Journal of Algebra 640 (2024) 1–20

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Research Paper

Transfer theorems for finitely subdirectly irreducible algebras

Wesley Fussner *, George Metcalfe

Mathematical Institute, University of Bern, Switzerland

A R T I C L E   I N F O

Article history:
Received 9 June 2022
Available online 13 November 2023
Communicated by Franz-Viktor Kuhlmann

Keywords:
Variety
Quasivariety
Congruence-distributive
Finitely subdirectly irreducible
Congruence extension property
Amalgamation property
BL-algebra

A B S T R A C T

We show that under certain conditions, well-studied algebraic properties transfer from the class $\mathcal{Q}_{\mathrm{RFSI}}$ of the relatively finitely subdirectly irreducible members of a quasivariety $\mathcal{Q}$ to the whole quasivariety, and, in certain cases, back again. First, we prove that if $\mathcal{Q}$ is relatively congruence-distributive, then it has the $\mathcal{Q}$-congruence extension property ($\mathcal{Q}$-CEP) if and only if $\mathcal{Q}_{\mathrm{RFSI}}$ has this property. We then prove that if $\mathcal{Q}$ has the $\mathcal{Q}$-CEP and $\mathcal{Q}_{\mathrm{RFSI}}$ is closed under subalgebras, then $\mathcal{Q}$ has a one-sided amalgamation property (for quasivarieties, equivalent to the amalgamation property) if and only if $\mathcal{Q}_{\mathrm{RFSI}}$ has this property. We also establish similar results for the transferable injections property and strong amalgamation property. For each property considered, we specialize our results to the case where $\mathcal{Q}$ is a variety — so that $\mathcal{Q}_{\mathrm{RFSI}}$ is the class of finitely subdirectly irreducible members of $\mathcal{Q}$ and the $\mathcal{Q}$-CEP is the usual congruence extension property — and prove that when $\mathcal{Q}$ is finitely generated and congruence-distributive, and $\mathcal{Q}_{\mathrm{RFSI}}$ is closed under subalgebras, possession of the property is decidable. Finally, as a case study, we provide a complete description of the subvarieties of a notable variety of BL-algebras that have the amalgamation property.

* Corresponding author.
E-mail addresses: wesley.fussner@unibe.ch (W. Fussner), george.metcalfe@unibe.ch (G. Metcalfe).

https://doi.org/10.1016/j.jalgebra.2023.11.003

1. Introduction

This paper studies a cluster of interrelated properties of general interest in algebra: the congruence extension property, (strong) amalgamation property, transferable injections property, and surjective epimorphisms property. The latter have each been investigated in a range of algebraic contexts, with studies spanning groups, rings, lattices, Lie algebras, and numerous other algebraic structures (see [27] for an extensive survey). In this paper, we adopt a general vantage point and develop a widely applicable toolkit for studying these properties. Our main contribution consists of transfer theorems that characterize each property for a class of algebraic structures satisfying certain hypotheses in terms of a smaller and more easily studied subclass, namely, the (relatively) finitely subdirectly irreducible members of the class. These theorems generalize other transfer results in the literature (found in, e.g., [9,10,19,26,29]), while simultaneously providing a more uniform treatment.

We freely make use of basic notions of universal algebra throughout our discussion, but first recall several key definitions, referring to [5,18] for more detailed treatments. Let $\mathcal{Q}$ be a *quasivariety*: a class of similar algebras (i.e., a class of algebras of the same signature) defined by quasiequations, or, equivalently, closed under isomorphisms, subalgebras, direct products, and ultraproducts. A congruence Θ of an algebra $\mathbf{A} \in \mathcal{Q}$ is called a *$\mathcal{Q}$-congruence* if $\mathbf{A}/\Theta \in \mathcal{Q}$. When ordered by inclusion, the set of $\mathcal{Q}$-congruences of $\mathbf{A}$ forms an algebraic lattice $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$. Arbitrary meets coincide in $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$ with those taken in the algebraic lattice $\mathrm{Con}\,\mathbf{A}$ of all congruences of $\mathbf{A}$, as do joins of chains, but this may not be the case for arbitrary joins. If $\mathcal{Q}$ is a *variety* — that is, a class of similar algebras defined by equations, or, equivalently, closed under homomorphic images, subalgebras, and direct products — then every congruence of $\mathbf{A}$ is a $\mathcal{Q}$-congruence and $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$ and $\mathrm{Con}\,\mathbf{A}$ coincide.

An algebra $\mathbf{A}$ is said to be *(finitely) subdirectly irreducible* if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a (non-empty finite) set of algebras, it is isomorphic to one of these algebras. Equivalently, $\mathbf{A}$ is finitely subdirectly irreducible if the least congruence $\Delta_A := \{\langle a, a\rangle \mid a \in A\}$ is meet-irreducible in $\mathrm{Con}\,\mathbf{A}$, and subdirectly irreducible if Δ_A is completely meet-irreducible in $\mathrm{Con}\,\mathbf{A}$.[1] When $\mathbf{A}$ belongs to $\mathcal{Q}$, it is convenient to relativize these notions. In this case, $\mathbf{A}$ is said to be *(finitely) $\mathcal{Q}$-subdirectly irreducible* if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a (non-empty finite) set of algebras in $\mathcal{Q}$, it is isomorphic to one of them. Equivalently, $\mathbf{A}$ is finitely $\mathcal{Q}$-subdirectly irreducible if and only if Δ_A is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$, and $\mathcal{Q}$-subdirectly irreducible if and only if Δ_A is completely meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$.

For any $\mathbf{A} \in \mathcal{Q}$ and $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$, it follows from the correspondence theorem for universal algebra that the quotient algebra $\mathbf{A}/\Theta \in \mathcal{Q}$ is finitely $\mathcal{Q}$-subdirectly irreducible

[1] An element a of a lattice $\mathbf{L}$ is *meet-irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$, and this is true of any greatest element $\top$ of $\mathbf{L}$; however, a is *completely meet-irreducible* if $a = \bigwedge B$ implies $a \in B$ for any $B \subseteq L$, which is not the case for $\top = \bigwedge \emptyset$.

if and only if Θ is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$, and $\mathcal{Q}$-subdirectly irreducible if and only if it is completely meet-irreducible. Clearly, if $\mathcal{Q}$ is a variety, the properties of being (finitely) $\mathcal{Q}$-subdirectly irreducible and (finitely) subdirectly irreducible coincide. When $\mathcal{Q}$ is clear from the context, we call a (finitely) $\mathcal{Q}$-subdirectly irreducible algebra $\mathbf{A}$ *relatively (finitely) subdirectly irreducible*.

Let $\mathcal{Q}_{\mathrm{FSI}}$, $\mathcal{Q}_{\mathrm{SI}}$, $\mathcal{Q}_{\mathrm{RFSI}}$, and $\mathcal{Q}_{\mathrm{RSI}}$ denote the classes of finitely subdirectly irreducible, subdirectly irreducible, relatively finitely subdirectly irreducible, and relatively subdirectly irreducible members of $\mathcal{Q}$, respectively. A sizeable number of results in the universal algebra literature state that, under certain conditions, well-studied algebraic properties transfer from $\mathcal{Q}_{\mathrm{RSI}}$ to $\mathcal{Q}$, at least when $\mathcal{Q}$ is a variety (see, e.g., [9,10,19,26,29]). The aim of this paper is to determine conditions under which these properties transfer from $\mathcal{Q}_{\mathrm{RFSI}}$ to $\mathcal{Q}$ and, in some cases, back again. A key motivation for considering $\mathcal{Q}_{\mathrm{RFSI}}$ rather than $\mathcal{Q}_{\mathrm{RSI}}$ is that it is often easier to state or check conditions for the larger class. Notably, if $\mathcal{Q}$ has equationally definable relative principal congruence meets (a common property for quasivarieties corresponding to non-classical logics), then $\mathcal{Q}_{\mathrm{RFSI}}$ is a universal class [8, Theorem 2.3]. Moreover, if $\mathcal{V}$ is a variety such that $\mathcal{V}_{\mathrm{FSI}}$ is a universal class, then $\mathcal{V}_{\mathrm{FSI}}$ is a positive universal class if and only if $\mathrm{Con}\,\mathbf{A}$ is a chain (totally ordered set) for each $\mathbf{A} \in \mathcal{V}_{\mathrm{FSI}}$.[2]

An algebra $\mathbf{B} \in \mathcal{Q}$ is said to be *$\mathcal{Q}$-congruence-distributive* if $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ is a distributive lattice. If every member of $\mathcal{Q}$ is $\mathcal{Q}$-congruence-distributive, then $\mathcal{Q}$ is said to be *relatively congruence-distributive* and in this case, as shown in [12, Theorem 2.3], $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$. Clearly, a variety is relatively congruence-distributive if and only if it is congruence-distributive in the usual sense.

An algebra $\mathbf{B} \in \mathcal{Q}$ is said to have the *$\mathcal{Q}$-congruence extension property* (for short, $\mathcal{Q}$-CEP) if for any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$, there exists a $\Phi \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$. A class $\mathcal{K}$ of algebras in $\mathcal{Q}$ is said to have the $\mathcal{Q}$-CEP if every member of $\mathcal{K}$ has the $\mathcal{Q}$-CEP. In Section 2, we prove that if $\mathcal{Q}$ is relatively congruence-distributive, it has the $\mathcal{Q}$-CEP if and only if $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ has the $\mathcal{Q}$-CEP (Theorem 2.3). When $\mathcal{Q}$ is a variety, the $\mathcal{Q}$-CEP is the usual congruence extension property (for short, CEP) and if it is congruence-distributive, $\mathcal{Q}$ has the CEP if and only if $\mathcal{Q}_{\mathrm{FSI}}$ has the CEP (Corollary 2.4). This result yields also the known fact that a congruence-distributive variety $\mathcal{V}$ such that $\mathcal{V}_{\mathrm{SI}}$ is elementary has the CEP if and only if $\mathcal{V}_{\mathrm{SI}}$ has the CEP [10, Theorem 3.3]. Note, however, that the requirement that $\mathcal{V}_{\mathrm{SI}}$ is elementary may fail or be difficult to establish, and this result may therefore be significantly harder to apply than Theorem 2.3.

When $\mathcal{Q}_{\mathrm{FSI}}$ is closed under subalgebras, Theorem 2.3 can be reformulated in terms of commutative diagrams. Let $\mathcal{K}$ be any class of similar algebras. A *span* in $\mathcal{K}$ is a 5-tuple $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ consisting of $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and homomorphisms $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$. We call this span *injective* if φ_B is an embedding, *doubly injective* if both

[2] Suppose that $\mathcal{V}_{\mathrm{FSI}}$ is a universal class. Then it is a positive universal class if and only if for any $\mathbf{A} \in \mathcal{V}_{\mathrm{FSI}}$ and $\Theta \in \mathrm{Con}\,\mathbf{A}$, also $\mathbf{A}/\Theta \in \mathcal{V}_{\mathrm{FSI}}$, i.e., Θ is meet-irreducible in $\mathrm{Con}\,\mathbf{A}$. But a lattice is a chain if and only if all its elements are meet-irreducible, so $\mathcal{V}_{\mathrm{FSI}}$ is a positive universal class if and only if $\mathrm{Con}\,\mathbf{A}$ is a chain for each $\mathbf{A} \in \mathcal{V}_{\mathrm{FSI}}$.

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, dashed, hook, "\psi_C"] & \\
\mathbf{A} \arrow[ur, two heads, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
& \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}$$

(i) EP

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, dashed, hook, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
& \mathbf{B} \arrow[ur, dashed, hook, "\psi_B"'] &
\end{tikzcd}$$

(ii) AP

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, dashed, hook, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
& \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}$$

(iii) 1AP

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, dashed, hook, "\psi_C"] & \\
\mathbf{A} \arrow[ur, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
& \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}$$

(iv) TIP

Fig. 1. Commutative diagrams for algebraic properties.

φ_B and φ_C are embeddings, and *injective-surjective* if φ_B is an embedding and φ_C is surjective. The class $\mathcal{K}$ has the *extension property* (for short, EP) if for any injective-surjective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$, that is, the diagram in Fig. 1(i) is commutative. We prove that $\mathcal{Q}$ has the $\mathcal{Q}$-CEP if and only if it has the EP (Corollary 2.11, generalizing [3, Lemma 1.2]) and that if $\mathcal{Q}$ is relatively congruence-distributive and $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ is closed under subalgebras, then the EP and $\mathcal{Q}$-CEP for $\mathcal{Q}$ and $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ all coincide (Theorem 2.12).

Now let $\mathcal{K}$ and $\mathcal{K}'$ be two classes of algebras of the same signature. An *amalgam in* $\mathcal{K}'$ of a doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$ is a triple $\langle \mathbf{D}, \psi_B, \psi_C \rangle$ where $\mathbf{D} \in \mathcal{K}'$ and ψ_B, ψ_C are embeddings of $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$, respectively, such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Fig. 1(ii)). The class $\mathcal{K}$ has the *amalgamation property* (for short, AP) if every doubly injective span in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. We also say that $\mathcal{K}$ has the *one-sided amalgamation property* (for short, 1AP) if for any doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Fig. 1(iii)). It is easy to see that a class $\mathcal{K}$ of similar algebras that is closed under finite products has the 1AP if and only if it has the AP,[3] but this is not always the case for other classes, in particular, the class of relatively finitely subdirectly irreducible members of a quasivariety. In Section 3, we prove that when $\mathcal{Q}$ has the $\mathcal{Q}$-CEP and $\mathcal{Q}_{\mathrm{RFSI}}$ is closed under subalgebras, $\mathcal{Q}$ has the 1AP (equivalently, the AP) if and only if $\mathcal{Q}_{\mathrm{RFSI}}$ has the 1AP (Theorem 3.4).

In Section 4, we consider consequences of our results for three further properties. First, a class $\mathcal{K}$ of similar algebras is said to have the *transferable injections property* (for short, TIP) if for any injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Fig. 1(iv)). A variety has the TIP if and only if it has the CEP and AP ([3, Lemma 1.7]). More generally, we show here that a class of similar algebras that is closed under subalgebras has the TIP if and only if it has the EP and 1AP. It then follows from our previous results that when $\mathcal{Q}$ is relatively congruence-distributive and $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ is closed under subalgebras, $\mathcal{Q}$ has the TIP if and only if $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ has the TIP (Theorem 4.3).

[3] For the non-trivial direction, observe that an amalgam of a doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$ may be obtained as the product of the one-sided amalgams of $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ and $\langle \mathbf{A}, \mathbf{C}, \mathbf{B}, \varphi_C, \varphi_B \rangle$.

Next, let $\mathcal{K}$ be a class of similar algebras and $\mathbf{A}, \mathbf{B} \in \mathcal{K}$. A homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ is an *epimorphism in* $\mathcal{K}$ if for all $\mathbf{C} \in \mathcal{K}$ and all homomorphisms $\psi_1, \psi_2\colon \mathbf{B} \to \mathbf{C}$, if $\psi_1\varphi = \psi_2\varphi$, then $\psi_1 = \psi_2$. Surjective homomorphisms are always epimorphisms, but the converse does not hold in general. If all epimorphisms in $\mathcal{K}$ are surjections, $\mathcal{K}$ is said to have *surjective epimorphisms* (for short, SE). In [6, Theorem 22], it was proved that an arithmetical (i.e., congruence-distributive and congruence-permutable) variety $\mathcal{V}$ such that $\mathcal{V}_{\text{FSI}}$ is a universal class has SE if and only if $\mathcal{V}_{\text{FSI}}$ has SE.

For a class of algebras $\mathcal{K}'$ of the same signature as $\mathcal{K}$, an amalgam $\langle \mathbf{D}, \psi_B, \psi_C \rangle$ in $\mathcal{K}'$ of a doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$ is called *strong* if $\psi_B\varphi_B[A] = \psi_B[B] \cap \psi_C[C]$. The class $\mathcal{K}$ is said to have the *strong amalgamation property* (for short, SAP) if every doubly injective span in $\mathcal{K}$ has a strong amalgam in $\mathcal{K}$. Using the fact that a quasivariety has the SAP if and only if it has SE and the AP [22], it follows from our previous results and [6, Theorem 22] that an arithmetical variety $\mathcal{V}$ with the CEP such that $\mathcal{V}_{\text{FSI}}$ is a universal class has the SAP if and only if $\mathcal{V}_{\text{FSI}}$ has SE and the 1AP (Corollary 4.6). We also show that such a variety has the SAP if every doubly injective span in $\mathcal{V}_{\text{FSI}}$ has a strong amalgam in $\mathcal{V}$ (Theorem 4.8).

In Section 5, we conclude that possession of all the properties mentioned above is decidable for certain finitely generated varieties. More precisely, we obtain effective algorithms to decide if a congruence-distributive variety $\mathcal{V}$ that is generated by a given finite set of finite algebras, such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras, has the CEP, AP, or TIP (Theorem 5.1). In the case where $\mathcal{V}$ is arithmetical, we obtain also effective algorithms to decide if $\mathcal{V}$ has SE or the SAP. Finally, in Section 6, we provide a complete description of the subvarieties of a notable variety of BL-algebras (those generated by a class of "one-component" totally ordered BL-algebras) that have the AP.

2. The congruence extension property

We first recall some basic facts about extending congruences, denoting the $\mathcal{Q}$-congruence of an algebra $\mathbf{A} \in \mathcal{Q}$ generated by a set $R \subseteq A^2$ by $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R)$.

Lemma 2.1. *Let $\mathcal{Q}$ be any quasivariety and let $\mathbf{B} \in \mathcal{Q}$.*

(a) (cf. [26, Lemma 1.3]) Suppose that for any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and completely meet-irreducible $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$, there exists a $\Psi \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ such that $\Psi \cap A^2 = \Theta$. Then $\mathbf{B}$ has the $\mathcal{Q}$-CEP.

(b) (cf. [4, p. 392]) Let $\mathbf{A}$ be a subalgebra of $\mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$ such that $\Psi \cap A^2 = \Theta$ for some $\Psi \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$. Then $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$.

Proof. (a) Consider any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$. Since $\mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$ is an algebraic lattice, there exists a set $\{\Theta_i\}_{i \in I}$ of $\mathcal{Q}$-congruences of $\mathbf{A}$ such that $\Theta = \bigcap_{i \in I} \Theta_i$ and Θ_i is completely meet-irreducible for each $i \in I$ (see, e.g., [18, Lemma 1.3.2]). By assumption, there exists for each $i \in I$, a $\Psi_i \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ such that $\Psi_i \cap A^2 = \Theta_i$. It follows that $\Psi := \bigcap_{i \in I} \Psi_i \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ satisfies $\Psi \cap A^2 = \bigcap_{i \in I}(\Psi_i \cap A^2) = \bigcap_{i \in I} \Theta_i = \Theta$.

(b) Since $\Theta \subseteq \Psi$ and $\Psi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$, also $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \subseteq \Psi$. Hence, using the assumption, $\Theta \subseteq \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 \subseteq \Psi \cap A^2 = \Theta$. $\square$

We will also make use of the following consequence of the correspondence theorem of universal algebra.

Lemma 2.2 *(cf. [4, Lemma 2]). Let $\mathcal{Q}$ be any quasivariety and let $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$. For any surjective homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ and $R \subseteq A \times A$,*

$$\varphi^{-1}[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\varphi[R])] = \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \vee \ker(\varphi),$$

where the join on the right hand side is taken in $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$ and $\varphi[R]$ abbreviates $\{\langle \varphi(x), \varphi(y)\rangle \mid \langle x, y\rangle \in R\}$.

Proof. Let $\Theta := \varphi^{-1}[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\varphi[R])]$. Since $R \subseteq \Theta$ and $\Theta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$, also $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \subseteq \Theta$. Moreover, $\ker(\varphi) \subseteq \Theta$, so $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \vee \ker(\varphi) \subseteq \Theta$. For the converse inclusion, since $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \vee \ker(\varphi) \in [\ker(\varphi), A^2]$, it follows using the correspondence theorem that

$$\varphi^{-1}[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\varphi[R])] \subseteq \varphi^{-1}[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\varphi[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \vee \ker(\varphi)])] = \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{A}}(R) \vee \ker(\varphi). \quad \square$$

We now establish the first main result of this section, recalling from the introduction that if $\mathcal{Q}$ is a relatively congruence-distributive quasivariety, then $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ [12, Theorem 2.3].

Theorem 2.3. *Let $\mathcal{Q}$ be any relatively congruence-distributive quasivariety. Then $\mathcal{Q}$ has the $\mathcal{Q}$-congruence extension property if and only if $\mathcal{Q}_{\mathit{RFSI}} = \mathcal{Q}_{\mathit{FSI}}$ has the $\mathcal{Q}$-congruence extension property.*

Proof. Suppose for the non-trivial direction that $\mathcal{Q}_{\mathrm{RFSI}}$ has the $\mathcal{Q}$-CEP. Let $\mathbf{A}$ be a subalgebra of some $\mathbf{B} \in \mathcal{Q}$ and let $\Theta \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$. We assume towards a contradiction that $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 \neq \Theta$, that is, there exists an ordered pair $\langle a, b\rangle \in \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2$ satisfying $\langle a, b\rangle \notin \Theta$. Define

$$T := \{\Psi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B} \mid \langle a, b\rangle \notin (\Psi \cap A^2) \vee \Theta\}.$$

Then $\Delta_B \in T$, so $T \neq \emptyset$. Moreover, every chain in $\langle T, \subseteq\rangle$ has an upper bound (its union) in T, so, by Zorn's Lemma, $\langle T, \subseteq\rangle$ has a maximal element Ψ^*.

Claim. *Ψ^* is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$ and hence $\mathbf{B}/\Psi^* \in \mathcal{Q}_{\mathit{RFSI}}$.*

Proof of Claim. Suppose that $\Psi^* = \Psi_1 \cap \Psi_2$ for some $\Psi_1, \Psi_2 \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$. Then, since $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$ is distributive by assumption,

$$((\Psi_1 \cap A^2) \vee \Theta) \cap ((\Psi_2 \cap A^2) \vee \Theta) = (\Psi_1 \cap \Psi_2 \cap A^2) \vee \Theta = (\Psi^* \cap A^2) \vee \Theta.$$

But $\langle a,b\rangle \notin (\Psi^* \cap A^2) \vee \Theta$, so $\langle a,b\rangle \notin (\Psi_1 \cap A^2) \vee \Theta$ or $\langle a,b\rangle \notin (\Psi_2 \cap A^2) \vee \Theta$. Hence $\Psi_1 \in T$ or $\Psi_2 \in T$ and, by the maximality of Ψ^* in $\langle T, \subseteq\rangle$, either $\Psi^* = \Psi_1$ or $\Psi^* = \Psi_2$. So Ψ^* is meet-irreducible. □

Observe next that $\mathbf{A}/(\Psi^* \cap A^2)$ embeds into $\mathbf{B}/\Psi^*$ and can be identified with a subalgebra of $\mathbf{B}/\Psi^*$ with universe A/Ψ^*. We consider the congruence $((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2)$ of $\mathbf{A}/(\Psi^* \cap A^2)$. By the second isomorphism theorem of universal algebra,

$$(\mathbf{A}/(\Psi^* \cap A^2))/((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2)) \cong \mathbf{A}/((\Psi^* \cap A^2) \vee \Theta) \in \mathcal{Q}.$$

In particular, $(\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2) \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}/(\Psi^* \cap A^2)$. Moreover, since $\langle a,b\rangle \notin (\Psi^* \cap A^2) \vee \Theta$,

$$\langle [a]_{\Psi^* \cap A^2}, [b]_{\Psi^* \cap A^2}\rangle \notin ((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2).$$

Recall now that $\langle a,b\rangle \in \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2$, so $\langle a,b\rangle \in \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}((\Psi^* \cap A^2) \vee \Theta) \vee \Psi^*$. Letting $R := (\Psi^* \cap A^2) \vee \Theta$ and φ be the canonical homomorphism from $\mathbf{B}$ to $\mathbf{B}/\Psi^*$ with $\ker(\varphi) = \Psi^*$, an application of Lemma 2.2 yields

$$\varphi^{-1}[\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}/\Psi^*}(\varphi[(\Psi^* \cap A^2) \vee \Theta])] = \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}((\Psi^* \cap A^2) \vee \Theta) \vee \Psi^*.$$

Hence, identifying the congruence $((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2)$ of $\mathbf{A}/(\Psi^* \cap A^2)$ with the corresponding subset of B/Ψ^*,

$$\langle [a]_{\Psi^*}, [b]_{\Psi^*}\rangle \in \mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}/\Psi^*}(((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2)).$$

But, by Lemma 2.1, this contradicts the assumption that $\mathbf{B}/\Psi^* \in \mathcal{Q}_{\mathrm{RFSI}}$ has the $\mathcal{Q}$-CEP. So $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$ as required. □

Corollary 2.4. *Let $\mathcal{V}$ be any congruence-distributive variety. Then $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}_{FSI}$ has the congruence extension property.*

Note that if $\mathcal{Q}$ is a relatively congruence-distributive quasivariety, then every member of $\mathcal{Q}_{\mathrm{RFSI}}$ embeds into an ultraproduct of members of $\mathcal{Q}_{\mathrm{RSI}}$, by the Relativized Jónsson Lemma [8, Lemma 1.5]. Hence, since the $\mathcal{Q}$-CEP is preserved under subalgebras, Theorem 2.3 yields also a characterization of the $\mathcal{Q}$-CEP for $\mathcal{Q}$ in terms of the members of $\mathcal{Q}_{\mathrm{RSI}}$.

Corollary 2.5. *Let $\mathcal{Q}$ be any relatively congruence-distributive quasivariety. Then $\mathcal{Q}$ has the $\mathcal{Q}$-congruence extension property if and only if the class of ultraproducts of members of $\mathcal{Q}_{RSI}$ has the $\mathcal{Q}$-congruence extension property.*

In particular, we obtain Davey's result that for any congruence-distributive variety $\mathcal{V}$ such that $\mathcal{V}_{\mathrm{SI}}$ is an elementary class, $\mathcal{V}$ has the CEP if and only if $\mathcal{V}_{\mathrm{SI}}$ has the CEP [10, Theorem 3.3].[4]

We now turn our attention to the relationship between the $\mathcal{Q}$-CEP and the EP, establishing first a simple lemma and useful corollary for investigating relatively finitely subdirectly irreducible algebras.

Lemma 2.6. *Let $\mathcal{Q}$ be any quasivariety and let $\mathbf{A}$ be a subalgebra of some $\mathbf{B} \in \mathcal{Q}$. For any meet-irreducible $\Theta \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$ satisfying $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$, there exists a meet-irreducible $\Phi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$ such that $\Phi \cap A^2 = \Theta$.*

Proof. Consider any meet-irreducible $\Theta \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$ satisfying $\mathrm{Cg}^{\mathcal{Q}}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$. By assumption, $T := \{\Psi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B} \mid \Psi \cap A^2 = \Theta\} \neq \emptyset$, and, since every chain in $\langle T, \subseteq\rangle$ has an upper bound (its union) in T, by Zorn's Lemma, $\langle T, \subseteq\rangle$ has a maximal element $\Phi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$.

It remains to show that Φ is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$, so let $\Phi = \Phi_1 \cap \Phi_2$ for some $\Phi_1, \Phi_2 \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$. Then

$$(\Phi_1 \cap A^2) \cap (\Phi_2 \cap A^2) = \Phi_1 \cap \Phi_2 \cap A^2 = \Phi \cap A^2 = \Theta$$

and, since Θ is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{A}$, either $\Phi_1 \cap A^2 = \Theta$ or $\Phi_2 \cap A^2 = \Theta$. So $\Phi_1 \in T$ or $\Phi_2 \in T$. Hence, by the maximality of Φ in $\langle T, \subseteq\rangle$, either $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$. So Φ is meet-irreducible in $\mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$. $\square$

Corollary 2.7. *Let $\mathcal{Q}$ be any quasivariety and suppose that $\mathbf{A} \in \mathcal{Q}_{\mathrm{RFSI}}$ is a subalgebra of some $\mathbf{B} \in \mathcal{Q}$. Then there exists a meet-irreducible $\Phi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$ such that $\Phi \cap A^2 = \Delta_A$, and hence there exist also a $\mathbf{C} \in \mathcal{Q}_{\mathrm{RFSI}}$ and surjective homomorphism $\varphi\colon \mathbf{B} \to \mathbf{C}$ such that $\ker(\varphi) \cap A^2 = \Delta_A$.*

The following result provides a general sufficient criterion for a subclass $\mathcal{K}$ of a quasivariety $\mathcal{Q}$ to have the EP.

Proposition 2.8. *Let $\mathcal{K}$ be a subclass of a quasivariety $\mathcal{Q}$ satisfying*

(i) $\mathcal{K}$ is closed under isomorphisms;
(ii) for any $\mathbf{B} \in \mathcal{Q}$ and subalgebra $\mathbf{A} \in \mathcal{K}$ of $\mathbf{B}$, there exists a $\Phi \in \mathrm{Con}_{\mathcal{Q}}\, \mathbf{B}$ such that $\mathbf{B}/\Phi \in \mathcal{K}$ and $\Phi \cap A^2 = \Delta_A$;
(iii) $\mathcal{K}$ has the $\mathcal{Q}$-congruence extension property.

Then $\mathcal{K}$ has the extension property.

[4] This result also follows from a more general theorem of Kiss [26, Theorem 2.3] for congruence-modular varieties; however, the latter does not imply, or seem to be implied by, our Theorem 2.3.

Proof. Consider any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and surjective homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$. Since $\mathcal{K}$ is closed under isomorphisms, by (i), we may assume without loss of generality that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and that $\mathbf{C} = \mathbf{A}/\Theta$ for some $\Theta \in \mathrm{Con}_{\mathcal{Q}} \mathbf{A}$. Since $\mathcal{K}$ has the $\mathcal{Q}$-CEP, by (iii), there exists a $\Psi \in \mathrm{Con}_{\mathcal{Q}} \mathbf{B}$ such that $\Psi \cap A^2 = \Theta$. Let $\mathbf{D} := \mathbf{B}/\Psi \in \mathcal{Q}$ and let ψ_B be the canonical homomorphism from $\mathbf{B}$ to $\mathbf{D}$ mapping each $b \in B$ to $[b]_\Psi \in B/\Psi$. Observe also that for any $a, b \in A$,

$$[a]_\Theta = [b]_\Theta \iff \langle a, b\rangle \in \Theta \iff \langle a, b\rangle \in \Psi \iff [a]_\Psi = [b]_\Psi.$$

Hence we obtain an embedding ψ_C of $\mathbf{C}$ into $\mathbf{D}$ mapping each $[a]_\Theta \in A/\Theta$ to $[a]_\Psi \in B/\Psi$ such that $\psi_B\varphi_B = \psi_C\varphi_C$. Finally, by (ii), there exist a $\mathbf{D}^* \in \mathcal{K}$, a surjective homomorphism $\psi_B^* \colon \mathbf{D} \to \mathbf{D}^*$, and an embedding $\psi_C^* \colon \mathbf{C} \to \mathbf{D}^*$ such that $\psi_B^*\psi_B\varphi_B = \psi_C^*\varphi_C$. So $\mathcal{K}$ has the EP. $\square$

In particular, combining Proposition 2.8 with Corollary 2.7, we obtain the following result for the class of relatively finitely subdirectly irreducible members of a quasivariety.

Corollary 2.9. *Let $\mathcal{Q}$ be any quasivariety. If $\mathcal{Q}_{\mathrm{RFSI}}$ has the $\mathcal{Q}$-congruence extension property, then $\mathcal{Q}_{\mathrm{RFSI}}$ has the extension property.*

Next, we provide a sufficient criterion for a subclass $\mathcal{K}$ of a quasivariety $\mathcal{Q}$ to have the $\mathcal{Q}$-CEP.

Proposition 2.10. *Let $\mathcal{K}$ be a subclass of a quasivariety $\mathcal{Q}$ satisfying*

(i) $\mathcal{K}$ is closed under subalgebras;
(ii) every relatively subdirectly irreducible member of $\mathcal{Q}$ belongs to $\mathcal{K}$;
(iii) $\mathcal{K}$ has the extension property.

Then $\mathcal{K}$ has the $\mathcal{Q}$-congruence extension property.

Proof. Consider any $\mathbf{B} \in \mathcal{K}$. To show that $\mathbf{B}$ has the $\mathcal{Q}$-CEP, it suffices, by Lemma 2.1(1), to prove that for any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and completely meet-irreducible $\Theta \in \mathrm{Con}_{\mathcal{Q}} \mathbf{A}$, there exists a $\Psi \in \mathrm{Con}_{\mathcal{Q}} \mathbf{B}$ such that $\Psi \cap A^2 = \Theta$. Note first that $\mathbf{A} \in \mathcal{K}$, by (i), and $\mathbf{A}/\Theta \in \mathcal{Q}_{\mathrm{RSI}} \subseteq \mathcal{K}$, by (ii). Now let $\varphi_C \colon \mathbf{A} \to \mathbf{A}/\Theta$ be the canonical homomorphism mapping $a \in A$ to $[a]_\Theta \in A/\Theta$ and let $\varphi_B \colon \mathbf{A} \to \mathbf{B}$ be the inclusion map. Since $\mathcal{K}$ has the EP, by (iii), there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{A}/\Theta \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$. Let $\Psi := \ker(\psi_B)$. By the homomorphism theorem, $\mathbf{B}/\Psi$ is isomorphic to a subalgebra of $\mathbf{D} \in \mathcal{K}$, so $\mathbf{B}/\Psi \in \mathcal{Q}$ and $\Psi \in \mathrm{Con}_{\mathcal{Q}} \mathbf{B}$. Moreover, for any $a, b \in A$, using the injectivity of ψ_C for the third equivalence,

$$\begin{aligned} \langle a, b\rangle \in \Psi &\iff \psi_B\varphi_B(a) = \psi_B\varphi_B(b) \\ &\iff \psi_C\varphi_C(a) = \psi_C\varphi_C(b) \\ &\iff \varphi_C(a) = \varphi_C(b) \end{aligned}$$

$$\iff \langle a, b\rangle \in \ker(\varphi_C) = \Theta.$$

That is, $\Psi \cap A^2 = \Theta$. So $\mathbf{B}$ has the $\mathcal{Q}$-CEP. □

In particular, we obtain the following generalization of [3, Lemma 1.2].

Corollary 2.11. *Let $\mathcal{Q}$ be any quasivariety. Then $\mathcal{Q}$ has the $\mathcal{Q}$-congruence extension property if and only if $\mathcal{Q}$ has the extension property.*

We can now combine these results to obtain the second main result of this section.

Theorem 2.12. *Let $\mathcal{Q}$ be a relatively congruence-distributive quasivariety such that $\mathcal{Q}_{RFSI} = \mathcal{Q}_{FSI}$ is closed under subalgebras. The following are equivalent:*

(1) $\mathcal{Q}$ has the $\mathcal{Q}$-congruence extension property.
(2) $\mathcal{Q}$ has the extension property.
(3) $\mathcal{Q}_{RFSI} = \mathcal{Q}_{FSI}$ has the $\mathcal{Q}$-congruence extension property.
(4) $\mathcal{Q}_{RFSI} = \mathcal{Q}_{FSI}$ has the extension property.

Proof. The equivalence of (1) and (2) is a special case of Corollary 2.11, and the equivalence of (1) and (3), and the implications from (3) to (4), and from (4) to (3), follow from Theorem 2.3, Corollary 2.9, and Proposition 2.10, respectively. □

Corollary 2.13. *Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{FSI}$ is closed under subalgebras. The following are equivalent:*

(1) $\mathcal{V}$ has the congruence extension property.
(2) $\mathcal{V}$ has the extension property.
(3) $\mathcal{V}_{FSI}$ has the congruence extension property.
(4) $\mathcal{V}_{FSI}$ has the extension property.

Remark 2.14. Even for a congruence-distributive variety $\mathcal{V}$, it is possible for $\mathcal{V}_{FSI}$ to have the EP but not the CEP. For example, let $\mathcal{V}$ be the variety generated by the lattice-ordered monoid $\mathbf{C}_4 = \langle\{-2, -1, 1, 2\}, \min, \max, \cdot, 1\rangle$ with multiplication table

$\cdot$	-2	-1	1	2
-2	-2	-2	-2	-2
-1	-2	-1	-1	2
1	-2	-1	1	2
2	-2	2	2	2

The proper subuniverses of $\mathbf{C}_4$ are $C_1 = \{1\}$, $C_2 = \{-1, 1\}$, $C_2^\delta = \{1, 2\}$, $C_3 = \{-1, 1, 2\}$, $C_3^\delta = \{-2, 1, 2\}$, and $C_3^* = \{-2, -1, 1\}$, up to isomorphism, and any homomorphic image of $\mathbf{C}_4$ is isomorphic to one of its subalgebras. As shown in [33], the algebra $\mathbf{C}_4$, and hence $\mathcal{V}_{\mathrm{FSI}}$, does not have the CEP: just observe that $\Theta := \Delta_{C_3^*} \cup \{\langle -1,-2\rangle, \langle -2,-1\rangle\} \in \operatorname{Con}\mathbf{C}_3^*$, but $\mathrm{Cg}_{\mathbf{C}_4}(\Theta) = C_4 \times C_4$. On the other hand, using the fact that $\mathbf{C}_3^* \notin \mathcal{V}_{\mathrm{FSI}}$, it is easy to confirm that $\mathcal{V}_{\mathrm{FSI}}$ has the EP.

3. The amalgamation property

We first recall a useful necessary and sufficient condition for the existence of amalgams.

Lemma 3.1 *([19, Lemma 2]). Let $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ be a doubly injective span in a class of similar algebras $\mathcal{K}$ and suppose that*

(i) for any distinct $x, y \in B$, there exist a $\mathbf{D}_B^{xy} \in \mathcal{K}$ and homomorphisms $\psi_B^{xy} : \mathbf{B} \to \mathbf{D}_B^{xy}$ and $\psi_C^{xy} : \mathbf{C} \to \mathbf{D}_B^{xy}$ satisfying $\psi_B^{xy}\varphi_B = \psi_C^{xy}\varphi_C$ and $\psi_B^{xy}(x) \neq \psi_B^{xy}(y)$;

(ii) for any distinct $x, y \in C$, there exist a $\mathbf{D}_C^{xy} \in \mathcal{K}$ and homomorphisms $\chi_B^{xy} : \mathbf{B} \to \mathbf{D}_C^{xy}$ and $\chi_C^{xy} : \mathbf{C} \to \mathbf{D}_C^{xy}$ satisfying $\chi_B^{xy}\varphi_B = \chi_C^{xy}\varphi_C$ and $\chi_C^{xy}(x) \neq \chi_C^{xy}(y)$.

Then $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ has an amalgam $\langle \mathbf{D}, \psi_B, \psi_C\rangle$, where $\mathbf{D}$ is the product of the algebras in the set $\{\mathbf{D}_B^{xy} \mid x, y \in B, x \neq y\} \cup \{\mathbf{D}_C^{xy} \mid x, y \in C, x \neq y\}$.

Next, we show that to establish the AP for a universal class of algebras, we can restrict our attention to the finitely generated members of the class.[5]

Lemma 3.2. *Let $\mathcal{K}$ be a universal class of algebras such that every doubly injective span of finitely generated algebras in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. Then $\mathcal{K}$ has the amalgamation property.*

Proof. Let $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ be any doubly injective span in $\mathcal{K}$, assuming without loss of generality that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$, and φ_B and φ_C are inclusion maps. Referring to [21] for basic notions of model theory, let Σ be the union of the theory of $\mathcal{K}$ and the atomic diagrams of $\mathbf{B}$ and $\mathbf{C}$. Then the span has an amalgam in $\mathcal{K}$ if and only if Σ has a model. To show that Σ has a model, consider any union Σ' of the theory of $\mathcal{K}$ and arbitrary finite subsets of the atomic diagrams of $\mathbf{B}$ and $\mathbf{C}$, and let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ be the respective subalgebras of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ generated by the finitely many elements named in Σ'. By assumption, the doubly injective span $\langle \mathbf{A}', \mathbf{B}', \mathbf{C}', \varphi'_B, \varphi'_C\rangle$, where φ'_B and φ'_C are inclusion maps, has an amalgam in $\mathcal{K}$. So Σ' has a model. Hence, by the compactness theorem, Σ has a model. □

The following result will play a key role in the proof of our Theorem 3.4. A slightly weaker version (applying only to varieties) was first proved in [29] (see also [30]) and used to establish a special case of Theorem 3.4 where $\mathcal{V}$ is a variety of semilinear residuated lattices with the CEP and $\mathcal{V}_{\mathrm{FSI}}$ is the class of totally ordered members of $\mathcal{V}$.

[5] We thank the anonymous referee for providing a proof of this lemma.

Proposition 3.3 *(cf. [29, Theorem 9]). Let $\mathcal{K}$ be a subclass of a quasivariety $\mathcal{Q}$ satisfying*

(i) $\mathcal{K}$ is closed under isomorphisms and subalgebras;

(ii) every relatively subdirectly irreducible member of $\mathcal{Q}$ belongs to $\mathcal{K}$;

(iii) for any $\mathbf{B} \in \mathcal{Q}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$ and $\mathbf{A}/\Theta \in \mathcal{K}$, then there exists a $\Phi \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in \mathcal{K}$;

(iv) every doubly injective span of finitely generated algebras in $\mathcal{K}$ has an amalgam in $\mathcal{Q}$.

Then $\mathcal{Q}$ has the amalgamation property.

Proof. By Lemma 3.2, it suffices to show that any doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ of finitely generated members of $\mathcal{Q}$ has an amalgam in $\mathcal{Q}$, assuming, without loss of generality, that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$, and φ_B and φ_C are inclusion maps. We check condition (i) of Lemma 3.1 for the existence of an amalgam, condition (ii) being completely symmetrical. Consider any distinct $x, y \in B$ and let Ψ be a $\mathcal{Q}$-congruence of $\mathbf{B}$ that is maximal with respect to $\langle x, y \rangle \notin \Psi$. Then $\mathbf{B}/\Psi$ is a relatively subdirectly irreducible member of $\mathcal{Q}$ and belongs to $\mathcal{K}$, by (ii). Define $\Theta := \Psi \cap A^2$. The map φ'_B sending $[a]_\Theta \in \mathbf{A}/\Theta$ to $[a]_\Psi$ is an embedding of $\mathbf{A}/\Theta$ into $\mathbf{B}/\Psi$, so $\mathbf{A}/\Theta \in \mathcal{K}$, by (i), and $\Theta \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}$. Hence, by (iii), there exists a $\Phi \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{C}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{C}/\Phi \in \mathcal{K}$. Moreover, the map φ'_C sending any $[a]_\Theta \in \mathbf{A}/\Theta$ to $[a]_\Phi$ is an embedding of $\mathbf{A}/\Theta$ into $\mathbf{C}/\Phi$.

Since $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are finitely generated, $\langle \mathbf{A}/\Theta, \mathbf{B}/\Psi, \mathbf{C}/\Phi, \varphi'_B, \varphi'_C \rangle$ is a doubly injective span of finitely generated members of $\mathcal{K}$ and, by (iv), has an amalgam $\langle \mathbf{D}_{xy}, \chi_B, \chi_C \rangle$ in $\mathcal{Q}$. We define homomorphisms

$$\psi_B^{xy} \colon \mathbf{B} \to \mathbf{D}_{xy};\ \ b \mapsto \chi_B([b]_\Psi) \quad \text{and} \quad \psi_C^{xy} \colon \mathbf{C} \to \mathbf{D}_{xy};\ \ c \mapsto \chi_C([c]_\Phi).$$

Then $\psi_B^{xy}(x) \neq \psi_B^{xy}(y)$ (as χ_B is injective and $[x]_\Psi \neq [y]_\Psi$) and for any $a \in A$,

$$\begin{aligned} \psi_B^{xy}(\varphi_B(a)) &= \chi_B([a]_\Psi) \\ &= \chi_B(\varphi'_B([a]_\Theta)) \\ &= \chi_C(\varphi'_C([a]_\Theta)) \\ &= \chi_C([a]_\Phi) \\ &= \psi_C^{xy}(\varphi_C(a)). \quad \square \end{aligned}$$

We now prove the main result of this section.

Theorem 3.4. *Let $\mathcal{Q}$ be any quasivariety with the $\mathcal{Q}$-congruence extension property such that $\mathcal{Q}_{\scriptscriptstyle RFSI}$ is closed under subalgebras. The following are equivalent:*

(1) $\mathcal{Q}$ has the amalgamation property.

(2) $\mathcal{Q}$ has the one-sided amalgamation property.

(3) $\mathcal{Q}_{RFSI}$ *has the one-sided amalgamation property.*
(4) *Every doubly injective span in* $\mathcal{Q}_{RFSI}$ *has an amalgam in* $\mathcal{Q}_{RFSI} \times \mathcal{Q}_{RFSI}$*.*
(5) *Every doubly injective span of finitely generated algebras in* $\mathcal{Q}_{RFSI}$ *has an amalgam in* $\mathcal{Q}$*.*

Proof. (1) ⇒ (2). Immediate.

(2) ⇒ (3). Suppose that $\mathcal{Q}$ has the 1AP and let $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ be a doubly injective span in $\mathcal{Q}_{\mathrm{RFSI}}$. By assumption, there exist a $\mathbf{D}' \in \mathcal{Q}$, a homomorphism $\psi'_B \colon \mathbf{B} \to \mathbf{D}'$, and an embedding $\psi'_C \colon \mathbf{C} \to \mathbf{D}'$ such that $\psi'_B\varphi_B = \psi'_C\varphi_C$. We may assume without loss of generality that $\mathbf{C}$ is a subalgebra of $\mathbf{D}'$. By Corollary 2.7, there exist a $\mathbf{D} \in \mathcal{Q}_{\mathrm{RFSI}}$ and a surjective homomorphism $\chi \colon \mathbf{D}' \to \mathbf{D}$ such that $\ker(\chi) \cap C^2 = \Delta_C$. Hence $\psi_B := \chi\psi'_B$ is a homomorphism from $\mathbf{B}$ to $\mathbf{D}$, and $\psi_C := \chi\psi'_C$ is an embedding of $\mathbf{C}$ into $\mathbf{D}$ satisfying $\psi_B\varphi_B = \chi\psi'_B\varphi_B = \chi\psi'_C\varphi_C = \psi_C\varphi_C$.

(3) ⇒ (4). Suppose that $\mathcal{Q}_{\mathrm{RFSI}}$ has the 1AP and let $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ be any doubly injective span in $\mathcal{Q}_{\mathrm{RFSI}}$. By assumption, there exist a $\mathbf{D}_C \in \mathcal{Q}_{\mathrm{RFSI}}$, a homomorphism $\psi^C_B \colon \mathbf{B} \to \mathbf{D}_C$, and an embedding $\psi^C_C \colon \mathbf{C} \to \mathbf{D}_C$ such that $\psi^C_B\varphi_B = \psi^C_C\varphi_C$. However, $\langle \mathbf{A}, \mathbf{C}, \mathbf{B}, \varphi_C, \varphi_B\rangle$ is also a doubly injective span in $\mathcal{Q}_{\mathrm{RFSI}}$, so there exist a $\mathbf{D}_B \in \mathcal{Q}_{\mathrm{RFSI}}$, a homomorphism $\psi^B_C \colon \mathbf{C} \to \mathbf{D}_B$, and an embedding $\psi^B_B \colon \mathbf{B} \to \mathbf{D}_B$ such that $\psi^B_C\varphi_C = \psi^B_B\varphi_B$. Hence $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ has an amalgam $\langle \mathbf{D}, \psi_B, \psi_C\rangle$, where $\mathbf{D} = \mathbf{D}_B \times \mathbf{D}_C \in \mathcal{Q}_{\mathrm{RFSI}} \times \mathcal{Q}_{\mathrm{RFSI}}$, ψ_B maps $x \in B$ to $\langle \psi^B_B(x), \psi^C_B(x)\rangle$, and ψ_C maps $x \in C$ to $\langle \psi^B_C(x), \psi^C_C(x)\rangle$.

(4) ⇒ (5). Immediate.

(5) ⇒ (1). Suppose that every doubly injective span of finitely generated algebras in $\mathcal{Q}_{\mathrm{RFSI}}$ has an amalgam in $\mathcal{Q}$. Since $\mathcal{Q}_{\mathrm{RSI}} \subseteq \mathcal{Q}_{\mathrm{RFSI}}$ and $\mathcal{Q}_{\mathrm{RFSI}}$ is closed under subalgebras by assumption, it suffices to apply Proposition 3.3 with $\mathcal{K} := \mathcal{Q}_{\mathrm{RFSI}}$, observing that condition (iii) is satisfied by Lemma 2.6. □

Corollary 3.5. *Let* $\mathcal{V}$ *be any variety with the congruence extension property such that* $\mathcal{V}_{FSI}$ *is closed under subalgebras. The following are equivalent:*

(1) $\mathcal{V}$ *has the amalgamation property.*
(2) $\mathcal{V}$ *has the one-sided amalgamation property.*
(3) $\mathcal{V}_{FSI}$ *has the one-sided amalgamation property.*
(4) *Every doubly injective span in* $\mathcal{V}_{FSI}$ *has an amalgam in* $\mathcal{V}_{FSI} \times \mathcal{V}_{FSI}$*.*
(5) *Every doubly injective span of finitely generated algebras in* $\mathcal{V}_{FSI}$ *has an amalgam in* $\mathcal{V}$*.*

Remark 3.6. The 1AP cannot be replaced by the AP in condition (3) of Theorem 3.4 or Corollary 3.5. For example, the variety $\mathcal{DL}$ of distributive lattices is congruence-distributive and has the CEP and AP, but $\mathcal{DL}_{\mathrm{FSI}}$, which up to isomorphism contains only the trivial lattice and two-element lattice, does not have the AP. Just observe that any amalgam of a doubly injective span embedding the trivial lattice into the two-element lattice in two different ways must have at least three elements and hence cannot belong to $\mathcal{DL}_{\mathrm{FSI}}$.

The following result is useful for the study of joins of varieties with the AP (see, e.g., the proof of Theorem 6.2). Recall that a subalgebra $\mathbf{A}$ of an algebra $\mathbf{B}$ is a *retract* of $\mathbf{B}$ if there exists a homomorphism $\psi\colon \mathbf{B} \to \mathbf{A}$ such that ψ is the identity on A.

Proposition 3.7. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be varieties of the same signature such that $\mathcal{V}_1 \vee \mathcal{V}_2$ is congruence-distributive, and suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ have the AP and CEP, $(\mathcal{V}_1)_{FSI}$ and $(\mathcal{V}_2)_{FSI}$ are closed under subalgebras, and whenever $\mathbf{A} \in (\mathcal{V}_1)_{FSI} \cap (\mathcal{V}_2)_{FSI}$ is a subalgebra of $\mathbf{B} \in (\mathcal{V}_1)_{FSI} \cup (\mathcal{V}_2)_{FSI}$, it is a retract of $\mathbf{B}$. Then $\mathcal{V}_1 \vee \mathcal{V}_2$ has the amalgamation property.*

Proof. Note first that, since $\mathcal{V}_1 \cup \mathcal{V}_2$ is a positive universal class, Jónsson's Lemma [24] yields $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{SI}} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$ and hence $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{SI}} = (\mathcal{V}_1)_{\mathrm{SI}} \cup (\mathcal{V}_2)_{\mathrm{SI}}$. However, by the Relativized Jónsson Lemma [8, Lemma 1.5], every finitely subdirectly irreducible member of a variety embeds into an ultraproduct of its subdirectly irreducible members. So also $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{FSI}} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$, and $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{FSI}} = (\mathcal{V}_1)_{\mathrm{FSI}} \cup (\mathcal{V}_2)_{\mathrm{FSI}}$. Now consider any doubly-injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{FSI}}$. Since $\mathcal{V}_1$ and $\mathcal{V}_2$ have the AP, we may assume that $\mathbf{A} \in (\mathcal{V}_1)_{\mathrm{FSI}} \cap (\mathcal{V}_2)_{\mathrm{FSI}}$ is a subalgebra of $\mathbf{B}, \mathbf{C} \in (\mathcal{V}_1)_{\mathrm{FSI}} \cup (\mathcal{V}_2)_{\mathrm{FSI}}$. By assumption, there exists a homomorphism $\psi_B\colon \mathbf{B} \to \mathbf{C}$ (since $\mathbf{A}$ is a subalgebra of $\mathbf{C}$) such that ψ_B is the identity on A. Let ψ_C be the identity map on $\mathbf{C}$. Clearly, $\psi_B\varphi_B = \psi_C\varphi_C$. Hence $(\mathcal{V}_1 \vee \mathcal{V}_2)_{\mathrm{FSI}}$ has the 1AP and so $\mathcal{V}_1 \vee \mathcal{V}_2$ has the AP, by Theorem 3.4. $\square$

Let us conclude this section by remarking that by considering completely meet-irreducible $\mathcal{Q}$-congruences in the proof of Theorem 3.4, we obtain the same result with $\mathcal{Q}_{\mathrm{RFSI}}$ replaced by the class $\mathcal{Q}^{+}_{\mathrm{RSI}}$ of trivial or $\mathcal{Q}$-subdirectly irreducible members of $\mathcal{Q}$. In particular, we obtain [19, Theorem 3], which states that a variety $\mathcal{V}$ with the CEP such that the class $\mathcal{V}^{+}_{\mathrm{SI}}$ of trivial or subdirectly irreducible members of $\mathcal{V}$ is closed under subalgebras has the AP if and only if every doubly injective span in $\mathcal{V}^{+}_{\mathrm{SI}}$ has an amalgam in $\mathcal{V}$. Note, however, that while the property that $\mathcal{Q}_{\mathrm{RFSI}}$ is closed under subalgebras follows from the existence of equationally definable relative principal congruence meets (satisfied by many quasivarieties serving as algebraic semantics for non-classical logics), a similarly general condition guaranteeing closure under subalgebras is not known for $\mathcal{Q}^{+}_{\mathrm{RSI}}$.

4. Transferable injections and strong amalgamation

We first establish a generalization for classes of similar algebras closed under subalgebras of the well-known fact that a variety has the TIP if and only if it has the CEP and AP [3, Lemma 1.7].

Proposition 4.1. *Let $\mathcal{K}$ be a class of similar algebras that is closed under subalgebras. Then $\mathcal{K}$ has the transferable injections property if and only if it has the one-sided amalgamation property and extension property.*

Proof. The left-to-right direction is immediate. For the converse, suppose that $\mathcal{K}$ has the 1AP and EP and consider any injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$. Let $\mathbf{C}' := \varphi_C[\mathbf{A}]$. By assumption, $\mathbf{C}' \in \mathcal{K}$. Moreover, $\varphi_C \colon \mathbf{A} \to \mathbf{C}'$ is surjective, so, since $\mathcal{K}$ has the EP, there exist a $\mathbf{D}' \in \mathcal{K}$, a homomorphism $\psi'_B \colon \mathbf{B} \to \mathbf{D}'$, and an embedding $\psi'_C \colon \mathbf{C}' \to \mathbf{D}'$ such that $\psi'_B \varphi_B = \psi'_C \varphi_C$. Let $\nu \colon \mathbf{C}' \to \mathbf{C}$ be the inclusion map. Since $\mathcal{K}$ has the 1AP, there exist for the doubly injective span $\langle \mathbf{C}', \mathbf{D}', \mathbf{C}, \psi'_C, \nu \rangle$, a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\chi \colon \mathbf{D}' \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\chi \psi'_C = \psi_C \nu$. Let $\psi_B := \chi \psi'_B$. Then $\psi_B \varphi_B = \chi \psi'_B \varphi_B = \chi \psi'_C \varphi_C = \psi_C \nu \varphi_C = \psi_C \varphi_C$. Hence $\mathcal{K}$ has the TIP. $\square$

Combining now Proposition 4.1 with our earlier results for the 1AP and EP, we are able to transfer results for the TIP from a quasivariety to the class of its relatively finitely subdirectly irreducible members, and back again.

Lemma 4.2. *If a quasivariety $\mathcal{Q}$ has the transferable injections property, then $\mathcal{Q}_{\mathrm{RFSI}}$ has the transferable injections property.*

Proof. Suppose that $\mathcal{Q}$ has the TIP and let $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ be an injective span in $\mathcal{Q}_{\mathrm{RFSI}}$. Then there exist a $\mathbf{D} \in \mathcal{Q}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$. Without loss of generality, we may assume that $\mathbf{C}$ is a subalgebra of $\mathbf{D}$ and ψ_C is the inclusion map. By Corollary 2.7, there exist a $\mathbf{D}^* \in \mathcal{Q}_{\mathrm{RFSI}}$ and a surjective homomorphism $\chi \colon \mathbf{D} \to \mathbf{D}^*$ such that $\ker(\chi) \cap C^2 = \Delta_C$. Let $\psi^*_B := \chi \psi_B$ and $\psi^*_C := \chi \psi_C$. Then $\ker(\psi^*_C) = \ker(\chi) \cap C^2 = \Delta_C$, so ψ^*_C is an embedding, and $\psi^*_B \varphi_B = \chi \psi_B \varphi_B = \chi \psi_C \varphi_C = \psi^*_C \varphi_C$. Hence $\mathcal{Q}_{\mathrm{RFSI}}$ has the TIP. $\square$

Theorem 4.3. *Let $\mathcal{Q}$ be a relatively congruence-distributive quasivariety such that $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ is closed under subalgebras. Then $\mathcal{Q}$ has the transferable injections property if and only if $\mathcal{Q}_{\mathrm{RFSI}} = \mathcal{Q}_{\mathrm{FSI}}$ has the transferable injections property.*

Proof. The left-to-right direction follows directly from Lemma 4.2. For the converse, suppose that $\mathcal{Q}_{\mathrm{RFSI}}$ has the TIP. Then $\mathcal{Q}_{\mathrm{RFSI}}$ has the 1AP and the EP, by Proposition 4.1, and hence $\mathcal{Q}$ has the $\mathcal{Q}$-CEP, by Theorem 2.12. Moreover, $\mathcal{Q}$ has the AP, by Theorem 3.4. So $\mathcal{Q}$ has the TIP, by Proposition 4.1. $\square$

We now recall a useful characterization of the SAP and a transfer theorem for SE.

Theorem 4.4 *([22]). Let $\mathcal{Q}$ be a quasivariety. Then $\mathcal{Q}$ has the strong amalgamation property if and only if $\mathcal{Q}$ has the amalgamation property and surjective epimorphisms.*

Theorem 4.5 *([6, Theorem 22]). Let $\mathcal{V}$ be an arithmetical variety such that $\mathcal{V}_{\mathrm{FSI}}$ is a universal class. Then $\mathcal{V}$ has surjective epimorphisms if and only if $\mathcal{V}_{\mathrm{FSI}}$ has surjective epimorphisms.*

Combining Theorems 4.4 and 4.5 with Corollary 3.5 yields the following result.

Corollary 4.6. *Let $\mathcal{V}$ be an arithmetical variety with the congruence extension property such that $\mathcal{V}_{FSI}$ is a universal class. Then $\mathcal{V}$ has the strong amalgamation property if and only if $\mathcal{V}_{FSI}$ has the one-sided amalgamation property and surjective epimorphisms.*

These results do not provide a transfer theorem for the SAP, however. To obtain such a theorem, at least in one direction, we make use of a well-known characterization of the SE property. Let $\mathcal{K}$ be a class of similar algebras and consider any $\mathbf{A} \in \mathcal{K}$. We say that $\mathbf{A}$ is a $\mathcal{K}$-*epic subalgebra* of $\mathbf{B} \in \mathcal{K}$ if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and for every $\mathbf{C} \in \mathcal{K}$ and all homomorphisms $\psi_1, \psi_2 \colon \mathbf{B} \to \mathbf{C}$, if ψ_1 and ψ_2 are equal on their restriction to A, then $\psi_1 = \psi_2$. It is easy to see that $\mathbf{A}$ is a $\mathcal{K}$-epic subalgebra of $\mathbf{B}$ if and only if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and the inclusion homomorphism of $\mathbf{A}$ into $\mathbf{B}$ is an epimorphism in $\mathcal{K}$. The following result is folklore (see, e.g., [32, Lemma 3.1]), but we include a proof for completeness.

Lemma 4.7. *Let $\mathcal{K}$ be a class of similar algebras closed under subalgebras. Then $\mathcal{K}$ has surjective epimorphisms if and only if no member of $\mathcal{K}$ has a proper $\mathcal{K}$-epic subalgebra.*

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and suppose that $\varphi \colon \mathbf{A} \to \mathbf{B}$ is a non-surjective $\mathcal{K}$-epimorphism. Then $\varphi[\mathbf{A}]$ is a proper subalgebra of $\mathbf{B}$. Observe that if $\mathbf{C} \in \mathcal{K}$ and $\psi_1, \psi_2 \colon \mathbf{B} \to \mathbf{C}$ are homomorphisms that coincide on their restriction to $\varphi[A]$, then $\psi_1\varphi = \psi_2\varphi$ and hence $\psi_1 = \psi_2$ since φ is an epimorphism. It follows that $\varphi[\mathbf{A}]$ is a proper $\mathcal{K}$-epic subalgebra of $\mathbf{B}$.

For the converse, let $\mathbf{B} \in \mathcal{K}$ and suppose that $\mathbf{A}$ is a proper $\mathcal{K}$-epic subalgebra of $\mathbf{B}$. Then $\mathbf{A} \in \mathcal{K}$ since $\mathcal{K}$ is closed under subalgebras. Hence the inclusion map of $\mathbf{A}$ into $\mathbf{B}$ is a non-surjective $\mathcal{K}$-epimorphism between members of $\mathcal{K}$. $\square$

The following theorem strengthens a result from [16].

Theorem 4.8. *Let $\mathcal{V}$ be an arithmetical variety with the congruence extension property such that $\mathcal{V}_{FSI}$ is a universal class. If every doubly injective span in $\mathcal{V}_{FSI}$ has a strong amalgam in $\mathcal{V}$, then $\mathcal{V}$ has the strong amalgamation property.*

Proof. Suppose that every doubly injective span in $\mathcal{V}_{\mathrm{FSI}}$ has a strong amalgam in $\mathcal{V}$. Then $\mathcal{V}$ has the AP, by Corollary 3.5. Hence, by Theorems 4.4 and 4.5, it is enough to show that $\mathcal{V}_{\mathrm{FSI}}$ has SE, or, equivalently, by Lemma 4.7, that no member of $\mathcal{V}_{\mathrm{FSI}}$ has a proper $\mathcal{V}_{\mathrm{FSI}}$-epic subalgebra. Let $\mathbf{B} \in \mathcal{V}_{\mathrm{FSI}}$, let $\mathbf{A}$ be a proper subalgebra of $\mathbf{B}$, and let $\iota \colon \mathbf{A} \to \mathbf{B}$ be the inclusion map. By assumption, the span $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}, \iota, \iota \rangle$ has a strong amalgam $\langle \mathbf{D}, \psi_1, \psi_2 \rangle$ in $\mathcal{V}$. Let $\chi \colon \mathbf{D} \to \prod_{i \in I} \mathbf{D}_i$ be a subdirect representation of $\mathbf{D}$, so that in particular $\mathbf{D}_i \in \mathcal{V}_{\mathrm{FSI}}$ for each $i \in I$. Since $\mathbf{D}$ is a strong amalgam, there exists an $i \in I$ such that $\chi\psi_1(i) \neq \chi\psi_2(i)$. On the other hand, since $\mathbf{D}$ is an amalgam, it follows that $\chi\psi_1\iota = \chi\psi_2\iota$. This implies that ι is not a $\mathcal{K}$-epimorphism, proving the theorem. $\square$

5. Decidability

To fix terminology, let us call a variety *finitely generated* if it is generated as a variety by some given finite set of finite algebras of finite signature, and *residually small* if there exists a bound on the size of its subdirectly irreducible members. It is known that a residually small congruence-distributive variety that has the AP also has the CEP [25, Corollary 2.11].

Consider any finitely generated congruence-distributive variety $\mathcal{V}$ such that $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras. By Jónsson's Lemma [24], there exists and can be constructed a finite set $\mathcal{V}_{\mathrm{FSI}}^* \subseteq \mathcal{V}_{\mathrm{FSI}}$ of finite algebras such that each $\mathbf{A} \in \mathcal{V}_{\mathrm{FSI}}$ is isomorphic to some $\mathbf{A}^* \in \mathcal{V}_{\mathrm{FSI}}^*$. Hence, by Corollary 2.4, it can be decided if $\mathcal{V}$ has the CEP by checking if each member of $\mathcal{V}_{\mathrm{FSI}}^*$ has the CEP. Since $\mathcal{V}$ is clearly residually small, if $\mathcal{V}$ does not have the CEP, it cannot have the AP. Otherwise, $\mathcal{V}$ has the CEP and, by Corollary 3.5, it can be decided if $\mathcal{V}$ has the AP by checking — by considering the finitely many finite algebras in $\mathcal{V}_{\mathrm{FSI}}^*$ — if $\mathcal{V}_{\mathrm{FSI}}$ has the 1AP. Finally, if $\mathcal{V}$ is also arithmetical, to check if $\mathcal{V}$ has SE, it suffices to check — again, considering the algebras in $\mathcal{V}_{\mathrm{FSI}}^*$ — if $\mathcal{V}_{\mathrm{FSI}}$ has SE [6, Theorem 22], and $\mathcal{V}$ then has the SAP if and only if it has SE and the AP [22]. Hence we have established the following result.

Theorem 5.1. *Let $\mathcal{V}$ be a finitely generated congruence-distributive variety such that $\mathcal{V}_{FSI}$ is closed under subalgebras. There exist effective algorithms to decide if $\mathcal{V}$ has the congruence extension property, amalgamation property, or transferable injections property. If $\mathcal{V}$ is also arithmetical, then there exist effective algorithms to decide if $\mathcal{V}$ has surjective epimorphisms or the strong amalgamation property.*

6. A case study: varieties of BL-algebras

BL-algebras, introduced by Hájek in [20] as an algebraic semantics for his basic fuzzy logic of continuous t-norms, have been studied intensively over the past twenty five years, largely in the framework of substructural logics and residuated lattices (see, e.g., [1,2,14,15,23,30,31]). In this section, we use the tools developed in previous sections to contribute to the development of a (still incomplete) description of the varieties of BL-algebras that have the AP.

A *BL-algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ satisfying

(i) $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with order $a \le b :\Longleftrightarrow a \wedge b = a$;
(ii) $\langle A, \cdot, 1\rangle$ is a commutative monoid;
(iii) $\to$ is the residual of $\cdot$, i.e., $a \cdot b \le c \iff b \le a \to c$ for all $a, b, c \in A$;
(iv) $a \wedge b = a \cdot (a \to b)$ and $(a \to b) \vee (b \to a) = 1$ for all $a, b \in A$.

The class of BL-algebras forms a congruence-distributive variety $\mathcal{BL}$ with the CEP, and $\mathcal{BL}_{\mathrm{FSI}}$ is the positive universal class consisting of all totally ordered BL-algebras (i.e., such that $\langle A, \le\rangle$ in the preceding definition is a chain). BL-algebras hence form a subvariety of the variety of semilinear integral bounded residuated lattices (also known as MTL-algebras) [13,17].

Notable subvarieties of $\mathcal{BL}$ include (i) the variety $\mathcal{MV}$ of *MV-algebras* consisting of BL-algebras satisfying $a \vee b = (a \to b) \to b$ for all $a, b \in A$; (ii) the variety $\mathcal{G}$ of *Gödel algebras* consisting of BL-algebras satisfying $a \wedge b = a \cdot b$ for all $a, b \in A$; (iii) the variety $\mathcal{P}$ of *product algebras* consisting of BL-algebras satisfying $a \wedge (a \to 0) = 0$ and $(a \to 0) \vee ((a \to (a \cdot b)) \to b) = 1$ for all $a, b \in A$. The varieties $\mathcal{MV}$, $\mathcal{G}$, and $\mathcal{P}$ are generated by algebras $\mathbf{Ł}$, $\mathbf{G}$, and $\mathbf{P}$ of the form $\langle [0,1], \min, \max, \star, \to_\star, 0, 1\rangle$, where $x \star y$ is $\max(0, x+y-1)$, $\min(x,y)$, and xy, respectively. Every member of $\mathcal{BL}_{\text{FSI}}$ (i.e., every totally ordered BL-algebra) can be constructed as a certain ordinal sum of members of $\mathcal{MV}_{\text{FSI}}$ and $\mathcal{P}_{\text{FSI}}$ [1, Theorem 3.7].

Let us briefly recall some further relevant facts about $\mathcal{MV}$, $\mathcal{G}$, and $\mathcal{P}$. First, given any totally ordered Abelian group $\mathbf{L} = \langle L, +, -, 0, \leq\rangle$ and $u \in L$ with $u \geq 0$, defining $a \cdot b := \max(a + b - u, 0)$ and $a \to b := \min(u - a + b, u)$ yields a totally ordered MV-algebra $\Gamma(\mathbf{L}, u) := \langle [0, u], \min, \max, \cdot, \to, 0, u\rangle$. Each proper non-trivial subvariety of $\mathcal{MV}$ is generated by a non-empty finite set of algebras of the form $\mathbf{S}_n := \Gamma(\mathbf{Z}, n)$ or $\mathbf{S}_n^\omega := \Gamma(\mathbf{Z} \times_{\text{lex}} \mathbf{Z}, \langle n, 0\rangle)$, where $\mathbf{Z}$ is the ordered group of integers, $\mathbf{Z} \times_{\text{lex}} \mathbf{Z}$ is the lexicographic product of two copies of $\mathbf{Z}$, and $n \in \mathbb{N}^{>0}$ [28, Theorem 4.11]. Notably, $\mathbf{S}_1$ generates the variety $\mathcal{BA}$ of Boolean algebras, and $\mathbf{S}_1^\omega$, known as the Chang algebra, generates a variety denoted by $\mathcal{C}$. Each proper non-trivial subvariety $\mathcal{G}_n$ of $\mathcal{G}$ is generated by the algebra $\mathbf{G}_n := \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \min, \max, \min, \to, 0, 1\rangle$, where $n \in \mathbb{N}^{>0}$ and $a \to b = 1$ if $a \leq b$, otherwise $a \to b = b$. Finally, the only proper non-trivial subvariety of $\mathcal{P}$ is $\mathcal{BA}$ (which coincides with $\mathcal{G}_2$).

The non-trivial subvarieties of $\mathcal{MV}$, $\mathcal{G}$, and $\mathcal{P}$ that have the AP are precisely the varieties generated by $\mathbf{S}_n$ ($n \in \mathbb{N}^{>0}$), $\mathbf{S}_n^\omega$ ($n \in \mathbb{N}^{>0}$), $\mathcal{MV}$, $\mathcal{G}$, $\mathcal{G}_3$, and $\mathcal{P}$ [11, Theorem 13]. Note also that $\mathcal{BL}$ has the AP [31, Theorem 3.7], and [2, Theorem 6] gives a complete description of the varieties of BL-algebras having the AP that are generated by a totally ordered BL-algebra built as an ordinal sum of finitely many members of $\mathcal{MV}_{\text{FSI}}$ and $\mathcal{P}_{\text{FSI}}$. Here, we consider subvarieties of $\mathcal{BL}_1 := \mathcal{MV} \vee \mathcal{G} \vee \mathcal{P}$, each of which is generated by a class of "one-component" totally ordered BL-algebras, i.e., members of $\mathcal{MV}_{\text{FSI}}$, $\mathcal{G}_{\text{FSI}}$, and $\mathcal{P}_{\text{FSI}}$.

Lemma 6.1. *Let $\mathcal{V}$ be a subvariety of $\mathcal{BL}_1$ satisfying $\mathcal{V} \not\subseteq \mathcal{MV}$ and $\mathbf{S}_n \in \mathcal{V}$ for some $n \in \mathbb{N}^{>1}$. Then $\mathcal{V}$ does not have the amalgamation property.*

Proof. Consider a doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C\rangle$ in $\mathcal{V}_{\text{FSI}}$, where $\mathbf{A}$ is the two-element Boolean algebra, $\mathbf{B}$ is $\mathbf{S}_n \in \mathcal{V}$ for some $n \in \mathbb{N}^{>1}$, and $\mathbf{C} \notin \mathcal{MV}$. Suppose that there exist a $\mathbf{D} \in \mathcal{V}_{\text{FSI}}$ and an embedding ψ_C of $\mathbf{C}$ into $\mathbf{D}$. Then also $\mathbf{D} \notin \mathcal{MV}$, so $\mathbf{D} \in \mathcal{G}_{\text{FSI}} \cup \mathcal{P}_{\text{FSI}}$. Since $\mathbf{B}$ is a finite totally ordered MV-algebra, it is simple. Hence any homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$ is either trivial (i.e., maps all elements of $\mathbf{B}$ to one element in $\mathbf{D}$), which is not possible, since $\mathbf{D}$ is non-trivial, or injective, which is not possible, since $\mathbf{B}$ does not embed into any totally ordered Gödel algebra or product algebra. So

$\mathcal{V}_{\mathrm{FSI}}$ does not have the 1AP and it follows, by Theorem 3.4, that $\mathcal{V}$ does not have the AP. □

Theorem 6.2. *In addition to the varieties of MV-algebras generated by one totally ordered MV-algebra, there are precisely ten non-trivial subvarieties of $\mathcal{BL}_1$ that have the amalgamation property: $\mathcal{G}$, $\mathcal{G}_3$, $\mathcal{P}$, $\mathcal{G} \vee \mathcal{P}$, $\mathcal{G} \vee \mathcal{C}$, $\mathcal{G}_3 \vee \mathcal{P}$, $\mathcal{G}_3 \vee \mathcal{C}$, $\mathcal{P} \vee \mathcal{C}$, $\mathcal{G}_3 \vee \mathcal{P} \vee \mathcal{C}$, and $\mathcal{G} \vee \mathcal{P} \vee \mathcal{C}$.*

Proof. Let $\mathcal{V}$ be any non-trivial subvariety of $\mathcal{BL}_1$. Then $\mathcal{V}_{\mathrm{FSI}}$ consists of members of $\mathcal{G}_{\mathrm{FSI}}$, $\mathcal{MV}_{\mathrm{FSI}}$, and $\mathcal{P}_{\mathrm{FSI}}$. The cases where $\mathcal{V}_{\mathrm{FSI}}$ is included in one of these classes are clear from the previous remarks. Moreover, it follows from [2, Theorem 3.3] that if $\mathcal{V}$ contains $\mathbf{G}_n$ for some $n > 3$ but not $\mathbf{G}$, then $\mathcal{V}$ does not have the AP. Suppose now that $\mathcal{V} \nsubseteq \mathcal{MV}$ and $\mathcal{V} \cap \mathcal{MV}$ is non-trivial. Then $\mathcal{V} \cap \mathcal{MV}$ is either $\mathcal{MV}$ or generated by a non-empty finite set of algebras of the form $\mathbf{S}_n$ or $\mathbf{S}_n^\omega$ ($n \in \mathbb{N}^{>0}$). If $\mathcal{V} \cap \mathcal{MV} \notin \{\mathcal{G}_2, \mathcal{C}\}$, then, since $\mathbf{S}_n$ is a subalgebra of $\mathbf{S}_n^\omega$ for each $n \in \mathbb{N}$, it follows that $\mathbf{S}_n \in \mathcal{V}$ for some $n \in \mathbb{N}^{>1}$, and hence, by Lemma 6.1, that $\mathcal{V}$ does not have the AP.

It remains therefore to show that each of $\mathcal{G} \vee \mathcal{P}$, $\mathcal{G} \vee \mathcal{C}$, $\mathcal{G}_3 \vee \mathcal{P}$, $\mathcal{G}_3 \vee \mathcal{C}$, $\mathcal{P} \vee \mathcal{C}$, $\mathcal{G}_3 \vee \mathcal{P} \vee \mathcal{C}$, and $\mathcal{G} \vee \mathcal{P} \vee \mathcal{C}$ has the AP. Clearly, the two-element Boolean algebra $\mathbf{G}_2$ is the only non-trivial algebra common to the finitely subdirectly irreducible members of the varieties in these joins. Hence, by Proposition 3.7, it suffices to observe that $\mathbf{G}_2$ is a retract of any non-trivial member of $\mathcal{G}_{\mathrm{FSI}}$, $\mathcal{P}_{\mathrm{FSI}}$, and $\mathcal{C}_{\mathrm{FSI}}$. This follows directly from a general result of [7, Theorem 4.5] describing BL-algebras with a Boolean retract, but we may also define a suitable retraction explicitly. Given any non-trivial $\mathbf{A} \in \mathcal{G}_{\mathrm{FSI}} \cup \mathcal{P}_{\mathrm{FSI}} \cup \mathcal{C}_{\mathrm{FSI}}$, define $\varphi\colon A \to \{0, 1\}$ by mapping $a \in A$ to 0 if and only if $a^n = 0$ for some $n \in \mathbb{N}$, where $a^0 := 1$ and $a^{k+1} := a^k \cdot a$ ($k \in \mathbb{N}$). It is straightforward to verify that φ is a retraction from $\mathbf{A}$ onto $\mathbf{G}_2$. □

Data availability

No data was used for the research described in the article.

Acknowledgment

This research was supported by Swiss National Science Foundation grant 200021_165850. We would also like to thank the anonymous referee for several helpful suggestions that have led to significant improvements to the paper.

References

[1] P. Aglianò, F. Montagna, Varieties of BL-algebras I: general properties, J. Pure Appl. Algebra 181 (2003) 105–129.

[2] S. Aguzzoli, M. Bianchi, Amalgamation property for varieties of BL-algebras generated by one chain with finitely many components, in: Proc. 19th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 2021), 2021, pp. 1–18.

[3] P.D. Bacsich, Injectivity in model theory, Colloq. Math. 25 (1972) 165–176.
[4] W.J. Blok, D. Pigozzi, On the congruence extension property, Algebra Univers. 38 (1997) 391–394.
[5] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer, 1981.
[6] M. Campercholi, Dominions and primitive positive functions, J. Symb. Log. 83 (2018) 40–54.
[7] R. Cignoli, A. Torrens, Free algebras in varieties of BL-algebras with a Boolean retract, Algebra Univers. 48 (2002) 55–79.
[8] J. Czelakowski, W. Dziobiak, Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class, Algebra Univers. 27 (1990) 128–149.
[9] J. Czelakowski, D. Pigozzi, Amalgamation and interpolation in abstract algebraic logic, in: Models, Algebras, and Proofs, 1999, pp. 187–265.
[10] B.A. Davey, Weak injectivity and congruence extension in congruence-distributive equational classes, Can. J. Math. 29 (3) (1977) 449–459.
[11] A. Di Nola, A. Lettieri, One chain generated varieties of MV-algebras, J. Algebra 225 (2) (2000) 667–697.
[12] W. Dziobiak, Finitely generated congruence distributive quasivarieties of algebras, Fundam. Math. 133 (1989) 47–58.
[13] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets Syst. 124 (2001) 271–288.
[14] W. Fussner, Poset products as relational models, Stud. Log. 110 (2022) 95–120.
[15] W. Fussner, W. Zuluaga Botero, Some modal and temporal translations of generalized basic logic, in: Proc. 19th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 2021), 2021, pp. 176–191.
[16] W. Fussner, N. Galatos, Semiconic idempotent logic II: Beth definability and deductive interpolation, available to download from https://arxiv.org/abs/2208.09724v3, 2023.
[17] W. Fussner, S. Ugolini, A topological approach to MTL-algebras, Algebra Univers. 80 (2019) 38.
[18] V.A. Gorbunov, Algebraic Theory of Quasivarieties, Springer, 1998.
[19] G. Grätzer, H. Lakser, The structure of pseudocomplemented distributive lattices II: congruence extension and amalgamation, Trans. Am. Math. Soc. 156 (1971) 343–358.
[20] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, 1998.
[21] W. Hodges, A Shorter Model Theory, Cambridge University Press, Cambridge, 1997.
[22] J.R. Isbell, Epimorphisms and dominions, in: Proceedings of the Conference on Categorical Algebra, 1965, pp. 232–246.
[23] P. Jipsen, F. Montagna, Embedding theorems for classes of GBL-algebras, J. Pure Appl. Algebra 214 (9) (2010) 1559–1575.
[24] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967) 110–121 (1968).
[25] K.A. Kearnes, On the relationship between AP, RS and CEP, Proc. Am. Math. Soc. 105 (4) (1989) 827–839.
[26] E.W. Kiss, Injectivity and related concepts in modular varieties II: the congruence extension property, Bull. Aust. Math. Soc. 32 (1985) 45–53.
[27] E.W. Kiss, L. Márki, P. Pröhle, W. Tholen, Categorical algebraic properties. A compendium of amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity, Studia Sci. Math. Hung. 18 (1983) 79–141.
[28] Y. Komori, Super Łukasiewicz propositional logics, Nagoya Math. J. 84 (1981) 119–133.
[29] G. Metcalfe, F. Montagna, C. Tsinakis, Amalgamation and interpolation in ordered algebras, J. Algebra 402 (2014) 21–82.
[30] G. Metcalfe, F. Paoli, C. Tsinakis, Residuated Structures in Algebra and Logic, Mathematical Surveys and Monographs, vol. 277, American Mathematical Society, 2023.
[31] F. Montagna, Interpolation and Beth's property in many-valued logic: a semantic investigation, Ann. Pure Appl. Log. 141 (2006) 148–179.
[32] T. Moraschini, J.J. Wannenburg, Epimorphism surjectivity in varieties of Heyting algebras, Ann. Pure Appl. Log. 171 (2020).
[33] S. Santschi, Semilinear idempotent distributive l-monoids, available to download from https://arxiv.org/abs/2210.16576, 2022.