

AG  
T

*Algebraic & Geometric  
Topology*

Volume 23 (2023)

**The bridge number of arborescent links with many twigs**

SEBASTIAN BAADER

RYAN BLAIR

ALEXANDRA KJUCHUKOVA

FILIP MISEV





# The bridge number of arborescent links with many twigs

SEBASTIAN BAADER  
RYAN BLAIR  
ALEXANDRA KJUCHUKOVA  
FILIP MISEV

We prove the meridional rank conjecture for arborescent links associated to plane trees with the following property: all branching points carry a straight branch to at least three leaves. The proof involves obtaining an upper bound on the bridge number in terms of the maximal number of link components of the underlying tree, which is valid for all arborescent links.

57K10; 20F55

## 1 Introduction

The family of arborescent tangles can be defined as the minimal family of tangles containing all rational tangles, closed under horizontal and vertical tangle composition; see Conway [7]. Their closures — arborescent links — admit a description via weighted plane trees, where each vertex stands for a twisted band, and edges indicate how these bands are glued together. See Bonahon and Siebenmann [5] or Gabai [8] for a precise definition, and Figure 1 for an illustration (ignoring the additional labels and dots in the link diagram for the time being). These descriptions are not unique, since small weights typically allow for simplifications of the underlying tree, without changing the link type.

The meridional rank conjecture by Cappell and Shaneson posits an equality between the bridge number and the meridional rank of a link; see Kirby [9, Problem 1.11]. Early evidence towards this was derived by Boileau and Zimmermann, who showed that two-bridge links are the only links with meridional rank two [4], and by Rost and Zieschang, who proved the conjecture for torus links [10].

We recall that the bridge number  $\beta(L)$  of a link  $L \subset \mathbb{R}^3$  is the minimal number of local maxima of  $L$  with respect to a fixed direction, minimized over all isotopic

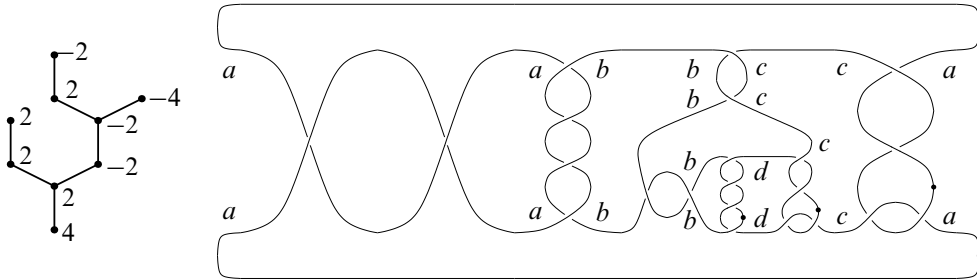


Figure 1: Example of an arborescent knot and underlying weighted tree.

representatives of  $L$ . The meridional rank  $\mu(L)$  is the minimal number of generators of the fundamental group  $\pi_1(\mathbb{R}^3 \setminus L)$ , where all generators are required to be conjugate to a standard meridional loop of the link  $L$ . The bridge number of a link is bounded below by its meridional rank.

Given a fixed plane tree  $T$ , each choice of weights for its vertices determines an arborescent link  $L(T)$ . Define  $m(T)$  to be the maximal number of components of  $L(T)$  over all links obtained by assigning weights to the vertices of  $T$ . We will show that  $m(T)$  admits a description in terms of the combinatorics of the tree.

**Theorem 1** *For every arborescent link  $L(T)$  determined by a weighted plane tree  $T$ , the bridge number of  $L(T)$  is bounded above by the maximal component number of  $T$ ,*

$$\beta(L(T)) \leq m(T).$$

This bound is sharp for a class of trees, defined next. A twig is a straight branch connecting a leaf to a branching point. A tree  $T$  is said to have many twigs if it is obtained from a subtree  $T' \subset T$  by adding at least three twigs to every vertex of  $T'$ .

**Theorem 2** *Let  $L(T)$  be an arborescent link associated to a plane tree  $T$  with many twigs and all weights  $\neq 0, \pm 1$ . The meridional rank conjecture holds for  $L(T)$  and*

$$\mu(L(T)) = \beta(L(T)) = m(T).$$

To evaluate the maximal component number  $m(T)$ , we shall use  $f(T)$ , the flattening number of  $T$ . Define a subset of edges of  $T$  to be flattening if the complement of their interiors is a subforest of  $T$  with no vertex of valency bigger than two. The natural number  $f(T)$  is the minimal number of edges among all flattening subsets; the first author gives this definition in the context of braid indices of fibered arborescent links

in [1]. In the special case of trees with a bipartite ramification structure, where all vertices of valency bigger than two are even distance apart, the number  $f(T)$  is easily seen to coincide with the number of leaves of  $T$  minus two. The first three authors proved the meridional rank conjecture for links associated to bipartite trees in [2], establishing Theorems 1 and 2 for this class of links, although the formulation in terms of the maximal component number is new. Trees with many twigs and trees with a bipartite ramification structure have a small intersection, consisting of star-like trees. The links corresponding to these trees are known as Montesinos links.

Our proof is inspired by the technique developed in [2]. We construct Coxeter quotients of the groups  $\pi_1(\mathbb{R}^3 \setminus L(T))$  of rank  $f(T) + 2$  for trees with many twigs. This is done in the next section and establishes the inequality

$$f(T) + 2 \leq \mu(L(T)).$$

In the third and final section, we show that the bridge number of arborescent links (without restriction) is bounded above by  $f(T) + 2$ ,

$$\beta(L(T)) \leq f(T) + 2.$$

This in turn is done by computing the Wirtinger number of arborescent diagrams, a combinatorial version of the bridge number introduced by Blair, Kjuchukova, Velazquez and Villanueva in [3]. Moreover, for all plane trees we establish the equality

$$m(T) = f(T) + 2.$$

## 2 Coxeter quotients for arborescent links

Coxeter groups are encoded by finite simple weighted graphs. Let  $\Gamma$  be a finite simple graph with  $v(\Gamma)$  vertices, whose edges carry integer weights  $\geq 2$ . The corresponding Coxeter group  $C(\Gamma)$  is generated by  $v(\Gamma)$  elements of order two, one for each vertex of  $\Gamma$ . Every edge with weight  $k$  stands for a relation of the form  $(st)^k = 1$ , where  $s, t$  is the pair of generators associated with the two vertices of that edge. Elements of  $C(\Gamma)$  conjugate to these generators are called reflections. The minimal number of reflections needed to generate  $C(\Gamma)$  is called the reflection rank of  $C(\Gamma)$ ; it is known to equal  $v(\Gamma)$ . The following elementary lower bound for the meridional rank  $\mu(L)$  of links in terms of the reflection rank was derived in [2, Section 2, Proposition 1].

**Proposition 1** *Let  $L$  be a link whose fundamental group surjects onto a Coxeter group  $C(\Gamma)$ , so that all meridians are mapped to reflections. Then  $\mu(L) \geq v(\Gamma)$ .*

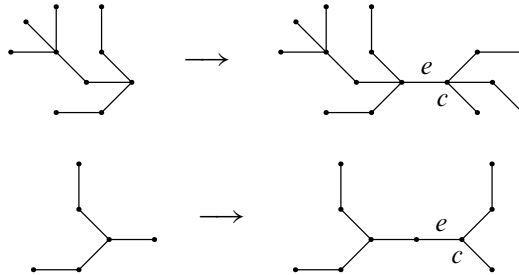


Figure 2: Adding ramification points of valency 4 and 3.

We will use the term Coxeter quotient for quotients of link groups that arise by sending all meridians of  $L$  to reflections of a Coxeter group. These were introduced by Brunner in [6] as homomorphisms onto Artin groups rather than Coxeter groups.

An important class of links that admit noncyclic Coxeter quotients are two-bridge links, which can be encoded by rational numbers  $\alpha/\beta$  with relatively prime integers  $\alpha, \beta$  and  $-\alpha < \beta < \alpha$ . As explained in [2], the two-bridge link  $L(\alpha/\beta)$  admits a rank two Coxeter quotient generated by two reflections  $s, t$  satisfying the relation  $(st)^\alpha = 1$ .

This is simply the quotient of the link group by the square of all meridians, which is maximal among its Coxeter quotients. Any Coxeter quotient of a two-bridge link is therefore a dihedral group, the only possible cyclic quotient being  $\mathbb{Z}/2\mathbb{Z}$ .

The goal of this section is to construct Coxeter quotients of reflection rank  $f(T) + 2$  for all arborescent links  $L(T)$  with the restrictions stated in Theorem 2. For this purpose, we need a recursive formula for the flattening number  $f(T)$ . We say that a tree  $\bar{T}$  is obtained from  $T$  by adding a ramification point if  $\bar{T}$  contains an edge  $e$  whose complement is the union of  $T$  and a star-like tree, whose central vertex  $c$  is adjacent to  $e$ . Figure 2 illustrates this operation and serves as a hint for the proof of the following easy fact.

**Lemma 1** *If  $\bar{T}$  is obtained from  $T$  by adding a ramification point of valency  $k \geq 2$ , then  $f(\bar{T}) = f(T) + k - 2$ .*

Adding a ramification point of valency  $k$  to a tree  $T$  has the effect of inserting  $k - 1$  rational tangles to the arborescent link  $L(T)$ . This is illustrated in Figure 3 for  $k = 4$ , where each of the three boxes labeled  $A, B$  and  $C$  stands for a rational tangle determined by the branches incident to the new ramification point, and the number of twists in the central band is given by the weight of that point. Recall that a tree  $T$  satisfying the hypotheses of Theorem 2 is obtained from a subtree  $T' \subset T$  by adding at least three

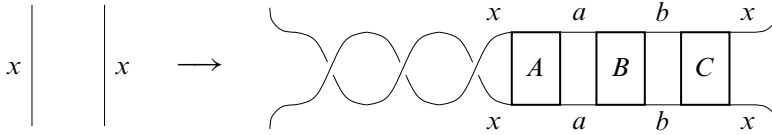


Figure 3: Extending a system of Coxeter generators.

straight branches, or twigs, to every vertex of  $T'$ . For this reason,  $T$  can be constructed inductively from a star-shaped tree by adding ramification points of valency at least four to branching points, ie to vertices of valency at least three, as in the upper part of Figure 2.

We will construct a Coxeter quotient of rank  $f(T) + 2$  by induction on the number of vertices of  $T'$ . An important element of this construction is that each twist region of the arborescent diagram corresponding to a vertex of  $T'$  — that is, to a branching point of  $T$  — carries a single Coxeter generator. Coincidentally, the base case and the inductive step can be understood in the same diagram: Figure 3 illustrates the base case of a star-shaped tree with three branches, as well as the addition of a ramification point of valency four to an existing branching point. The three labels  $x$ ,  $a$  and  $b$  stand for labels of a Coxeter group. Here a label  $z$  means that the meridian around the labeled string gets mapped to the generator  $z$  of a Coxeter group determined by the link diagram. Our assumption on the weights makes sure that all the rational tangles have nontrivial numerators, hence give rise to Coxeter relations  $(st)^\alpha = 1$  with  $\alpha \geq 2$  (compare the discussion in the second paragraph after Proposition 1).

For the base case, a star-shaped tree, we observe that  $L(T)$  admits a Coxeter quotient with  $f(T) + 2$  generators — as many as the number of branches. Indeed, the defining diagram of  $L(T)$  consists of a single twist region associated with the center of the star, with a chain of  $f(T) + 2$  rational tangles attached, one for each branch. This diagram comes with a canonical Coxeter quotient with  $f(T) + 2$  generators and equally many relations. The corresponding Coxeter graph is a cycle, whose edges are labeled with the nominators of the rational tangles, in the cyclic order given by the branches of the tree.

The case of a star with three branches is illustrated on the right side of Figure 3, with  $A$ ,  $B$  and  $C$  the rational tangles corresponding to the branches and  $x$ ,  $a$  and  $b$  the Coxeter generators of the quotient.

For the inductive step, we construct a Coxeter quotient of rank

$$f(\bar{T}) + 2 = f(T) + 2 + k - 2$$

for the link  $L(\bar{T})$  by adding  $k - 2$  new reflection generators and  $k - 1$  new Coxeter type relations determined by the new rational tangles. This is again illustrated in Figure 3 for  $k = 4$ , where the label  $x$  stands for the generator of the branching point, to which we add the new ramification point. The new generators are labeled  $a$  and  $b$ . At this point, it is essential that the new ramification point has at least three twigs. If it had only two twigs, the single new generator  $a$  would satisfy two Coxeter type relations with  $x$ , which are possibly in contradiction. For example, if  $p, q \in \mathbb{N}$  are coprime, then the two relations  $(ax)^p = 1 = (ax)^q$  enforce  $a = x$ .

This inductive construction, together with Proposition 1, proves the desired lower bound on the meridional rank of arborescent links  $L(T)$  associated to trees  $T$  with many twigs and all weights  $\neq 0, \pm 1$ ,

$$\mu(L(T)) \geq f(T) + 2.$$

As pointed out above, the inductive step does not work for arbitrary trees. However, the class of trees  $T$  admitting a Coxeter quotient of rank  $f(T) + 2$  is bigger than the class of trees with many twigs. For example, the labels  $a, b, c$  and  $d$  of the arborescent knot  $L(T)$  in Figure 1 generate a Coxeter quotient of rank  $f(T) + 2 = 4$ , obtained by a similar procedure. The Coxeter relations satisfied by these generators are

$$(ab)^4 = (ac)^3 = (bc)^2 = (bd)^4 = (cd)^5 = 1.$$

### 3 Wirtinger and bridge number of arborescent links

The Wirtinger number of a link, introduced in [3], is a combinatorial version of the bridge number. We fix a connected link diagram  $D$  with  $n$  crossings, whose complement is a union of  $n$  arcs. Marking  $k$  of these arcs — called seeds — by a dot, we obtain a partial coloring of the diagram. We allow the coloring to propagate over crossings by the rule depicted in Figure 4, motivated by the Wirtinger calculus. The idea is that the meridians of all strands marked with a dot are in the subgroup of the link group generated by the meridians of the initially dotted strands, or seeds.

The Wirtinger number  $\omega(D)$  is the minimal number of seeds whose coloring propagates to a coloring of the entire diagram  $D$ . The main result in [3] states that the bridge

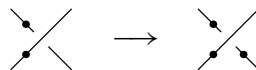


Figure 4: Propagation rule for colors.



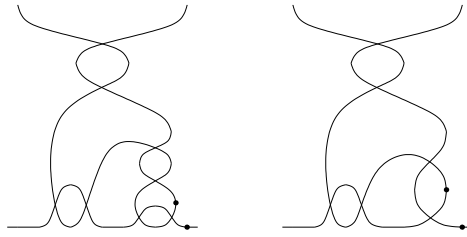


Figure 5: Initial seeds for rational tangles.

number  $\beta(L)$  of a link  $L$  coincides with the Wirtinger number of  $L$ , that is the minimum value of  $\omega(D)$  among all diagrams of  $L$ . In particular, the Wirtinger number of any diagram  $D$  of a link  $L$  is an upper bound for the bridge number,

$$\beta(L) \leq \omega(D).$$

In this section, we will prove that a suitable choice of  $f(T) + 2$  seeds in a diagram of the arborescent link  $L(T)$  propagates to a coloring of the entire diagram, without any restriction on the tree  $T$  and its weights. This implies that for all arborescent links  $L(T)$ ,

$$\mu(L(T)) \leq \beta(L(T)) \leq f(T) + 2.$$

Combining this with the inequality of the previous section,  $\mu(L(T)) \geq f(T) + 2$ , valid for all arborescent links  $L(T)$  with the restrictions stated in Theorem 2, we obtain the two desired equalities,

$$\beta(L(T)) = \mu(L(T)) = f(T) + 2.$$

We are left to construct diagram colorings for  $L(T)$  with  $f(T) + 2$  seeds, starting with the base case  $f(T) = 0$ , or trees  $T$  without ramification points. The corresponding links  $L(T)$  are precisely the two-bridge links, or closures of rational tangles. For each of the two rational tangles shown in Figure 5, two initial seeds are enough to propagate to a coloring of the entire tangle. Note that no information on over- and undercrossings is needed in these diagrams.

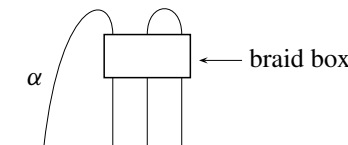


Figure 6: Rational tangle with a distinguished string  $\alpha$ .

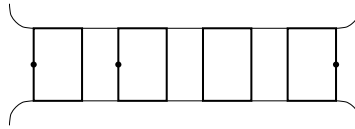


Figure 7: Adding three seeds.

This does not only work in these two illustrative examples, but generally for any rational tangle. In fact, an even stronger statement holds: a single seed on any of the four outgoing strings of a rational tangle can be complemented by a second seed, so that the coloring propagates to a coloring of the entire tangle. Indeed, for any rational tangle  $R$  and choice of incoming string  $\alpha$ , a suitable isotopy will bring  $R$  into the position shown in Figure 6. See the claim in the proof of Theorem 1 in [11] for details.

Now, if  $\alpha$  is a given seed, then choosing the string corresponding to the other maximum in Figure 6 as a second seed results in a coloring that propagates across the entire tangle.

We are now set for an inductive construction of a diagram coloring for  $L(T)$  with  $f(T) + 2$  initial seeds, making once again use of Lemma 1. Let  $\bar{T}$  be a tree obtained from  $T$  by adding a ramification point of valency  $k$  to  $T$ . Suppose that a link diagram of  $L(T)$  admits a coloring by  $f(T) + 2$  initial seeds that propagate to a coloring of the entire link diagram. This step potentially requires an ambient isotopy of  $L(T)$  fixing the boundaries of all the rational tangles corresponding to branches of  $T$ , whereby each of these tangles is positioned like in Figure 6. We obtain a diagram of  $L(\bar{T})$  by adding a twist region and  $k - 1$  rational tangles to the diagram of  $L(T)$ . The twist region corresponding to the new ramification point can be included in any of these rational tangles. For example, in Figure 3 the three crossings together with tangle  $A$  form a single rational tangle. Adding  $k - 2$  suitable seeds, one for each rational tangle except one, as shown schematically in Figure 7 for  $k = 5$ , gives rise to a set of  $f(T) + k = f(\bar{T}) + 2$  seeds that propagate to a coloring of a link diagram of  $L(\bar{T})$ . This completes the proof of the inequality  $\beta(L(T)) \leq f(T) + 2$ .

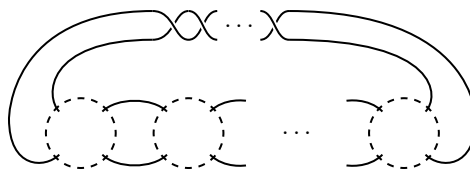


Figure 8: Montesinos link on  $n$  rational tangles.

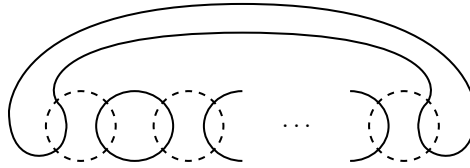


Figure 9: Connectedness diagram for a Montesinos link on  $n$  rational tangles with suitably chosen weights.

To obtain Theorems 1 and 2, it remains to show that  $m(T) = f(T) + 2$  for all plane trees. Once again we consider star-like trees as a first step.

**Lemma 2** *Given a star-like plane tree  $T$ , there exists a vertex labeling of  $T$  such that the corresponding link  $L(T)$  has  $f(T) + 2$  components.*

**Proof** Suppose the center vertex of  $T$  has valency  $n \geq 2$ . Then  $f(T) = n - 2$  and  $L(T)$  is a Montesinos link on  $n$  rational tangles as in Figure 8. It now suffices to show that we can choose the labels of  $T$  to achieve the connectedness diagram given in Figure 9, so that  $L(T)$  has  $n = f(T) + 2$  components. Let  $R$  be any of the  $n$  rational tangles. If  $R$  has length one — that is, it consists of a single twist region — then we may choose any even label to achieve the desired connectedness. If  $R$  has length at least two, all three ways to connect the four endpoints in pairs can be achieved.  $\square$

**Proposition 2** *For any plane tree  $T$ , the flattening number  $f(T)$  and the maximal component number  $m(T)$  are related by*

$$m(T) = f(T) + 2.$$

**Proof** Assign weights to  $T$  such that  $L(T)$  is a link realizing the maximal component number  $m(T)$ . We then have

$$f(T) + 2 \geq \beta(L(T)) \geq m(T),$$

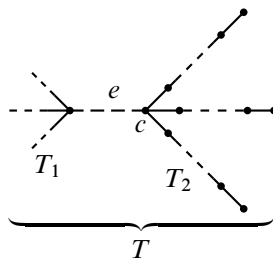


Figure 10: New ramification point,  $c$ .

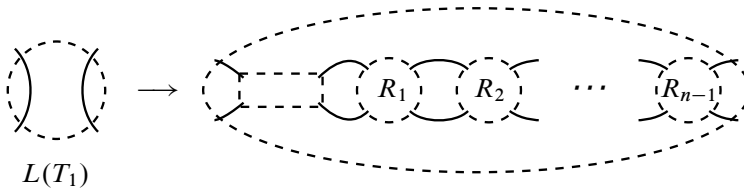


Figure 11: Tangle substitution corresponding to the new ramification point.

since the inequality  $f(T) + 2 \geq \beta(L(T))$  holds for any choice of weights, and the number of components of the link  $L(T)$  can not exceed its bridge number.

It remains to show that  $f(T) + 2 \leq m(T)$ . We do this by choosing labelings on  $T$  that result in a link  $L(T)$  with  $f(T) + 2$  components. We induct on the number of ramifications needed to construct  $T$ . The base case,  $T$  is star-like, is treated in Lemma 2. Now assume  $T$  is the result of adding a ramification point — that is, a star-like tree  $T_2$  — to an arbitrary plane tree  $T_1$ . Note that  $T_1$  can be constructed using strictly fewer ramifications than  $T$ . By induction, there exists a labeling of  $T_i$  such that  $L(T_i)$  has  $f(T_i) + 2$  components for  $i = 1, 2$ . These labels of  $T_1$  and  $T_2$  induce a labeling of  $T$  which results in the connectedness diagram for  $L(T)$  as in Figure 11. Here, the  $R_i$  are rational tangles and the ramification point has valence  $n$ .

Figure 12 shows the connectedness diagram induced by the chosen labels for  $T_2$ , with  $n - 2$  components of  $L(T)$  contained therein. Hence, the number of components of  $L(T)$  is

$$(f(T_1) + 2) + (n - 2) = f(T_1) + n.$$

By Lemma 1,  $f(T) = f(T_1) + n - 2$ , so  $L(T)$  has  $f(T) + 2$  components. □

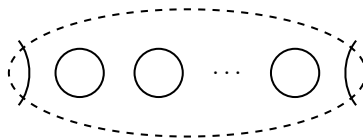


Figure 12: Connectedness diagram of inserted tangle induced by the choice of labels on  $T_2$ .

## References

[1] **S Baader**, *Hopf plumbing and minimal diagrams*, Comment. Math. Helv. 80 (2005) 631–642 MR Zbl

- [2] **S Baader, R Blair, A Kjachukova**, *Coxeter groups and meridional rank of links*, Math. Ann. 379 (2021) 1533–1551 MR Zbl
- [3] **R Blair, A Kjachukova, R Velazquez, P Villanueva**, *Wirtinger systems of generators of knot groups*, Comm. Anal. Geom. 28 (2020) 243–262 MR Zbl
- [4] **M Boileau, B Zimmermann**, *The  $\pi$ -orbifold group of a link*, Math. Z. 200 (1989) 187–208 MR Zbl
- [5] **F Bonahon, L C Siebenmann**, *New geometric splittings of classical knots and the classification and symmetries of arborescent knots*, book project (2016) Available at <https://dornsife.usc.edu/assets/sites/1191/docs/Preprints/BonSieb.pdf>
- [6] **A M Brunner**, *Geometric quotients of link groups*, Topology Appl. 48 (1992) 245–262 MR Zbl
- [7] **J H Conway**, *An enumeration of knots and links, and some of their algebraic properties*, from “Computational problems in abstract algebra” (J Leech, editor), Pergamon, Oxford (1970) 329–358 MR Zbl
- [8] **D Gabai**, *Genera of the arborescent links*, Mem. Amer. Math. Soc. 339, Amer. Math. Soc., Providence, RI (1986) MR Zbl
- [9] **R Kirby**, *Problems in low-dimensional topology*, from “Geometric topology” (W H Kazez, editor), AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 35–473 MR Zbl
- [10] **M Rost, H Zieschang**, *Meridional generators and plat presentations of torus links*, J. London Math. Soc. 35 (1987) 551–562 MR Zbl
- [11] **J Schultens**, *Additivity of bridge numbers of knots*, Math. Proc. Cambridge Philos. Soc. 135 (2003) 539–544 MR Zbl

*Mathematisches Institut, Universität Bern  
Bern, Switzerland*

*Department of Mathematics and Statistics, California State University, Long Beach  
Long Beach, CA, United States*

*Department of Mathematics, University of Notre Dame  
Notre Dame, IN, United States*

*Fakultät für Mathematik, Universität Regensburg  
Regensburg, Germany*

sebastian.baader@math.unibe.ch, ryan.blair@csulb.edu, akjachuk@nd.edu,  
filip.misev@mathematik.uni-regensburg.de

Received: 26 August 2020      Revised: 28 July 2021



# ALGEBRAIC & GEOMETRIC TOPOLOGY

[msp.org/agt](http://msp.org/agt)

## EDITORS

### PRINCIPAL ACADEMIC EDITORS

John Etnyre  
etnyre@math.gatech.edu  
Georgia Institute of Technology

Kathryn Hess  
kathryn.hess@epfl.ch  
École Polytechnique Fédérale de Lausanne

### BOARD OF EDITORS

Matthew Ando	University of Illinois mando@math.uiuc.edu	Tsuyoshi Kobayashi	Nara Women's University tsuyoshi09@gmail.com
Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Joan Birman	Columbia University jb@math.columbia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Norihiko Minami	Nagoya Institute of Technology nori@nitech.ac.jp
Indira Chatterji	CNRS & Université Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Andrés Navas Flores	Universidad de Santiago de Chile andres.navas@usach.cl
Fred Cohen	University of Rochester cohf@math.rochester.edu	Thomas Nikolaus	University of Münster nikolaus@uni-muenster.de
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris-Nord bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Luis Paris	Université de Bourgogne lparis@u-bourgogne.fr
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
David Futер	Temple University dfuter@temple.edu	Peter Scott	University of Michigan pscott@umich.edu
Soren Galatius	Stanford University galatius@math.stanford.edu	Zoltán Szabó	Princeton University szabo@math.princeton.edu
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Ulrike Tillmann	Oxford University tillmann@maths.ox.ac.uk
J. Elisenda Grigsby	Boston College grigsbyj@bc.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Daniel T. Wise	McGill University, Canada daniel.wise@mcgill.ca
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu		

---

See inside back cover or [msp.org/agt](http://msp.org/agt) for submission instructions.


The subscription price for 2023 is US \$650/year for the electronic version, and \$940/year (+ \$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

---

AGT peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers

# ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23    Issue 1 (pages 1–508)    2023

---

Proper 2–equivalences between infinite ended finitely presented groups	1
MANUEL CÁRDENAS, FRANCISCO FERNÁNDEZ LASHERAS, ANTONIO QUINTERO and RANJA ROY	
Groups acting on CAT(0) cube complexes with uniform exponential growth	13
RADHIKA GUPTA, KASIA JANKIEWICZ and THOMAS NG	
Admissible replacements for simplicial monoidal model categories	43
HALDUN ÖZGÜR BAYINDIR and BORIS CHORNY	
The bridge number of arborescent links with many twigs	75
SEBASTIAN BAADER, RYAN BLAIR, ALEXANDRA KJUCHUKOVA and FILIP MISEV	
$G_\infty$ –ring spectra and Moore spectra for $\beta$ –rings	87
MICHAEL STAHLHAUER	
External Spanier–Whitehead duality and homology representation theorems for diagram spaces	155
MALTE LACKMANN	
On the surjectivity of the $\mathrm{tmf}$ –Hurewicz image of $A_1$	217
VIET-CUONG PHAM	
The handlebody group and the images of the second Johnson homomorphism	243
QUENTIN FAES	
Recognition of connective commutative algebra spectra through an idempotent quasiadjunction	295
RENATO VASCONCELLOS VIEIRA	
$\mathrm{SL}_2$ quantum trace in quantum Teichmüller theory via writhe	339
HYUN KYU KIM, THANG T Q LÊ and MIRI SON	
Constraints on families of smooth 4–manifolds from $\mathrm{Pin}^-(2)$ –monopole	419
HOKUTO KONNO and NOBUHIRO NAKAMURA	
Suspension homotopy of 6–manifolds	439
RUIZHI HUANG	
Time-periodic solutions of Hamiltonian PDEs using pseudoholomorphic curves	461
OLIVER FABERT and NIEK LAMOREE	