$ ). To arrive at  $\lambda$  constant along the fibre we need to solve

$$\lambda + D_\phi \Lambda = \lambda_0 \tag{3.22}$$

for some  $\lambda_0$  constant in  $t$  to be determined. Writing

$$\partial_t \left( e^{t\phi} \Lambda e^{-t\phi} \right) = e^{t\phi} (D_\phi \Lambda) e^{-t\phi} \tag{3.23}$$

one sees that this equation has the solution.

$$h(t)\Lambda(t)h^{-1}(t) = \int_0^t e^{s\phi} (\lambda_0 - \lambda(s)) e^{-s\phi} ds + \Lambda(0) \tag{3.24}$$

where

$$h(t) = e^{t\phi}, \quad h = e^\phi \tag{3.25}$$

Periodicity in the gauge parameter  $\Lambda(1) = \Lambda(0)$  gives us the equation

$$(\text{Ad}(h) - 1)\Lambda(0) = \int_0^1 e^{s\phi} (\lambda_0 - \lambda(s)) e^{-s\phi} ds \tag{3.26}$$

Before going on we note that a shift of  $\Lambda$  by a constant (in  $t$ )  $\Lambda_0$  simply amounts to a shift in  $\lambda_0$ ,

$$\Lambda \rightarrow \Lambda + \Lambda_0 \quad \Rightarrow \quad \lambda_0 \rightarrow \lambda_0 + [\phi, \Lambda_0] \tag{3.27}$$



Without loss of generality we can therefore assume that  $\Lambda(0) = 0$ , so that the periodicity condition (3.26) implies

$$\int_0^1 e^{s\phi} (\lambda_0 - \lambda(s)) e^{-s\phi} ds = 0. \quad (3.28)$$

The integral involving  $\lambda_0$  can be done explicitly, with the result

$$\frac{e^{\text{ad}(\phi)} - 1}{\text{ad}(\phi)} \lambda_0 = \int_0^1 e^{s \text{ad}(\phi)} \lambda(s) ds. \quad (3.29)$$

For *generic* values of  $\phi$ , the operator on the left-hand side can be inverted to give

$$\lambda_0 = \text{ad}(\phi) (e^{\text{ad}(\phi)} - 1)^{-1} \int_0^1 e^{s \text{ad}(\phi)} \lambda(s) ds \quad (3.30)$$

which tells us in which sense  $\lambda_0$  can be regarded as a  $\phi$ -weighted average of  $\lambda$  over the fibre.

However, we see that this solution fails when  $\phi$  is not suitably generic, so that the operator  $\exp \text{ad}(\phi) - 1$  has zero-modes. This should not come as a surprise, as this is precisely the condition that the operator  $D_\phi = \partial_t + \text{ad}(\phi)$  has zero-modes, so that the gauge fixing condition (3.22) cannot be solved for all  $\lambda(t)$ . Indeed, for a zero mode  $\psi$  of  $D_\phi$  one has (see also the discussion in section 2.3.4 of [1])

$$D_\phi \psi = 0 \quad \Rightarrow \quad \psi(t) = e^{t \text{ad}(\phi)} \psi(0), \quad (3.31)$$

and periodicity  $\psi(1) = \psi(0)$  implies

$$\psi(1) = \psi(0) \quad \Rightarrow \quad \left( e^{\text{ad}(\phi)} - 1 \right) \psi(0) = 0. \quad (3.32)$$

For  $\phi$  taking values in the Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ , this is simply the condition that  $\phi$  has a component that is an element of the integral lattice  $I(G)$  of  $G$ .

Normally this kind of constraint arising only for highly non-generic field configurations would not be an issue as one is integrating over  $\phi$  and the points where  $\phi$  is integral have measure zero. Unfortunately, in  $BF$  theory we get delta function support onto particular values of  $\phi$ . In particular we will see when we are considering  $\mathbb{Z}$  Homology Spheres that these integral points are the ones that are selected (cf. (4.6)).

### 3.4 Gauge fixing 2: Abelianisation and emergence of non-trivial line bundles

After having chosen the gauges (3.18), we still have gauge transformations available that are constant along the fibre. For a compact gauge group, one can locally use this residual gauge freedom to conjugate the field  $\phi$  into the Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ , i.e. one can impose the gauge condition

$$\phi^{\mathfrak{k}} = 0 \quad \Leftrightarrow \quad \phi = \phi^{\mathfrak{t}}, \quad (3.33)$$

where  $\mathfrak{k}$  is the orthogonal complement of the Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ ,

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}. \quad (3.34)$$

We will come back to the global issues involved in this choice of gauge below.

However, first of all we need to address the question if we can do something analogous in the case at hand, with gauge group  $TG$ . Since  $TG$  is neither compact nor semi-simple, this is not a priori obvious. However, we show in appendix A.2 that one can use the adjoint action of  $TG$  on its Lie algebra to conjugate any element into an element of  $\mathfrak{t} \oplus \mathfrak{t}$ , which plays the role of the Cartan subalgebra of the Lie algebra of  $TG$ . From one perspective, the reason we are able to do this is that  $TG$  arises as the Inönü-Wigner contraction of a group where this is certainly possible, namely  $G \times G$ .

Explicitly, this means that we can use the gauge transformation

$$(\phi, \lambda) \mapsto (g, \Lambda)^{-1} \cdot (\phi, \lambda) \cdot (g, \Lambda) = (g^{-1} \phi g, g^{-1}(\lambda + [\phi, \Lambda])g) \tag{3.35}$$

to (locally) impose simultaneously the Abelianising gauge conditions

$$\phi^{\mathfrak{t}} = 0, \quad \lambda^{\mathfrak{t}} = 0 \quad \Leftrightarrow \quad \phi = \phi^{\mathfrak{t}}, \quad \lambda = \lambda^{\mathfrak{t}} \tag{3.36}$$

Now let us turn to the global issues regarding this gauge choice, which implies that we will be dealing with an Abelian gauge theory on  $\Sigma_V$ . As has been described in detail in [14, 31], there are topological obstructions to imposing the gauge choice on  $\phi$  globally; in particular, conjugating  $\phi$  to the Cartan subalgebra  $\mathfrak{t}$  on  $\Sigma_V$  is not possible with smooth gauge transformations. However, if we insist on doing so anyway, the price to be paid is that we must sum over all available line bundles on  $\Sigma_V$ . The reason for the emergence of this sum has been explained a number of times [13–16, 18, 31].

Since  $TG$  is contractible to  $G$  (and there are no topological issues involved in the shift transformation  $\lambda \rightarrow \lambda + [\phi, \Lambda]$  beyond those involving  $\phi$  itself), there are no additional topological obstructions arising from diagonalisation in  $TG$ .

In practice there are two ways to introduce the non-trivial line bundles and the sum over them. They both have merits and in any case are equivalent. The first makes contact with flat connections rather straightforwardly while the other is computationally easier.

We begin by describing the method using background fields described in more detail in [18]. The available line  $V$ -bundles on  $\Sigma_V$  are

$$\mathcal{L} = \mathcal{L}_0^{\otimes n_0} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_N^{\otimes n_N} \tag{3.37}$$

and the basic idea is to allow for a non-trivial background connection on patches about each divisor, with the divisors being the singular points and with one divisor being a regular point of  $\Sigma_V$ . It was shown in [18] that in the tubular neighbourhood  $V_{(a_i, b_i)}$  of the  $i$ -th singular point one could introduce a connection 1-form  $\kappa_i$  which would glue to a global one form  $\kappa$  on  $M$ . In terms of these local forms and the local surgery data

$$a_i s_i - b_i r_i = -1, \quad \text{with} \quad (a_i, b_i) = 1 \tag{3.38}$$

the local orbifold data, the background connection takes the simple form<sup>2</sup>

$$\mathcal{A}_B = 2\pi i \left( \mathbf{n}_0 \kappa_0 + \sum_{i=1}^N \mathbf{n}_i r_i \kappa_i \right) \tag{3.39}$$

---

<sup>2</sup>A word on notation. Throughout we write connections in a basis of simple roots  $\alpha_i$ . For semi-simple simply laced Lie algebras we have that the natural inner product  $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ . The bold facing of integers indicates that we mean  $\mathbf{m} = m^i \alpha_i$  and for each  $i$  we have  $m^i \in \mathbb{Z}$ . We will write that  $\mathbf{m} \in I(G)$  where  $I(G)$  is the integral lattice of  $G$ . As  $G$  is simply connected we have identified the root lattice with  $I(G)$ .

where the  $\mathbf{n}$  are Hermitian and lie in  $\mathfrak{t}$ . With an abuse of notation, we may write the background as

$$\kappa \phi_B = 2\pi i \left( \mathbf{n}_0 \kappa_0 + \sum_{i=1}^N \mathbf{n}_i r_i \kappa_i \right) \quad (3.40)$$

Each of the local forms  $\kappa_i$  extends outside of the tubular neighbourhood to  $d\theta$  on the fibre. One also has that

$$\int_{V_{(a_i, b_i)}} \kappa_i d\kappa_i = \frac{b_i}{a_i} \quad (3.41)$$

with the regular point having weight  $a_0 = 1$ . Glued together the  $\kappa_i$  form a global, smooth 1-form  $\kappa = \kappa_0 \cup \kappa_1 \cup \dots \cup \kappa_N$  that defines a principal bundle structure on  $M$  with

$$c_1(\mathcal{L}_M) = \int_M \kappa \wedge d\kappa = b_0 + \sum_{i=1}^N \frac{b_i}{a_i} \quad (3.42)$$

With the background field in place, one makes the substitution

$$\kappa \phi \rightarrow \kappa \phi + \kappa \phi_B \quad (3.43)$$

everywhere in the path integral, with the second term on the right given by (3.40).

Alternatively [13–16] one sets all the  $\mathbf{n}_i$   $i \neq 0$  to zero and  $\kappa_0 \rightarrow \kappa$  (and this is the natural procedure from the point of view of obstruction theory outlined in [31] and extended here to the manifolds that we are considering). Otherwise one keeps the substitution (3.43) in the path integral. In appendix B we show that there is enough symmetry in the theory to pass between these different formulations.

### 3.5 Reduction of the partition function to a finite-dimensional integral

With all the preliminaries in place we can now outline how to perform the path integral for the partition function.

Firstly the action (3.14), taking into account the non-trivial bundles that arise on Abelianisation, goes over to

$$I_{BF} = \int_M \text{Tr} \left( B_\Sigma \wedge \kappa D_{\phi + \phi_B} A_\Sigma + \kappa B_\Sigma \wedge d^\Sigma \phi + \lambda \kappa \wedge F_{A_\Sigma} + \kappa \wedge d\kappa(\phi + \phi_B)\lambda \right) \quad (3.44)$$

with  $\kappa \phi_B$  and  $\kappa d\kappa \phi_B$  suitably understood. Indeed we note that the part of  $B_\Sigma$  that is constant along the fibre and takes values in the Cartan subalgebra  $\mathfrak{t}$  only appears through the term

$$\int_M \text{Tr} \left( \kappa B_\Sigma \wedge d^\Sigma \phi \right). \quad (3.45)$$

Thus integrating out these modes of  $B_\Sigma$  imposes the condition that  $d^\Sigma \phi = 0$ . Given the gauge conditions  $\partial_t \phi = 0$ , this implies that  $\phi$  is constant on  $M$ . Likewise the component of  $A_\Sigma$  constant along the fibre and in the Cartan subalgebra only appears in

$$\int_M \text{Tr} \left( \kappa A_\Sigma \wedge d^\Sigma \lambda \right) \quad (3.46)$$

so integrating over that mode of  $A_\Sigma$  fixes  $d^\Sigma \lambda = 0$  which, together with  $\partial_t \lambda = 0$ , tells us that  $\lambda$  is also constant on  $M$ . The other components of  $A_\Sigma^\xi$  and  $B_\Sigma^\xi$  are paired with each other and lead to an overall constant when integrated over.

We should also take care of any possible harmonic modes of the components of  $A$  and  $B$  that we have integrated over. If there any such harmonic modes of  $B$ , they would be non-compact directions and would lead to divergences. In order to avoid this we fix on the genus of  $\Sigma_V$  to be zero, as in (3.2) and (3.3) thus ensuring that there are no  $B_\Sigma^\xi$  zero modes to contend with. The stronger choice that we have made of having  $M$  be a  $\mathbb{Z}$  or  $\mathbb{Q}$  homology sphere is to ensure that there are no moduli of Abelian connections on  $M$  as then we would necessarily have Abelian  $B$  zero modes.

The form that we now have for the action is

$$I_{BF} = \int_M \text{Tr} \left( B_\Sigma^\xi \wedge \kappa D_{\phi+\phi_B} A_\Sigma^\xi + \kappa \wedge \lambda A_\Sigma^\xi A_\Sigma^\xi \right) + \text{Tr} \lambda (c_1(\mathcal{L}_M)\phi + 2\pi i \hat{\mathbf{q}}) \quad (3.47)$$

where

$$\hat{\mathbf{q}} = \sum_{i=0}^N \frac{1}{a_i} \mathbf{n}_i \quad (3.48)$$

We are now ready to integrate out the fields  $A_\Sigma^\xi$  and  $B_\Sigma^\xi$  that appear in (3.47). There are also ghost terms of the same type to take into account. The functional determinant that we obtain and need to evaluate is

$$\text{Det} \left( \begin{array}{cc} \text{ad}_\lambda & D_{\phi+\phi_B} \\ -D_{\phi+\phi_B} & 0 \end{array} \right) \Big|_{\Omega_H^1(M, \mathfrak{k}) \oplus \Omega_H^1(M, \mathfrak{k})}^{-1/2} \quad (3.49)$$

There is a similar determinant from the ghost terms. As usual we will expand in Fourier modes along the  $S^1$  fibre and we will regulate with a  $\zeta$ -function regularisation. What one finds is, even upon regularisation, that the  $\text{ad}_\lambda$  part of the determinant does not contribute. In this way what finally needs to be evaluated is

$$|\text{Det} (D_{\phi+\phi_B})|_{2\Omega_H^0(M, \mathfrak{k}) \oplus \Omega_H^1(M, \mathfrak{k})} \quad (3.50)$$

For every Fourier mode  $2\Omega_H^0(M, \mathfrak{k}) \oplus \Omega_H^1(M, \mathfrak{k})$  is roughly the Euler characteristic of  $\Sigma_V$ . This, almost, cancellation of all the modes in the functional determinant can be used to evaluate it. However, the field  $\phi_B$  is not constant on  $\Sigma_V$  unlike  $\phi$ . This issue was circumvented in [18] by using a density form of the index theorem originally found in [13]. The determinant in [18] is the square root of (3.50), so we may straightforwardly borrow the result from there. In this way we see that the absolute value (3.50) is

$$\tau_M(\phi; \mathbf{n}_i) = \tau_{S^1}(\phi + 2\pi i \mathbf{n}_0)^{2-2g-N} \cdot \prod_{i=1}^N \tau_{S^1}((\phi + 2\pi i r_i \mathbf{n}_i)/a_i) \quad (3.51)$$

where  $\tau_M$  is the Ray-Singer torsion of  $M$ . The form of  $\tau_M$  shows that it is given by circles that is in terms of the Ray-Singer torsion of  $S^1$ 's [29]. For completeness we note that

$$\tau_{S^1}(\varphi) = \det_{\mathfrak{k}}(I - \text{Ad}(\exp \varphi)) \quad (3.52)$$

so that  $\mathbf{n}_0$  drops out of (3.51) since it has integral (lattice) entries.

We regularise the Ray-Singer Torsion by introducing a mass [1]. In the present context the mass is introduced into the determinants (3.49). This is not consistent with the  $TG$  symmetry but does preserve the Abelian part of it. One exchanges (3.52) with

$$\tau_{S^1}(\varphi, \mu) = \det_{\mathfrak{g}} \left( I - e^{-2\pi\mu} \text{Ad}(\exp \varphi) \right) \tag{3.53}$$

and it is understood that we use the massive form for the Ray-Singer Torsion of the torsion on the circles on the right hand side of (3.51). For the details of how this comes about see Example 2.5 in [1]. We should note we have introduced extra factors of  $\mu$  (namely by the factor  $(1 - e^{-2\pi\mu})^{\dim T}$ ) by passing to (3.53) and hence we have also changed the normalisation of the path integral

Putting everything together, we see that we have now managed to reduce the path integral for the  $BF$  partition function to a finite dimensional integral over the (by now constant) fields  $\phi$  and  $\lambda$ ,

$$Z_{BF}[M, G] = \sum_{\mathbf{n}_0} \cdots \sum_{\mathbf{n}_N} \int_{\mathfrak{t} \oplus \mathfrak{t}} \tau_M(\phi; \mathbf{n}_i) \cdot \exp(i \text{Tr} \lambda (c_1(\mathcal{L}_M)\phi + 2\pi i \hat{\mathbf{q}})) \tag{3.54}$$

This is the integral that we will look at in detail in the following section.

#### 4 Definition 1: direct evaluation of the $BF$ (path) integral

As we saw in the last section we are able to perform the functional integral representing the partition function  $Z_{BF}$  to the point that we are left with finite dimensional integrals. In this section we analyse the remaining integrals and notice that there are issues with them. Indeed this prompts us to search for a definition for  $Z_{BF}$ . This definition does not agree with the perturbative result of Witten [11] in the case of isolated and irreducible connections and we end the section by explaining where there could be possible issues with one of the gauge conditions that we have chosen. The formulae obtained capture those Abelian connections in  $\mathbb{Q}$  Homology Spheres and in these cases reproduce the expected result (2.25).

##### 4.1 Symmetries and other properties of the finite dimensional integrals

At this point we have collected all the pieces and the  $BF$  partition function, for the 3-manifolds under consideration, is now a finite dimensional integral

$$Z_{BF}[M, G] = \sum_{\mathbf{n}_0} \cdots \sum_{\mathbf{n}_N} \int_{\mathfrak{t} \oplus \mathfrak{t}} \tau_M(\phi; \mathbf{n}_i) \cdot \exp(i \text{Tr} \lambda (d\phi/P + 2\pi i \hat{\mathbf{q}})) \tag{4.1}$$

where the integral over  $\mathfrak{t} \oplus \mathfrak{t}$  is that over the constant Abelian “fields”  $\phi$  and  $\lambda$  and we have used the fact that  $c_1(\mathcal{L}_M) = d/P$  for the 3-manifolds we are considering as given in (3.2) and (3.3) which in particular means that  $g = 0$ . There are no sums over the  $\mathbf{n}_i$   $i \neq 0$  if we have set them to zero, in which case we also have that  $\hat{\mathbf{q}} = \mathbf{n}_0$ . Note that as the gauge group  $G$  is compact connected and simply connected the  $G$  bundles over  $M$  are trivial and, on Abelianisation, the line V bundles at the smooth point are ‘honest’ line bundles so

that  $\mathbf{n}_0 \in \mathbb{Z}^{\mathbf{rk}}$ . Consequently, in this case,  $\mathbf{n}_0$  only appears in the action in (4.1) (for either formulation of background).

In order to understand and ultimately evaluate the integral (4.1), we note the following properties of the integrand:

1. both the action and the Ray-Singer torsion are invariant under the discrete symmetry

$$\phi \rightarrow \phi + 2\pi i P \mathbf{r}, \quad \mathbf{n}_0 \rightarrow \mathbf{n}_0 - d\mathbf{r} \quad \text{with } \mathbf{r} \in \mathbb{Z}^{\mathbf{rk}(G)}, \quad P = \prod_{i=1}^N a_i \quad (4.2)$$

2. the Ray-Singer torsion with  $N \geq 3$  has poles at  $\alpha(\phi) = n\pi$  with  $n \neq Pm$  (while for  $N \leq 2$  there are no poles)
3. the Ray-Singer torsion has zeros at  $\alpha(\phi) = m\pi P$
4. the Ray-Singer torsion is an even function of its arguments, in the sense that

$$\tau_M(\phi, \mathbf{n}_i) = \tau_M(-\phi, a_i - \mathbf{n}_i) = \tau_M(-\phi, -\mathbf{n}_i) \quad (4.3)$$

Using the symmetry (4.2), we may either ‘compactify’ the field  $\phi$  or limit the range of  $\mathbf{n}_0$ . Either the way, the integral over  $\lambda$  leads to a delta function constraint on  $\phi$ , namely

$$\phi = \phi_C \equiv -\frac{2\pi i P}{d} \hat{\mathbf{q}} \quad (4.4)$$

and so it appears that only these values contribute to the path integral. These values correspond to reducible connections as we have seen in the past [17] and so one might conclude that the only flat connections that contribute to the  $BF$  partition function are those that are Abelian. However, because of the presence of poles, this conclusion is a bit quick, and we will take a closer look at this issue in section 4.4.

Before moving on, it is worthwhile reminding ourselves that there are restrictions on  $M$  so the analysis we are performing holds. As a counter-example, presume that  $M$  is a product  $\Sigma \times S^1$ , or more generally a Seifert mapping torus, i.e. with  $c_1(\mathcal{L}_M) = 0$  (so definitely not one of the manifolds that we are considering). Then there would be no symmetry such as (4.2) for the action and the integral over  $\lambda$  would not lead to (4.4).

## 4.2 The $\phi_C$ contribution to $\mathbb{Z}$ homology spheres

Here we specialise to the  $\phi_C$  contribution to the path integral for  $M$  either an  $\mathbb{Z}$  or a  $\mathbb{Q}$  Homology Sphere. We will use the symmetries available not to limit the range of  $\phi$  but rather to set  $\mathbf{n}_0 \in I(G)/dI(G)$  and we will keep the  $\mathbf{n}_i$  and so take the classical field contribution to be

$$\phi_C = -2\pi i \frac{P}{d} \sum_{i=0}^N \frac{\mathbf{n}_i}{a_i} \quad (4.5)$$

and any other choice leads to the same conclusions that we arrive at presently. As we only want the  $\phi_C$  contribution we must ensure that  $\tau_M$  has no poles and this is achieved by passing to the massive Ray-Singer torsion  $\tau_M(\phi, \mu)$  which we do.

If  $M$  is a  $\mathbb{Z}$  homology sphere then  $d = 1$  and the symmetry allows us to set  $\mathbf{n}_0 = 0$  so that

$$\phi_C = -2\pi i P \sum_{i=1}^N \frac{\mathbf{n}_i}{a_i} \in I(G) \tag{4.6}$$

Note that these are precisely the (potentially problematic) integral values of  $\phi$  discussed at the end of section 3.3. We will (have to) come back to this below. A tiny bit of gymnastics<sup>3</sup> shows that

$$\phi_C + 2\pi i \mathbf{n}_j r_j = 0 \pmod{2\pi i a_j I(G)} \tag{4.7}$$

One immediately sees that the Ray-Singer Torsion (without a mass) on the circle vanishes and the overall ratio in (3.51) is zero. The poles and the zeros of the finite dimensional integral arise as the zeros of the (inverse) determinant of the  $D_\phi$  operator. The poles are there when  $D_\phi$  acts on 1-forms and the zeros arise when acting on 0-forms at the special values (4.4) of  $\phi$ .

In order to understand the contributions we turn on the mass regulator, as in (3.53). Then we find

$$\tau_M(\phi, \mu) = \det_{\mathfrak{g}} \left( I - e^{-2\pi\mu} \right)^{2-N} \prod_{i=1}^N \det_{\mathfrak{g}} \left( I - e^{-2\pi\mu/a_i} \right) \tag{4.8}$$

The small mass limit gives us, as in (2.22),

$$\tau_M(\phi, \mu) = (2\pi\mu)^{2 \dim G} \cdot (a_1 \dots a_N)^{-\dim G} + \dots \tag{4.9}$$

The exponent of  $\mu$  is understood as follows: the torsion of  $M$  has been pushed to that on  $S^1$  and as on  $S^1$  the twisted cohomology groups satisfy  $H_\omega^0 = H_\omega^1$  the combination  $3 \dim H_\omega^0 - \dim H_\omega^1 = 2 \dim H_\omega^0$  and  $\dim H_\omega^0 = \dim G$  for the trivial connection which is the maximally reducible connection.

This result is quite far from the known Reidemeister Torsion for regular elements of a Brieskorn Sphere  $\Sigma(a_1, a_2, a_3)$  [32],

$$(a_1 a_2 a_3)^{-1} \prod_{i=1}^3 4 \sin^2 \left( \frac{n_i r_i \pi}{a_i} \right) \tag{4.10}$$

in the case of  $SU(2)$ . The semi-classical analysis of Witten would have suggested

$$Z_{BF}[\Sigma(a_1, a_2, a_3) SU(2)] = (a_1 a_2 a_3)^{-1} \prod_{i=1}^3 \sum_{n_i=1}^{a_i-1} 4 \sin^2 \left( \frac{n_i r_i \pi}{a_i} \right) + \text{reducible terms} \tag{4.11}$$

rather than the result that we obtained.

From (4.9) we understand that the  $\phi_C$  contributions that we are looking at come from reducible connections, for these manifolds the trivial connection as  $G = SU(2)$ , and so it appears that we see only the non acyclic terms in (4.11). For example for  $S^3$  the only flat

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<sup>3</sup>By (3.2)  $Pb_j/a_j = 1 - a_j t_j$ ,  $t_j \in \mathbb{Z}$  and  $Pb_j r_j/a_j = r_j - r_j a_j t_j = (1 - a_j s_j)P/a_j$ ,  $t_j \in \mathbb{Z}$  whence  $r_j - P/a_j = 0 \pmod{a_j \mathbb{Z}}$ . Consequently,  $\phi_C + 2\pi i \mathbf{n}_j r_j = -2\pi i P \sum_{i \neq j} \mathbf{n}_i/a_i + 2\pi i \mathbf{n}_j \cdot (r_j - P/a_j) = 0 \pmod{2\pi i a_j I(G)}$ .



connection is the trivial connection and, as it has maximal reducibility, one sees that a formula of the type (4.9) correctly captures this fact. Indeed, for any (trivial)  $G$  bundle over a  $\mathbb{Z}$  Homology sphere the trivial connection is flat and isolated and would give a contribution as above. What are missing are the contributions of non-reducible or, for higher rank groups, less reducible connections.

### 4.3 The $\phi_C$ contribution to $\mathbb{Q}$ homology spheres

The zeros that were faced when  $M$  is an  $\mathbb{Z}$  Homology Sphere are avoided to some extent when the manifold is a  $\mathbb{Q}$  Homology Sphere. The first advantage is that the points at which we wish to evaluate the torsion are no longer necessarily at integer values but rather at

$$\phi_C = -2\pi i \frac{P}{d} \sum_{i=0}^N \frac{\mathbf{n}_i}{a_i} \tag{4.12}$$

Even if  $\phi_C$  does at some points take integer values, the mass regularised torsion remains well defined and that is what we will use. We will give an example of this later on. Recall that here we fix the symmetry (4.2) by insisting that  $\mathbf{n}_0 \in I(G)/d.I(G)$ .

The  $\phi_C$  contribution to the partition function is, with the masses re-instated so that we may follow the reducibility of the connections,

$$\begin{aligned} & Z_{BF}[M, G, \mu]_{\phi_C} \\ &= \sum_{\mathbf{n}_0} \cdots \sum_{\mathbf{n}_N} \tau_{S^1}(-2\pi i(P/d)\widehat{\mathbf{q}} + 2\pi i\mathbf{n}_0, \mu)^{2-N} \cdot \prod_{i=1}^N \tau_{S^1}((-2\pi i(P/d)\widehat{\mathbf{q}} + 2\pi i r_i \mathbf{n}_i)/a_i, \mu/a_i) \end{aligned} \tag{4.13}$$

This result can be understood in terms of the homotopy representations for flat connections on the Seifert  $\mathbb{Q}$  Homology Sphere to which we turn shortly. Notice that the contribution from these connections is precisely of the form that was anticipated in (2.20). However, for present purposes the most useful form for us is where we have set the  $\mathbf{n}_i$  for  $i \geq 1$  to zero and restrict  $\mathbf{n}_0 \in I(G)/d.I(G)$

$$Z_{BF}[M, G, \mu]_{\phi_C} = \sum_{\mathbf{n}_0 \in I(G)/d.I(G)} \tau_{S^1}(-2\pi i(P/d)\mathbf{n}_0, \mu)^{2-N} \cdot \prod_{i=1}^N \tau_{S^1}(-2\pi i(P/d)\mathbf{n}_0/a_i, \mu/a_i) \tag{4.14}$$

The classical examples of  $\mathbb{Q}$  Homology Spheres are of course the Lens spaces  $L(p, q)$ . The simplest Seifert presentation of the Lens space  $L(p, q)$  is with  $N = 1$ ,  $d = p$  and  $a = q^*$  where  $qq^* = 1 \pmod p$ ,  $p = b_0q^* + b_1$ . Substituting these values into (4.14)

$$Z_{BF}[L(p, q), G, \mu] = \sum_{\mathbf{n} \in I(G)/p.I(G)} \tau_{S^1}(2\pi i q^* \mathbf{n}/p + \mu) \cdot \tau_{S^1}(2\pi i \mathbf{n}/p + \mu/q^*) \tag{4.15}$$

To specialise to  $G = \text{SU}(2)$  we note that the massive Ray-Singer Torsion (3.53) takes the form

$$\tau_{S^1}(\phi, \mu) = (1 - e^{-2\pi\mu}) \cdot (1 + e^{-4\pi\mu} - e^{-2\pi\mu} 2 \cos 2\phi) \tag{4.16}$$

so that on specialising (4.15) to  $G = \text{SU}(2)$  one obtains

$$\begin{aligned}
 Z_{BF}[L(p, q), \text{SU}(2), \mu] &= \left(\frac{\mu^2}{q^*}\right)^3 + \dots \\
 &+ \left(\frac{\mu^2}{q^*}\right) \sum_{n=1}^{p-1} \sin^2(2\pi i q^* n/p) \cdot \sin^2(2\pi i n/p) + \dots \quad (4.17)
 \end{aligned}$$

We note that happily (4.15) and (4.17) have the form that we expected from the covariant analysis in section 2 and in particular agree with the general form (2.22).

The first term in (4.17) corresponds to the trivial connection with reducibility of order 3 while the second term is the sum over the Abelian connections with reducibility of order 1 and the torsions agree with the formulae obtained by Ray [33] for Lens spaces. This should also be compared with the Chern-Simons large  $k$  limit formula (5.7) in [34] and we see that there is complete agreement with the expression for the Ray-Singer Torsion of the Lens spaces for the Abelian connections (the only difference with Jeffrey's formula is that we are summing over the Ray-Singer Torsion not its square root).

We note that  $\phi_C$  in this case is the complete contribution to the integral (4.1) for, as noted in item 2 after (4.2), when  $N = 1$  there are no poles in the integrand and the integral is well defined as is.

Now consider quotients of the Poincare sphere  $\Sigma(2, 3, 5) = M[-1, g = 0, (2, 1), (3, 1), (5, 1)]$ . The fundamental group of  $\Sigma(2, 3, 5)$  is  $\pi_1(\Sigma(2, 3, 5)) = I^*$  the binary icosahedral group of order 120 while the quotient manifold  $\Sigma(2, 3, 5)_d = S(\mathcal{L}_{\Sigma(2,3,5)}^{\otimes d})$  has fundamental group  $\mathbb{Z}_d \times I^*$  (Theorem 2 (v) page 112 in [35]). This means that  $\Sigma(2, 3, 5)_d$  comes equipped with the non-Abelian flat connections of  $\Sigma(2, 3, 5)$  through the  $I^*$  factor of the fundamental group and with Abelian connections through the  $\mathbb{Z}_d$  factor of  $\pi_1$ . In this case we have

$$\begin{aligned}
 Z_{BF}[\Sigma(2, 3, 5)_d, G, \mu]|_{\phi_C} &\simeq \\
 \sum_{0 \leq \mathbf{n} \leq d-1} &\tau_{S^1}(60\pi i \mathbf{n}/d + \mu)^{-1} \cdot \tau_{S^1}(30\pi i \mathbf{n}/d + \mu/2) \cdot \tau_{S^1}(20\pi i \mathbf{n}/d + \mu/3) \cdot \tau_{S^1}(12\pi i \mathbf{n}/d + \mu/5)
 \end{aligned} \quad (4.18)$$

Apart from having the standard form we note that we can expect, at least in the case of  $\text{SU}(2)$  that this formula correctly captures the contributions of the reducible connections.

Indeed all the so called small Seifert manifolds with genus zero and orientable base have finite fundamental groups with cyclic factors. Hence quite generally the space of flat connections for these manifolds naturally splits into the non-Abelian and Abelian parts. The cyclic group is generated by the fibre generator  $h$ . For large Seifert manifolds the fibre generator also generates the cyclic group as a normal subgroup of the infinite order fundamental group. Then  $\mathbb{Z}_d \subset \pi_1(M_d)$  and  $\mathbb{Z}_d = (\pi_1/[\pi_1, \pi_1])(M_d)$ .

The interpretation of the formulae at  $\phi_C$  that we arrived at above in terms of homotopy representations has been given in [17]. At these reducible connections the Chern-Simons invariant agrees with that given by Kirk and Klassen in [36], Auckly [37] and Nishi [38] and is consistent with the identification of the connections being reducible and flat. Also the known relationship between the Ray-Singer Torsion for flat connections and Reidemeister torsion corroborates the identification of the contributions that we find.

#### 4.4 A residue formula for SU(2)

As we saw in section 4.3 that Abelian reducible connections are accounted for by  $\phi_C$  we now take another and closer look at the finite-dimensional integral to see where the contributions from non-Abelian (ir)reducibles could come from.

The integrand of (4.1) has poles and zeros and so we must give meaning to the integral. A similar situation arises in Chern-Simons theory [13] where one explicitly removes the poles with the addition of a mass term. There is a good reason for this in Chern-Simons theory as, say on  $\Sigma \times S^1$ , one is counting the finite number of states in the theory. Here, if we use a mass regularisation then there is the same effect, namely the poles do not contribute. However, this then excludes important contributions to the  $BF$  partition function. Also, it should be said, that for  $BF$  theory one does not have a finite dimensional Hilbert space of states and so there is no ‘directive’ to ensure that the poles should not be counted in some way.

One could rotate the  $\phi$  integral to say  $\exp(-i\epsilon) \times \mathfrak{t}$  which would ensure that one avoids the poles altogether. However, one would need to simultaneously require a different path of integration for  $\lambda$  to ensure convergence of the integrals. Passing to the massive Ray-Singer Torsion is another way of avoiding the poles as the mass pushes the poles off the real  $\mathfrak{t}$  axis and into the complex plane  $\mathbb{C} \times \mathfrak{t}$ .

Short of a guiding principle we use the first symmetry in the Properties of the Integrand to restrict the range of each component of the  $\phi$  field to lie in  $(-\pi P, \pi P)$ . In this way we do not have to worry about convergence issues with respect to integration over  $\phi$ . However, there are still the poles to contend with and the integration over  $\lambda$ .

In order to be concrete and to fix ideas we now focus on  $G = \text{SU}(2)$ . From the outset we are tasked with having to make sense of

$$Z_{BF}[M, \text{SU}(2)] \sim \sum_{n \in \mathbb{Z}} \int_{\mathbb{R} \times \mathbb{R}} \tau_M(\phi) \cdot \exp\left(i\lambda \left(\phi \frac{d}{P} - 2\pi n\right)\right) \quad (4.19)$$

We firstly sum over  $n$  to exchange  $\lambda$  with the integers and exchange the integral with the sum

$$Z_{BF}[M, \text{SU}(2)] \sim \int_{\mathbb{R}} \tau_M(\phi) \cdot \sum_{n \in \mathbb{Z}} \exp\left(in\phi \frac{d}{P}\right) \quad (4.20)$$

As  $\tau_M(\phi)$  is even in  $\phi$  we have

$$\begin{aligned} Z_{BF}[M, \text{SU}(2)] &\sim \int_{-\pi P}^{\pi P} \tau_M(\phi) \cdot \frac{(1 + \exp(id\phi/P))}{(1 - \exp(id\phi/P))} \\ &\sim \int_{-\pi P}^{\pi P} \tau_M(\phi) \cdot \frac{f(\phi)}{(1 - \exp(id\phi/P))} \end{aligned} \quad (4.21)$$

and we have used the symmetries available to limit the range of  $\phi$  to lie in the range  $(-\pi P, \pi P)$ . Here the function  $f$  is

$$f(\phi) = (1 + \exp(id\phi/P)) \quad (4.22)$$

There are now poles on the real axis coming both from the Ray-Singer Torsion as well as from the result of the geometric sum. We push all of these, in the complex plane, to lie above the real line (say by use of an  $i\epsilon$  prescription).

The contour that we choose in the complex plane in order to evaluate the integral is as follows: travel along the real axis from  $-\pi P$  to  $\pi P$  then straight up along the imaginary axis to  $\pi P + iR$  followed by an arc to  $-\pi P + iR$  then straight down to  $-\pi P$  and finally take the  $R \rightarrow \infty$  limit. For the integral over the arc to make sense one regularises the integrand by multiplying by  $\exp(i\epsilon\phi^2)$  with  $\epsilon > 0$ . The integrals over the other parts of the contour are convergent as we will see below. The fact that the prescription for the poles on the interval  $(-\pi P, \pi P)$  are now such that they all lie inside the contour (in the upper half plane) means that the contour integral is just given by the residue of all of those poles.

Denote the contribution of each segment of the integral by the start and end points as  $I(x, y)$  so that we have, with  $R \rightarrow \infty$

$$I(-\pi P, \pi P) + I(\pi P, \pi P + iR) - I(-\pi P, -\pi P + iR) = 2\pi i \text{Res} \left( \tau_M(\phi) \cdot \frac{(1 + \exp(id\phi/P))}{(1 - \exp(id\phi/P))} \right) \quad (4.23)$$

as the integral over the arc is zero. Had we preserved the symmetry  $\phi \rightarrow \phi + 2\pi P$  with our regulator then we would have been able to declare that  $I(\pi P, \pi P + iR) = I(-\pi P, -\pi P + iR)$ . However as  $\phi$  becomes large the integrand of  $I(\pm\pi P, \pm\pi P + iR)$  goes as

$$\exp \left( i\epsilon(i\phi \pm \pi P)^2 - 2\phi \left( N - 2 - \sum_{i=1}^N 1/a_i \right) \right) + \dots \quad (4.24)$$

and for  $N \geq 3$  (but not for  $\Sigma(2, 3, 5)$  the Poincare sphere) there is convergence without the need for the Gaussian oscillatory behaviour so that one may take the  $\epsilon \rightarrow 0$  limit and the integrals over the two vertical parts of the contour cancel each other in the limit. In short, we have that with this prescription for handling the poles that

$$Z_{BF}[M, \text{SU}(2)] = 2\pi i \text{Res} \left( \tau_M(\phi) \cdot \frac{(1 + \exp(id\phi/P))}{(1 - \exp(id\phi/P))} \right) \quad (4.25)$$

One would also like to understand a little better what the residue formula implies. To that end let

$$\begin{aligned} \tau_M(\phi) &= \prod_{\alpha > 0} (\sin(\alpha(\phi)))^{4-2N} \cdot \prod_{\alpha > 0} \prod_{i=1}^N (\sin(\alpha(\phi)/a_i))^2 \\ &= \prod_{\alpha > 0} (\sin(\alpha(\phi)))^{4-2N} \cdot \tilde{\tau}_M(\phi) \end{aligned} \quad (4.26)$$

The residue formula in the case of  $\text{SU}(2)$  takes the simple form

$$\begin{aligned} Z_{BF}[M, \text{SU}(2)] &= Z_{BF}[M, \text{SU}(2)]|_{\phi_C} \\ &+ \sum_{r=0}^{2N-5} \binom{2N-5-r}{r} \sum_{m=1-P}^{P-1} \left( f(m\pi) \cdot \frac{1}{1 - e^{im\pi d/P}} \right)^{(2N-5-r)} \tilde{\tau}_M(m\pi)^{(r)} \end{aligned} \quad (4.27)$$

where the sum over  $m$  indicates the poles of  $(\sin(\alpha(\phi)))^{4-2N}$ . One sees that this is essentially a sum over derivatives of the Reidemeister torsion  $\tilde{\tau}_M$  (up to normalisation).

### 4.5 SU(2) BF theory on Brieskorn spheres $\Sigma(a_1, a_2, a_3)$

We now look concretely at SU(2) BF Theory on Brieskorn Spheres  $\Sigma(a_1, a_2, a_3)$ , in order to see if the suggested contour reproduces the expectations from perturbation theory as derived by Witten [11]. Even though the result for the path integral is not exactly that predicted by Witten, we will see that this choice does have contributions from irreducible flat connections which is quite pleasing in its own right though, as we explain below, not completely unexpected.

With  $N = 3$  and  $d = 1$  the path integral becomes, apart from the  $\phi_C$  contributions of the previous sections

$$\begin{aligned} Z_{BF}[\Sigma(a_1, a_2, a_3), \text{SU}(2)] & \\ & \simeq \sum_{m=1}^{P-1} \left( \frac{1}{P} \csc^2(m\pi/2P) - 4 \cot(m\pi/P) \sum_{i=1}^3 \frac{1}{a_i} \cot(m\pi/a_i) \right) \tilde{\tau}_{\Sigma(a_1, a_2, a_3)}(m\pi) \\ & \simeq \frac{1}{a_1 a_2 a_3} \sum_{m=1}^{P-1} h(m) \tilde{\tau}_{\Sigma(a_1, a_2, a_3)}(m\pi) \end{aligned} \tag{4.28}$$

To compare with (4.11) we note that the integer  $m$  can be written in various ways in particular

$$m = a_1 \alpha_1 + n_1 r_1 = a_2 \alpha_2 + n_2 r_2 = a_3 \alpha_3 + n_3 r_3, \quad \alpha_i, n_i \in \mathbb{Z} \tag{4.29}$$

to arrive at

$$Z_{BF}[\Sigma(a_1, a_2, a_3), \text{SU}(2)] \simeq \frac{1}{a_1 a_2 a_3} \sum_{n_i} g(n_i) \prod_{i=1}^3 4 \sin^2 \left( \frac{\pi n_i r_i}{a_i} \right) \tag{4.30}$$

which would agree with (4.11) if the function  $g$  is  $n_i$  independent.

As our first example (even though as we noted the derivation provided fails) consider the Poincaré homology sphere  $\Sigma(2, 3, 5)$ . The Ray-Singer-Torsion is the same for the values of  $(n_1, n_2, n_3)$  corresponding to  $(1, 1, r)$ ,  $(1, 1, 5 - r)$  and  $(1, 2, r)$ . In this way we have, apart from the trivial connection, the two possible non-Abelian connections  $(1, 1, 1)$  and  $(1, 1, 2)$ . By Proposition 2.8 of [39], this count is correct for the flat connections on  $\Sigma(2, 3, 5)$ .

As our next example let us fix on  $\Sigma(2, 3, 7)$  for which there are 2 irreducible isolated flat connections. The Ray-Singer-Torsion is the same for the values of  $(n_1, n_2, n_3)$  corresponding to  $(1, 1, r)$ ,  $(1, 1, 7 - r)$  and  $(1, 2, r)$ . If we identify those contributions as corresponding to the same isolated and irreducible connection then we get 4 copies each of three different connections, say of  $(1, 1, 1)$ ,  $(1, 1, 2)$  and  $(1, 1, 3)$ . In that case the sum in (4.30) is over the 3 different connections each of which corresponds to 4 different values of  $m$  as tabulated below (the values of  $g$  are approximate)

$m$	$(n_1, n_2, n_3)$	$g/P = \sum_m h(m)/P$
1, 13, 29, 41	(1, 1, 1)	-34.45
5, 19, 23, 37	(1, 1, 2)	6.96
11, 17, 25, 31	(1, 1, 3)	3.35

(4.31)

Hence the partition function does not just give the sum of these connections with equal weight.

The situation is somewhat less clear than just indicated and rather more puzzling as the third contribution  $m = 11, 17, 25, 31$  does not correspond to a flat  $SU(2)$  connection on  $\Sigma(2, 3, 7)$ . Rather this connection is one identified in [27] as a flat  $SL(2, \mathbb{C})$  connection that is conjugate to  $SL(2, \mathbb{R})$  and not to  $SU(2)$  (cf. also [40]).

The holonomies of the isolated irreducible flat connections are elements of the group and the holonomies can, independently, be conjugated into a preferred maximal torus (even though they are not reducible). The contribution to the Reidemeister Torsion of these holonomies is then determined by the integers  $\mathbf{n}$  as described in [32] equation 2.8 for Brieskorn spheres. To apply this to an actual flat connection there are conditions on the integers arising from the presentation of the fundamental group. What we have in our situation is that all such integers arise as the poles of the Ray-Singer Torsion and, unfortunately, our residue formula (4.25) does not restrict only to the ones that correspond to honest flat  $SU(2)$  connections.

#### 4.6 A residue formula for higher rank and outlook

Here we would like to briefly give a formula in terms of residues for the partition function (4.1) to use as a definition for general  $G$ .

A possible definition, is

$$Z_{BF}[M, G] \simeq 2\pi i \text{Res} \left( f(\phi) \tau_M(\phi) \cdot \prod_{j=1}^{\dim t} \frac{1}{1 - e^{i\phi_j d/P}} \right) \tag{4.32}$$

where  $\phi = \phi_j \alpha^j$  and the function  $f$  is such that  $f(2\pi \mathbf{n} P/d) = 1$  for all  $\mathbf{n} \in I(G)$ ,

$$f(\phi) = \sum_{\mathbf{n} \in \Lambda} e^{in^j \phi_j d/P} f_{\mathbf{n}} \tag{4.33}$$

The poles of  $\prod_j (1 - \exp(i\phi_j d/P))^{-1}$  are at  $\phi_C$ , (4.4), so summing over its residues at these poles is equivalent to performing the  $\lambda$  integral. The properties of  $f(\phi)$  are so that at those poles  $f$  is unity (whence the contribution is exactly the same as performing the  $\lambda$  integral). The function  $f$  is obtained, as in the  $SU(2)$  case, by first performing the sum over  $\mathbf{n}$  to restrict the form of  $\lambda$  and symmetry properties within the integral

$$\sum_{\mathbf{n}} \int_{\mathfrak{t}} d\lambda \exp(i \text{Tr} \lambda (d\phi/P + 2\pi i \mathbf{n})) \rightarrow \prod_{j=1}^{\dim t} \frac{f(\phi)}{1 - e^{i\phi_j d/P}} \tag{4.34}$$

It must also be remembered that the range of  $\phi$  is constrained.

The poles of  $\tau_M$  may also contribute to the residue. If one takes the attitude that one should use the massive Ray-Singer Torsion directly then there are no extra poles and (4.32) agrees with the naive evaluation of the path integral. Alternatively, (4.32) allows one to take the poles of  $\tau_M$  into account after which one may reinstate the mass.

We have not shown that there is a contour that leads to (4.32) but one may reasonably hope that a generalisation of that used in the  $SU(2)$  case will be available.

Before ending this section we would like to suggest a possible a priori reason for why the definition adopted here has shortcomings. We start by observing that it is somewhat

surprising that the possible isolated irreducible flat connections are determined by poles in the Ray-Singer Torsion. This is certainly counterintuitive as the poles of the Ray-Singer Torsion are understood to arise when one has moduli, i.e. 1-form zero modes. A possible reason for this is a shortcoming in the gauge-fixing choice that has been made. As already noted in order to gauge fix  $\lambda$  to be constant on the fibre we require that  $\phi$  be generic ‘enough’. To see that this may be the cause of concern, imagine that one wishes, as we did in section 2, to expand around an isolated irreducible flat connection. We do this but we still insist on gauge fixing the quantum fields as we did in section 3 so that  $\lambda$  is constant along the fibre. Then we would need to solve the equivalent of (3.22)

$$\lambda + D_{\phi_f + \phi} \Lambda = \lambda_0 \tag{4.35}$$

where  $\phi_f$  is the fibre component of the background flat connection and  $\phi$  is the fibre component of the quantum field. If one thinks of the quantum field as having a  $\hbar$  in front of it then to lowest order the equation to be solved is

$$\lambda + D_{\phi_f} \Lambda \simeq \lambda_0 \tag{4.36}$$

and the field  $\phi_f$  is certainly not generic. The discussion in section 3 tells us that precisely for these  $\phi_f$ , as explained around (3.31)–(3.32), the operator  $D_{\phi_f}$  is not invertible and the gauge  $\partial_t \lambda = 0$  cannot be achieved.

Given this question about the gauge choice and the appearance of a complex flat connection in the SU(2) theory one may wonder if resurgence is an approach that may demystify the situation. The *BF* theory, as we have approached it here, arises as one possible limit of  $G_{\mathbb{C}}$  Chern-Simons theory (see the next section). In the SU(2) case the reducible connections are Abelian and one may hope that the resurgence programme will yield the correct non-Abelian connections as they do in [27] with the correct contribution to the partition function.

## 5 Large $k$ asymptotics of $G \times G$ and $G_{\mathbb{C}}$ Chern-Simons theory

Our aim in this section is to see if embedding *BF* theory in a ‘bigger’ theory may act to give a suitable definition which can be used in case  $M$  is an  $\mathbb{Z}$  Homology Sphere. To that end we consider some naive large  $k$  limits of Chern-Simons theory with the compact gauge group  $G \times G$  (with levels  $k$  and  $-k$  for the two factors) and the complexified group  $G_{\mathbb{C}}$ . The ‘fattened’ groups  $G_{-k} \times G_k$  and  $G_{\mathbb{C}}$ , as has already been explained (see appendix A), have been chosen as in the limits to be discussed they both contract to  $TG$ .

The advantage of such an approach is that we avoid the possible issues we had with the generic nature (or not) of  $\phi$  as the values of  $\phi$  signalled out prior to taking the large  $k$  limit allow for the gauge where  $\lambda$  is constant along the fibre.

In the case of  $G_{-k} \times G_k$  we find that the theory does not appear to ‘decompactify’ enough in the large  $k$  limit to capture all the features of the  $TG$  theory. However, in examples, we show that it does actually pick up those flat connections which are both isolated and irreducible correctly. The  $G_{\mathbb{C}}$  Chern-Simons theory, on the otherhand, has all



the features of the  $TG$  theory from the outset. We show that the partition function has the formal properties that would allow an application of the large  $k$  analysis that have worked in the case of the compact groups.

**5.1 Definition 2:  $BF$  and the large  $k$  limit of  $G_k \times G_{-k}$  Chern-Simons theory**

Here we consider the Chern-Simons action for two connections on two copies of the same bundle for the product gauge group  $G \times G$  and for the following gymnastics we require that the bundle in question is trivial. The action under consideration is

$$CS(A_+, A_-) = \frac{1}{4\pi} \int_M \left[ \text{Tr} \left( A_+ \wedge dA_+ + \frac{2}{3} A_+ \wedge A_+ \wedge A_+ \right) - \text{Tr} \left( A_- \wedge dA_- + \frac{2}{3} A_- \wedge A_- \wedge A_- \right) \right] \quad (5.1)$$

The relative sign means that the two gauge theories have opposite levels and we denote such a theory, with action  $k CS(A_+, A_-)$  as a  $G_k \times G_{-k}$  Chern-Simons theory. If we denote the level  $k$  Chern-Simons theory partition function for group  $G$  by  $Z_{CS}[M, G_k]$  then the partition function for the  $G_k \times G_{-k}$  theory is

$$Z_{CS}[M, G_k \times G_{-k}] = Z_{CS}[M, G_k].Z_{CS}[M, G_{-k}] \quad (5.2)$$

Now to show how this is related to  $BF$  theory make the substitutions

$$A_{\pm} = A \pm \frac{\pi}{k} B, \quad A = \frac{1}{2}(A_+ + A_-), \quad B = \frac{k}{2\pi}(A_+ - A_-) \quad (5.3)$$

to arrive at

$$CS(A_+, A_-) = \int_M \text{Tr} \left( B \wedge F_A + \frac{\pi^2}{k^2} B \wedge B \wedge B \right) \quad (5.4)$$

There may be a Jacobian  $J$ , depending on  $k$  in passing to the new variables in the path integral, so the relationship that we expect is finally

$$Z_{BF}[M, G] = \lim_{k \rightarrow \infty} J(k) Z_{CS}[M, G_k \times G_{-k}] \quad (5.5)$$

where the limit as  $k \rightarrow \infty$  formally ensures that the cubic term in  $B$  may be neglected. We allow  $J$  to also soak up any other factors of  $k$  that may be present due to zero modes and so on.

This formal result says that the  $k \rightarrow \infty$  limit of a  $G_k \times G_{-k}$  Chern-Simons theory is equivalent to a pure BF theory. However, this statement certainly requires some elaboration as adding and subtracting connections at will as in (5.3) is not usually an operation that makes sense in the theory of bundles. This is mirrored in the gauge transformations that one obtains for  $A$  and  $B$  which are also not what one might call ‘standard’. To emphasise that the limit in question is by no means obvious consider the 3-manifold  $\Sigma \times S^1$ . Chern-Simons theory on such a manifold is perfectly sensible and the large  $k$  limit is well understood, the partition function having leading term  $k^{n/2} \text{Vol}(\mathfrak{M}[\Sigma, G])$  the symplectic volume of the moduli space of  $G$  connections on  $\Sigma$  where  $n = \dim(\mathfrak{M}[\Sigma, G])$  [42]. The large  $k$  limit of

the  $G_k \times G_{-k}$  partition function is then proportional to  $k^n \text{Vol}(\mathfrak{M}[\Sigma, G])^2$  and not what one would expect from  $BF$  theory, say  $k^n \text{Vol}(T\mathfrak{M}[\Sigma, G])$  (even though this is ill defined).

So we need to better understand when the limit will indeed correspond to  $BF$  theory. At large  $k$ , the stationary phase approximation tells us that the functional integral [12] localises around the critical points of the action  $CS(A)$ , which in this case are precisely the flat connections. We then have, for isolated flat connections, as  $k \rightarrow \infty$ ,

$$Z_k[M] \sim \sum_{\alpha} e^{2\pi i k CS(\alpha)} e^{-\frac{i\pi}{2} I(\alpha)} \sqrt{\tau_M(\alpha)} \tag{5.6}$$

where  $\alpha$ ,  $\tau_M(\alpha)$  and  $I(\alpha)$  are the flat connections, the Ray-Singer torsion at the flat connection and the spectral flow to the flat connection respectively.

The  $G_k \times G_{-k}$  theory then has the asymptotic form

$$Z_k[M] Z_{-k}[M] \sim \sum_{\alpha, \beta} e^{ik (CS(\alpha) - CS(\beta))} f_{\alpha\beta} \sqrt{\tau_M(\alpha)} \sqrt{\tau_M(\beta)} \tag{5.7}$$

where

$$f_{\alpha\beta} = \exp\left(-\frac{i\pi}{2} [I(A_{\alpha}) - I(A_{\beta})]\right) \tag{5.8}$$

One would expect therefore, that in the limit as  $k \rightarrow \infty$ , that providing that flat connections  $\alpha$  and  $\beta$  having the same Chern-Simons invariant implies they are the same flat connection,

$$CS(\alpha) = CS(\beta) \pmod{2\pi\mathbb{Z}} \implies \alpha = \beta \tag{5.9}$$

then the oscillations in (5.7) would ensure that we would indeed only need to sum over  $\alpha = \beta$ . In such a situation then one could reasonably expect that (5.5) holds. If this condition does not hold then there may be ‘non-diagonal’ contributions to the sum (5.7) and so the relationship with  $BF$  theory becomes more tenuous.

Indeed for any connected component of non-isolated flat connections all the flat connections in that component have the same Chern-Simons invariant and fail our test (5.9). The flat connections on  $\Sigma \times S^1$  that we discussed previously are of this type and this, to some extent, explains why they do not match the expectations for  $BF$  theory. The lesson here is that we must concentrate on isolated flat connections. Fortunately for us Fintushil and Stern [39] have shown that the Brieskorn spheres (the Seifert  $\mathbb{Z}$  homology spheres with 3 exceptional fibres) have moduli spaces of flat  $SU(2)$  connections made up of a finite number of discrete points.

We shall now illustrate when (5.9) holds with a few examples in the literature. Consider firstly the Poincaré Homology Sphere,  $M = \Sigma(2, 3, 5)$ . The Poincaré  $\mathbb{Z}$  homology sphere has three flat  $SU(2)$  connections, one of which is the reducible trivial connection and two which are non-Abelian. Freed and Gompf [19] determine the large  $k$  behaviour for the Chern-Simons partition function and find

$$Z_k[\Sigma(2, 3, 5)] \sim \sqrt{\frac{2}{5}} e^{-3\pi i/4} \left[ \sin\left(\frac{\pi}{5}\right) e^{-\pi i(k+2)/60} + \sin\left(\frac{2\pi}{5}\right) e^{-49\pi i(k+2)/60} \right] \tag{5.10}$$

Taking the modulus square, and presuming the limit of the norm agrees with the norm of the limit, one gets

$$\lim_{k \rightarrow \infty} Z_k[\Sigma(2, 3, 5)]Z_{-k}[\Sigma(2, 3, 5)] \sim \frac{2}{5} \left[ \sin^2 \left( \frac{\pi}{5} \right) + \sin^2 \left( \frac{2\pi}{5} \right) \right] \quad (5.11)$$

this corresponds to the sum of the Ray-Singer torsions of the irreducible connections for  $\Sigma(2, 3, 5)$  up to a finite normalisation. Similarly [19] provides us with the example of  $\Sigma(2, 3, 17)$  and, with the same caveat on exchanging limits and norms, we find that the norm square of the Chern-Simons partition function is the sum of the Ray-Singer Torsion over the 6 irreducible flat  $SU(2)$  connections.

There are manifolds, however, where you find two or more isolated flat connections giving the same Chern-Simons invariant so that (5.9) does not hold. Lens spaces yield examples of this phenomena. As we have seen for Lens spaces  $L(p, q)$  the Ray-Singer Torsion is given by

$$\tau_{L(p,q)} = \frac{16}{p} \sin^2 \left( \frac{2\pi n}{p} \right) \sin^2 \left( \frac{2\pi q^* n}{p} \right) \quad (5.12)$$

Here one integer is enough to describe the flat (Abelian) connections.

A typical example where two different connections have the same Chern-Simons invariant is afforded by the Lens space  $L(12, 5)$ . For  $L(12, 5)$  we have  $q^* = 5$  and one can see for example that  $n = 1$  and  $n = 5$  give the same Chern-Simons invariant (mod  $2\pi$ ), namely

$$CS(n) = q^* n^2 / p = \frac{5}{12} \quad (5.13)$$

so we are not free to use the norm of the Chern-Simons theory in this case as a way to define the  $BF$  theory.

The discussion of this section shows that this definition of  $BF$  theory as a limit of the  $G_{-k} \times G_k$  Chern-Simons theory needs to be handled with care and we might have to know more about the moduli space itself than we would care to. However, for the part of the moduli space where the connections are isolated and irreducible this appears to be an appropriate definition.

### 5.2 Definition 3: $BF$ and the large $k$ or $s$ limit of $G_{\mathbb{C}}$ Chern-Simons theory

The final definition that we give involves the gauge group  $G_{\mathbb{C}}$ . Note that the classical groups  $TG$  and  $G_{\mathbb{C}}$  are diffeomorphic (as spaces). The Inönü-Wigner contraction that establishes the Lie algebra relationship is given in appendix A. Furthermore, the  $G_{\mathbb{C}}$  connection is

$$A_{\mathbb{C}} = A + iB \quad (5.14)$$

and as  $G_{\mathbb{C}}$  contracts to the compact group  $G$  the connection  $A$  may be considered a  $G$  connection while  $B$  is then a Lie algebra  $\mathfrak{g}$  valued one-form. This is a much clearer decomposition than the mixing of objects that appeared in the  $G_{-k} \times G_k$  theory of the previous section.

The  $G_{\mathbb{C}}$  Chern-Simons action is

$$I(k, s) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A - B \wedge d_A B \right) - \frac{s}{2\pi} \int_M \text{Tr} \left( B \wedge F_A - \frac{1}{3} B \wedge B \wedge B \right) \quad (5.15)$$

with  $k \in \mathbb{Z}$  and we take  $s \in \mathbb{R}$  [20]. One can see that the complex Chern-Simons action has a number of limits which formally lead to the  $BF$  action. The first, and most straightforward way, to arrive at the  $BF$  theory is to set  $k = 0$  and consider the  $s \rightarrow \infty$  limit where one sends  $B \rightarrow B/s$  to formally arrive at

$$I(k, s) \rightarrow -\frac{1}{2\pi} \int_M \text{Tr} (B \wedge F_A) \quad (5.16)$$

Alternatively one can consider the large  $k$  limit, for finite  $s$ , and this time by scaling  $B \rightarrow B/\sqrt{k}$ . One does not get the  $BF$  action directly, but rather

$$I(k, s) \rightarrow \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{1}{4\pi} \int_M \text{Tr} (B \wedge d_A B) \quad (5.17)$$

About an isolated flat connection this would be the same as the large  $k$  limit of Chern-Simons theory except that the integral over the field  $B$  has, formally the effect of replacing the square root of the Ray-Singer torsion of the flat connection with the Ray-Singer torsion itself and so would, in the limit, reproduce the partition function of  $BF$  theory with a Chern-Simons action contribution of the flat connection. Both limits (and indeed any other limits of  $k$  and  $s$ ) need to be carefully taken as in principle one would most likely land on the moduli space of flat  $G_{\mathbb{C}}$  connections and not just the flat  $G$  connections.

For finite  $k$  and with  $s = 0$  Gukov and Pei [23] have evaluated the partition function on  $M = \Sigma \times S^1$ . Even though the Hilbert space is also infinite dimensional in this case, they ‘filter’ it by giving the  $B$  field a mass and so are able to extract meaningful results. Unlike the situation with the  $G_{-k} \times G_k$ , where one gets essentially the square of the compact case, here one sees that the results come from a non-compact group and formally diverges as  $k \rightarrow \infty$  as we would expect for the  $BF$  theory.

Returning to the case at hand note that the corresponding finite dimensional integral to be performed in the case of complex Chern-Simons after Abelianisation [17] has as its action

$$I(k, s) = -\frac{k}{4\pi} \text{Tr}(\phi^2 - \lambda^2) c_1(\mathcal{L}_M) + k \sum_{i=1}^N \frac{1}{a_i} \text{Tr}(-i\phi \mathbf{n}_i + r_i \pi \mathbf{n}_i^2) + k \text{Tr}(-i\phi \mathbf{n}_0) + \frac{s}{2\pi} \text{Tr} \left( \lambda \left( \phi c_1(\mathcal{L}_M) + 2\pi i \sum_{i=1}^N \frac{1}{a_i} \mathbf{n}_i + 2\pi i \mathbf{n}_0 \right) \right) \quad (5.18)$$

while the Ray-Singer Torsion goes over to the complex Ray-Singer Torsion

$$\tau_M(\phi) \rightarrow \sqrt{\tau_M(\phi + i\lambda) \tau_M(\phi - i\lambda)} \quad (5.19)$$

Formally the complex Ray-Singer Torsion under either scaling goes back to the ‘real’ Ray-Singer Torsion. The  $k = 0$  and  $s \rightarrow \infty$  limit lands us on the finite dimensional action that we have been using for  $BF$  theory. Here we would like to investigate the other limit.

The limit of interest then is large  $k$  and for  $\lambda \rightarrow \lambda/\sqrt{k}$ . As  $\lambda$  now only appears in the action quadratically one may integrate it out. In this limit the finite action goes over to

$$I(k, s) \rightarrow -\frac{k}{4\pi} \text{Tr}(\phi^2)_{C_1}(\mathcal{L}_M) + k \sum_{i=1}^N \frac{1}{a_i} \text{Tr}(-i\phi \mathbf{n}_i + r_i \pi \mathbf{n}_i^2) + k \text{Tr}(-i\phi \mathbf{n}_0) \quad (5.20)$$

which is just the standard Chern-Simons action for compact group  $G$ . The partition function, however, includes an extra factor of the square root of the Ray-Singer Torsion (arising from the integral over the field  $B$  in (5.17)) in the integral over  $\mathfrak{t}$  so this is not the partition function of compact Chern-Simons theory.

As the action agrees with that of  $G$  Chern-Simons theory then all the properties of the integrand (see appendix B for the symmetry properties) and hence the considerations that appear in [24] and [25] apply here too. This includes the choice of contour to take the large  $k$  limit, the only difference being that of considering a slightly different function that takes into account that we have the Ray-Singer Torsion not its square root in the integrand. The resulting large  $k$  asymptotics are, for Chern-Simons theory, just as predicted by Witten in [12], contributions around flat connections. Consequently, in the limit that we are considering in the  $G_{\mathbb{C}}$  theory, this means that the asymptotic form will have contributions from flat connections in  $BF$  theory as anticipated in section 2.

Specifically in the case of  $SU(2)$  the large  $k$  limit is that of the flat connection contributions on page 302 of [24] but with  $F(y)$  there replaced with  $F(y)^2$  in order to pass to the  $BF$  theory (and one should not include the framing as it ought not to arise in complex Chern-Simons theory) while for a general compact group one can refer to (4.17) in [25] from which one can deduce the large  $k$  behaviour. At this point this particular definition of  $BF$  theory is the most complete that we have.

If one would like to obtain more explicit formulae for the  $BF$  theory one could follow the explanation of [24] on how to arrive at Rozansky’s formula for the large  $k$  expansion of  $SU(2)$  Chern-Simons theory [41] which reproduces Witten’s expansion. Of course this also encodes some of the large  $k$  structure of the  $G_{\mathbb{C}}$  theory itself so that by following different parts of the asymptotic expansion we would be able to obtain parts of the perturbation theory. In particular we also have in mind the contributions about the trivial connection. We leave these issues, and the question of how and whether complex connections contribute to the  $BF$  limit for the future.

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## A The tangent bundle group $TG$ and its Lie algebra $\mathfrak{tg}$

### A.1 Basic properties of $TG$ and $\mathfrak{tg}$

let  $G$  be a compact Lie group and (to not unnecessarily complicate things) assume that  $G$  is connected, simply-connected and semi-simple. Via a right-invariant trivialisation of the tangent bundle, the group  $TG$  can be identified with the semi-direct product of  $G$  with its Lie algebra  $\mathfrak{g}$ ,

$$TG \simeq G \ltimes \mathfrak{g} \equiv G \times_{\text{Ad}} \mathfrak{g}, \quad (\text{A.1})$$

with multiplication

$$(g, v)(h, w) = (gh, v + \text{Ad}_g w), \quad (\text{A.2})$$

where the adjoint action of  $g \in G$  on  $w \in \mathfrak{g}$  is  $\text{Ad}_g w = gwg^{-1}$ . Correspondingly, its Lie algebra has the form

$$\text{Lie}(TG) \equiv \mathfrak{tg} \simeq \mathfrak{g} \oplus_{\text{ad}} \mathfrak{g}_{Ab} \quad (\text{A.3})$$

where  $\mathfrak{g}_{Ab}$  is the Abelian Lie algebra based on the underlying vector space  $\mathfrak{g}$ , i.e. the commutator is

$$[(x, v), (y, w)] = ([x, y], [x, w] + [v, y]) = ([x, y], \text{ad}_x w - \text{ad}_y v) \quad (\text{A.4})$$

and the (inverse) adjoint action of  $TG$  on its Lie algebra is

$$(g, v)^{-1}(y, w)(g, v) = (\text{Ad}_{g^{-1}} y, \text{Ad}_{g^{-1}}(w + [y, v])) \quad (\text{A.5})$$

Let  $\ell_a$  be a (real) basis of the (real) Lie algebra  $\mathfrak{g}$ , with

$$[\ell_a, \ell_b] = f_{ab}^c \ell_c. \quad (\text{A.6})$$

Then

$$j_a = (\ell_a, 0), \quad p_a = (0, \ell_a) \quad (\text{A.7})$$

are a basis of  $\mathfrak{tg}$ , and the Lie algebra commutation relations take the form

$$[j_a, j_b] = f_{ab}^c j_c, \quad [j_a, p_b] = [p_a, j_b] = f_{ab}^c p_c, \quad [p_a, p_b] = 0. \quad (\text{A.8})$$

Here are two useful properties of  $\mathfrak{tg}$ :

1. The Lie algebra  $\mathfrak{tg}$  of  $TG$  can be obtained as a contraction of the Lie algebra of the group  $G \times G$

- Consider the  $G \times G$  Lie algebra

$$\mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{g} \oplus \mathfrak{g} \quad (\text{A.9})$$

with generators

$$[j_a^\pm, j_b^\pm] = f_{ab}^c j_c^\pm, \quad [j_a^+, j_b^-] = 0. \quad (\text{A.10})$$

- Perform the redefinition

$$j_a^\pm = \frac{1}{2}(j_a \pm p_a/\epsilon) \quad \Leftrightarrow \quad j_a = j_a^+ + j_a^-, \quad p_a = \epsilon(j_a^+ - j_a^-) \quad (\text{A.11})$$

- Then the algebra  $\mathfrak{g} \oplus \mathfrak{g}$  takes the form

$$[j_a, j_b] = f_{ab}^c j_c, \quad [j_a, p_b] = [p_a, j_b] = f_{ab}^c p_c, \quad [p_a, p_b] = \epsilon^2 f_{ab}^c j_c. \quad (\text{A.12})$$

- In the limit  $\epsilon \rightarrow 0$ , this reduces to (A.8), the Lie algebra of  $TG$ .
- If one sets  $\epsilon = i\delta$ , the same reasoning shows that  $\mathfrak{tg}$  can be obtained as the contraction of the complexification  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$  of  $\mathfrak{g}$ .

## 2. Existence of Invariant Scalar Products on $\mathfrak{tg}$

Let us consider the case where the Lie algebra  $\mathfrak{g}$  is simple. In that case,  $\mathfrak{g}$  has a preferred (and up to a choice of scale unique) non-degenerate and  $\mathfrak{ad}$ -invariant metric scalar product, namely the Killing-Cartan form  $\text{Tr ad}_x \text{ad}_y$  (the trace in the adjoint representation). With respect to the basis  $\ell_a$  of generators, this metric has the components

$$g_{ab} \equiv \langle \ell_a, \ell_b \rangle = \text{Tr ad}_{\ell_a} \text{ad}_{\ell_b} = f_{ac}^d f_{bd}^c \quad (\text{A.13})$$

Turning to the Lie algebra

$$\mathfrak{tg} \simeq \mathfrak{g} \oplus_{\mathfrak{ad}} \mathfrak{g}_{Ab}, \quad (\text{A.14})$$

as it is not semi-simple, its Killing-Cartan form will be  $\mathfrak{tg}$ -invariant (by construction, i.e. by the Jacobi identity) but degenerate. Indeed, in terms of the generators  $(j_a, p_a)$  one has

$$\text{Tr ad}_{j_a} \text{ad}_{j_b} = 2g_{ab} = 2f_{ac}^d f_{bd}^c \quad (\text{A.15})$$

(from the adjoint action of  $\text{ad}_{j_a} \text{ad}_{j_b}$  on  $j_c$  and on  $p_c$ ), and

$$\text{Tr ad}_{j_a} \text{ad}_{p_b} = \text{Tr ad}_{p_a} \text{ad}_{p_b} = 0 \quad (\text{A.16})$$

(because both  $\text{ad}_{j_a} \text{ad}_{p_b}$  and  $\text{ad}_{p_a} \text{ad}_{p_b}$  only act non-trivially on a  $j_c$  and take it to a linear combination of the  $p_c$ ). This scalar product can therefore also be written as

$$\langle j_a, j_b \rangle = 2g_{ab}, \quad \langle j_a, p_b \rangle = \langle p_a, p_b \rangle = 0. \quad (\text{A.17})$$

In addition to the (degenerate) Killing-Cartan form,  $\mathfrak{tg}$  exceptionally also possesses a non-degenerate  $\mathfrak{ad}$ -invariant scalar product given by

$$\ll j_a, j_b \gg = \ll p_a, p_b \gg = 0, \quad \ll j_a, p_b \gg = g_{ab}. \quad (\text{A.18})$$

It is easily verified that this is indeed both invariant and non-degenerate.

The existence of this second invariant scalar product, or of the overall two-parameter family of invariant scalar products, can be understood in terms of the contraction of the Lie algebra  $\mathfrak{g} \oplus \mathfrak{g}$  to  $\mathfrak{tg}$  mentioned above. Indeed, starting with the non-degenerate Killing-Cartan metric on the Lie algebra  $\mathfrak{g} \oplus \mathfrak{g}$  (A.10) of  $G \times G$ , with coefficients  $c^\pm$ ,

$$\langle j_a^\pm, j_b^\pm \rangle = c^\pm g_{ab}, \quad \langle j_a^+, j_b^- \rangle = 0, \quad (\text{A.19})$$



and performing the redefinition (A.11), one finds

$$\langle j_a, j_b \rangle = (c^+ + c^-)g_{ab}, \quad \langle j_a, p_b \rangle = \epsilon(c^+ - c^-)g_{ab}, \quad \langle p_a, p_b \rangle = \epsilon^2(c^+ + c^-)g_{ab} \quad (\text{A.20})$$

Taking the contraction  $\epsilon \rightarrow 0$  with  $c^+ = c^- = 1$ , one finds (A.17), while taking  $c^+ = -c^- = 1/2\epsilon$  one finds (A.18),

$$\begin{aligned} c^+ = c^- = 1 &\Rightarrow \langle \cdot, \cdot \rangle \\ c^+ = -c^- = 1/2\epsilon &\Rightarrow \ll \cdot, \cdot \gg \end{aligned} \quad (\text{A.21})$$

## A.2 Diagonalisation / Abelianisation in $TG$ and $\mathfrak{t}_{\mathbf{g}}$

Let  $G$  be compact,  $T_G$  be a maximal torus,  $\mathfrak{t}_G$  its Lie algebra, a Cartan subalgebra. Then the following assertions are true:

1. For any  $h \in G$  one can find a  $g \in G$  such that

$$h \in G \Rightarrow \exists g \in G : \text{Ad}_g h = ghg^{-1} \in T_G. \quad (\text{A.22})$$

2. For any  $y \in \mathfrak{g}$  one can find a  $g \in G$  such that

$$y \in \mathfrak{g} \Rightarrow \exists g \in G : \text{Ad}_g y = gyg^{-1} \in \mathfrak{t}_G. \quad (\text{A.23})$$

As  $TG$  is not compact and not semi-simple, a priori the corresponding statements do not necessarily hold (and would usually be far from true for a generic such group). Nevertheless, it turns out that the above statements carry over literally to the case of the group  $TG$ , provided that we replace the maximal torus  $T_G$  of  $G$  and the Cartan subalgebra  $\mathfrak{t}_G$  of  $\mathfrak{g}$  by

$$T_{TG} = T(T_G) \simeq T_G \times_{\text{Ad}} \mathfrak{t}_G = T_G \times \mathfrak{t}_G \quad (\text{A.24})$$

$$\mathfrak{t}_{TG} = \text{Lie}(T_G \times \mathfrak{t}_G) = \mathfrak{t}_G \oplus \mathfrak{t}_G \quad (\text{A.25})$$

(since  $\mathfrak{t}_G$  is already an Abelian Lie algebra, here it is not necessary to write  $(\mathfrak{t}_G)_{Ab}$  in the second factor / summand). Indeed, we can now prove the following two statements:

1. Diagonalisation / Abelianisation in  $TG$

$$\forall (h, w) \in TG \simeq T_G \times_{\text{Ad}} \mathfrak{g} \quad \exists (g, v) \in TG : (g, v)^{-1}(h, w)(g, v) = (t, \tau_2) \in T_G \times \mathfrak{t}_G \quad (\text{A.26})$$

2. Diagonalisation / Abelianisation in  $\mathfrak{t}_{\mathbf{g}}$

$$\forall (y, w) \in \mathfrak{t}_{\mathbf{g}} \simeq \mathfrak{g} \oplus_{\text{ad}} \mathfrak{g} \quad \exists (g, v) \in TG : (g, v)^{-1}(y, w)(g, v) = (\tau_1, \tau_2) \in \mathfrak{t}_G \oplus \mathfrak{t}_G \quad (\text{A.27})$$

Here are the proofs of these assertions:

1. Diagonalisation / Abelianisation in  $TG$

- The (inverse) Ad-action of  $TG$  on itself is

$$(g, v)^{-1}(h, w)(g, v) = (\text{Ad}_{g^{-1}}h, \text{Ad}_{g^{-1}}(w + \text{Ad}_h v - v)) \quad (\text{A.28})$$

In particular, for the action of  $(g, 0)$  one finds

$$(g, 0) : \quad h \mapsto \text{Ad}_{g^{-1}}h, \quad w \mapsto \text{Ad}_{g^{-1}}w \quad (\text{A.29})$$

and for that of  $(e, v)$  one has

$$(e, v) : \quad h \mapsto h, \quad w \mapsto w + \text{Ad}_h v - v. \quad (\text{A.30})$$

- First of all, we can and will choose  $g \in G$  such that

$$h \mapsto g^{-1}hg = t \in T_G. \quad (\text{A.31})$$

- Next we define  $S$  to be the stabiliser of  $t$  under the adjoint action of  $G$ ,

$$S := \text{Stab}_G(t) = \{g \in G : g^{-1}tg = t\}. \quad (\text{A.32})$$

One has

$$S \supseteq T_G, \quad (\text{A.33})$$

with equality if  $t$  is generic (regular). In the non-generic case,  $S$  will be a product of simple factors and  $U(1)$ s. We use the convention that  $T_S = T_G$ , and likewise at the Lie algebra level,

$$T_S = T_G, \quad \mathfrak{t}_S = \mathfrak{t}_G. \quad (\text{A.34})$$

At the Lie algebra level one has the (reductive, because  $\mathfrak{g}$  is compact) decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m} \quad \text{with} \quad [\mathfrak{s}, \mathfrak{m}] \subset \mathfrak{m}. \quad (\text{A.35})$$

- Now let us turn to the second entry in (A.28),

$$w \mapsto \text{Ad}_{g^{-1}}(w + \text{Ad}_h v - v). \quad (\text{A.36})$$

With  $h = t \in T_G$  and thus  $g = s \in S$ , this becomes

$$w \mapsto \text{Ad}_{s^{-1}}(w + \text{Ad}_t v - v). \quad (\text{A.37})$$

Decomposing  $v = v^{\mathfrak{s}} + v^{\mathfrak{m}}$  into its components in  $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}$ , we see that only the  $\mathfrak{m}$ -component of  $v$  contributes, since  $\text{Ad}_t v^{\mathfrak{s}} = v^{\mathfrak{s}}$ ,

$$\text{Ad}_t v - v = (\text{Ad}_t - 1)v^{\mathfrak{m}} \in \mathfrak{m}, \quad (\text{A.38})$$

and  $v^{\mathfrak{m}}$  can be chosen to cancel the  $\mathfrak{m}$ -component  $w^{\mathfrak{m}}$  of  $w$  (because by definition of  $\mathfrak{m}$  the operator  $(\text{Ad}_t - 1)$  is invertible on  $\mathfrak{m}$ ).

- We are then left with

$$w \mapsto \text{Ad}_{s^{-1}} w^{\mathfrak{s}}, \tag{A.39}$$

and we can now choose  $s \in S$  such that

$$w \mapsto \text{Ad}_{s^{-1}} w^{\mathfrak{s}} = \tau_2 \in \mathfrak{t}_S = \mathfrak{t}_G. \tag{A.40}$$

- Altogether, we have thus managed to conjugate

$$(h, w) \mapsto (t, \tau_2) \in T_G \times \mathfrak{t}_G, \tag{A.41}$$

as announced.

## 2. Diagonalisation / Abelianisation in $\mathfrak{t}_g$

- The inverse adjoint action of  $TG$  on its Lie algebra is

$$(g, v)^{-1}(y, w)(g, v) = (\text{Ad}_{g^{-1}} y, \text{Ad}_{g^{-1}}(w + [y, v])) \tag{A.42}$$

In particular, for the action of  $(g, 0)$  one finds

$$(g, 0) : \quad y \mapsto \text{Ad}_{g^{-1}} y, \quad w \mapsto \text{Ad}_{g^{-1}} w \tag{A.43}$$

and for that of  $(e, v)$  one has

$$(e, v) : \quad y \mapsto y, \quad w \mapsto w + [y, v]. \tag{A.44}$$

- First of all, we can and will choose  $g \in G$  such that

$$y \mapsto g^{-1} y g = \tau_1 \in \mathfrak{t}_G. \tag{A.45}$$

- Next we define  $S$  to be the stabiliser of  $\tau_1$  under the adjoint action of  $G$ ,

$$S := \text{Stab}_G(\tau_1) = \{g \in G : g^{-1} \tau_1 g = \tau_1\}. \tag{A.46}$$

One has

$$S \supseteq T_G, \tag{A.47}$$

with equality if  $\tau_1$  is generic (regular). As above, we set  $T_S = T_G$  and  $\mathfrak{t}_S = \mathfrak{t}_G$ , with  $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}$  and  $[\mathfrak{s}, \mathfrak{m}] \subset \mathfrak{m}$ .

- Now let us turn to the second entry in (A.42),

$$w \mapsto \text{Ad}_{g^{-1}}(w + [y, v]) \tag{A.48}$$

With  $y = \tau_1 \in \mathfrak{t}_G$ , and thus  $g = s \in S$ , this becomes

$$w \mapsto \text{Ad}_{s^{-1}}(w + [\tau_1, v]) \tag{A.49}$$

Decomposing  $v = v^{\mathfrak{s}} + v^{\mathfrak{m}}$ , only  $v^{\mathfrak{m}}$  contributes to the commutator,

$$[\tau_1, v] = [\tau_1, v^{\mathfrak{m}}] \in \mathfrak{m}, \tag{A.50}$$

and  $v^{\mathfrak{m}}$  can be chosen to cancel  $w^{\mathfrak{m}}$  (because by definition of  $\mathfrak{m}$  the operator  $\text{ad}_{\tau_1}$  is invertible on  $\mathfrak{m}$ ).

- We are then left with

$$w \mapsto \text{Ad}_{s^{-1}} w^{\mathfrak{s}}, \tag{A.51}$$

and as in the proof above we can now choose  $s \in S$  such that

$$w \mapsto \text{Ad}_{s^{-1}} w^{\mathfrak{s}} = \tau_2 \in \mathfrak{t}_S = \mathfrak{t}_G. \tag{A.52}$$

- Altogether, we have thus managed to conjugate

$$(y, w) \mapsto (\tau_1, \tau_2) \in \mathfrak{t}_G \oplus \mathfrak{t}_G, \tag{A.53}$$

as announced.

Remarks:

1. We see that for any  $(h, w) \in TG$  or  $(y, w) \in \mathfrak{t}\mathfrak{g}$ 
  - one needs  $g \in G/S$  (meaning  $g \in G$  modulo elements in  $S$ ) to conjugate  $h$  or  $y$  into  $T_G$
  - one needs  $v^{\mathfrak{m}} \in \mathfrak{m} \subset \mathfrak{g}$  to map  $w \mapsto w^{\mathfrak{s}} \in \mathfrak{s}$
  - one then needs  $s \in S/T_G$  to conjugate  $w^{\mathfrak{s}} \mapsto \tau_2 \in \mathfrak{t}_G$

Thus in total, for any  $g$  or  $y$  (regardless of regular or not) the parameters  $(g, v)$  lying in  $TT_G \simeq T_G \times \mathfrak{t}_G \subset TG$  are not needed to accomplish the Abelianisation / diagonalisation.

2. In particular, in the generic regular case ( $g$  or  $y$  regular), one has  $S = T_G$  and  $\mathfrak{s} = \mathfrak{t}_G$ . Thus the last step in the proof (consisting of conjugating  $w^{\mathfrak{s}}$  into  $\mathfrak{t}_S = \mathfrak{t}_G$ ) is empty. The entire ambiguity in the construction then lies in the first step, the conjugation of  $h$  into  $T_G$  (or  $y$  into  $\mathfrak{t}_G$ ). The possible choices of  $t$  or  $\tau_1$  are related by the action of the Weyl Group  $W$ ,

$$W = N_G(T_G)/T_G, \tag{A.54}$$

where  $N_G(T_G)$  is the normaliser of  $T_G$  in  $G$ .

3. In the opposite extreme case of  $g = e$  or  $y = 0$ , say, evidently the first step of the argument is empty, the stabiliser of  $g = e$  or  $y = 0$  is of course all of  $G$ ,  $S = G$ , and then in the last step there is then a  $W$ -fold degeneracy in the choice  $\tau_2 \in \mathfrak{t}_G$ .

## B A symmetry of complex Chern-Simons theory on Seifert 3-manifolds

Here we establish that the symmetry that we need in section 3.4 to pass from having a background field and summing over integers at each orbifold point to having a background field with just one integer summation to perform exists at the level of Complex Chern-Simons theory.  $BF$  theory is then just a special limit.<sup>4</sup> One, possibly surprising, outcome is that the two approaches do not agree at the level of holomorphic factorisation.

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<sup>4</sup>It is straightforward to show that the symmetry is available directly in  $BF$  theory but we wanted to establish the more general result here.

We should also point out that these symmetries are a variation of those found by Lawrence and Rozansky [24] and by Mariño [25]. However, here the symmetries have a clear geometric meaning which we explain.

### B.1 Discrete symmetries

The exponent in the finite dimensional integral that one arrives at in the case of complex Chern-Simons theory is

$$\begin{aligned}
 I(k, s) = & -\frac{k}{4\pi} \text{Tr}(\phi^2 - \lambda^2) c_1(\mathcal{L}_M) + k \sum_1^N \frac{1}{a_i} \text{Tr}(-i\phi \mathbf{n}_i + r_i \pi \mathbf{n}_i^2) + k \text{Tr}(-i\phi \mathbf{n}_0) \\
 & + \frac{s}{2\pi} \text{Tr} \left( \lambda \left( \phi c_1(\mathcal{L}_M) + 2\pi i \sum_{i=1}^N \frac{1}{a_i} \mathbf{n}_i + 2\pi i \mathbf{n}_0 \right) \right)
 \end{aligned} \tag{B.1}$$

(just use this instead of the exponent in (4.1)).

The exponential of this action as well as the complex Ray-Singer torsion enjoy a number of symmetries. We will present them here, but before doing that we note that the transformations below do not depend on the coupling constants  $k$  and  $s$  and so those parts of the action are independently invariant. Also one can see that the  $k$ -dependent part of the action (B.1), apart from the  $\lambda^2$  term, is just what one gets from Chern-Simons theory on  $M$  with compact gauge group  $G$ . Furthermore, the field  $\lambda$  does not transform, so that the  $k$ -dependent part has the same invariance as for the  $G$  Chern-Simons theory.

First, we have the symmetry

$$\mathbf{n}_i \rightarrow \mathbf{n}_i + a_i \mathbf{m}_i, \quad \mathbf{n}_0 \rightarrow \mathbf{n}_0 - \sum_{i=1}^N \mathbf{m}_i \tag{B.2}$$

It is clear that (B.1) is not changed by these transformations (and it is consistent with our limit on the range of summation over the  $\mathbf{n}_i$ ).

**Remark.** Note that geometrically this symmetry is the statement that, as the appropriate powers of the V-line bundle  $\mathcal{L}_i$  of degree  $1/a_i$  at the  $i$ 'th orbifold point is an honest line bundle

$$\mathcal{L}_i^{\otimes m_i a_i} \simeq \mathcal{L}_0^{\otimes m_i} \tag{B.3}$$

any orbifold line bundle  $\mathcal{L}$  satisfies

$$\begin{aligned}
 \mathcal{L} = & \mathcal{L}_0^{\otimes n_0} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_N^{\otimes n_N} \\
 \simeq & \mathcal{L}_0^{\otimes (n_0 - \sum_i m_i)} \otimes \mathcal{L}_1^{\otimes (n_1 + a_1 m_1)} \otimes \dots \otimes \mathcal{L}_N^{\otimes (n_N + a_N m_N)}
 \end{aligned} \tag{B.4}$$

so one has not changed the bundle but just expressed it in a different way.

Second, we also have the transformations

$$\mathbf{n}_i \rightarrow \mathbf{n}_i + b_i \mathbf{v}, \quad \phi \rightarrow \phi - 2\pi i \mathbf{v}, \quad \mathbf{n}_0 \rightarrow \mathbf{n}_0 + b_0 \mathbf{v} \tag{B.5}$$

which form a symmetry of the exponential of the action.

**Remark.** Geometrically this says that the pair  $(\mathcal{L}, \phi)$  of an orbifold bundle  $\mathcal{L}$  with curvature  $\phi d\kappa$  satisfy

$$(\mathcal{L}, \phi) \simeq (\mathcal{L} \otimes \mathcal{L}_M^{\otimes v}, \phi - 2\pi i v) \quad (\text{B.6})$$

The first Chern class of  $\mathcal{L}_M^{\otimes v}$  is

$$c_1(\mathcal{L}_M^{\otimes v}) = v.c_1(\mathcal{L}_M) \quad (\text{B.7})$$

while

$$\frac{i}{2\pi} \int_{\Sigma_V} -2\pi i v d\kappa = v \int_{\Sigma_V} d\kappa = -v.c_1(\mathcal{L}_M) \quad (\text{B.8})$$

It is now possible to combine the symmetries and consider

$$\mathbf{n}_i \rightarrow \mathbf{n}_i + a_i \mathbf{u} + b_i \mathbf{v}, \quad \phi \rightarrow \phi - 2\pi i v, \quad \mathbf{n}_0 \rightarrow \mathbf{n}_0 + b_0 \mathbf{v} - N \mathbf{u} \quad (\text{B.9})$$

One then has a constructive proof that this symmetry allows one to set all the  $\mathbf{n}_i$  for  $i \neq 0$  to zero. Recall that if the  $\text{gcd}(a, b) = 1$  then any integer  $n$  can be expressed as

$$n = au + bv \quad (\text{B.10})$$

for some integers  $u$ . Furthermore, if  $\text{gcd}(a, b) = 1$  and  $\text{gcd}(a, c) = 1$  then  $\text{gcd}(a, bc) = \text{gcd}(a, b) \cdot \text{gcd}(a, c) = 1$ .

As a first step let  $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_1, \mathbf{v}_1)$  so that

$$\mathbf{n}_1 + a_1 \mathbf{u}_1 + b_1 \mathbf{v}_1 = 0 \quad (\text{B.11})$$

which is guaranteed to have a solution by (B.10). All the other  $\mathbf{n}_i$  are changed by this but are brought back into their appropriate ranges by the use of (B.2). Now we want to use (B.9) again to set  $\mathbf{n}_2 = 0$  while keeping  $\mathbf{n}_1 = 0$ . If the transformation  $\mathbf{v}$  is proportional to  $a_1$

$$\begin{aligned} \mathbf{n}_1 &\rightarrow 0 + a_1 \mathbf{u}_2 + b_1 a_1 \mathbf{v}_2 \\ \mathbf{n}_2 &\rightarrow \mathbf{n}_2 + a_2 \mathbf{u}_2 + b_2 a_1 \mathbf{v}_2 \\ \mathbf{n}_3 &\rightarrow \mathbf{n}_3 + a_3 \mathbf{u}_2 + b_3 a_1 \mathbf{v}_2 \\ &\dots \quad \dots \quad \dots \\ \mathbf{n}_N &\rightarrow \mathbf{n}_N + a_N \mathbf{u}_2 + b_N a_1 \mathbf{v}_2 \end{aligned} \quad (\text{B.12})$$

then the change in  $\mathbf{n}_1$  is proportional to  $a_1$  and by (B.2) is zero. We note that  $\text{gcd}(a_2, b_2 a_1) = 1$  so by (B.10) we can choose  $(\mathbf{u}_2, \mathbf{v}_2)$  so the  $\mathbf{n}_2$  maps to zero. Now we would like to set  $\mathbf{n}_3 = 0$  without changing  $\mathbf{n}_1$  and  $\mathbf{n}_2$ . To not change  $\mathbf{n}_1$  the vector  $\mathbf{v}$  must be proportional to  $a_1$ , as we saw before, and not to change  $\mathbf{n}_2$  it must also be proportional to  $a_2$  so we perform a transformation

$$\begin{aligned} \mathbf{n}_1 &\rightarrow 0 + a_1 \mathbf{u}_3 + b_1 a_1 a_2 \mathbf{v}_3 \simeq 0 \\ \mathbf{n}_2 &\rightarrow 0 + a_2 \mathbf{u}_3 + b_2 a_1 a_2 \mathbf{v}_3 \simeq 0 \\ \mathbf{n}_3 &\rightarrow \mathbf{n}_3 + a_3 \mathbf{u}_3 + b_3 a_1 a_2 \mathbf{v}_3 \\ &\dots \quad \dots \quad \dots \\ \mathbf{n}_N &\rightarrow \mathbf{n}_N + a_N \mathbf{u}_3 + b_N a_1 a_2 \mathbf{v}_3 \end{aligned} \quad (\text{B.13})$$

Now  $\text{gcd}(a_3, b_3 a_1 a_2) = 1$  so we can set  $\mathbf{n}_3 = 0$  by a suitable choice of  $(\mathbf{u}_3, \mathbf{v}_3)$ .

Clearly the procedure can be repeated till we have set all the  $\mathbf{n}_i$  to zero and the parameters we use are

$$\mathbf{u} = \sum_{i=1}^N \mathbf{u}_i, \quad \mathbf{v} = \mathbf{v}_1 + a_1 \mathbf{v}_2 + \dots + a_1 \dots a_{N-1} \mathbf{v}_N \tag{B.14}$$

## B.2 Holomorphic factorisation

The finite dimensional action for complex Chern-Simons theory (B.1) can be expressed in terms of complex fields  $\phi + i\lambda$  and a complex coupling constant  $t = k + is$  as

$$\begin{aligned} I(t, \bar{t}) = & -\frac{t}{8\pi} c_1(\mathcal{L}_M) \text{Tr} \Phi^2 - i\frac{t}{2} \text{Tr}(\Phi \hat{\mathbf{q}}) + \frac{t\pi}{2} \sum_{i=1}^N \text{Tr}(\mathbf{n}_i^2) \\ & -\frac{\bar{t}}{8\pi} c_1(\mathcal{L}_M) \text{Tr} \bar{\Phi}^2 + i\frac{\bar{t}}{2} \text{Tr}(\bar{\Phi} \hat{\mathbf{q}}) + \frac{\bar{t}\pi}{2} \sum_{i=1}^N \text{Tr}(\mathbf{n}_i^2) \end{aligned} \tag{B.15}$$

Now we note that the holomorphic part of the action

$$I(t) = -\frac{t}{8\pi} c_1(\mathcal{L}_M) \text{Tr} \Phi^2 - i\frac{t}{2} \text{Tr}(\Phi \hat{\mathbf{q}}) + \frac{t\pi}{2} \sum_{i=1}^N \text{Tr}(\mathbf{n}_i^2) \tag{B.16}$$

is not invariant under the first symmetry namely under

$$\mathbf{n}_i \rightarrow \mathbf{n}_i + a_i \mathbf{m}_i, \quad \mathbf{n}_0 \rightarrow \mathbf{n}_0 - \sum_{i=1}^N \mathbf{m}_i \tag{B.17}$$

even though both  $\Phi$  and  $\hat{\mathbf{q}}$  are invariant. The quadratic term  $\mathbf{n}_i^2$  is not invariant. Furthermore, the exponential  $\exp(iI(t))$  is also not invariant, unless  $s = 0$ , as the coupling constant is complex and so one does not just get a phase.

This implies that we can have two inequivalent holomorphic factorizations. The first is to take the partition function with all the  $\mathbf{n}_i$  switched on and then factorise with holomorphic action (B.16) while the second is to set all the  $\mathbf{n}_i = 0$  for  $i \neq 0$  and then get the factorisation in [17].

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