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Maximal sectorial operators and invariant operator ranges [☆]

Yury Arlinskiĭ [a], Christiane Tretter [b,*]

[a] *Volodymyr Dahl East Ukrainian National University, Ukraine*
[b] *Mathematisches Institut, Universität Bern, Sidlerstr. 5, 3012, Bern, Switzerland*

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For unbounded maximal sectorial operators we establish necessary and sufficient conditions for the domain equality $\operatorname{dom} A = \operatorname{dom} A^*$ and for the equality $\operatorname{Re} A = A_R$ of operator real part $\operatorname{Re} A$ and form real part A_R. Here $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ is half of the operator sum defined on $\operatorname{dom} A \cap \operatorname{dom} A^*$, whereas $A_R = \frac{1}{2}(A \dot{+} A^*)$ is the self-adjoint operator given by half of the form-sum of A and A^* so that, in general, $\operatorname{Re} A \subseteq A_R$. The natural question posed in [6], *whether for a maximal sectorial operator A the equality $\operatorname{dom} A = \operatorname{dom} A^*$ implies the equality $\operatorname{Re} A = A_R$*, is answered negatively in this paper. We construct families of unbounded coercive m-sectorial operators A such that $\operatorname{dom} A = \operatorname{dom} A^*$ for which $\operatorname{Re} A$ is a closed symmetric non-selfadjoint operator or a non-closed essentially selfadjoint operator. Moreover, we show that the domain equalities $\operatorname{dom} A = \operatorname{dom} A^*$ and $\operatorname{dom} \operatorname{Re} A = \operatorname{dom} A_R$ are equivalent to problems of invariant operator ranges of bounded selfadjoint or unitary operators as well as to the existence of bounded operators with specific operator range properties.

Contents

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* Corresponding author.
E-mail addresses: yury.arlinskii@gmail.com (Y. Arlinskiĭ), tretter@math.unibe.ch (C. Tretter).

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1. Introduction

In contrast to bounded operators the decomposition of an unbounded operator A in a Hilbert space into real and imaginary part may, in general, pose serious problems. For example, it may happen that $\operatorname{dom} A \cap \operatorname{dom} A^* = \{0\}$ ruling out the obvious candidate $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ or the operator A may not be sectorial so that the next obvious candidate $A_R = \frac{1}{2}(A\dot{+}A^*)$ taken in form sum sense may not be available either. However, even if both $\operatorname{Re} A$ and A_R can be formed and hence $\operatorname{Re} A \subseteq A_R$, the problem whether, for maximal sectorial A, the equality $\operatorname{dom} A = \operatorname{dom} A^*$ implies the equality $\operatorname{Re} A = A_R$, see [6], has remained open.

In this paper we give a negative answer to this question. In addition, for an unbounded maximal sectorial (m-sectorial, for short) operator A we establish various equivalent conditions that guarantee the inclusion $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ as well as the equalities $\operatorname{dom} A = \operatorname{dom} A^*$ and $\operatorname{Re} A = A_R$. Here the *operator real part* $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ is defined as the operator sum on $\operatorname{dom} \operatorname{Re} A = \operatorname{dom} A \cap \operatorname{dom} A^*$, while *the form real part* $A_R = \frac{1}{2}(A\dot{+}A^*)$ is defined via the form-sum of the operators A and A^*, i.e. A_R is associated with the real part of the closed sesquilinear form induced by A, see [28, Sect. VI.3.1]. The operator A_R is always selfadjoint (and nonnegative) and, in general, $\operatorname{Re} A \subseteq A_R$.

Only recently, in [6], it was proved that the domain intersections $\operatorname{dom} A \cap \operatorname{dom} A^*$ may be everything in between being $\{0\}$ and being dense but no core of A, and that these properties may occur for nice operator classes such as m-sectorial ones. In particular, in [6, Sect. 3, 5] we showed that for any $n \in \mathbb{N} \cup \{0\}$ there is an m-sectorial operator A such that $\dim(\operatorname{dom} A \cap \operatorname{dom} A^*) = n$ or $\operatorname{codim}(\operatorname{dom} A \cap \operatorname{dom} A^*) = n$, respectively. Further, in [6, Sect. 7] we constructed examples of m-sectorial operators A such that $\operatorname{dom} A = \operatorname{dom} A^*$. We also proved that in all these examples the equality $\operatorname{dom} A_R = \operatorname{dom} A$ or, equivalently, $\operatorname{Re} A = A_R$, holds true. Moreover, in [1], it was shown how to construct an unbounded operator S with nonempty resolvent set and $\operatorname{dom} S = \operatorname{dom} S^*$ such that the operators $\operatorname{Re} S$, $\operatorname{Im} S = \frac{1}{2\mathrm{i}}(S - S^*)$ are symmetric with prescribed deficiency indices; in the corresponding construction in [1] the operator S is neither accretive nor dissipative. On the other hand, it is known, see [25], that if A is a maximal accretive (m-accretive, for short) operator, then

- for $\gamma \in [0, 1)$ the m-accretive power A^γ can be defined and A^γ is m-sectorial with semi-angle $\pi\gamma/2$,
- for $\gamma < 1/2$ we have $\operatorname{dom} A^\gamma = \operatorname{dom} A^{*\gamma} = \operatorname{dom}(A^\gamma)_R$, while for $\gamma = 1/2$ we have $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} A^{*\frac{1}{2}} = \operatorname{dom}(A^{\frac{1}{2}})_R$ and hence $\operatorname{Re} A^{\frac{1}{2}} = (A^{\frac{1}{2}})_R$.

Consequently, for m-sectorial A with semi-angle $\alpha \le \pi/4$, the following are true:

a) if A^2 is accretive, then $\operatorname{Re} A = A_R$;
b) if $\operatorname{Re} A \ne A_R$, then A^2 is not accretive.

Moreover, in [31] McIntosh constructed an abstract example of an unbounded m-sectorial operator B with $\operatorname{dom} B^{\frac{1}{2}} \ne \operatorname{dom} B^{*\frac{1}{2}}$; other abstract examples may be found in [4]. On the other hand, the equality $\operatorname{dom} B^{\frac{1}{2}} = \operatorname{dom} B^{*\frac{1}{2}}$ was established for some classes of m-sectorial second order elliptic differential and differential-difference operators B in [8], [9], [10], [23], [38], see also the references therein, hence $\operatorname{dom}(B^{\frac{1}{2}})_R = \operatorname{dom}(B_R)^{\frac{1}{2}}$ and $\operatorname{Re} B^{\frac{1}{2}} = (B^{\frac{1}{2}})_R$ for these operators. Note that due to Kato's result [27, Cor. 2] for m-sectorial operators B, the equality $\operatorname{dom} B = \operatorname{dom} B^*$ implies the equalities

$$\operatorname{dom} B^{\frac{1}{2}} = \operatorname{dom} B^{*\frac{1}{2}} = \operatorname{dom}(B_R)^{\frac{1}{2}}.$$

In general, even in the case when $\operatorname{dom} A \cap \operatorname{dom} A^*$ is dense, it is possible that $\operatorname{Re} A \neq A_R$. For example, set

$$\mathcal{A}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \operatorname{dom} \mathcal{A}_0 = \left\{ f \in H^2(\mathbb{R}_+) : f'(0) = f(0) = 0 \right\},$$

$$\mathcal{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \operatorname{dom} \mathcal{A} = \left\{ f \in H^2(\mathbb{R}_+) : f'(0) = hf(0) \right\} \text{ with } h \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{Re} h > 0.$$

Then the operator $\mathcal{A}_0$ is densely defined, nonnegative and symmetric with deficiency indices $\langle 1, 1 \rangle$ in the Hilbert space $L^2(\mathbb{R}_+)$, while $\mathcal{A}$ is m-sectorial, see [5], and

$$\mathcal{A}^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \operatorname{dom} \mathcal{A}^* = \left\{ f \in H^2(\mathbb{R}_+) : f'(0) = \overline{h} f(0) \right\}.$$

Clearly, $\operatorname{Re} \mathcal{A} = \mathcal{A}_0$. Hence $\operatorname{Re} \mathcal{A} \subsetneqq \mathcal{A}_R$ since $\mathcal{A}_R$ is selfadjoint.

In this paper we prove that, for coercive m-sectorial A, acting in a Hilbert space $\mathfrak{H}$, the range inclusion $\operatorname{ran}(\operatorname{Im} A^{-1}) \subseteq \operatorname{ran}(\operatorname{Re} A^{-1})$ is necessary and sufficient for the equality $\operatorname{Re} A = A_R$, see Theorem 3.11. Moreover, we show that if for an m-accretive operator A having bounded inverse the above range inclusion is fulfilled, then A is m-sectorial, and the equality $\operatorname{dom} A = \operatorname{dom} A^*$ holds if and only if $\operatorname{ran}\left(I + ((\operatorname{Re} A^{-1})^{-1}(\operatorname{Im} A^{-1}))^2\right) = \mathfrak{H}$, see Theorem 3.13. We also construct holomorphic families $A(\lambda)$, $\operatorname{Re} \lambda > 0$, of m-sectorial operators with the property $\operatorname{Re}(A(\lambda)) = (A(\lambda))_R$, see Theorem 3.16.

Moreover, we establish several equivalent necessary and sufficient conditions for the equalities $\operatorname{dom} A = \operatorname{dom} A^*$ and $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$ for m-sectorial A, see Theorem 3.2, Corollary 3.3, Corollary 3.4 and Theorem 3.11, and we give new abstract examples of operators for which $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$; here we will often assume that $0 \in \rho(A)$ or that A is coercive which simplifies the treatments.

Besides, for a given bounded nonnegative selfadjoint operator Q with dense range we construct, see Theorem 3.16, Corollary 4.4, Theorem 4.8, holomorphic families of m-sectorial operators $A(\lambda)$, $\operatorname{Re} \lambda > 0$, such that

$$\operatorname{dom} A(\lambda) = \operatorname{dom} A(\lambda)^* = \operatorname{dom} A(\lambda)_R = \operatorname{ran} Q, \quad \operatorname{dom}(A(\lambda)_R)^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}},$$

and others such that

$$\operatorname{dom} A(\lambda) \neq \operatorname{dom} A(\lambda)^*, \quad \operatorname{Re} A(\lambda) = A(\lambda)_R, \quad \operatorname{dom}(A(\lambda)_R)^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}.$$

Further, we solve the problem formulated in our paper [6]: *does the equality* $\operatorname{dom} A = \operatorname{dom} A^*$ *imply the equality* $\operatorname{Re} A = A_R$ *for an m-sectorial operator A?* To this end, we provide an abstract construction of families of coercive m-sectorial dissipative operators A such that

$$\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{ran} Q, \quad \operatorname{dom}(A_R)^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}},$$

but the real part $\operatorname{Re} A$ is

- a closed symmetric operator with arbitrary defect number $n \in \mathbb{N} \cup \{\infty\}$, in which case A_R is the Friedrichs extension of $\operatorname{Re} A$;
- a non-closed essentially selfadjoint operator, in which case A_R is the closure of $\operatorname{Re} A$,

see Theorem 4.9 and Corollary 4.13.

Finally, we show that the above mentioned problems of domain equalities are equivalent to problems concerning invariant operator ranges of bounded selfadjoint or unitary operators and to the existence of bounded operators with specific operator range properties.

Throughout this paper we assume that all Hilbert spaces are separable and we use the following notations. The symbols $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ denote the domain, range and kernel of a linear operator T, respectively, and we write $\overline{\operatorname{ran}}\, T := \overline{\operatorname{ran} T}$ for the closure of the range of T. For a contraction K we will use the notation $D_K := (I - K^*K)^{\frac{1}{2}}$ for the unique nonnegative square root of the nonnegative operator $I - K^*K$. The Banach space of all bounded operators acting in a Hilbert space $\mathcal{H}$ is denoted by $\mathbf{B}(\mathcal{H})$; the cone of all bounded self-adjoint nonnegative operators in a complex Hilbert space $\mathcal{H}$ is denoted by $\mathbf{B}^+(\mathcal{H})$. The spectrum and resolvent set of a linear operator T are denoted by $\sigma(T)$ and $\rho(T)$, respectively. If $\mathfrak{L}$ is a subspace, i.e. a closed linear manifold of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T{\upharpoonright}\mathcal{N}$ means the restriction of a linear operator T to the linear manifold $\mathcal{N} \subset \operatorname{dom} T$. The open right/left complex half-plane are denoted by $\mathbb{C}_\pm := \{z \in \mathbb{C} : \operatorname{Re} z \gtrless 0\}$. For an interval $I \subseteq (-\pi, \pi)$ we denote $\mathcal{S}_I := \{z \in \mathbb{C} \setminus \{0\} : \arg z \in I\}$.

2. Preliminaries

2.1. Maximal sectorial operators

A linear operator A with domain $\operatorname{dom} A$ in a Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$ is called *accretive* if its numerical range, see e.g. [20], $W(A) = \{(Au,u)\colon u \in \operatorname{dom} A,\ \|u\| = 1\}$ lies in the closed right half-plane, i.e. $W(A) \subset \overline{\mathbb{C}_+}$; it is called *maximal accretive*, if it is densely defined and $\mathbb{C}_- \cap \rho(A) \neq \emptyset$ or, equivalently, A is densely defined, closed and A^* is accretive, see [25], [28, Sect. V.3.10]. If A is m-accretive, then $\ker A = \ker A^*$ and hence $\ker A \subseteq \operatorname{dom} A \cap \operatorname{dom} A^*$. An accretive operator A is called *coercive* if there exists $m > 0$ with $\operatorname{Re}(Af, f) \geq m(f, f)$, $f \in \operatorname{dom} A$.

An accretive operator A is called *sectorial* with vertex $z = 0$ and semi-angle $\alpha \in [0, \pi/2)$, or *α-sectorial* for short, if $W(A)$ lies in the closed sector with vertex 0 and semi-angle α, i.e.

$$W(A) \subset \overline{\mathcal{S}_{(-\alpha,\alpha)}}, \quad \mathcal{S}_{(-\alpha,\alpha)} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\},$$

see [25], [27], [28, Sect. V.3.10], [21], [37], [40] and *maximal α-sectorial*, or *m-α-sectorial* for short, if it is m-accretive.

The closed sesquilinear form associated with an m-sectorial operator A is denoted by $A[\cdot,\cdot]$ and its domain by $\mathcal{D}[A]$; note that $A[\cdot,\cdot]$ is the closure of the form $\mathfrak{a}[f,g] := (Af, g)$, $f, g \in \operatorname{dom} A$, see [28, Sect. VI.2.1]. By the second representation theorem,

$$\mathcal{D}[A] = \operatorname{dom}(A_R)^{\frac{1}{2}}, \quad \operatorname{Re} A[u,v] = ((A_R)^{\frac{1}{2}}u, (A_R)^{\frac{1}{2}}v), \quad u, v \in \mathcal{D}[A]. \tag{2.1}$$

Moreover, the form $A[\cdot,\cdot]$ and the operator A admit the canonical representations, see [28, Thm. VI.3.2],

$$A[u,v] = ((I + \mathrm{i}G)(A_R)^{\frac{1}{2}}u, (A_R)^{\frac{1}{2}}v), \quad u, v \in \mathcal{D}[A], \quad A = (A_R)^{\frac{1}{2}}(I + \mathrm{i}G)(A_R)^{\frac{1}{2}}, \tag{2.2}$$

where G is a bounded selfadjoint operator in the subspace $\overline{\operatorname{ran}}\,(A_R)^{\frac{1}{2}}$ and, if α is the semi-angle of A, then $\|G\| \leq \tan\alpha$.

In the sequel we introduce several operators associated with an m-sectorial operator A, the operator real part $\operatorname{Re} A$, the form real part A_R and the harmonic mean $h(A, A^*)$.

Given an m-α-sectorial operator A, by (2.2) there exist a bounded selfadjoint $G \in \mathfrak{H}$ with $\|G\| \leq \tan\alpha$ and a closed densely defined operator L such that

$$A = L^*(I + \mathrm{i}G)L$$

as a product of unbounded operators. Vice versa, if G is a bounded selfadjoint operator in $\mathfrak{H}$ and L is a closed densely defined operator, then the sesquilinear form

$$\mathfrak{a}[u,v] := ((I+\mathrm{i}G)Lu, Lv)\,, \quad u,v \in \operatorname{dom}\mathfrak{a} = \operatorname{dom} L, \tag{2.3}$$

is densely defined, closed and sectorial with adjoint form $\mathfrak{a}^*$ given by

$$\mathfrak{a}^*[\phi,\psi] = ((I-\mathrm{i}G)L\phi, L\psi), \quad \phi,\psi \in \operatorname{dom}\mathfrak{a}^* = \operatorname{dom} L. \tag{2.4}$$

By (2.3), (2.4) and the first representation theorem [28, Thm. VI.2.1], the associated m-sectorial operators A and A^* are given by

$$A = L^*(I+\mathrm{i}G)L, \quad \operatorname{dom} A = \{u \in \operatorname{dom} L : (I+\mathrm{i}G)Lu \in \operatorname{dom} L^*\}\,, \tag{2.5}$$

$$A^* = L^*(I-\mathrm{i}G)L, \quad \operatorname{dom} A^* = \{\phi \in \operatorname{dom} L : (I-\mathrm{i}G)L\phi \in \operatorname{dom} L^*\}\,. \tag{2.6}$$

These relations yield that the *operator real part* $\operatorname{Re} A := \frac{1}{2}(A+A^*)$ is given by

$$\operatorname{dom}\operatorname{Re} A = \operatorname{dom} A \cap \operatorname{dom} A^* = \{u \in \operatorname{dom} L^*L :\ GLu \in \operatorname{dom} L^*\}\,, \tag{2.7}$$

$$(\operatorname{Re} A)u = \frac{1}{2}(A+A^*)u = L^*Lu, \quad u \in \operatorname{dom} A \cap \operatorname{dom} A^*. \tag{2.8}$$

Note that a selfadjoint operator G is bounded in $\mathfrak{H}$ if and only if the unitary operator

$$U_G := (I-\mathrm{i}G)(I+\mathrm{i}G)^{-1} = -I + 2(I+\mathrm{i}G)^{-1} \tag{2.9}$$

in $\mathfrak{H}$ (the Cayley transform of G) satisfies $-1 \in \rho(U_G)$. Since $I+\mathrm{i}G = 2(I+U_G)^{-1}$, and by (2.5), we have

$$A = 2L^*(I+U_G)^{-1}L.$$

The *form real part* A_R of A is the nonnegative selfadjoint operator associated with the real part of $\mathfrak{a}$ given by $\operatorname{Re}\mathfrak{a} := (\mathfrak{a}+\mathfrak{a}^*)/2$ according to the second representation theorem, see [28, Thm. VI.2.23], i.e. A_R is half the form-sum (denoted by $\dotplus$) of A and A^*, see [28, Sect. VI.1],

$$A_R = \frac{1}{2}(A \dotplus A^*)$$

and

$$\operatorname{dom} A_R = \operatorname{dom} L^*L, \quad A_R = L^*L, \qquad \operatorname{dom} A_R^{\frac{1}{2}} = \operatorname{dom} L, \quad A_R^{\frac{1}{2}} = (L^*L)^{\frac{1}{2}}\,. \tag{2.10}$$

Clearly,

$$\operatorname{dom} A_R \supseteq \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom}\operatorname{Re} A, \quad A_R \supseteq \operatorname{Re} A. \tag{2.11}$$

While both the operator real part and the form real part of A are versions of the 'arithmetic mean' of A and A^*, the third nonnegative selfadjoint operator associated with an m-sectorial operator A is defined if $\ker A = \{0\}$, which implies that also $\ker A^* = \{0\}$. In this case, the inverses A^{-1} and $A^{-*} := (A^*)^{-1}$ exist and are possibly unbounded operators with domains $\operatorname{dom} A^{-1} = \operatorname{ran} A$ and $\operatorname{dom} A^{-*} = \operatorname{ran} A^*$. Then the '*harmonic mean*' of A and A^* is defined as

$$h(A,A^*) := \left((A^{-1})_R\right)^{-1} = \left(\frac{1}{2}\left(A^{-1}\dot{+}A^{-*}\right)\right)^{-1}.$$

Suppose that $L = L^*$, $\ker L = \{0\}$, and $A = L(I + \mathrm{i}G)L$. Since $\ker A = \ker L = \{0\}$ and

$$A^{-1} = L^{-1}(I + \mathrm{i}G)^{-1}L^{-1} = L^{-1}(I + G^2)^{-\frac{1}{2}}(I - \mathrm{i}G)(I + G^2)^{-\frac{1}{2}}L^{-1},$$

it follows that

$$(A^{-1})_R = L^{-1}(I + G^2)^{-1}L^{-1}, \quad ((A^{-1})_R)^{-1} = L(I + G^2)L.$$

Therefore

$$\begin{aligned} &\operatorname{dom} h(A,A^*)^{\frac{1}{2}} = \operatorname{dom} L, \\ &\left(h(A,A^*)^{\frac{1}{2}}u, h(A,A^*)^{\frac{1}{2}}v\right) = \left((I + G^2)^{\frac{1}{2}}Lu, (I + G^2)^{\frac{1}{2}}Lv\right), \quad u,v \in \operatorname{dom} L, \end{aligned} \tag{2.12}$$

and hence, by (2.10) and (2.12),

$$\begin{aligned} &\operatorname{dom}(A_R)^{\frac{1}{2}} = \operatorname{dom}(h(A,A^*))^{\frac{1}{2}}, \\ &A_R = \frac{1}{2}(A\dot{+}A^*) \le h(A,A^*) = ((A^{-1})_R)^{-1} \le \frac{1}{\cos^2\alpha}A_R; \end{aligned} \tag{2.13}$$

here, for the last estimate, we have used that $\|G\| \le \tan\alpha$ and so $||I + G^2|| \le 1 + \tan^2\alpha = 1/\cos^2\alpha$.

An m-sectorial operator A is coercive if and only if the operator A_R is positive definite, i.e. $(A_R f, f) \ge m(f,f)$, $f \in \operatorname{dom} A_R$ with $m > 0$. Then $0 \in \rho(A)$, $(A^{-1})_R = \operatorname{Re} A^{-1}$ and

$$\operatorname{dom} A_R^{\frac{1}{2}} = \operatorname{dom}((A^{-1})_R)^{-\frac{1}{2}} = \operatorname{dom}(\operatorname{Re}(A^{-1}))^{-\frac{1}{2}} = \operatorname{ran}(\operatorname{Re}(A^{-1}))^{\frac{1}{2}}. \tag{2.14}$$

If $B \in \mathbf{B}(\mathcal{H})$ is a bounded, accretive and coercive operator, then due to the estimate

$$|\operatorname{Im}(Bf,f)| \le \|B\|\|f\|^2 \le \|B\|\frac{1}{m}\operatorname{Re}(Bf,f), \quad f \in \mathcal{H},$$

the operator B is m-sectorial. If, further, L is a possibly unbounded, closed and densely defined operator acting between Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$ with $\ker L = \{0\}$, then the sesquilinear form

$$\mathfrak{a}[u,v] := (BLu, Lv), \quad \mathcal{D}[A] = \operatorname{dom} L,$$

is closed and sectorial. The associated m-sectorial operator A in $\mathfrak{H}$ and its adjoint A^* are given by, see [9, Prop. 1],

$$\begin{aligned} A &= L^*BL, & \operatorname{dom} A &= L^{-1}\left(\operatorname{ran} L \cap B^{-1}\operatorname{dom} L^*\right), \\ A^* &= L^*B^*L, & \operatorname{dom} A^* &= L^{-1}\left(\operatorname{ran} L \cap B^{-*}\operatorname{dom} L^*\right). \end{aligned} \tag{2.15}$$

Further, the real part of $\mathfrak{a}$ and the corresponding associated nonnegative selfadjoint operator A_R are given by

$$\begin{aligned} &\operatorname{Re}\mathfrak{a}[u,v] = ((\operatorname{Re} B)Lu, Lv), \quad u,v \in \operatorname{dom} L, \\ &A_R = L^*(\operatorname{Re} B)L, \quad \operatorname{dom} A_R = L^{-1}\left(\operatorname{ran} L \cap (\operatorname{Re} B)^{-1}\operatorname{dom} L^*\right). \end{aligned} \tag{2.16}$$

Note that if $0 \in \rho(L)$, we can write

$$\operatorname{dom} A = L^{-1}(B^{-1}\operatorname{dom} L^*), \ \operatorname{dom} A^* = L^{-1}(B^{-*}\operatorname{dom} L^*), \ \operatorname{dom} A_R = L^{-1}((\operatorname{Re} B)^{-1}\operatorname{dom} L^*). \tag{2.17}$$

2.2. The class $\widetilde{C}_{\mathfrak{H}}$

Let $\alpha \in (0, \pi/2)$. A linear operator T in Hilbert space $\mathfrak{H}$ defined everywhere is said to belong to the class $C_{\mathfrak{H}}(\alpha)$ if

$$\|T \sin \alpha \pm \mathrm{i} \cos \alpha I_{\mathfrak{H}}\| \leq 1, \tag{2.18}$$

see [2]. Condition (2.18) is equivalent to

$$2|\mathrm{Im}\,(Tf, f)| \leq \tan \alpha\, ||f||^2 - ||Tf||^2, \quad f \in \mathfrak{H}. \tag{2.19}$$

Therefore, if $T \in C_{\mathfrak{H}}(\alpha)$, then T is a contraction and $T^* \in C_{\mathfrak{H}}(\alpha)$. Vice versa, if $\|T\| = \rho < 1$, then $T \in C_{\mathfrak{H}}(\alpha_\rho)$ with $\alpha_\rho = 2 \tan^{-1} \rho$. In view of (2.19), in the limit $\alpha \searrow 0$, it is natural to consider $C_{\mathfrak{H}}(0)$ as the set of all selfadjoint contractions. Then, since $C_{\mathfrak{H}}(\alpha_1) \subseteq C_{\mathfrak{H}}(\alpha_2)$ if $0 < \alpha_1 \leq \alpha_2 < \pi/2$ by (2.19), one can write $C_{\mathfrak{H}}(0) = \bigcap_{\alpha \in (0,\pi/2)} C_{\mathfrak{H}}(\alpha)$.

It is easy to see that the numerical range of $T \in C_{\mathfrak{H}}(\alpha)$ satisfies

$$W(T) \subset C_{\mathbb{C}}(\alpha) := \{z \in \mathbb{C} : |z \sin \alpha + \mathrm{i} \cos \alpha| \leq 1 \wedge |z \sin \alpha - \mathrm{i} \cos \alpha| \leq 1\},$$

where $C_{\mathbb{C}}(\alpha)$ is a lens-shaped region inside the unit disc with $\pm 1 \in C_{\mathbb{C}}(\alpha)$ and that, hence, the operators $I + T$ and $I - T$ are sectorial with vertex 0 and semi-angle α.

Operators of the class $\widetilde{C}_{\mathfrak{H}}$ defined as

$$\widetilde{C}_{\mathfrak{H}} := \bigcup_{\alpha \in [0,\pi/2)} C_{\mathfrak{H}}(\alpha)$$

and their properties were studied in [2,3,7]. In particular, in Section 4 we will need the following, see [2]:

1) $T \in C_{\mathfrak{H}}(\alpha)$ if and only if the operator $S := (I + T)(I - T^*)$ is bounded and sectorial with semi-angle α;
2) $T \in C_{\mathfrak{H}}(\alpha)$ implies that $T^n \in C_{\mathfrak{H}}(\alpha)$ for all $n \in \mathbb{N}$,

$$\mathrm{ran}\, D_{T^n} = \mathrm{ran}\, D_{T^{*n}} = \mathrm{ran}\, D_{\mathrm{Re}\, T}, \quad n \in \mathbb{N}, \tag{2.20}$$

   and $T{\upharpoonright} \ker D_T = T^*{\upharpoonright} \ker D_{T^*}$ (recall that if K is a contraction, $D_K := (I - K^*K)^{\frac{1}{2}}$ is the unique nonnegative square root of the nonnegative operator $I - K^*K$);
3) if A is m-α-sectorial, then the semi-group $T(t) = \exp(-tA)$, $t \geq 0$, has a holomorphic continuation into the open sector $\mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)} := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \frac{\pi}{2} - \alpha\right\}$ to the contractive semigroup $T(\lambda) = \exp(-\lambda A) \in C_{\mathfrak{H}}(\alpha + |\arg \lambda|)$, $\lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}$, see [2], [28].

2.3. Douglas' Lemma and Heinz' inequality

The next two results on operator ranges by Douglas and a generalization of Heinz' inequality by Kato will be important tools in the following.

Theorem 2.1. (*Douglas' Lemma [14]*) *Let F, G be bounded operators on a Hilbert space $\mathfrak{H}$. Then the following are equivalent:*

(i) $\mathrm{ran}\, F \subseteq \mathrm{ran}\, G$;
(ii) $FF^* \leq c\, GG^*$ *with some* $c > 0$;

(iii) *there exists a bounded operator Y on $\mathfrak{H}$ so that $F = GY$.*

Moreover, if one of (i), (ii), (iii) *holds, there exists a unique bounded operator Y in $\mathfrak{H}$ so that*

a) $\|Y\|^2 = \inf\{c : FF^* \le cGG^*\}$;
b) $\ker F = \ker Y$;
c) $\operatorname{ran} Y \subseteq \overline{\operatorname{ran}}\, G^*$.

Theorem 2.2. (*Generalized Heinz' inequality [26, Thm. 1]*) *Let A, B be m-accretive operators on Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$, respectively, T a bounded operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and $\gamma \in [0,1]$. If $T\operatorname{dom} A \subseteq \operatorname{dom} B$ and $\|BTu\| \le M\|Au\|$, $u \in \operatorname{dom} A$, for some $M \ge 0$, then $T\operatorname{dom} A^\gamma \subseteq \operatorname{dom} B^\gamma$ and there exists $c \ge 0$ such that*

$$\|B^\gamma Tu\| \le \exp\left(c\gamma(1-\gamma)\right) M^\gamma \|T\|^{1-\gamma} ||A^\gamma u||, \quad u \in \operatorname{dom} A^\gamma;$$

if A, B are selfadjoint and nonnegative, we can choose $c = 0$.

2.4. Invariant operator ranges

If $\mathcal{R}$ is an operator range, i.e. the range of bounded selfadjoint operator, see [16], and if $S\mathcal{R} \subseteq \mathcal{R}$ for a bounded operator S, then $\mathcal{R}$ is called *invariant operator range* of S, see [32]. Clearly, if $S\mathcal{R} \subseteq \mathcal{R}$, then $(S + \lambda I)\mathcal{R} \subseteq \mathcal{R}$ for all $\lambda \in \mathbb{C}$. If $\mathcal{R} = \operatorname{ran} C$ with a bounded selfadjoint nonnegative operator C, then Douglas' Lemma, see Theorem 2.1, implies that

$$S\operatorname{ran} C \subseteq \operatorname{ran} C \iff SC = CW$$

where W is a bounded operator in $\mathfrak{H}$ and $S\operatorname{ran} C = \operatorname{ran} C$ if $\operatorname{ran} W = \mathfrak{H}$.

Now assume, in addition, that $\ker C = \{0\}$ and $\operatorname{ran} C \ne \mathfrak{H}$; then $\overline{\operatorname{ran}}\, C = \mathfrak{H}$, i.e. $\operatorname{ran} C$ is not closed and hence C^{-1} is unbounded. Because C is bounded and C^{-1} is closed, the operator range $\mathcal{R} = \operatorname{ran} C$ becomes a Hilbert space $\mathfrak{H}_C$ with the inner product

$$(u,v)_C = (C^{-1}u, C^{-1}v), \quad u,v \in \mathfrak{H}_C, \tag{2.21}$$

and the operator C maps the Hilbert space $\mathfrak{H}$ unitarily onto the Hilbert space $\mathfrak{H}_C$.

By the Closed Graph Theorem, each operator $S \in \mathbf{B}(\mathfrak{H})$ leaving $\mathfrak{H}_C$ invariant, i.e. $S\operatorname{ran} C \subseteq \operatorname{ran} C$, is a bounded operator in $\mathfrak{H}_C$ and $W := C^{-1}SC$ is a bounded operator in $\mathfrak{H}$. An interpolation theoretic argument, see [30, Thm. I.5.1]), yields that S leaves $\operatorname{ran} C^\alpha$ invariant for each $\alpha \in (0,1)$ and is bounded in the interpolation space $\mathfrak{H}_{C^\alpha} = \operatorname{ran} C^\alpha$.

Given a bounded selfadjoint operator C with $0 \le C \le I$, $\ker C = \{0\}$ and $\operatorname{ran} C \ne \mathfrak{H}$, the algebra $\mathcal{A}(C)$ of operators leaving $\mathcal{R} = \operatorname{ran} C$ invariant was constructed in [32].

3. Domain and range inclusions and equalities

In this section, for m-sectorial operators A, we establish a series of equivalent conditions for domain inclusion $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, domain equalities $\operatorname{dom} A = \operatorname{dom} A^*$, $\operatorname{dom} \operatorname{Re} A = \operatorname{dom} A_R$, as well as for the stronger domain equalities $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$ which imply that $\operatorname{Re} A = A_R$.

Theorem 3.1. *Let L be a possibly unbounded, closed and densely defined operator acting between Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$ with $\ker L = \{0\}$ and let B be a bounded, accretive and coercive operator in $\mathcal{H}$. If $A = L^*BL$ is the corresponding m-sectorial operator in $\mathfrak{H}$, see Subsection 2.1, then*

1) *for the conditions*
   (a) $A_R = \operatorname{Re} A$,
   (b) $(\operatorname{Im} B)(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$,
   (b) *implies* (a) *and, if additionally* $\operatorname{ran} L = \mathcal{H}$, (a) *and* (b) *are equivalent;*
2) *for the conditions*
   (a) $\operatorname{dom} A \subseteq \operatorname{dom} A^*$,
   (b) $B^*B^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$,
   (b) *implies* (a) *and, if additionally* $\operatorname{ran} L = \mathcal{H}$, (a) *and* (b) *are equivalent.*

Proof. 1) (b) $\Longrightarrow$ (a): It is easy to see that

$$B^*(\operatorname{Re} B)^{-1} = I - \mathrm{i}(\operatorname{Im} B)(\operatorname{Re} B)^{-1}, \quad B(\operatorname{Re} B)^{-1} = I + \mathrm{i}(\operatorname{Im} B)(\operatorname{Re} B)^{-1}. \tag{3.1}$$

If $(\operatorname{Im} B)(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$, then (3.1) yields that

$$B^*(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*, \quad B(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*. \tag{3.2}$$

Hence, by (2.15) and (2.16), if $u \in \operatorname{dom} L$ and $Lu \in (\operatorname{Re} B)^{-1}\operatorname{dom} L^*$, then $Lu \in B^{-1}\operatorname{dom} L^*$ and $Lu \in B^{-*}\operatorname{dom} L^*$. Therefore

$$\begin{aligned}\operatorname{dom} A \cap \operatorname{dom} A^* &= \{u \in \operatorname{dom} L : Lu \in B^{-1}\operatorname{dom} L^* \cap B^{-*}\operatorname{dom} L^*\}\\ &\supseteq \{u \in \operatorname{dom} L : Lu \in (\operatorname{Re} B)^{-1}\operatorname{dom} L^*\} = \operatorname{dom} A_R,\end{aligned}$$

i.e. $\operatorname{Re} A \supseteq A_R$. Since $A_R \supseteq \operatorname{Re} A$ by (2.11), we conclude $\operatorname{Re} A = A_R$.

(a) $\Longrightarrow$ (b): Now assume that $\operatorname{ran} L = \mathcal{H}$. By (2.15) and (2.16), the equivalences

$$\begin{aligned}\operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A_R &\iff B^{-*}\operatorname{dom} L^* \cap B^{-1}\operatorname{dom} L^* = (\operatorname{Re} B)^{-1}\operatorname{dom} L^*\\ &\iff \operatorname{dom} L^* \cap B^*B^{-1}\operatorname{dom} L^* = B^*(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \qquad (3.3)\\ &\iff \operatorname{dom} L^* \cap BB^{-*}\operatorname{dom} L^* = B(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \qquad (3.4)\end{aligned}$$

hold. If $\operatorname{Re} A = A_R$, then $\operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A_R$ and hence (3.3), (3.4) are satisfied. The latter imply that (3.2) holds. Now either the first or the second identity in (3.1) together with (3.2) imply that $(\operatorname{Im} B)(\operatorname{Re} B)^{-1}\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$.

2) (b) $\Longrightarrow$ (a): Condition (b) implies that $B^{-1}\operatorname{dom} L^* \subseteq B^{-*}\operatorname{dom} L^*$ and therefore (a) follows from (2.15).

(a) $\Longrightarrow$ (b): If we assume that $\operatorname{ran} L = \mathcal{H}$, then $\operatorname{dom} L^{-1} = \mathcal{H}$. Hence (2.15) implies that $\operatorname{dom} A = L^{-1}B^{-1}\operatorname{dom} L^*$ and $\operatorname{dom} A^* = L^{-1}B^{-*}\operatorname{dom} L^*$. Therefore, in this case (a) also implies (b). □

Theorem 3.2. *Suppose that L is a closed densely defined operator in the Hilbert space $\mathfrak{H}$ and $0 \in \rho(L)$. Let G be a bounded selfadjoint operator in $\mathfrak{H}$, U_G the Cayley transform of G given by* (2.9)*, and let A be the m-sectorial operator associated with the closed form given by* (2.3)*. Then*

$$\begin{aligned}\operatorname{dom} A &= L^{-1}(I + \mathrm{i}G)^{-1}\operatorname{dom} L^* = L^{-1}(U_G + I)\operatorname{dom} L^*,\\ \operatorname{dom} A^* &= L^{-1}(I - \mathrm{i}G)^{-1}\operatorname{dom} L^* = L^{-1}(U_G^{-1} + I)\operatorname{dom} L^*,\end{aligned} \tag{3.5}$$

and

1) *the following are equivalent:*

(a) $\operatorname{Re} A = A_R$,
(b) $G\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$,
(c) $(U_G + I)^{-1}\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$;

2) *the following are equivalent:*
(a) $\mathrm{dom}\, A \subseteq \mathrm{dom}\, A^*$,
(b) $(I + \mathrm{i}G)^{-1}\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$,
(c) $U_G\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$,
(d) $(I + \mathrm{i}G)^{-1}\mathrm{dom}\, L^* \subseteq (I - \mathrm{i}G)^{-1}\mathrm{dom}\, L^*$,

Proof. We apply Theorem 3.1 for the case $B = I + \mathrm{i}G$. Then $\operatorname{Re} B = I$, $\operatorname{Im} B = G$ and

$$U_G = (I - \mathrm{i}G)(I + \mathrm{i}G)^{-1} = B^*B^{-1}, \quad (U_G + I)^{-1} = \frac{1}{2}(I + \mathrm{i}G) = \frac{1}{2}B, \quad (U_G^{-1} + I)^{-1} = \frac{1}{2}(I - \mathrm{i}G) = \frac{1}{2}B^*.$$

Now Theorem 3.1 yields the equivalences of (a) and (c) in 1) and 2).

Since L has bounded inverse L^{-1} and $0 \in \rho(I \pm \mathrm{i}G)$, the operators A and A^* defined by (2.5), (2.6) have bounded inverses

$$A^{-1} = L^{-1}(I + \mathrm{i}G)^{-1}L^{-*}, \quad A^{-*} = L^{-1}(I - \mathrm{i}G)^{-1}L^{-*}.$$

Then the relations in (3.5) follow from (2.15).

The equivalence of (b) and (c) in 1) is obvious from the identity $(U_G + I)^{-1} = \frac{1}{2}(I + \mathrm{i}G)$. The equivalences of (b), (c) and (d) in 2) follow since $U_G = -I + 2(I + \mathrm{i}G)^{-1}$. □

If we apply Theorem 3.2 also to A^*, the following corollary is immediate.

Corollary 3.3. *Under the hypothesis of Theorem 3.2, the following are equivalent:*

(a) $\mathrm{dom}\, A = \mathrm{dom}\, A^*$,
(b) $(I + \mathrm{i}G)^{-1}\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$ *and* $(I - \mathrm{i}G)^{-1}\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*$,
(c) $U_G\mathrm{dom}\, L^* = \mathrm{dom}\, L^*$,
(d) $(I + \mathrm{i}G)^{-1}\mathrm{dom}\, L^* = (I - \mathrm{i}G)^{-1}\mathrm{dom}\, L^*$.

Observe that if an m-sectorial operator A is associated with a closed sesquilinear sectorial form (2.3) and if $\mathrm{dom}\, A \subseteq \mathrm{dom}\, A^*$, then $\mathrm{dom}\, A \subseteq \mathrm{dom}\, L^*L = \mathrm{dom}\, A_R$ and, since $\mathrm{dom}\, A$ is a core of L, the Friedrichs extension of $\operatorname{Re} A$ coincides with A_R.

Corollary 3.4. *Under the hypothesis of Theorem 3.2, the following are equivalent:*

(a) $\mathrm{dom}\, A = \mathrm{dom}\, A_R$ $(= \mathrm{dom}\, L^*L)$,
(b) $(I + \mathrm{i}G)\mathrm{dom}\, L^* = \mathrm{dom}\, L^*$,

Proof. Since $\mathrm{dom}\, A = L^{-1}(I + \mathrm{i}G)^{-1}\mathrm{dom}\, L^*$, we have $\mathrm{dom}\, A = \mathrm{dom}\, L^*L \Longleftrightarrow (I + \mathrm{i}G)\mathrm{dom}\, L^* = \mathrm{dom}\, L^*$. □

By Corollary 3.4, Corollary 3.3 applied to A^* and because $U_G + I = 2(I + \mathrm{i}G)^{-1}$, we obtain the following.

Corollary 3.5. *Under the hypothesis of Theorem 3.2, the following are equivalent:*

(a) $\mathrm{dom}\, A = \mathrm{dom}\, A^* = \mathrm{dom}\, A_R$,

(b) $(I+\mathrm{i}G)\mathrm{dom}\,L^* = (I-\mathrm{i}G)\mathrm{dom}\,L^* = \mathrm{dom}\,L^*$,
(c) $U_G\mathrm{dom}\,L^* = (U_G+I)\mathrm{dom}\,L^* = \mathrm{dom}\,L^*$.

Corollary 3.6. *Under the hypothesis of Theorem 3.2, let* $\widetilde{A} := L^*(I+\mathrm{i}G)^{-1}L$. *Then*

1) *the following are equivalent:*
   (a) $\mathrm{dom}\,\mathrm{Re}\,A = \mathrm{dom}\,A_R$,
   (b) $\mathrm{dom}\,\mathrm{Re}\,\widetilde{A} = \mathrm{dom}\,\widetilde{A}_R$;
2) *the following are equivalent:*
   (a) $\mathrm{dom}\,A = \mathrm{dom}\,A^*$,
   (b) $\mathrm{dom}\,\widetilde{A} = \mathrm{dom}\,\widetilde{A}^*$;
3) *if* $\mathrm{dom}\,A = \mathrm{dom}\,A^*$, *then* $\mathrm{ran}\,(\mathrm{Re}\,\widetilde{A}) = \mathrm{ran}\,(\mathrm{Re}\,A)$, $\quad \mathrm{ran}\,(\mathrm{Im}\,\widetilde{A}) = \mathrm{ran}\,(\mathrm{Im}\,A)$.

Proof. The operator $\widetilde{A}$ can be rewritten as

$$\widetilde{A} = L^*(I+G^2)^{-\frac{1}{2}}(I-\mathrm{i}G)(I+G^2)^{-\frac{1}{2}}L = \widetilde{L}^*(I+\mathrm{i}\widetilde{G})\widetilde{L}, \tag{3.6}$$

where

$$\widetilde{L} := (I+G^2)^{-\frac{1}{2}}L, \quad \mathrm{dom}\,\widetilde{L} = \mathrm{dom}\,L, \quad \widetilde{G} := -G. \tag{3.7}$$

Then

$$\mathrm{dom}\,\widetilde{L}^* = (I+G^2)^{\frac{1}{2}}\mathrm{dom}\,L^*. \tag{3.8}$$

1) Equality (3.8) yields $\widetilde{G}\mathrm{dom}\,\widetilde{L}^* = G(I+G^2)^{\frac{1}{2}}\mathrm{dom}\,L^* = (I+G^2)^{\frac{1}{2}}G\mathrm{dom}\,L^*$. This implies the equivalences

$$G\mathrm{dom}\,L^* \subseteq \mathrm{dom}\,L^* \iff (I+G^2)^{\frac{1}{2}}G\mathrm{dom}\,L^* \subseteq (I+G^2)^{\frac{1}{2}}\mathrm{dom}\,L^* \iff \widetilde{G}\mathrm{dom}\,\widetilde{L}^* \subseteq \mathrm{dom}\,\widetilde{L}^*.$$

Applying Theorem 3.2 1), we obtain that $\mathrm{dom}\,\mathrm{Re}\,A = \mathrm{dom}\,A_R \iff \mathrm{dom}\,\mathrm{Re}\,\widetilde{A} = \mathrm{dom}\,\widetilde{A}_R$.

2) Since, again by (3.8),

$$(I\pm\mathrm{i}\widetilde{G})^{-1}\mathrm{dom}\,\widetilde{L}^* = (I\mp\mathrm{i}G)^{-1}(I+G^2)^{\frac{1}{2}}\mathrm{dom}\,L^* = (I+G^2)^{\frac{1}{2}}(I\mp\mathrm{i}G)^{-1}\mathrm{dom}\,L^*,$$

we deduce the equivalences

$$\begin{aligned}
&(I+\mathrm{i}G)^{-1}\mathrm{dom}\,L^* = (I-\mathrm{i}G)^{-1}\mathrm{dom}\,L^* \\
&\iff (I+G^2)^{\frac{1}{2}}(I+\mathrm{i}G)^{-1}\mathrm{dom}\,L^* = (I+G^2)^{\frac{1}{2}}(I-\mathrm{i}G)^{-1}\mathrm{dom}\,L^* \\
&\iff (I-\mathrm{i}\widetilde{G})^{-1}\mathrm{dom}\,\widetilde{L}^* = (I+\mathrm{i}\widetilde{G})^{-1}\mathrm{dom}\,\widetilde{L}^*.
\end{aligned}$$

Now Corollary 3.3 shows that $\mathrm{dom}\,A = \mathrm{dom}\,A^* \iff \mathrm{dom}\,\widetilde{A} = \mathrm{dom}\,\widetilde{A}^*$.

3) Assume that $\mathrm{dom}\,A = \mathrm{dom}\,A^*$. Then, by 2), we have $\mathrm{dom}\,\widetilde{A} = \mathrm{dom}\,\widetilde{A}^*$. By (2.8), we have $\mathrm{ran}\,\mathrm{Re}\,A = L^*L\,\mathrm{dom}\,A$ and, analogously, $\mathrm{ran}\,\mathrm{Im}\,A = L^*GL\,\mathrm{dom}\,A$. Due to Corollary 3.3 and (3.5), it then it follows that

$$\begin{aligned}
\mathrm{ran}\,\mathrm{Re}\,A &= L^*L\,\mathrm{dom}\,A = L^*LL^{-1}(I+\mathrm{i}G)^{-1}\mathrm{dom}\,L^* \\
&= L^*(I+\mathrm{i}G)^{-1}\mathrm{dom}\,L^* = L^*(I-\mathrm{i}G)^{-1}\mathrm{dom}\,L^*,
\end{aligned}$$

$$\begin{aligned}\operatorname{ran}\operatorname{Im} A &= L^*GL\operatorname{dom} A = L^*GLL^{-1}(I+\mathrm{i}G)^{-1}\operatorname{dom} L^*\\ &= L^*G(I+\mathrm{i}G)^{-1}\operatorname{dom} L^* = L^*G(I-\mathrm{i}G)^{-1}\operatorname{dom} L^*.\end{aligned}$$

Analogously, for $\widetilde{A}$ we obtain, using (3.6), (3.7) and (3.8),

$$\begin{aligned}\operatorname{ran}(\operatorname{Re}\widetilde{A}) &= \widetilde{L}^*(I+\mathrm{i}\widetilde{G})^{-1}\operatorname{dom}\widetilde{L}^* = L^*(I+G^2)^{-\frac12}(I-\mathrm{i}G)^{-1}(I+G^2)^{\frac12}\operatorname{dom} L^*\\ &= L^*(I-\mathrm{i}G)^{-1}\operatorname{dom} L^* = \operatorname{ran}(\operatorname{Re} A),\\ \operatorname{ran}(\operatorname{Im}\widetilde{A}) &= \widetilde{L}^*\widetilde{G}(I+\mathrm{i}\widetilde{G})^{-1}\operatorname{dom}\widetilde{L}^* = L^*(I+G^2)^{-\frac12}G(I-\mathrm{i}G)^{-1}(I+G^2)^{\frac12}\operatorname{dom} L^*\\ &= L^*G(I-\mathrm{i}G)^{-1}\operatorname{dom} L^* = \operatorname{ran}(\operatorname{Im} A). \quad \square\end{aligned}$$

Remark 3.7. The form domains of $A = L^*(I+\mathrm{i}G)L$ and $\widetilde{A} = L^*(I+\mathrm{i}G)^{-1}L$ satisfy

$$\mathcal{D}[A] = \mathcal{D}[\widetilde{A}] = \operatorname{dom} L.$$

In fact, $\mathcal{D}[A] = \operatorname{dom} L$ by (2.1), (2.3) and, analogously, $\mathcal{D}[\widetilde{A}] = \operatorname{dom}\widetilde{L} = \operatorname{dom} L$ by (3.6), (3.7).

Corollary 3.8. *Under the hypothesis of Theorem 3.2, if $A = L^*(I+\mathrm{i}G)L$ satisfies $\operatorname{Re} A = A_R$, then so do all operators $A_n := L^*(I+\mathrm{i}G^n)L$ and $\widetilde{A}_n := L^*(I+\mathrm{i}G^n)^{-1}L$ for arbitrary $n \in \mathbb{N}$, i.e.*

$$\operatorname{Re} A_n = (A_n)_R, \quad \operatorname{Re}\widetilde{A}_n = (\widetilde{A}_n)_R.$$

Proof. By Theorem 3.2 1), the condition $G\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$ is fulfilled. Then $G^n\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$ for arbitrary $n \in \mathbb{N}$ and hence, again by Theorem 3.2 1), $\operatorname{Re} A_n = (A_n)_R$. The last claim now follows from Corollary 3.6 1). $\square$

Note that if $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ for $A = L^*(I+\mathrm{i}G)L$, then the Cayley transform U_G of G given by (2.9) satisfies $U_G\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$ and, if $0 \in \rho(L)$, the operator

$$K := L^*U_GL^{-*} \tag{3.9}$$

is well defined and bounded in $\mathfrak{H}$ by the Closed Graph Theorem.

Proposition 3.9. *Let $A = L^*(I+\mathrm{i}G)L$ be a coercive m-sectorial operator such that $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ and let K be given by (3.9). Then*

$$A^*u = KAu, \quad u \in \operatorname{dom} A.$$

Moreover, $\operatorname{Re} A = A_R$ if and only if $-1 \in \rho(K)$; in particular, the latter holds if $\|K\| < 1$.

Proof. Since A is coercive, we have $0 \in \rho(L)$; since A is sectorial, G is bounded and hence $-1 \in \rho(U_G)$ for U_G given by (2.9). Further, as noted above, $U_G\operatorname{dom} L^* \subseteq \operatorname{dom} L^*$, K in (3.9) is bounded in $\mathfrak{H}$ and $K+I$ is injective. By (2.5), we have $(I+\mathrm{i}G)L\operatorname{dom} A \subset \operatorname{dom} L^*$ and so we can write

$$A^*u = L^*(I-\mathrm{i}G)Lu = L^*(I-\mathrm{i}G)(I+\mathrm{i}G)^{-1}L^{-*}L^*(I+\mathrm{i}G)Lu = KAu, \quad u \in \operatorname{dom} A.$$

Since $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, it follows that $\operatorname{dom}\operatorname{Re} A = \operatorname{dom} A$, $\operatorname{Re} A = \frac12(A+A^*) = \frac12(I+K)A$ and

$$\operatorname{ran}(\operatorname{Re} A) = (I+K)\operatorname{ran} A = (I+K)\mathfrak{H}. \tag{3.10}$$

Because A is coercive, $\operatorname{Re} A$ is positive definite and hence injective. Thus (3.10) shows that $\operatorname{Re} A$ is selfadjoint if and only if $\operatorname{ran}(I+K) = \mathfrak{H}$. □

Remark 3.10. A closed densely defined operator A is called *q-hyponormal*, see [33,34], if

$$\operatorname{dom} A \subseteq \operatorname{dom} A^* \quad \text{and} \quad \|A^* u\| \leq \sqrt{q}\, \|Au\|, \quad u \in \operatorname{dom} A.$$

If $q = 1$, then A is called *hyponormal*, see [24]; if A is hyponormal, then $\lambda A + \mu I$ is hyponormal for any $\lambda, \mu \in \mathbb{C}$, see [24, Rem. 1]. For a nice overview on q-hyponormal operators and related operator classes we refer to [35, Fig. 1].

Proposition 3.9 implies that if A is a coercive m-sectorial operator such that $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, then

$$\|A^* u\| \leq \|K\| \|Au\|, \quad u \in \operatorname{dom} A,$$

and therefore A is q-hyponormal with $q = \|K\|^2$; if $\|K\| < 1$, then A^* is A-bounded with A-bound < 1 and hence $\operatorname{Re} A$ and $\operatorname{Im} A$ are closed since so is A, see also [33, Prop. 8.1].

Theorem 3.11. *Under the assumptions of Theorem 3.2, let $T := A^{-1}$ and write it as*

$$T = C(I + \mathrm{i}F)C \quad \text{with} \quad F = F^* \in \mathbf{B}(\mathfrak{H}), \quad C \in \mathbf{B}^+(\mathfrak{H}), \quad \ker C = \{0\}, \quad \operatorname{ran} C \neq \mathfrak{H}. \tag{3.11}$$

Then

1) *for* $\operatorname{dom} A_R^{\frac{1}{2}}$, $A[\cdot,\cdot]$, $A_R[\cdot,\cdot]$, $\operatorname{dom} A_R$ *and* A_R *the following hold:*

$$\begin{aligned}
&\operatorname{dom} A_R^{\frac{1}{2}} = \mathcal{D}[A] = \operatorname{ran} C, \quad A[f,g] = ((I+\mathrm{i}F)^{-1}C^{-1}f, C^{-1}g), \quad f,g \in \operatorname{ran} C,\\
&A_R[f,g] = \left((I+\mathrm{i}F)^{-1}C^{-1}f, (I+\mathrm{i}F)^{-1}C^{-1}g\right), \quad f,g \in \operatorname{ran} C,\\
&\operatorname{dom} A_R = \operatorname{ran}\left(C(I+F^2)C\right), \quad A_R = C^{-1}(I+F^2)^{-1}C^{-1};
\end{aligned} \tag{3.12}$$

2) *the following are equivalent:*
   (a) $\operatorname{dom} A \subseteq \operatorname{dom} A^*$,
   (b) $\operatorname{ran} T \subseteq \operatorname{ran} T^*$,
   (c) $(I+\mathrm{i}F)\operatorname{ran} C \subseteq (I-\mathrm{i}F)\operatorname{ran} C$,
   (d) $(I+\mathrm{i}F)(I-\mathrm{i}F)^{-1}\operatorname{ran} C \subseteq \operatorname{ran} C$,
   (e) $(I-\mathrm{i}F)^{-1}\operatorname{ran} C \subseteq \operatorname{ran} C$;
3) *the following are equivalent:*
   (a) $\operatorname{dom} A = \operatorname{dom} A^*$,
   (b) $\operatorname{ran} T = \operatorname{ran} T^*$,
   (c) $(I+\mathrm{i}F)(I-\mathrm{i}F)^{-1}\operatorname{ran} C = \operatorname{ran} C$;
4) *the following are equivalent:*
   (a) $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$,
   (b) $\operatorname{ran} T = \operatorname{ran} T^* = \operatorname{ran}(\operatorname{Re} T)$,
   (c) $(I-\mathrm{i}F)\operatorname{ran} C = (I+\mathrm{i}F)\operatorname{ran} C = \operatorname{ran} C$;
5) *the following are equivalent:*
   (a) $\operatorname{Re} A = A_R$,
   (b) $F\operatorname{ran} C \subseteq \operatorname{ran} C$,
   (c) $\operatorname{ran}(\operatorname{Im} T) \subseteq \operatorname{ran}(\operatorname{Re} T)$.

Proof. 1) Since A^{-1} is bounded by assumption (3.11),

$$(A^{-1})_R = \operatorname{Re} A^{-1} = \operatorname{Re} T = C^2, \quad \operatorname{ran}(\operatorname{Re} T) = \operatorname{ran} C^2.$$

Because

$$A = T^{-1} = C^{-1}(I + \mathrm{i}F)^{-1}C^{-1}, \quad A^* = T^{-*} = C^{-1}(I - \mathrm{i}F)^{-1}C^{-1},$$

analogously as in (3.6), (3.7), we conclude that $A = L^*(I + \mathrm{i}G)L$ where

$$L = (I + F^2)^{-\frac{1}{2}}C^{-1}, \quad G = -F,$$

and thus $A_R = L^*L$. The preceding formulas imply that all claims in (3.12) hold.

2), 3) and 4) In order to prove the sets of equivalences in 2)–4), we first note that

$$\operatorname{dom} A = \operatorname{ran} T = \operatorname{ran}\left(C(I + \mathrm{i}F)C\right), \quad \operatorname{dom} A^* = \operatorname{ran} T^* = \operatorname{ran}\left(C(I - \mathrm{i}F)C\right) \tag{3.13}$$

which implies the equivalence of (a), (b), (c) in 2) and in 3). The equivalence of (c), (d) and (e) in 2) follows if we note that $(I + \mathrm{i}F)(I - \mathrm{i}F)^{-1} = (I - \mathrm{i}F)^{-1}(I + \mathrm{i}F) = -I + 2(I - \mathrm{i}F)^{-1}$. Further, (3.13) yields the equivalences

$$\begin{aligned} \operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R &\iff (I + \mathrm{i}F)\operatorname{ran} C = (I - \mathrm{i}F)\operatorname{ran} C = (I + F^2)\operatorname{ran} C \\ &\iff \operatorname{ran} C = (I + \mathrm{i}F)\operatorname{ran} C = (I - \mathrm{i}F)\operatorname{ran} C \\ &\iff \operatorname{ran} C^2 = \operatorname{ran}\left(C(I + \mathrm{i}F)C\right) = \operatorname{ran}\left(C(I - \mathrm{i}F)C\right) \\ &\iff \operatorname{ran}(\operatorname{Re} T) = \operatorname{ran} T = \operatorname{ran} T^*. \end{aligned} \tag{3.14}$$

5) Next we prove the equivalence of (a) and (b) in 5). Note that we always have the inclusion $\operatorname{dom} A \cap \operatorname{dom} A^* \subseteq \operatorname{dom} A_R$. Due to (3.13), (3.12), this is equivalent to

$$\operatorname{ran}(C(I + \mathrm{i}F)C) \cap (\operatorname{ran}(C(I - \mathrm{i}F)C) \subseteq \operatorname{ran}\left(C(I + F^2)C\right), \tag{3.15}$$

while $\operatorname{dom} \operatorname{Re} A = \operatorname{dom} A_R = \operatorname{dom} A \cap \operatorname{dom} A^*$ is equivalent to

$$\operatorname{ran}(C(I + \mathrm{i}F)C) \cap (\operatorname{ran}(C(I - \mathrm{i}F)C) = \operatorname{ran}\left(C(I + F^2)C\right).$$

(a) $\Longrightarrow$ (b): Let $\psi \in \mathfrak{H}$. Then

$$f := C(I + F^2)C\psi \in \operatorname{ran}(C(I + \mathrm{i}F)C) \cap (\operatorname{ran}(C(I - \mathrm{i}F)C)$$

and hence there exist x, $y \in \mathfrak{H}$ such that

$$f = C(I + F^2)C\psi = C(I + \mathrm{i}F)Cx = C(I - \mathrm{i}F)Cy.$$

Therefore, because $I + F^2 = (I + \mathrm{i}F)(I - \mathrm{i}F) = (I - \mathrm{i}F)(I + \mathrm{i}F)$ and C is injective, we conclude

$$Cx = (I - \mathrm{i}F)C\psi, \quad Cy = (I + \mathrm{i}F)C\psi.$$

It follows that $FC\psi = \mathrm{i}C(x - y) \in \operatorname{ran} C$ and thus $F\operatorname{ran} C \subseteq \operatorname{ran} C$, as required.

(b) $\Longrightarrow$ (a): Conversely, if $F\operatorname{ran} C \subseteq \operatorname{ran} C$, then for any $\psi \in \mathfrak{H}$ one can find $x, y \in \mathfrak{H}$ so that $(I - \mathrm{i}F)C\psi = Cx$ and $(I + \mathrm{i}F)C\psi = Cy$. It follows that, for $f := C(I + F^2)C\psi$,

$$f = C(I + \mathrm{i}F)Cx = C(I - \mathrm{i}F)Cy.$$

Since $\psi \in \mathfrak{H}$ was arbitrary, this implies $\operatorname{ran}\big(C(I + F^2)C\big) \subseteq \operatorname{ran}(C(I + \mathrm{i}F)C) \cap \operatorname{ran}(C(I - \mathrm{i}F)C)$ which was equivalent to (a) by (3.15), (3.12), see above.

(b) $\Longleftrightarrow$ (c): Because C is injective, the inclusion $F\operatorname{ran} C \subseteq \operatorname{ran} C$ is equivalent to the inclusion $\operatorname{ran}(CFC) \subseteq \operatorname{ran} C^2$. Now the claim follows since $CFC = \operatorname{Im} T$ and $C^2 = \operatorname{Re} T$. □

Remark 3.12. 1) The operator

$$U_F := (I - \mathrm{i}F)^{-1}(I + \mathrm{i}F)$$

is unitary. Because F is bounded, we have $-1 \in \rho(U_F)$ and

$$U_F + I = 2(I - \mathrm{i}F)^{-1}.$$

Hence the operator $T = C(I + \mathrm{i}F)C$ takes the form $T = 2C(I + U_F^*)^{-1}C$ and, further, the equality $(I - \mathrm{i}F)^{-1}(I + \mathrm{i}F)\operatorname{ran} C = \operatorname{ran} C$ can be rewritten as

$$(I - \mathrm{i}F)^{-1}(I + \mathrm{i}F)\operatorname{ran} C = \operatorname{ran} C \iff U_F\operatorname{ran} C = \operatorname{ran} C,$$
$$(I - \mathrm{i}F)\operatorname{ran} C = (I + \mathrm{i}F)\operatorname{ran} C = \operatorname{ran} C \iff U_F\operatorname{ran} C = (U_F + I)\operatorname{ran} C = \operatorname{ran} C.$$

2) The operator $((A^{-1})_R)^{-1} = (T_R)^{-1} = (\operatorname{Re} T)^{-1} = C^{-2}$ is the harmonic mean of A and A^*; recall that $A_R \le ((A^{-1})_R)^{-1}$ due to (2.13) or, equivalently, $(A_R)^{-1} \ge (A^{-1})_R$. Hence $\operatorname{dom}((A^{-1})_R)^{-1} = \operatorname{ran} C^2$ and so Theorem 3.11 4), together with (3.13), yields that $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$ if and only if $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom}((A^{-1})_R)^{-1}$.

3) If a bounded operator T has the property $\operatorname{ran} T \subseteq \operatorname{ran} T^*$, then Douglas' Lemma, see Theorem 2.1, yields that $T = T^*V$ with a bounded operator V. Hence $T^* = V^*T$, i.e. the operator T is q-hyponormal with $q = \|V\|^2$. Further, if $0, -1 \in \rho(V)$, then $\operatorname{ran} T = \operatorname{ran} T^* = \operatorname{ran}(\operatorname{Re} T)$.

4) If $F\operatorname{ran} C \subseteq \operatorname{ran} C$, then $F^n\operatorname{ran} C \subseteq \operatorname{ran} C$ for every $n \in \mathbb{N}$.

Let T be a bounded sectorial operator in $\mathfrak{H}$. Due to the canonical representation of m-sectorial operators, see (2.2), the imaginary part $\operatorname{Im} T$ takes the form

$$\operatorname{Im} T = (\operatorname{Re} T)^{\frac{1}{2}} X (\operatorname{Re} T)^{\frac{1}{2}}$$

with some bounded selfadjoint operator X. Hence the range inclusion $\operatorname{ran}\operatorname{Im} T \subseteq \operatorname{ran}(\operatorname{Re} T)^{\frac{1}{2}}$ holds. If T is bounded and accretive, $\operatorname{ran}\operatorname{Re} T$ is dense and does not coincide with $\mathfrak{H}$, then the latter condition does not imply, in general, that T is sectorial; as an example consider e.g. $T = Q + \mathrm{i}Q^{\frac{1}{2}}$ with bounded $Q \ge 0$ so that $\operatorname{ran} Q$ is dense, but not equal to $\mathfrak{H}$. The next theorem shows that the stronger condition $\operatorname{ran}(\operatorname{Im} T) \subseteq \operatorname{ran}(\operatorname{Re} T)$ is sufficient for the sectoriality of T.

Theorem 3.13. *Let A be an unbounded m-accretive operator having bounded inverse A^{-1} and suppose that*

$$\operatorname{ran}(\operatorname{Im} A^{-1}) \subseteq \operatorname{ran}(\operatorname{Re} A^{-1}). \tag{3.16}$$

Then the operator A is m-sectorial and $\operatorname{Re} A = A_R$. *Moreover,*

$$\begin{aligned} \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A_R &= (\operatorname{Re} A^{-1}) \operatorname{ran}\left(I + \left((\operatorname{Re} A^{-1})^{-1}(\operatorname{Im} A^{-1})\right)^2\right) \\ &= \operatorname{ran}\left(\operatorname{Re} A^{-1} + (\operatorname{Im} A^{-1})(\operatorname{Re} A^{-1})^{-1}(\operatorname{Im} A^{-1})\right) \end{aligned} \tag{3.17}$$

and the following are equivalent:

(i) $\operatorname{dom} A = \operatorname{dom} A^*$,
(ii) $\operatorname{ran}\left(I + ((\operatorname{Re} A^{-1})^{-1}(\operatorname{Im} A^{-1}))^2\right) = \mathfrak{H}$.
(iii) $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R = \operatorname{ran}(\operatorname{Re} A^{-1})$.

Proof. Set $Q := \operatorname{Re} A^{-1}$, $S := \operatorname{Im} A^{-1}$. By Douglas' Lemma, see Theorem 2.1, the range inclusion (3.16) yields that there exists a bounded operator Y in $\mathfrak{H}$ with $S = QY$. Since A is m-accretive, Q is injective and we can write

$$Z := \mathrm{i}Y = \mathrm{i}Q^{-1}S. \tag{3.18}$$

Since Q and S are self-adjoint, it follows that

$$QZ = \mathrm{i}S = -Z^*Q, \tag{3.19}$$

which implies that $\|QZf\| \le \|Z\|\|Qf\|$, $f \in \mathfrak{H}$, and hence, due to the generalized Heinz' Inequality, see Theorem 2.2,

$$\|Q^{\frac12}Zf\| \le \|Z\|\|Q^{\frac12}f\|, \quad f \in \mathfrak{H}.$$

This shows that the operator $X := -\mathrm{i}Q^{\frac12}ZQ^{-\frac12}\upharpoonright \operatorname{ran} Q^{\frac12}$ is bounded with $\|X\| \le \|Z\|$. Together with (3.19), we conclude that, for all f, $g \in \mathfrak{H}$,

$$(XQ^{\frac12}g, Q^{\frac12}h) = -\mathrm{i}(QZg, h) = \mathrm{i}(Z^*Qg, h) = \mathrm{i}(Qg, Zh) = \mathrm{i}(Q^{\frac12}g, Q^{\frac12}Zh) = (Q^{\frac12}g, XQ^{\frac12}h),$$

and hence the operator X is essentially selfadjoint. Because $\mathrm{i}Q^{\frac12}XQ^{\frac12} = QZ$,

$$A^{-1} = Q + \mathrm{i}S = Q + QZ = Q - Z^*Q = Q + \mathrm{i}Q^{\frac12}XQ^{\frac12}$$

satisfies

$$\left|\operatorname{Im}(A^{-1}f, f)\right| = \left|(XQ^{\frac12}f, Q^{\frac12}f)\right| \le \|X\|\|Q^{\frac12}f\|^2 = \|X\| \operatorname{Re}(A^{-1}f, f), \quad f \in \mathfrak{H}. \tag{3.20}$$

Thus A^{-1} is sectorial with semi-angle $\alpha \le \arctan\|X\| \le \arctan\|Z\|$. Since $\ker A^{-1} = \{0\}$, the operator $A = (A^{-1})^{-1}$ is m-sectorial. Theorem 3.11 5) yields that assumption (3.16) is equivalent to the equality $\operatorname{Re} A = A_R$, which completes the proof of the first two claims.

Due to (3.19), we can write

$$A^{-1} = Q + \mathrm{i}S = Q(I + Z), \quad A^{-*} = Q + \mathrm{i}S = Q(I - Z). \tag{3.21}$$

Then $\ker(I + Z) = \{0\}$ and $\ker(I - Z) = \{0\}$. It follows that $\operatorname{ran}(I + Z) \cap \operatorname{ran}(I - Z) = \operatorname{ran}(I - Z^2)$, see [4, Lemma 3.1]. Hence

$$\operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{ran} A^{-1} \cap \operatorname{ran} A^{-*} = Q \operatorname{ran}(I - Z^2) = (\operatorname{Re} A^{-1}) \operatorname{ran}(I - Z^2)$$

which proves (3.17) if we recall that $Z = \mathrm{i}(\operatorname{Re} A^{-1})^{-1}(\operatorname{Im} A^{-1})$ by (3.18).

Further, (3.21) shows that $\operatorname{ran} A^{-1} = \operatorname{ran} A^{-*}$ is equivalent to $\operatorname{ran}(I+Z) = \operatorname{ran}(I-Z)$. Because $\ker(I \pm Z) = \{0\}$, the latter holds if and only if $\operatorname{ran}(I - Z^2) = \mathfrak{H}$, see [4, Lemma 3.1]. Thus, under condition (3.16), the equality $\operatorname{dom} A = \operatorname{dom} A^*$ is equivalent to $\operatorname{ran}\left(I - Z^2\right) = \mathfrak{H}$ and, if this is the case, then $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{ran}(\operatorname{Re} A^{-1}) = \operatorname{dom} A_R$. This proves the equivalence of (i), (ii) and (iii). □

The next theorem is related to the Kato square root problem [25].

Theorem 3.14. *[4, Prop. 4.4, Thm. 4.7]. Let A be an unbounded m-accretive operator having bounded inverse A^{-1}. The following are equivalent:*

(i) *the operator A^2 is accretive (α-sectorial);*
(ii) *the operator $A^{-*}A$ is accretive (α-sectorial);*
(iii) *the operator $Z := \mathrm{i}(\operatorname{Re} A^{-1})^{-1}\operatorname{Im} A^{-1}$ is a contraction (belongs to the class $C_{\mathfrak{H}}(\alpha)$).*

If one of the conditions (i), (ii), (iii) is fulfilled, then A is m-$\pi/4$-sectorial and $\operatorname{Re} A = A_R$.

Proof. Set $T := A^{-1}$. Then $T = Q + \mathrm{i}S$ with $Q = \operatorname{Re} T$, $S = \operatorname{Im} T$ and the square $T^2 = Q^2 - S^2 + \mathrm{i}(QS + SQ)$ is accretive if and only if $Q^2 \geq S^2$. By Douglas's Lemma, see Theorem 2.1, $Q^2 \geq S^2$ is equivalent to $\|Q^{-1}S\| \leq 1$. Because $A^2 = (T^2)^{-1}$, we conclude that A^2 is accretive if and only if $Z := \mathrm{i}(\operatorname{Re} A^{-1})^{-1}\operatorname{Im} A^{-1}$ is a contraction. Since $QZ = -Z^*Q$, compare (3.19), we have

$$T = Q(I + Z) = (I - Z^*)Q,$$

and thus $T^2 = Q(I+Z)(I-Z^*)Q$. It follows that T^2 is α-sectorial if and only if $Z \in C_{\mathfrak{H}}(\alpha)$, see Subsection 2.2 1). Therefore A^2 is m-α-sectorial if and only if $Z \in C_{\mathfrak{H}}(\alpha)$.

The operator $M := A^{-*}A = T^*T^{-1}$ satisfies

$$M(Tf) = T^*f, \quad f \in \mathfrak{H},$$

and hence

$$\begin{aligned} &(Mh, h) = (T^*f, Tf) = (f, T^2 f) = (Qf, (I+Z)(I-Z^*)Qf), \\ &\operatorname{Re}(Mh, h) = (Qf, (I - ZZ^*)Qf), \quad |\operatorname{Im}(Mh, h)| = 2|\operatorname{Im}(ZQf, Qf)|, \end{aligned} \qquad h = Tf,\ f \in \mathfrak{H}.$$

Consequently, $A^{-*}A$ is accretive if and only if Z is a contraction and $A^{-*}A$ is α-sectorial if and only if $Z \in C_{\mathfrak{H}}(\alpha)$, see (2.19) in Subsection 2.2.

If Z is a contraction, then T, and thus A, is m-$\pi/4$-sectorial, compare (3.20) in the proof of Theorem 3.13. □

Remark 3.15. 1) Since $Mh + h = 2(\operatorname{Re} T)f$ for $h = Tf$, $f \in \mathfrak{H}$, we have $\operatorname{ran}(M + I) = \operatorname{ran} Q \neq \mathfrak{H}$. Hence, if the operator $M = A^{-*}A$ is accretive, then it is not closed. Because $T^* = Q(I - Z)$, the adjoint M^* takes the form $M^* = A^*A^{-1} = (I-Z)^{-1}(I+Z) = (I+Z)(I-Z)^{-1}$ with $\operatorname{dom} A^*A^{-1} = A(\operatorname{dom} A \cap \operatorname{dom} A^*)$, see [4, Prop. 4.4].

2) Let B be an m-accretive operator. As we mentioned in the introduction, see [25, Thm. 5.1], Kato proved that the operator $B^{\frac{1}{2}}$ is m-$\pi/4$-sectorial and $\operatorname{Re} B^{\frac{1}{2}}$ is selfadjoint. Hence the equality $\operatorname{Re} B^{\frac{1}{2}} = (B^{\frac{1}{2}})_R$ holds. Therefore Theorem 3.11 5) and Theorem 3.13 may be considered as generalizations of Kato's result.

Besides, for B having bounded inverse, it was established in [19, Thm. 1 3)] that B is m-α-sectorial if and only if $B^{*\frac{1}{2}}B^{-\frac{1}{2}}$ is m-α-sectorial.

Theorem 3.16. *Let Q, S be bounded selfadjoint operators with Q nonnegative,* $\ker Q = \{0\}$, $\operatorname{ran} Q \neq \mathfrak{H}$ *and*

$$\operatorname{ran} S \subseteq \operatorname{ran} Q. \tag{3.22}$$

Set

$$A(\lambda) := (\lambda Q + \mathrm{i}S)^{-1}, \quad \lambda \in \mathbb{C}.$$

Then, for $\lambda \in \mathbb{C}_+$ the operator $A(\lambda)$ is m-sectorial and

$$\operatorname{Re} A(\lambda) = (A(\lambda))_R, \quad \lambda \in \mathbb{C}_+.$$

Moreover,

1) *$A(\lambda)$ forms a holomorphic family of type* (B) *in the open right half-plane and*

$$\operatorname{dom}((A(\lambda))_R)^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}, \quad \lambda \in \mathbb{C}_+;$$

2) *the following are equivalent:*
   (a) $\operatorname{dom} A(\lambda) = \operatorname{dom} A(\lambda)^*$,
   (b) $-\lambda, \overline{\lambda} \in \rho(\mathrm{i}Q^{-1}S)$.

Proof. By assumption, $\operatorname{ran} S \subseteq \operatorname{ran}(\lambda Q)$ for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $A(\lambda)$ has bounded inverse

$$T(\lambda) := \lambda Q + \mathrm{i}S = A(\lambda)^{-1}$$

such that $\ker T(\lambda) = \{0\}$. If $\operatorname{Re}\lambda > 0$, then $A(\lambda)$ is m-accretive and thus Theorem 3.13 shows that $A(\lambda)$, and hence $T(\lambda)$, is m-sectorial with $\operatorname{Re} A(\lambda) = (A(\lambda))_R$.

1) Since the operator $T(1) = Q + \mathrm{i}S$ is sectorial, the operator S admits the representation $S = Q^{\frac{1}{2}} X Q^{\frac{1}{2}}$ where X is a bounded selfadjoint operator in $\mathfrak{H}$. Hence, for $\lambda \in \mathbb{C}_+$,

$$T(\lambda) = Q^{\frac{1}{2}}(\lambda I + \mathrm{i}X)Q^{\frac{1}{2}}$$

and the closed sectorial form associated with the operator $A(\lambda) = (\lambda Q + \mathrm{i}S)^{-1}$ is

$$A(\lambda)[f,g] = ((\lambda I + \mathrm{i}X)^{-1} Q^{-\frac{1}{2}} f, Q^{-\frac{1}{2}} g), \quad f, g \in \operatorname{ran} Q^{\frac{1}{2}},$$

with constant domain $\operatorname{ran} Q^{\frac{1}{2}}$ Thus $A(\lambda)$, $\lambda \in \mathbb{C}_+$, is a holomorphic family of type (B).

2) As in the proof of Theorem 3.13, we conclude that by Douglas' Lemma, see Theorem 2.1, the range inclusion (3.22) yields that there exists a bounded operator Z in $\mathfrak{H}$ with $\mathrm{i}S = QZ$, i.e. $Z = \mathrm{i}Q^{-1}S$, compare (3.18), (3.19). Hence $T(\lambda) = Q(\lambda I + Z)$, $T(\lambda)^* = Q(\overline{\lambda} I - Z)$, $\operatorname{Re} T(\lambda) = (\operatorname{Re}\lambda)Q$, and therefore

$$\begin{aligned} &\operatorname{ran} T(\lambda) = Q \operatorname{ran}(\lambda I + Z), \quad \operatorname{ran} T(\lambda)^* = Q \operatorname{ran}(\overline{\lambda} I - Z), \\ &\operatorname{ran} \operatorname{Re}(T(\lambda)) = \operatorname{ran} Q \quad \text{if } \operatorname{Re}\lambda > 0. \end{aligned}$$

(b) $\Longrightarrow$ (a): If $-\lambda$, $\overline{\lambda} \in \rho(Z)$, then $\operatorname{ran}(\lambda I + Z) = \operatorname{ran}(\overline{\lambda} I - Z) = \mathfrak{H}$. Consequently,

$$\operatorname{ran} T(\lambda) = \operatorname{ran} T(\lambda)^* = \operatorname{ran} \operatorname{Re} T(\lambda) = \operatorname{ran} Q$$

and hence $\operatorname{dom} A(\lambda) = \operatorname{dom} A(\lambda)^* = \operatorname{dom}(A(\lambda))_R$ by Theorem 3.13.

(a) $\Longrightarrow$ (b): Assume $-\lambda$ or $\overline{\lambda} \in \sigma(Z)$. We claim that then

$$\operatorname{ran}(\lambda I + Z) \neq \operatorname{ran}(\overline{\lambda} I - Z). \tag{3.23}$$

Indeed, using the identity $\lambda I + Z = 2(\operatorname{Re}\lambda)I - (\overline{\lambda} I - Z)$, one can easily prove that

$$\operatorname{ran}(\lambda I + Z) \cap \operatorname{ran}(\overline{\lambda} I - Z) = \operatorname{ran}\left((\lambda I + Z)(\overline{\lambda} I - Z)\right). \tag{3.24}$$

Because $\ker(\lambda I + Z) = \ker(\overline{\lambda} I - Z) = \{0\}$, the assumption $\operatorname{ran}(\lambda I + Z) = \operatorname{ran}(\overline{\lambda} I - Z)$ and equality (3.24) yield that

$$\operatorname{ran}(\lambda I + Z) = \operatorname{ran}(\overline{\lambda} I - Z) = \mathfrak{H},$$

i.e. $-\lambda, \overline{\lambda} \in \rho(Z)$, a contradiction. Thus (3.23) holds. Hence $\operatorname{dom} A(\lambda) \neq \operatorname{dom} A(\lambda)^*$. □

In the next section, see Theorem 4.8 below, we consider a more concrete abstract example where the operator Z above is a non-unitary isometry.

4. Examples

4.1. m-sectorial operators A with $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$

The next proposition provides some sufficient conditions for an m-sectorial operator A to have the property $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$ and hence $\operatorname{Re} A = A_R$; it follows readily from well-known results, see Subsection 2.1 and cf. [22, Ex. 5.6] for the case of linear relations.

Proposition 4.1. *Let L be a closed unbounded densely defined operator in the Hilbert space $\mathfrak{H}$ and assume that G is a bounded selfadjoint operator in $\mathfrak{H}$ such that*

$$\operatorname{ran} G \subseteq \operatorname{dom} L^*. \tag{4.1}$$

Then the operator

$$A := L^*L + \mathrm{i}L^*GL, \quad \operatorname{dom} A = \operatorname{dom} L^*L,$$

is m-sectorial, the adjoint A^* is given by

$$A^* = L^*L - \mathrm{i}L^*GL, \quad \operatorname{dom} A^* = \operatorname{dom} A = \operatorname{dom} L^*L,$$

and

$$\operatorname{Re} A = A_R = L^*L.$$

Proof. With the sesquilinear form $\mathfrak{a}[u, v]$ given by (2.3), all claims of the proposition are immediate from (4.1), (2.5) and (2.7). □

Proposition 4.2. *Let C be a bounded nonnegative selfadjoint operator with* $\ker C = \{0\}$, $\operatorname{ran} C \neq \mathfrak{H}$ *and let M be a bounded selfadjoint operator. Then*

$$T := C(I + \mathrm{i}CMC)C$$

is a sectorial operator with $\ker T = \{0\}$ *and*

$$\operatorname{ran} T = \operatorname{ran} T^* = \operatorname{ran}(\operatorname{Re} T) \quad (= \operatorname{ran} C^2) \tag{4.2}$$

and the m-sectorial operator $A := T^{-1}$ *satisfies*

$$\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R \quad (= \operatorname{ran} C^2),$$

and the operator $\operatorname{Im} A$ *defined on* $\operatorname{dom} A$ *extends to a bounded operator on* $\mathcal{H}$.

If $\|C^2M\| \le 1$, *then the operators* T^2, A^2 *are accretive, and* T, A *are* $\pi/4$*-sectorial.*

Proof. Set $F := CMC$. Then

$$(I \pm \mathrm{i}F)C = (I \pm \mathrm{i}CMC)C = C(I \pm \mathrm{i}MC^2).$$

Since $\pm 1 \in \rho(\mathrm{i}CMC)$, we have $\pm 1 \in \rho(\mathrm{i}MC^2)$ and hence $\operatorname{ran}(I \pm \mathrm{i}MC^2) = \mathfrak{H}$. Therefore $\operatorname{ran}\big(C(I \pm \mathrm{i}MC^2)\big) = \operatorname{ran} C$ and $(I \pm \mathrm{i}F)\operatorname{ran} C = \operatorname{ran} C$. Now Theorem 3.11 4) implies the first two claims.

If the operator C^2M is a contraction, then $I - MC^4M \ge 0$ and, therefore,

$$\operatorname{Re} T^2 = (\operatorname{Re} T)^2 - (\operatorname{Im} T)^2 = C^4 - C^2MC^4MC^2 = C^2(I - MC^4M)C^2 \ge 0$$

i.e. the operator T^2 is accretive. Because the operator T is the accretive square root of T^2, it is $\pi/4$-sectorial by [25]. The claims for A are then immediate.

Now we calculate the imaginary part $\operatorname{Im} A = (A - A^*)/2\mathrm{i}$ of the operator A. If we note that $1 \in \rho(\mathrm{i}MC^2)$, we find that, for $f \in \operatorname{dom} A = \operatorname{ran} C^2$,

$$\begin{aligned}(\operatorname{Im} A)f &= \frac{1}{2\mathrm{i}}(T^{-1} - T^{-*})f = T^{-*}(T^* - T)T^{-1}f \\ &= -(C^2 - \mathrm{i}C^2MC^2)^{-1}C^2MC^2(C^2 + \mathrm{i}C^2MC^2)^{-1}f = -(I - \mathrm{i}MC^2)^{-1}M(I + \mathrm{i}C^2M)^{-1}f.\end{aligned}$$

It follows that $\operatorname{Im} A$ defined on $\operatorname{dom} A$ is bounded, and since $\operatorname{dom} A$ is dense, $\operatorname{Im} A$ has a bounded extension to $\mathcal{H}$. □

Proposition 4.3. *Let* $Y \in \widetilde{C}_{\mathfrak{H}}$, $\|Y\| = 1$, *be such that* $\ker D_Y = \{0\}$ *for* $D_Y = (I - Y^*Y)^{\frac{1}{2}}$. *Define*

$$A_n := (D_Y)^{-1}(I + \mathrm{i}Y^{*n}Y^n)^{-1}(D_Y)^{-1}, \quad B_n := (D_Y)^{-1}(I + \mathrm{i}Y^{*n}Y^n)(D_Y)^{-1}, \quad n \in \mathbb{N}. \tag{4.3}$$

Then A_n, B_n *are m-sectorial operators with vertex* 0, *semi-angle* $\pi/4$ *and*

$$\operatorname{dom} A_n = \operatorname{dom} A_n^* = \operatorname{dom}(A_n)_R, \quad \operatorname{dom} B_n = \operatorname{dom} B_n^* = \operatorname{dom}(B_n)_R, \quad n \in \mathbb{N}.$$

Proof. Let $n \in \mathbb{N}$. If we set $T_n := A_n^{-1}$ and $C := D_Y$, $F_n := Y^{*n}Y^n$, then we can write

$$T_n = C(I + \mathrm{i}F_n)C = D_Y^2 + \mathrm{i}D_YY^{*n}Y^nD_Y$$

which shows that T_n is a bounded sectorial operator with vertex 0 and semi-angle $\pi/4$ since F_n is a contraction. If we note that the right hand side of (2.20) is independent of $n \in \mathbb{N}$, we conclude that $\operatorname{ran} D_{Y^n} = \operatorname{ran} D_Y$. By Douglas' Lemma, see Theorem 2.1, this range equality implies that there exists a bounded linear operator $\mathcal{N}_n$ with bounded inverse $\mathcal{N}_n^{-1}$ such that

$$D_Y = D_{Y^n}\mathcal{N}_n, \quad D_{Y^n} = D_Y\mathcal{N}_n^{-1}. \tag{4.4}$$

Then

$$\begin{aligned}(I \pm \mathrm{i}F_n)C &= (I \pm \mathrm{i}Y^{*n}Y^n)D_Y = (I \pm \mathrm{i}Y^{*n}Y^n)D_{Y^n}\mathcal{N}_n = D_{Y^n}(I \pm \mathrm{i}Y^{*n}Y^n)\mathcal{N}_n \\ &= D_Y\mathcal{N}_n^{-1}(I + \mathrm{i}Y^{*n}Y^n)\mathcal{N}_n = C\mathcal{N}_n^{-1}(I \pm \mathrm{i}F_n)\mathcal{N}_n.\end{aligned} \tag{4.5}$$

Because the operators $\mathcal{N}_n^{-1}$, $(I \pm \mathrm{i}F_n)$, $\mathcal{N}_n$ are all bijective, this means that

$$(I + \mathrm{i}F_n)\operatorname{ran} C = (I - \mathrm{i}F_n)\operatorname{ran} C = \operatorname{ran} C.$$

Now Theorem 3.11 4) yields the equalities $\operatorname{ran} T_n = \operatorname{ran} T_n^* = \operatorname{ran}(\operatorname{Re} T_n)$ and $\operatorname{dom} A_n = \operatorname{dom} A_n^* = \operatorname{dom}(A_n)_R$. This and Corollary 3.6 1), 2) imply $\operatorname{dom} B_n = \operatorname{dom} B_n^* = \operatorname{dom}(B_n)_R$. □

Notice that the operator $Z_n := \mathrm{i}(\operatorname{Re} A_n^{-1})^{-1}\operatorname{Im} A_n^{-1}$ associated with the operator A_n in (4.3), compare (3.18), takes the form $Z_n = \mathrm{i}D_Y^{-2}D_YY^{*n}Y^nD_Y = \mathrm{i}\mathcal{N}_n^{-1}Y^{*n}Y^n\mathcal{N}_n$, i.e. Z_n is similar to the operator $\mathrm{i}Y^{*n}Y^n$. Similar domain equalities hold for the following m-$\frac{\pi}{4}$-sectorial operators:

$$\begin{array}{lll}(D_{Y^m})^{-1}(I + \mathrm{i}Y^{*n}Y^n)^{\pm 1}(D_{Y^m})^{-1}, & (D_{Y^{*m}})^{-1}(I + \mathrm{i}Y^nY^{*n})^{\pm 1}(D_{Y^{*m}})^{-1}, & \\ (D_{Y^m})^{-1}(I + \mathrm{i}Y^nY^{*n})^{\pm 1}(D_{Y^m})^{-1}, & (D_{Y^{*m}})^{-1}(I + \mathrm{i}Y^{*n}Y^n)^{\pm 1}(D_{Y^{*m}})^{-1}, & n, m \in \mathbb{N},\ n \neq m, \\ (D_{Y^m})^{-1}(I + \mathrm{i}Y_R^n)^{\pm 1}(D_{Y^m})^{-1}, & (D_{Y^{*m}})^{-1}(I + \mathrm{i}Y_R^n)^{\pm 1}(D_{Y^{*m}})^{-1}, & \\ (D_{Y_R})^{-1}(I + \mathrm{i}Y^{*n}Y^n)^{\pm 1}(D_{Y_R})^{-1}, & (D_{Y_R})^{-1}(I + \mathrm{i}Y^nY^{*n})^{\pm 1}(D_{Y_R})^{-1}, & n \in \mathbb{N}.\end{array}$$

Corollary 4.4. *Let $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ be the open unit disc. Then*

$$A_n(\lambda) := (D_Y)^{-1}(I + \mathrm{i}\lambda Y^{*n}Y^n)^{-1}(D_Y)^{-1}, \quad \lambda \in \mathbb{C},\ |\lambda| \le 1,$$

forms a holomorphic family of m-sectorial operators of type (A) and (B) in $\mathbb{D}$. Moreover,

$$\operatorname{dom} A_n(\lambda) = \operatorname{dom} A_n(\lambda)^* = \operatorname{dom}(A_n(\lambda))_R = \operatorname{ran} D_Y^2, \quad \mathcal{D}[A_n(\lambda)] = \operatorname{ran} D_Y.$$

Proof. If we define

$$T_n(\lambda) := A_n(\lambda)^{-1} = D_Y(I + \mathrm{i}\lambda Y^{*n}Y^n)D_Y, \quad \lambda \in \mathbb{C},\ |\lambda| \le 1,$$

then, for $\lambda = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, $x^2 + y^2 \le 1$,

$$T_n(x + \mathrm{i}y) = D_Y((I - yY^{*n}Y^n) + \mathrm{i}xY^{*n}Y^n)D_Y.$$

If $|y| < 1$, the nonnegative selfadjoint operator $I - yY^{*n}Y^n$ has bounded inverse and hence $I - yY^{*n}Y^n + \mathrm{i}xY^{*n}Y^n$ is sectorial. If $y = \pm 1$, then $x = 0$ and the operator $T_n(\pm\mathrm{i}) = D_Y(I \mp Y^{*n}Y^n)D_Y$ is selfadjoint and nonnegative with $\ker T_n(\pm\mathrm{i}) = \{0\}$.

In analogy to (4.5), the operator $T_n(\lambda)$ can be represented in the form

$$T_n(\lambda) = D_Y^2\mathcal{N}_n^{-1}(I + \mathrm{i}\lambda Y^{*n}Y^n)\mathcal{N}_n, \quad \lambda \in \mathbb{C},\ |\lambda| \le 1,$$

with $\mathcal{N}_n$ as in (4.4). Clearly, for $\lambda \in \mathbb{D}$, we have $\operatorname{ran}(I + \mathrm{i}\lambda Y^{*n}Y^n) = \operatorname{ran}(I - \mathrm{i}\bar{\lambda}Y^{*n}Y^n) = \mathfrak{H}$ and therefore $\operatorname{ran} T_n(\lambda) = \operatorname{ran} T_n(\lambda)^* = \operatorname{ran} D_Y^2$. Besides,

$$\operatorname{Re} T_n(x+\mathrm{i}y) = D_Y^2 \mathcal{N}_n^{-1}(I - yY^{*n}Y^n)\mathcal{N}_n, \quad \operatorname{Im} T_n(x+\mathrm{i}y) = xD_Y^2 \mathcal{N}_n^{-1} Y^{*n}Y^n \mathcal{N}_n.$$

Since for $\lambda = x + \mathrm{i}y \in \mathbb{D}$, the inclusion $\operatorname{ran} Y^{*n}Y^n \subseteq \mathfrak{H} = \operatorname{ran}(I - yY^{*n}Y^n)$ holds, it follows that

$$\operatorname{ran}\big(\operatorname{Im} T_n(x+\mathrm{i}y)\big) \subseteq \operatorname{ran}\big(\operatorname{Re} T_n(x+\mathrm{i}y)\big).$$

Now Theorem 3.13 implies that $\operatorname{Re} A_n(\lambda) = (A_n(\lambda))_R$ for $\lambda \in \mathbb{D}$. □

The next theorem strengthens [6, Thm. 10.2] where the equality of operator real part and form real part, as in (4.7) below, could not yet be proved.

Theorem 4.5. *Let A be an m-α-sectorial operator in a Hilbert space $\mathfrak{H}$ with $\alpha \in (0, \frac{\pi}{2})$ and let*

$$T(\lambda) := \exp(-\lambda A), \quad \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)} := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \frac{\pi}{2} - \alpha\right\}$$

be the holomorphic contractive semigroup generated by $-A$, see [28, Thm. IX.1.24]. Then

$$\Psi(\lambda) := A^* \left(I + T(\lambda)\right) A, \quad \Phi(\lambda) := A^* \left(I + T(\lambda)\right)^{-1} A, \quad \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}, \tag{4.6}$$

are m-$(\alpha + |\arg \lambda|)$-sectorial and form holomorphic families of type (B). *If* $\operatorname{dom} A = \operatorname{dom} A^*$, *then*

$$\begin{aligned} &\operatorname{dom} \Psi(\lambda) = \operatorname{dom} \Psi(\lambda)^* = \operatorname{dom}(\Psi(\lambda))_R = \\ &= \operatorname{dom} \Phi(\lambda) = \operatorname{dom} \Phi(\lambda)^* = \operatorname{dom}(\Phi(\lambda))_R = \operatorname{dom} A^*A, \end{aligned} \quad \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}, \tag{4.7}$$

and both $\Psi(\lambda)$ and $\Phi(\lambda)$, $\lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}$, form holomorphic families of type (A).

Proof. By the first representation theorem, the operators $\Psi(\lambda)$ and $\Phi(\lambda)$ in (4.6) are associated with the closed sectorial sesquilinear forms

$$\begin{aligned} \psi(\lambda)[f,g] &:= \left((I + T(\lambda))Af, Ag\right), \\ \phi(\lambda)[f,g] &:= \left((I + T(\lambda))^{-1}Af, Ag\right), \end{aligned} \quad f, g \in \operatorname{dom} A, \ \ \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)},$$

respectively. Besides, $\Psi(\lambda)$ and $\Phi(\lambda)$ are coercive if and only if A is coercive.

All claims except for $\operatorname{dom}(\Psi(\lambda))_R = \operatorname{dom}(\Phi(\lambda))_R = \operatorname{dom} A^*A$, $\lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}$, follow from [6, Thm. 10.2]. To prove these remaining identities, we show that, if $\operatorname{dom} A = \operatorname{dom} A^*$, then

$$(I + T(\lambda))\operatorname{dom} A^* = (I + T(\lambda)^*)\operatorname{dom} A^* = (I + \operatorname{Re} T(\lambda))\operatorname{dom} A^* = \operatorname{dom} A^*, \ \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}. \tag{4.8}$$

First we note that $-1 \in \rho(T(\lambda)) \cap \rho(\operatorname{Re}(T(\lambda))$ according to the proof of [6, Thm. 10.2]. Since $\operatorname{ran} T(\lambda) \subset \operatorname{dom} A = \operatorname{dom} A^*$ by assumption and $\operatorname{ran} T(\lambda)^* \subset \operatorname{dom} A^*$, it follows that

$$\operatorname{ran}(\operatorname{Re} T(\lambda)) \subset \operatorname{dom} A^*, \quad \lambda \in \mathcal{S}_{(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)}, \tag{4.9}$$

and that $\operatorname{dom} A^*$ contains each of the first three sets in (4.8). Vice versa, if $g \in \operatorname{dom} A^*$, there exists $f \in \mathcal{H}$ such that $g = (I + T(\lambda))f$. Then $\operatorname{ran} T(\lambda) \subset \operatorname{dom} A^*$ implies $f \in \operatorname{dom} A^*$ which proves that $\operatorname{dom} A^* \subset (I + T(\lambda))\operatorname{dom} A^*$. The proof for the remaining two inclusions in (4.8) is analogous if we use $\operatorname{ran} T(\lambda) \subset \operatorname{dom} A^*$ and (4.9).

Clearly, we have

$$\begin{aligned}\operatorname{Re}\psi(\lambda)[f,f] &= \|(I+\operatorname{Re}T(\lambda))^{\frac{1}{2}}Af\|^2,\\ \operatorname{Re}\phi(\lambda)[f,f] &= \|(I+\operatorname{Re}T(\lambda))^{\frac{1}{2}}(I+T(\lambda))^{-1}Af\|^2,\end{aligned}\qquad f\in\operatorname{dom}A,\ \ \lambda\in\mathcal{S}_{(-\frac{\pi}{2}+\alpha,\frac{\pi}{2}-\alpha)},$$

and hence

$$\begin{aligned}(\Psi(\lambda))_R &= A^*(I+\operatorname{Re}T(\lambda))A,\\ (\Phi(\lambda))_R &= A^*(I+T(\lambda)^*)^{-1}(I+\operatorname{Re}T(\lambda))(I+T(\lambda))^{-1}A.\end{aligned}\qquad \lambda\in\mathcal{S}_{(-\frac{\pi}{2}+\alpha,\frac{\pi}{2}-\alpha)}.$$

This together with (4.8) implies $\operatorname{dom}(\Psi(\lambda))_R=\operatorname{dom}(\Phi(\lambda))_R=\operatorname{dom}A^*A$, as required. □

In the next theorem we show how, from one bounded m-sectorial operator T possessing property (4.2), one can construct infinitely many m-sectorial operators with the same property.

Theorem 4.6. *Let T be of the form (3.11), i.e.*

$$T=C(I+\mathrm{i}F)C\quad\textit{with}\quad F=F^*\in\mathbf{B}(\mathfrak{H}),\quad C\in\mathbf{B}^+(\mathfrak{H}),\quad \ker C=\{0\},\quad \operatorname{ran}C\neq\mathfrak{H}.\tag{4.10}$$

1) *If T satisfies*

$$\operatorname{ran}T=\operatorname{ran}T^*=\operatorname{ran}(\operatorname{Re}T),$$

then

a) *for any $\alpha\in(0,1)$ the operator $T_\alpha:=C^\alpha(I+\mathrm{i}F)C^\alpha$ satisfies the equalities*

$$\operatorname{ran}T_\alpha=\operatorname{ran}T_\alpha^*=\operatorname{ran}(\operatorname{Re}T_\alpha);$$

therefore, if $A_\alpha:=T_\alpha^{-1}$, then $\operatorname{dom}A_\alpha=\operatorname{dom}A_\alpha^*=\operatorname{dom}(A_\alpha)_R$;

b) *for any $\beta\in\mathbb{R}$ in some neighbourhood of 0 or of 1, the operator $\widetilde{T}_\beta:=C(I+\mathrm{i}\beta F)C$ satisfies the equalities*

$$\operatorname{ran}\widetilde{T}_\beta=\operatorname{ran}\widetilde{T}_\beta^*=\operatorname{ran}(\operatorname{Re}\widetilde{T}_\beta);$$

therefore, if $\widetilde{A}_\beta:=\widetilde{T}_\beta^{-1}$, then $\operatorname{dom}\widetilde{A}_\beta=\operatorname{dom}\widetilde{A}_\beta^*=\operatorname{dom}(\widetilde{A}_\beta)_R$;

c) *there exists $\varepsilon>0$ such that, for $\varphi\in(-\varepsilon,\varepsilon)$, the operator F_φ given by*

$$F_\varphi:=\left(F+\mathrm{i}\frac{1-\exp(-\mathrm{i}\varphi)}{1+\exp(-\mathrm{i}\varphi)}I\right)\left(I-\mathrm{i}\frac{1-\exp(-\mathrm{i}\varphi)}{1+\exp(-\mathrm{i}\varphi)}F\right)^{-1}=\left(F-\tan\frac{\varphi}{2}I\right)\left(I+\tan\frac{\varphi}{2}F\right)^{-1}$$

is well-defined, bounded and self-adjoint, and the operator

$$\widehat{T}_\varphi:=C(I+\mathrm{i}F_\varphi)C$$

satisfies the equalities

$$\operatorname{ran}\widehat{T}_\varphi=\operatorname{ran}\widehat{T}_\varphi^*=\operatorname{ran}(\operatorname{Re}\widehat{T}_\varphi);$$

therefore, if $\widehat{A}_\varphi:=\widehat{T}_\varphi^{-1}$, then A_φ is m-sectorial and $\operatorname{dom}\widehat{A}_\varphi=\operatorname{dom}\widehat{A}_\varphi^*=\operatorname{dom}(\widehat{A}_\varphi)_R$;

2) *The operator*

$$\widetilde{T} := C(I + \mathrm{i}F)^{-1}C \tag{4.11}$$

satisfies

a) $\operatorname{ran}\widetilde{T} = \operatorname{ran}\widetilde{T}^*$ *if and only if* $\operatorname{ran}T = \operatorname{ran}T^*$,

b) $\operatorname{ran}\widetilde{T} = \operatorname{ran}\widetilde{T}^* = \operatorname{ran}(\operatorname{Re}\widetilde{T})$ *if and only if* $\operatorname{ran}T = \operatorname{ran}T^* = \operatorname{ran}(\operatorname{Re}T)$.

Proof. 1) First we note that, by Theorem 3.11 4), assumption (4.10) on T yields that $\operatorname{ran}C$ is invariant for the operators $I \pm \mathrm{i}F$ and $(I \pm \mathrm{i}F)^{-1}$, i.e. $(I \pm \mathrm{i}F)\operatorname{ran}C = \operatorname{ran}C$.

a) This implies that, if $\alpha \in (0, 1)$, also $\operatorname{ran}C^\alpha$ is invariant range for $I \pm \mathrm{i}F$ and $(I \pm \mathrm{i}F)^{-1}$, see Section 2.4 and [17], [18], and hence $(I \pm \mathrm{i}F)\operatorname{ran}C^\alpha = \operatorname{ran}C^\alpha$. Now Theorem 3.11 4) yields that $\operatorname{ran}T_\alpha = \operatorname{ran}T_\alpha^* = \operatorname{ran}(\operatorname{Re}T_\alpha)$ for $\alpha \in (0, 1)$.

b) Because $(I \pm \mathrm{i}F)\operatorname{ran}C = \operatorname{ran}C$, Douglas' Lemma, see Theorem 2.1, yields that there are bounded operators $V_\pm$ with bounded inverses $V_\pm^{-1}$ such that $(I \pm \mathrm{i}F)C = CV_\pm$ or, equivalently,

$$\pm\mathrm{i}FC = C(V_\pm - I).$$

Then, for any $\beta \in \mathbb{R}$, $\beta \neq 0$,

$$(I \pm \mathrm{i}\beta F)C = C(\beta(V_\pm - I) + I) = \beta C(V_\pm - (1 - \beta^{-1})I). \tag{4.12}$$

Since $V_\pm$ are bounded and $0 \in \rho(V_-) \cap \rho(V_+)$, there exist ϵ_0, $\epsilon_1 > 0$ such that

$$\beta \in (-\epsilon_0, \epsilon_0) \cup (1 - \epsilon_1, 1 + \epsilon_1) \implies 1 - \beta^{-1} \in \rho(V_-) \cap \rho(V_+).$$

Together with (4.12), this implies that $(I \pm \mathrm{i}\beta F)\operatorname{ran}C = \operatorname{ran}C$ and thus, by Theorem 3.11 4), that $\operatorname{ran}\widetilde{T}_\beta = \operatorname{ran}\widetilde{T}_\beta^* = \operatorname{ran}(\operatorname{Re}\widetilde{T}_\beta)$ for such β.

c) Let $U := (I + \mathrm{i}F)(I - \mathrm{i}F)^{-1}$. Then $-1 \in \rho(U)$ since F is bounded and, due to (4.10) and Theorem 3.11 4), see also Remark 3.12,

$$U\operatorname{ran}C = \operatorname{ran}C, \quad U^*\operatorname{ran}C = \operatorname{ran}C, \quad (U + I)\operatorname{ran}C = \operatorname{ran}C.$$

It follows that $-1 \in \rho(U\restriction\mathfrak{H}_C)$ where $\mathfrak{H}_C = \operatorname{ran}C$ with inner product given by (2.21). Then, because $\rho(U) \cap \rho(U\restriction\mathfrak{H}_C)$ is an open set, there exists $\varepsilon > 0$ such that

$$\varphi \in (-\varepsilon, \varepsilon) \implies -\exp(\mathrm{i}\varphi) \in \rho(U) \cap \rho(U\restriction\mathfrak{H}_C). \tag{4.13}$$

Hence, the operator $U_\varphi := \exp(-\mathrm{i}\varphi)U$ is unitary in $\mathfrak{H}$ with $-1 \in \rho(U_\varphi)$ and satisfies

$$U_\varphi\operatorname{ran}C = \operatorname{ran}C, \quad U_\varphi^*\operatorname{ran}C = \operatorname{ran}C, \quad (U_\varphi + I)\operatorname{ran}C = \operatorname{ran}C; \tag{4.14}$$

here, for last identity we note that $(U_\varphi + I)\operatorname{ran}C = (U + \exp(\mathrm{i}\varphi)I)\operatorname{ran}C = \operatorname{ran}C$ by (4.13). Moreover, using that $\exp(-\mathrm{i}\varphi) = (1 - \mathrm{i}\tan\frac{\varphi}{2})(1 + \mathrm{i}\tan\frac{\varphi}{2})^{-1}$, it is not difficult to show that the operator F_φ defined in c) is related to U_φ by

$$F_\varphi = \mathrm{i}(I - U_\varphi)(I + U_\varphi)^{-1}$$

and hence, by (4.14) and Remark 3.12,

$$(I - iF_\varphi)\operatorname{ran} C = (I + iF_\varphi)\operatorname{ran} C = \operatorname{ran} C.$$

This and Theorem 3.11 4) now yield all claims for the bounded m-sectorial operator $\widehat{T}_\varphi$ and its m-sectorial inverse $\widehat{A}_\varphi = \widehat{T}_\varphi^{-1}$.

2) Let $\widetilde{T}$ be given by (4.11). By Theorem 3.11 3) and since $C : \mathfrak{H} \to \operatorname{ran} C$ is bijective, we have the equivalences

$$\begin{aligned}\operatorname{ran} T = \operatorname{ran} T^* &\iff (I + iF)(I - iF)^{-1}\operatorname{ran} C = \operatorname{ran} C \iff (I - iF)^{-1}\operatorname{ran} C = (I + iF)^{-1}\operatorname{ran} C\\ &\iff C(I - iF)^{-1}\operatorname{ran} C = C(I + iF)^{-1}\operatorname{ran} C \iff \operatorname{ran} \widetilde{T} = \operatorname{ran} \widetilde{T}^*.\end{aligned}$$

Finally, to prove the last claim, we first note that $\operatorname{Re} \widetilde{T} = C(I + F^2)^{-1}C$. Again by Theorem 3.11 4) and since $C : \mathfrak{H} \to \operatorname{ran} C$, $I \pm iF : \mathfrak{H} \to \mathfrak{H}$ are bijective, we conclude that

$$\begin{aligned}\operatorname{ran} T = \operatorname{ran} T^* = \operatorname{ran}(\operatorname{Re} T) &\iff \operatorname{ran} C = (I + iF)\operatorname{ran} C = (I - iF)\operatorname{ran} C\\ &\iff \operatorname{ran} C = (I + iF)^{-1}\operatorname{ran} C = (I - iF)^{-1}\operatorname{ran} C\\ &\iff (I - iF)^{-1}\operatorname{ran} C = (I + iF)^{-1}\operatorname{ran} C = (I + F^2)^{-1}\operatorname{ran} C\\ &\iff C(I - iF)^{-1}\operatorname{ran} C = C(I + iF)^{-1}\operatorname{ran} C = C(I + F^2)^{-1}\operatorname{ran} C\\ &\iff \operatorname{ran} \widetilde{T} = \operatorname{ran} \widetilde{T}^* = \operatorname{ran}(\operatorname{Re} \widetilde{T}). \quad \square\end{aligned}$$

In the following we denote by $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ the unit circle.

Proposition 4.7. *Let C be a bounded selfadjoint operator such that $C \geq 0$, $\ker C = \{0\}$, $\operatorname{ran} C \neq \mathfrak{H}$ and let U be a unitary operator in $\mathfrak{H}$. Assume that*

$$U\operatorname{ran} C = \operatorname{ran} C.$$

Then the following are equivalent:

(i) $\lambda \in \rho(U) \cap \mathbb{T}$ *and* $(U - \lambda I)\operatorname{ran} C = \operatorname{ran} C$,
(ii) $\lambda \notin \sigma_p(U)$, *the operator* $A_\lambda := C^{-1}(I - \lambda U^*)C^{-1}$, $\lambda \in \mathbb{T}$, *is m-sectorial and*

$$\operatorname{dom} A_\lambda = \operatorname{dom} A_\lambda^* = \operatorname{dom}(A_\lambda)_R.$$

Proof. (i) $\Longrightarrow$ (ii): Let $\lambda \in \mathbb{T}$ be such that (i) holds and set $U_\lambda := -\overline{\lambda}U$. Then U_λ is unitary, $-1 \in \rho(U_\lambda)$ and, since $U_\lambda + I = -\overline{\lambda}(U - \lambda I)$, we have

$$U_\lambda\operatorname{ran} C = \operatorname{ran} C, \quad (U_\lambda + I)\operatorname{ran} C = \operatorname{ran} C. \tag{4.15}$$

Further, the operator

$$F_\lambda := i(I - U_\lambda)(I + U_\lambda)^{-1} = -iI + 2i(I + U_\lambda)^{-1} \tag{4.16}$$

is selfadjoint, bounded and $I + iF_\lambda = 2(I + U_\lambda^*)^{-1}$.

Therefore the bounded operator $T_\lambda := C(I + U_\lambda^*)^{-1}C = \dfrac{1}{2}C(I + iF_\lambda)C$ is sectorial and $\ker T_\lambda = \{0\}$. Moreover, since by (4.15),

$$\operatorname{ran} C = (I + U_\lambda)^{-1}\operatorname{ran} C = (I + U_\lambda)^{-1}U_\lambda\operatorname{ran} C = (U_\lambda^* + I)^{-1}\operatorname{ran} C,$$

it follows that

$$\operatorname{ran} T_\lambda = \operatorname{ran} T_\lambda^* = \operatorname{ran}(\operatorname{Re} T_\lambda) = \operatorname{ran} C^2.$$

Then $A_\lambda = T_\lambda^{-1}$ is m-sectorial and, by Theorem 3.11 4),

$$\operatorname{dom} A_\lambda = \operatorname{dom} A_\lambda^* = \operatorname{dom}(A_\lambda)_R = \operatorname{ran} C^2.$$

(ii) $\Longrightarrow$ (i): Let $\lambda \in \mathbb{T}$ be such that (ii) holds and let again $U_\lambda := -\overline{\lambda} U$. Since U_λ is unitary, we have $\operatorname{ran}(I + U_\lambda) = \operatorname{ran}((I + U_\lambda^*)U_\lambda) = \operatorname{ran}(I + U_\lambda^*)$ and $\overline{\operatorname{ran}}(I + U_\lambda) = \overline{\operatorname{ran}}(U - \lambda I) = (\ker(U^* - \overline{\lambda} I))^\perp = (\ker(U - \lambda I))^\perp = \mathfrak{H}$ because $\lambda \notin \sigma_p(U)$. Thus the operator F_λ defined as in (4.16) is densely defined, selfadjoint and $I + \mathrm{i}F_\lambda = 2(I + U_\lambda^*)^{-1}$. Therefore, we can write

$$A_\lambda = C^{-1}(I + U_\lambda^*)C^{-1} = 2C^{-1}(I + \mathrm{i}F_\lambda)^{-1}C^{-1},$$

and hence, for all $f \in \operatorname{dom} A_\lambda$,

$$\begin{aligned}
&\operatorname{Re}(A_\lambda f, f) = 2\operatorname{Re}\left((I + \mathrm{i}F_\lambda)^{-1}C^{-1}f, C^{-1}f\right) = 2\left\|(I + \mathrm{i}F_\lambda)^{-1}C^{-1}f\right\|^2, \\
&\operatorname{Im}(A_\lambda f, f) = 2\operatorname{Im}\left((I + \mathrm{i}F_\lambda)^{-1}C^{-1}f, C^{-1}f\right) = -2\left(F_\lambda(I + \mathrm{i}F_\lambda)^{-1}C^{-1}f, (I + \mathrm{i}F_\lambda)^{-1}C^{-1}f\right).
\end{aligned}$$

Since $\operatorname{ran}(I + \mathrm{i}F_\lambda)^{-1}C^{-1} = \operatorname{dom} F_\lambda$, the m-sectoriality of A_λ implies that F_λ is a bounded operator. Because F_λ is closed and densely defined, it follows that it is everywhere defined, i.e. $\mathfrak{H} = \operatorname{dom} F_\lambda = \operatorname{ran}(I + U_\lambda) = \operatorname{ran}(U - \lambda I)$, and hence $\lambda \in \rho(U)$.

Moreover, Theorem 3.11 4) and Remark 3.12 apply to the operator

$$T_\lambda := A_\lambda^{-1} = \frac{1}{2}C(I + \mathrm{i}F_\lambda)C,$$

and thus the last assumption in (ii) implies that $(U - \lambda I)\operatorname{ran} C = (I + U_\lambda)\operatorname{ran} C = \operatorname{ran} C$. $\square$

4.1.1. Second order elliptic differential operators

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with infinitely smooth boundary $\partial\Omega$, and let $H^1(\Omega)$, $\mathring{H}^1(\Omega)$, $H^2(\Omega)$ be the Sobolev spaces associated with $\mathcal{L}^2(\Omega)$. In $\mathcal{L}^2(\Omega)$ we consider second order differential expressions

$$\tau_A(u) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(b_{jk}(\cdot)\frac{\partial u}{\partial x_k}\right)$$

with coefficients b_{jk} belonging to the class $C^\infty(\bar{\Omega})$. We suppose that τ_A satisfies the uniform ellipticity condition, i.e. that there exists a $c > 0$ with

$$\operatorname{Re}\left(\sum_{j,k=1}^{n} b_{jk}(x)\xi_k\overline{\xi_j}\right) \geqslant c\sum_{k=1}^{n}|\xi_k|^2, \quad x \in \Omega,\ \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{C}^n. \tag{4.17}$$

Here we consider operators A in $\mathcal{L}^2(\Omega)$ of the form $A = L^*BL$ where L is an unbounded operator from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)^n = \mathcal{L}^2(\Omega, \mathbb{C}^n)$, B denotes the bounded operator in $\mathcal{L}^2(\Omega)^n$ given by

$$\begin{aligned}
&Lu = \nabla u, \quad \operatorname{dom} L = \mathring{H}^1(\Omega), \\
&\left(B\vec{f}\right)(x) = B(x)\vec{f}(x), \quad \vec{f} = ((f_i)_{i=1}^n)^T \in \mathcal{L}^2(\Omega)^n,\ x \in \Omega,
\end{aligned}$$

with $B(\cdot) := \big(b_{jk}(\cdot)\big)_{j,k=1}^{n}$ and the operator L^* acting from $\mathcal{L}^2(\Omega)^n$ into $\mathcal{L}^2(\Omega)$ has the form, see [29, Thm. 6.2],

$$L^*\vec{f} = -\mathrm{div}\vec{f}, \quad \mathrm{dom}\, L^* = \big\{\vec{f} \in \mathcal{L}_2(\Omega)^n : \mathrm{div}\vec{f} \in \mathcal{L}_2(\Omega)\big\}. \tag{4.18}$$

Due to the classical Poincaré inequality, i.e.

$$\|\nabla u\|^2_{\mathcal{L}^2(\Omega)} \geq \gamma \|u\|^2_{\mathcal{L}^2(\Omega)}, \quad u \in \mathring{H}^1(\Omega),$$

with some $\gamma > 0$, the operator L has bounded inverse. Condition (4.17) implies that B is a bounded sectorial operator in $\mathcal{L}^2(\Omega)^n$ with bounded inverse. Hence A is a coercive m-sectorial operator in $\mathcal{L}^2(\Omega)$ associated with the closed sesquilinear form given by, see [12, Chapt. III, § 2], [15, Thm. 7.5.7],

$$\begin{aligned} &Au = \tau_A(u) = -\mathrm{div}\big(B(\cdot)\nabla u(\cdot)\big), \ \ \mathrm{dom}\, A = \mathring{H}^1(\Omega) \cap H^2(\Omega), \\ &A[u,v] = (BLu, Lv) = \int\limits_{\Omega} \big(B(x)\nabla u(x), \nabla v(x)\big)_{\mathbb{C}^n} \mathrm{d}x, \ \ u, v \in D[A] = \mathring{H}^1(\Omega). \end{aligned}$$

The adjoint operator A^* is of the form

$$\begin{aligned} &A^*u = L^*B^*Lu = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(\overline{b_{kj}(\cdot)}\frac{\partial u}{\partial x_k}\right) = -\mathrm{div}\big((B(\cdot))^*\nabla u(\cdot)\big), \\ &\mathrm{dom}\, A^* = \mathrm{dom}\, A = \mathring{H}^1(\Omega) \cap H^2(\Omega). \end{aligned}$$

It follows that

$$\mathrm{Re}\, A = L^*(\mathrm{Re}\, B)L = -\mathrm{div}\big((\mathrm{Re}\, B(\cdot))\nabla\big), \quad \mathrm{dom}\,\mathrm{Re}\, A = \mathring{H}^1(\Omega) \cap H^2(\Omega).$$

On the other hand, since

$$A_R[u,v] = ((\mathrm{Re}\, B)Lu, Lv) = \frac{1}{2}\int\limits_{\Omega} \big((B(x) + B(x)^*)\nabla u(x), \nabla v(x)\big)_{\mathbb{C}^n} \mathrm{d}x, \quad u, v \in D[A_R] = \mathring{H}^1(\Omega),$$

the operator A_R takes the form

$$A_R = L^*(\mathrm{Re}\, B)L, \quad \mathrm{dom}\, A_R = \mathring{H}^1(\Omega) \cap H^2(\Omega).$$

Thus we obtain the equalities

$$\mathrm{dom}\, A = \mathrm{dom}\, A^* = \mathrm{dom}\, A_R. \tag{4.19}$$

Note that, by Kato's result [27, Cor. 2], this implies that

$$\mathrm{dom}\, A^{\frac{1}{2}} = \mathrm{dom}\, A^{*\frac{1}{2}} = \mathrm{dom}\,(A_R)^{\frac{1}{2}} = \mathring{H}^1(\Omega).$$

We mention that the latter equalities were established in [10] by means of reducing to the case of $\Omega = \mathbb{R}^n$ considered in [8]; for more details and for the case of more general boundary conditions we refer e.g. to [11].

The domain equalities (4.19) and Theorem 3.1 2) now allow us to show the invariance of the domains of L and L^* for certain operators. For this, we recall that the range of L is the closed subspace $\mathrm{ran}\, L = \nabla\mathring{H}^1(\Omega)$ of $\mathcal{L}^2(\Omega)^n$ and we denote by P_L the orthogonal projection of $\mathcal{L}^2(\Omega)^n$ onto $\mathrm{ran}\, L$, note that

$$P_L\vec{f} = \nabla\big(\Delta^{-1}(\mathrm{div}\vec{f})\big), \quad \vec{f} \in \mathrm{dom}\, L^* \subset \mathcal{L}_2(\Omega)^n,$$

where $g := \Delta^{-1}(\mathrm{div}\vec{f}) \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ is the solution of the Dirichlet problem $\Delta g = \mathrm{div}\vec{f}$ for the Laplace operator. If Ω is connected, then $\mathcal{L}^2(\Omega)^n = \nabla\mathring{H}^1(\Omega) \oplus H(\mathrm{div}0, \Omega)$ where $H(\mathrm{div}\, 0, \Omega) := \{\vec{f} \in \mathcal{L}^2(\Omega)^n : \mathrm{div}\, \vec{f} = 0\}$, see [13, Chapt. IX, § 3, Prop. 1], and so $\mathrm{ran}\, L = \big(H(\mathrm{div}\, 0, \Omega)\big)^\perp$ and $P_L = P_{\nabla \mathring{H}^1(\Omega)}$.

If we now regard L as an operator from $\mathcal{L}^2(\Omega)$ to $\mathrm{ran}\, L$ and set $B_L := P_L B\upharpoonright \mathrm{ran}\, L$, then B_L acts in $\mathrm{ran}\, L$ and Theorem 3.1 applies, which yields that $\mathrm{dom}\, L^*$ given by (4.18) is invariant with respect to the operators $(\mathrm{Im}\, B_L)(\mathrm{Re}\, B_L)^{-1}$, $B_L^* B_L^{-1}$ and $B_L B_L^{-*} = (B_L^* B_L^{-1})^{-1}$, i.e.

$$(\mathrm{Im}\, B_L)(\mathrm{Re}\, B_L)^{-1}\mathrm{dom}\, L^* \subseteq \mathrm{dom}\, L^*, \quad B_L^* B_L^{-1}\mathrm{dom}\, L^* = \mathrm{dom}\, L^*.$$

4.2. m-sectorial operators A with $\mathrm{Re}\, A = A_R$

In the sequel we need the fact that *the spectrum $\sigma(Z)$ of an arbitrary non-unitary isometry Z coincides with the closed unit disc*, i.e.

$$\sigma(Z) = \overline{\mathbb{D}} = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}\,. \tag{4.20}$$

In fact, according to the Wold decomposition, see [39, Thm. I.1.1], the Hilbert space $\mathfrak{H}$ decomposes into an orthogonal sum $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ such that $\mathfrak{H}_0$ and $\mathfrak{H}_1$ reduce Z, the part of Z on $\mathfrak{H}_0$ is unitary and the part of Z on $\mathfrak{H}_1$ is a unilateral shift; the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are of the form

$$\mathfrak{H}_0 = \bigcap_{n=0}^{\infty} Z^n\mathfrak{H}, \quad \mathfrak{H}_1 = \mathfrak{H}_0^\perp = \bigoplus_{n=0}^{\infty} Z^n((\mathrm{ran}\, Z)^\perp).$$

Then, since the spectrum of the unilateral shift coincides with the closed unit disc $\overline{\mathbb{D}}$, the equality (4.20) holds.

Moreover, the linear manifold $\mathrm{ran}\,(\lambda I - Z)$ is a proper subspace of $\mathfrak{H}$ for every $\lambda \in \mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and

$$\dim\, (\mathrm{ran}\,(\lambda I - Z))^\perp = \dim\, \ker(\overline{\lambda} I - Z^*) = \dim\,(\mathrm{ran}\, Z)^\perp, \quad \lambda \in \mathbb{D}.$$

Theorem 4.8. *Let Q be a bounded nonnegative selfadjoint operator in the Hilbert space $\mathfrak{H}$, $\ker Q = \{0\}$ and $\mathrm{ran}\, Q \neq \mathfrak{H}$. Let $\mathfrak{M}$ be a proper subspace in $\mathfrak{H}$ such that*

$$\mathrm{ran}\, Q \cap \mathfrak{M}^\perp = \{0\}. \tag{4.21}$$

Then the operator

$$A(\lambda) := (\lambda Q + \mathrm{i}(QP_{\mathfrak{M}}Q)^{\frac{1}{2}})^{-1}, \quad \lambda \in \mathbb{C}_+, \tag{4.22}$$

is m-sectorial and

1) *$A(\lambda)$, $\lambda \in \mathbb{C}_+$, forms a holomorphic family of type (B) on the open right-half plane and*

$$\mathrm{Re}\, A(\lambda) = (A(\lambda))_R, \quad \mathrm{dom}\,((A(\lambda))_R)^{\frac{1}{2}} = \mathrm{ran}\, Q^{\frac{1}{2}};$$

2) *if $|\lambda| > 1$ and $\mathrm{Re}\, \lambda > 0$, then*

$$\mathrm{dom}\, A(\lambda) = \mathrm{dom}\, A(\lambda)^* = \mathrm{dom}\,(A(\lambda))_R = \mathrm{ran}\, Q;$$

3) *if* $|\lambda| \le 1$ *and* $\operatorname{Re}\lambda > 0$, *then*

$$\operatorname{dom} A(\lambda) \neq \operatorname{dom} A(\lambda)^*, \quad \operatorname{Re} A(\lambda) = (A(\lambda))_R.$$

Proof. If we define the bounded operator, see [4, Thm. 5.2],

$$T_0 := Q + \mathrm{i}\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}\,, \tag{4.23}$$

then $\operatorname{Re} T_0 = Q \ge 0$, i.e. T_0 is accretive and $\ker T_0 = \ker T_0^* = \{0\}$ since $\ker Q = \{0\}$. Because

$$\left\| (QP_{\mathfrak{M}}Q)^{\frac{1}{2}} f \right\|^2 = \|P_{\mathfrak{M}}Qf\|^2, \quad f \in \mathfrak{H},$$

there exists a partial isometry V in $\mathfrak{H}$ such that

$$(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = VP_{\mathfrak{M}}Q = QV^*.$$

Note that $V = W^*$ if $P_{\mathfrak{M}}Q = V^*\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = V^*((P_{\mathfrak{M}}Q)^*P_{\mathfrak{M}}Q)^{\frac{1}{2}}$ is the polar decomposition of $P_{\mathfrak{M}}Q$ and W is the partial isometry with initial space $(\ker(P_{\mathfrak{M}}Q))^\perp$ and final space $\overline{\operatorname{ran}}\,(P_{\mathfrak{M}}Q)$. Hence V has initial space $\overline{\operatorname{ran}}\,(P_{\mathfrak{M}}Q)$ and final space $(\ker(P_{\mathfrak{M}}Q))^\perp$. Since $\operatorname{ran} Q \cap \mathfrak{M}^\perp = \{0\}$, it follows that $\overline{\operatorname{ran}}\,(P_{\mathfrak{M}}Q) = \mathfrak{M}$ and $\operatorname{ran} V = (\ker(P_{\mathfrak{M}}Q))^\perp = (\ker Q)^\perp = \mathfrak{H}$. Thus the operator

$$Z_0 := \mathrm{i}V^* = \mathrm{i}Q^{-1}\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} \tag{4.24}$$

maps $\mathfrak{H}$ isometrically onto $\mathfrak{M}$ and, since $\mathfrak{M} \subsetneq \mathfrak{H}$ by assumption, Z_0 is a non-unitary isometry in $\mathfrak{H}$ and thus $\sigma(Z_0) = \overline{\mathbb{D}}$, see (4.20). Moreover, $\ker Z_0^* = \mathfrak{M}^\perp$ and Z_0 satisfies the equality

$$QZ_0 = -Z_0^*Q. \tag{4.25}$$

Therefore T_0 and T_0^* can be rewritten as

$$T_0 = Q(I + Z_0) = (I - Z_0^*)Q, \quad T_0^* = Q(I - Z_0) = (I + Z_0^*)Q.$$

Since $\operatorname{ran}(\operatorname{Im} T_0) = \operatorname{ran}\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = Q\,\mathfrak{M} \subset \operatorname{ran} Q = \operatorname{ran}(\operatorname{Re} T_0)$, Theorem 3.13, see also (3.20), yields that T_0 is $\pi/4$-sectorial and dissipative, i.e. $\operatorname{Im}(T_0 f, f) \ge 0$, $f \in \mathfrak{H}$. Moreover, since $\|Z_0\| = 1$, it follows that T_0^2 is accretive but non-sectorial. Consequently, T_0 is the accretive square root of the accretive operator T_0^2 and in [4, Thm. 5.2]) it was proved that $\operatorname{ran} T_0 \neq \operatorname{ran} T_0^*$.

If we set

$$T(\lambda) := A(\lambda)^{-1} = \lambda Q + \mathrm{i}(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}, \quad \lambda \in \mathbb{C}_+,$$

with $A(\lambda)$ in (4.22), then $T_0 = T(1)$ and thus $\operatorname{ran} T(1) \neq \operatorname{ran} T(1)^*$, $\operatorname{ran} \operatorname{Im} T(1) \subseteq \operatorname{ran} \operatorname{Re} T(1)$. By (4.24), we have

$$T(\lambda) = Q(\lambda I + Z_0), \quad T(\lambda)^* = Q(\overline{\lambda} I - Z_0). \tag{4.26}$$

1), 2) Since $\operatorname{Re} T(\lambda) = \operatorname{Re}(\lambda Q)$, relation (2.14) yields that the function $A(\lambda)$ is a holomorphic family of type (B) in the open right half-plane with $D[A(\lambda)] = \operatorname{ran} Q^{\frac{1}{2}}$.

For arbitrary $\lambda \in \mathbb{C}_+$, we can now apply Theorem 3.16 with $S = QV^*$ therein, which yields that $\operatorname{Re} A(\lambda) = (A(\lambda))_R$ and, since $-\lambda, \overline{\lambda} \in \rho(Q^{-1}S) = \rho(V^*)$ if $|\lambda| > 1$,

$$\operatorname{dom} A(\lambda) = \operatorname{dom} A(\lambda)^* = \operatorname{dom}(A(\lambda))_R = \operatorname{ran} Q \quad \text{if} \quad \lambda \in \mathbb{C}_+,\ |\lambda| > 1.$$

3) Let $\lambda \in \mathbb{C}_+$, $|\lambda| \le 1$. Then $-\lambda, \overline{\lambda} \in \sigma(Z_0)\,(= \overline{\mathbb{D}})$. Since Z_0 is an isometry, it is injective and so it follows that $\operatorname{ran}(\lambda I + Z_0) \ne \mathfrak{H}$, $\operatorname{ran}(\overline{\lambda} I - Z_0) \ne \mathfrak{H}$. Moreover, by (3.23),

$$\operatorname{ran}(\lambda I + Z_0) \ne \operatorname{ran}(\overline{\lambda} I - Z_0),$$

and hence, by (4.26),

$$\operatorname{dom} A(\lambda) = \operatorname{ran} T(\lambda) \ne \operatorname{ran} T(\lambda)^* = \operatorname{dom} A(\lambda)^*, \quad \lambda \in \mathbb{C}_+,\ |\lambda| \le 1. \quad \square$$

4.3. m-sectorial operators A with $\operatorname{dom} A = \operatorname{dom} A^*$, $\operatorname{Re} A \ne A_R$

Our next goal is to prove the following theorem.

Theorem 4.9. *Let Q be a bounded nonnegative selfadjoint operator in the Hilbert space $\mathfrak{H}$, $\ker Q = \{0\}$ and $\operatorname{ran} Q \ne \mathfrak{H}$. Let $\mathfrak{M}$ be a proper subspace in $\mathfrak{H}$ such that (4.21) holds, i.e. $\operatorname{ran} Q \cap \mathfrak{M}^\perp = \{0\}$, and let*

$$a \in (0,1), \quad \phi \in (0, \arctan a] \ \subset \left(0, \frac{\pi}{4}\right). \tag{4.27}$$

Then the operator defined by

$$\begin{aligned} \mathcal{A}_{a,\phi} &:= \exp(-\mathrm{i}\phi)\left(a\,(QP_{\mathfrak{M}}Q)^{\frac12} - \mathrm{i}Q\right)^{-1} \\ &= \left[\left(a\cos\phi\,(QP_{\mathfrak{M}}Q)^{\frac12} + \sin\phi\, Q\right) + \mathrm{i}\left(a\sin\phi\,(QP_{\mathfrak{M}}Q)^{\frac12} - \cos\phi\, Q\right)\right]^{-1}, \end{aligned} \tag{4.28}$$

has the following properties:

1) *$\mathcal{A}_{a,\phi}$ is unbounded, m-sectorial (with semi-angle $\frac{\pi}{2} - \phi$), coercive and dissipative,*
2) *$\operatorname{dom} \mathcal{A}_{a,\phi} = \operatorname{dom} \mathcal{A}^*_{a,\phi} = \operatorname{ran} Q$,*
3) *$\operatorname{dom}(\mathcal{A}_{a,\phi})_R^{\frac12} = \operatorname{ran} Q^{\frac12}$,*
4) *the imaginary part $\operatorname{Im} \mathcal{A}_{a,\phi}$ is a positive definite selfadjoint operator,*
5) *the real part $\operatorname{Re} \mathcal{A}_{a,\phi}$ is not selfadjoint, $\operatorname{Re} \mathcal{A}_{a,\phi} \subsetneqq (\mathcal{A}_{a,\phi})_R$, more precisely,*
   a) *for $\phi \in (0, \arctan a)$, the real part $\operatorname{Re} \mathcal{A}_{a,\phi}$ is a closed symmetric operator with defect index $n = \dim \mathfrak{M}^\perp$ and its Friedrichs extension is the operator $(\mathcal{A}_{a,\phi})_R$,*
   b) *for $\phi_a := \arctan a$, the operator $\operatorname{Re} \mathcal{A}_{a,\phi_a}$ is non-closed and essentially selfadjoint and its closure is the operator $(\mathcal{A}_{a,\phi_a})_R$;*
6) *the operator $\mathcal{A}^2_{a,\phi}$ is not accretive and, if $\operatorname{ran} Q \cap \mathfrak{M} = \{0\}$, then*

$$\operatorname{dom} \mathcal{A}^2_{a,\phi} \cap \operatorname{dom} \mathcal{A}^{*2}_{a,\phi} = \{0\}.$$

Proof. The proof is divided into several steps.

Step 1. In the proof of Theorem 4.8 we showed that the operator T_0 in (4.23) is $\pi/4$-sectorial. The same arguments yield that the condition $a \in (0,1)$ implies that the operator

$$T_a := Q + \mathrm{i}a\,(QP_{\mathfrak{M}}Q)^{\frac12}$$

is *sectorial with semi-angle* $\phi_a = \arctan a < \pi/4$. Moreover, since $\operatorname{Im} T_a \ge 0$, the numerical range $W(T_a)$ is contained in the closed sector

$$\overline{\mathcal{S}_{(0,\phi_a)}} = \{\lambda \in \mathbb{C} : 0 \le \arg\lambda \le \phi_a\}.$$

Taking into account (4.24) and (4.25), we conclude that T_a and T_a^* admit the representations

$$T_a = Q(I + aZ_0) = (I - aZ_0^*)Q, \quad T_a^* = Q(I - aZ_0) = (I + aZ_0^*)Q.$$

Since $|a| < 1$ and Z_0 is a partial isometry, we have $\operatorname{ran}(I \pm aZ_0) = \mathfrak{H}$ and hence

$$\operatorname{ran} T_a = \operatorname{ran} T_a^* = \operatorname{ran}(\operatorname{Re} T_a) = \operatorname{ran} Q. \tag{4.29}$$

Besides, since $\|aZ_0\| = a$, the operator T_a^2 is sectorial with semi-angle $\alpha = 2\arctan a = 2\phi_a$, see [4, Prop. 4.4].

Step 2. By (4.27), we have $\phi \in (0, \phi_a]$ with $\phi_a := \arctan a$. If we set

$$\begin{aligned} T_{a,\phi} &:= \exp(\mathrm{i}\phi)T_a = \exp(\mathrm{i}\phi)\left(Q + \mathrm{i}a\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}\right) = \exp(\mathrm{i}\phi)Q(I + aZ_0) \\ &= \cos\phi Q - a\sin\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} + \mathrm{i}\left(\sin\phi\,Q + a\cos\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}\right), \end{aligned} \tag{4.30}$$

then $\operatorname{ran} T_{a,\phi} = \operatorname{ran} T_a$ and $\operatorname{ran} T_{a,\phi}^* = \operatorname{ran} T_a^*$. Consequently, due to (4.29),

$$\operatorname{ran} T_{a,\phi} = \operatorname{ran} T_{a,\phi}^* = \operatorname{ran} Q, \quad \phi \in (0, \phi_a]. \tag{4.31}$$

Because $W(T_a) \subset \overline{\mathcal{S}_{(0,\phi_a)}}$, the numerical range $W(T_{a,\phi})$ satisfies

$$W(T_{a,\phi}) \subset \overline{\mathcal{S}_{(\phi,\phi_a+\phi)}} := \exp(\mathrm{i}\phi)\overline{\mathcal{S}_{(0,\phi_a)}} = \{\lambda \in \mathbb{C} : \phi \le \arg\lambda \le \phi_a + \phi\}. \tag{4.32}$$

Therefore, if $\phi \in (0, \phi_a]$ and hence

$$\phi_a < \phi_a + \phi \le 2\phi_a < \frac{\pi}{2},$$

the operator $T_{a,\phi}$ is dissipative and sectorial with semi-angle $\phi_a + \phi$.

According to the definition of $T_{a,\phi}$ in (4.30) and by (4.24), we have

$$\operatorname{Re} T_{a,\phi} = \cos\phi\,Q - a\sin\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = \cos\phi\,Q(I + \mathrm{i}a\tan\phi\,Z_0), \tag{4.33}$$

$$\begin{aligned} \operatorname{Im} T_{a,\phi} = \sin\phi\,Q + a\cos\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} &= -\mathrm{i}a\cos\phi\,Q\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right) \\ &= -\mathrm{i}a\cos\phi\left(\frac{\mathrm{i}\tan\phi}{a}I - Z_0^*\right)Q. \end{aligned} \tag{4.34}$$

Since $0 \le (QP_{\mathfrak{M}}Q)^{\frac{1}{2}} \le Q$, we have

$$\sin\phi\,Q \le \operatorname{Im} T_{a,\phi} \le (\sin\phi + a\cos\phi)Q.$$

Because $\ker Q = \{0\}$ and by Douglas' Lemma, see Theorem 2.1, these inequalities imply that $\ker \operatorname{Im} T_{a,\phi} = \{0\}$, and $\operatorname{ran}(\operatorname{Im} T_{a,\phi})^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}$. Since $\tan\phi \le \tan\phi_a = a$ for $\phi \in (0, \phi_a]$, we have

$$|a\tan\phi| \le a^2 < 1, \quad \left|\frac{\mathrm{i}\tan\phi}{a}\right| \le 1. \tag{4.35}$$

It follows from (4.33) and the first inequality in (4.35) that

$$\operatorname{Re} T_{a,\phi} \geq \cos\phi(1-a^2)Q \geq 0. \tag{4.36}$$

Together with the fact that Z_0 is a non-unitary isometry and thus $\sigma(Z_0) = \overline{\mathbb{D}}$, see (4.20), we obtain that $-\frac{1}{\mathrm{i}a\tan\phi} \in \rho(Z_0)$ and $-\frac{\mathrm{i}\tan\phi}{a} \in \sigma(Z_0)$. Consequently,

$$\operatorname{ran}(I + \mathrm{i}a\tan\phi\, Z_0) = \mathfrak{H}, \quad \operatorname{ran}\Big(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\Big) \neq \mathfrak{H}.$$

Therefore, by (4.33), (4.34) and since $\ker Q = \{0\}$,

$$\begin{aligned} \operatorname{ran}(\operatorname{Re} T_{a,\phi}) &= Q\operatorname{ran}(I + \mathrm{i}a\tan\phi Z_0) = \operatorname{ran} Q, \\ \operatorname{ran}(\operatorname{Im} T_{a,\phi}) &= Q\operatorname{ran}\Big(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\Big) \subsetneqq \operatorname{ran} Q. \end{aligned} \tag{4.37}$$

Thus, by (4.31) and Theorem 3.11 3), 4) and 5),

$$\operatorname{ran} T_{a,\phi} = \operatorname{ran} T^*_{a,\phi} = \operatorname{ran}(\operatorname{Re} T_{a,\phi}) = \operatorname{ran} Q, \quad \phi \in (0, \phi_a]. \tag{4.38}$$

Moreover, if $\phi < \phi_a = \arctan a$, then $\left|\frac{\mathrm{i}\tan\phi}{a}\right| < 1$ and hence, since the defect is locally constant and $\operatorname{ran} Z_0 = \mathfrak{M}$,

$$\dim\Big(\operatorname{ran}\Big(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\Big)\Big)^{\perp} = \dim\big(\operatorname{ran} Z_0\big)^{\perp} = \dim \mathfrak{M}^{\perp};$$

if $\phi = \phi_a$, then one can show that

$$\dim\Big(\operatorname{ran}\Big(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\Big)\Big)^{\perp} = \dim\,(\operatorname{ran}(\mathrm{i}I + Z_0))^{\perp} = 0.$$

To prove the latter equality, we first note that, by (4.34), $\operatorname{Im} T_{a,\phi_a} = -\mathrm{i}a\cos\phi_a\, Q\,(\mathrm{i}I + Z_0) = -\mathrm{i}a\cos\phi_a\,(\mathrm{i}I - Z_0^*)\,Q$ and thus $\ker(\mathrm{i}I + Z_0) = \ker(I - \mathrm{i}Z_0) = Q^{-1}(\ker \operatorname{Im} T_{a,\phi_a}) = \{0\}$. This implies that

$$(\operatorname{ran}(\mathrm{i}I + Z_0))^{\perp} = \ker(-\mathrm{i}I + Z_0^*) = \ker(I + \mathrm{i}Z_0^*) = \ker(I - \mathrm{i}Z_0) = \ker(\mathrm{i}I + Z_0) = \{0\},$$

where, for the third identity, we have used that $\ker(I - Y) = \ker(I - Y^*)$ for an arbitrary contraction Y in $\mathfrak{H}$, see [39, Prop. I.3.1], here with $Y = \mathrm{i}Z_0$.

Step 3. We define

$$\begin{aligned} \mathcal{T}_{a,\phi} := -\mathrm{i}T_{a,\phi} &= -\mathrm{i}\exp(\mathrm{i}\phi)Q(I + aZ_0) = -\mathrm{i}\exp(\mathrm{i}\phi)\Big(Q + \mathrm{i}a\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}\Big) \\ &= a\cos\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} + \sin\phi\, Q + \mathrm{i}\Big(a\sin\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} - \cos\phi\, Q\Big). \end{aligned} \tag{4.39}$$

Because $W(T_{a,\phi}) \subseteq \overline{\mathcal{S}_{(\phi,\phi_a+\phi)}}$ by (4.32), we have

$$W(\mathcal{T}_{a,\phi}) \subseteq \overline{\mathcal{S}_{(\phi-\pi/2,\phi_a+\phi-\pi/2)}} := -\mathrm{i}\overline{\mathcal{S}_{(\phi,\phi_a+\phi)}} = \{\lambda \in \mathbb{C} : \phi - \pi/2 \leq \arg\lambda \leq \phi_a + \phi - \pi/2\}.$$

Since $0 < \phi \leq \phi_a < \frac{\pi}{4}$, this means that *the operator* $\mathcal{T}_{a,\phi}$ *is sectorial with semi-angle* $\gamma = \pi/2 - \phi$ *and anti-dissipative, i.e.* $\operatorname{Im}\mathcal{T}_{a,\phi} \leq 0$. Further, by (4.38) we obtain

$$\operatorname{ran}\mathcal{T}_{a,\phi} = \operatorname{ran}(-\mathrm{i}T_{a,\phi}) = \operatorname{ran}(T_{a,\phi}) = \operatorname{ran}Q = \operatorname{ran}(T^*_{a,\phi}) = \operatorname{ran}((-\mathrm{i}T_{a,\phi})^*) = \operatorname{ran}\mathcal{T}^*_{a,\phi}.$$

On the other hand,

$$\begin{aligned}\operatorname{Re}\mathcal{T}_{a,\phi} &= \operatorname{Im}T_{a,\phi} = -\mathrm{i}\,a\cos\phi\,Q\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right) = a\cos\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} + \sin\phi\,Q,\\ \operatorname{Im}\mathcal{T}_{a,\phi} &= -\operatorname{Re}T_{a,\phi} = -\cos\phi\,Q(I + \mathrm{i}a\tan\phi\,Z_0) = a\sin\phi\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} - \cos\phi\,Q,\end{aligned} \tag{4.40}$$

and hence, due to (4.37) and (4.40),

$$\operatorname{ran}(\operatorname{Re}\mathcal{T}_{a,\phi}) = \operatorname{ran}(\operatorname{Im}T_{a,\phi}) = Q\operatorname{ran}\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right) \subsetneqq \operatorname{ran}Q. \tag{4.41}$$

Step 4. The operator $\mathcal{A}_{a,\phi}$ in (4.28) is related to the bounded operator $\mathcal{T}_{a,\phi}$ in (4.39) by

$$\mathcal{A}_{a,\phi} = \mathcal{T}^{-1}_{a,\phi} = \mathrm{i}\exp(-\mathrm{i}\phi)(I + aZ_0)^{-1}Q^{-1}. \tag{4.42}$$

Therefore, the operator $\mathcal{A}_{a,\phi}$ is *unbounded, m-sectorial with semi-angle $\pi/2 - \phi$, coercive and dissipative.* The adjoint operator $\mathcal{A}^*_{a,\phi}$ is of the form

$$\mathcal{A}^*_{a,\phi} = \mathcal{T}^{-*}_{a,\phi} = -\mathrm{i}\exp(\mathrm{i}\phi)Q^{-1}(I + aZ_0^*)^{-1} = -\mathrm{i}\exp(\mathrm{i}\phi)(I - aZ_0)^{-1}Q^{-1}. \tag{4.43}$$

Because

$$\operatorname{dom}\mathcal{A}_{a,\phi} = \operatorname{ran}\mathcal{T}_{a,\phi} = \operatorname{ran}Q, \quad \operatorname{dom}\mathcal{A}^*_{a,\phi} = \operatorname{ran}\mathcal{T}^*_{a,\phi} = \operatorname{ran}Q, \quad \operatorname{ran}(\operatorname{Re}\mathcal{T}_{a,\phi}) \subsetneqq \operatorname{ran}Q,$$

Theorem 3.11 3) and 4) yield $\operatorname{dom}\mathcal{A}_{a,\phi} = \operatorname{dom}\mathcal{A}^*_{a,\phi} \subsetneqq \operatorname{dom}(\mathcal{A}_{a,\phi})_R$ and thus $\operatorname{Re}\mathcal{A}_{a,\phi} \subsetneqq (\mathcal{A}_{a,\phi})_R$.

To calculate the defect index of $\operatorname{Re}\mathcal{A}_{a,\phi}$, we note that $\operatorname{ran}\mathcal{A}_{a,\phi} = \operatorname{ran}\mathcal{A}^*_{a,\phi} = \mathfrak{H}$ since $\mathcal{T}_{a,\phi}$ is bounded and hence, by (4.40), (4.42), (4.43) and (4.25),

$$\operatorname{Re}\mathcal{A}_{a,\phi} = \frac{1}{2}(\mathcal{A}_{a,\phi} + \mathcal{A}^*_{a,\phi}) = \frac{1}{2}\mathcal{A}_{a,\phi}\left((\mathcal{A}^{-1}_{a,\phi} + \mathcal{A}^{-*}_{a,\phi})\right)\mathcal{A}^*_{a,\phi} = \mathcal{A}_{a,\phi}\big(\operatorname{Re}(\mathcal{T}_{a,\phi})\mathcal{A}^*_{a,\phi} \tag{4.44}$$

$$= \phi(I + aZ_0)^{-1}Q^{-\frac{1}{2}}\big(a\cos\phi Q^{-\frac{1}{2}}(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}Q^{-\frac{1}{2}} + \sin\phi\,I\big)Q^{-\frac{1}{2}}(I + aZ_0^*)^{-1} \tag{4.45}$$

$$\begin{aligned}&= -\mathrm{i}a\cos\phi(I + aZ_0)^{-1}Q^{-1}Q\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right)(I - aZ_0)^{-1}Q^{-1}\\ &= -\mathrm{i}a\cos\phi\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right)(I - a^2Z_0^2)^{-1}Q^{-1}, \qquad \operatorname{dom}\operatorname{Re}\mathcal{A}_{a,\phi} = \operatorname{ran}Q,\end{aligned}$$

and hence

$$\begin{aligned}\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi}) &= \mathcal{A}_{a,\phi}\operatorname{ran}\big(\operatorname{Re}(\mathcal{T}_{a,\phi})\mathcal{A}^*_{a,\phi}\big)\\ &= \mathcal{A}_{a,\phi}\operatorname{ran}(\operatorname{Re}\mathcal{T}_{a,\phi}) = \mathcal{A}_{a,\phi}Q\operatorname{ran}\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right) = (I + aZ_0)^{-1}\left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right)\mathfrak{H}\\ &= \left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right)\left(\frac{1}{a}I + Z_0\right)^{-1}\mathfrak{H} = \left(\frac{\mathrm{i}\tan\phi}{a}I + Z_0\right)\mathfrak{H},\end{aligned}$$

where we have used that $a \in (0,1)$ and hence $\frac{1}{a} \in \rho(Z_0)$, see (4.20). It follows that

a) for $\phi \in (0, \phi_a)$ the linear manifold $\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi})$ is a proper subspace of $\mathfrak{H}$, $\operatorname{Re}\mathcal{A}_{a,\phi}$ is a positive definite closed symmetric operator with

$$(\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi}))^{\perp} = \ker\left(-\frac{\mathrm{i}\tan\phi}{a} I + Z_0^*\right),$$

and therefore

$$\dim(\operatorname{ran}\operatorname{Re}\mathcal{A}_{a,\phi})^{\perp} = \dim\ker\left(-\frac{\mathrm{i}\tan\phi}{a} I + Z_0^*\right) = \dim\mathfrak{M}^{\perp} > 0$$

which means that the defect index of the closed symmetric operator $\operatorname{Re}\mathcal{A}_{a,\phi}$ is $\dim\mathfrak{M}^{\perp} > 0$ and, since $\operatorname{Re}\mathcal{A}_{a,\phi} \subset (\mathcal{A}_{a,\phi})_R$, $\operatorname{dom}\operatorname{Re}\mathcal{A}_{a,\phi} = \operatorname{dom}\mathcal{A}_{a,\phi}$ and $\operatorname{dom}\mathcal{A}_{a,\phi}$ is a core of $\operatorname{dom}(\mathcal{A}_{a,\phi})_R^{\frac{1}{2}}$, the operator $(\mathcal{A}_{a,\phi})_R$ is the Friedrichs extension of $\operatorname{Re}\mathcal{A}_{a,\phi}$;

b) for $\phi = \phi_a$ the linear manifold $\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi_a}) = \operatorname{ran}(\mathrm{i}I + Z_0)$ is non-closed and dense in $\mathfrak{H}$; this means that $\operatorname{Re}\mathcal{A}_{a,\phi_a} \subsetneqq (\mathcal{A}_{a,\phi})_R$ is essentially selfadjoint with closure $(\mathcal{A}_{a,\phi})_R$.

In Step 2 we had shown that $\operatorname{ran}(\operatorname{Im}T_{a,\phi})^{\frac{1}{2}} = \operatorname{ran}Q^{\frac{1}{2}}$ which, by (4.39), implies that $\operatorname{ran}(\operatorname{Re}\mathcal{T}_{a,\phi})^{\frac{1}{2}} = \operatorname{ran}(\operatorname{Im}T_{a,\phi})^{\frac{1}{2}} = \operatorname{ran}Q^{\frac{1}{2}}$. Together with (2.14), it follows that

$$\operatorname{dom}((\mathcal{A}_{a,\phi})_R)^{\frac{1}{2}} = \operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi}^{-1})^{\frac{1}{2}} = \operatorname{ran}(\operatorname{Re}\mathcal{T}_{a,\phi})^{\frac{1}{2}} = \operatorname{ran}Q^{\frac{1}{2}}.$$

In the same way as for $\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi})$, we conclude that

$$\begin{aligned}\operatorname{ran}(\operatorname{Im}\mathcal{A}_{a,\phi}) &= \operatorname{ran}(\mathcal{A}_{a,\phi} - \mathcal{A}_{a,\phi}^*) = \operatorname{ran}\left(\mathcal{A}_{a,\phi}(\mathcal{A}_{a,\phi}^{-1} - \mathcal{A}_{a,\phi}^{-*})\mathcal{A}_{a,\phi}^*\right) = \mathcal{A}_{a,\phi}\operatorname{ran}[(\operatorname{Im}\mathcal{T}_{a,\phi})\mathcal{A}_{a,\phi}^*] \\ &= \mathcal{A}_{a,\phi}\operatorname{ran}(\operatorname{Im}\mathcal{T}_{a,\phi}) = \mathcal{A}_{a,\phi}Q\operatorname{ran}(I + \mathrm{i}a\tan\phi Z_0) = (I + aZ_0)^{-1}(I + \mathrm{i}a\tan\phi Z_0)\mathfrak{H} \\ &= (I + \mathrm{i}a\tan\phi Z_0)(I + aZ_0)^{-1}\mathfrak{H} = \mathfrak{H}.\end{aligned}$$

Moreover, since the operator $\operatorname{Im}\mathcal{A}_{a,\phi}$ is symmetric and $\operatorname{Re}T_{a,\phi} \geq 0$ by (4.36), we have

$$\operatorname{Im}\mathcal{A}_{a,\phi} = -\mathcal{A}_{a,\phi}^*(\operatorname{Im}\mathcal{T}_{a,\phi})\mathcal{A}_{a,\phi} = \mathcal{A}_{a,\phi}^*(\operatorname{Re}T_{a,\phi})\mathcal{A}_{a,\phi} \geq 0;$$

together with $\operatorname{ran}\operatorname{Im}\mathcal{A}_{a,\phi} = \mathfrak{H}$, it follows that $\operatorname{Im}\mathcal{A}_{a,\phi}$ is a positive definite selfadjoint operator for each $\phi \in (0, \phi_a]$.

Since $\operatorname{Re}\mathcal{A}_{a,\phi} \subsetneqq (\mathcal{A}_{a,\phi})_R$ so that $\operatorname{Re}\mathcal{A}_{a,\phi}$ is not selfadjoint, Kato's theorem, see [25, Thm. 5.1] applied to $A = \mathcal{A}_{a,\phi}^2$ therein yields that the operator $\mathcal{A}_{a,\phi}^2$ is not accretive.

Finally, assume that $\operatorname{ran}Q \cap \mathfrak{M} = \{0\}$. Since $\mathfrak{M} = \operatorname{ran}Z_0$, by [4, Prop. 4.1 7)] we obtain that $T_a = Q(I + aZ_0)$ satisfies $\operatorname{ran}T_a^2 \cap \operatorname{ran}T_a^{*2} = \{0\}$. Because $\mathcal{T}_{a.\phi} = -\mathrm{i}\exp(\mathrm{i}\phi)T_a$ by (4.30), (4.39) and $\mathcal{A}_{a,\phi}^2 = \mathcal{T}_{a,\phi}^{-1}$ by (4.42), it follows that

$$\operatorname{dom}\mathcal{A}_{a,\phi}^2 \cap \operatorname{dom}\mathcal{A}_{a,\phi}^{*2} = \operatorname{ran}\mathcal{T}_{a,\phi}^2 \cap \operatorname{ran}\mathcal{T}_{a,\phi}^{*2} = \{0\}. \quad \square$$

Remark 4.10. For $\mathcal{R} = \operatorname{ran}Q$ the co-dimension $n = \dim\mathfrak{M}^{\perp}$ of the subspace $\mathfrak{M}$ with $\operatorname{ran}Q \cap \mathfrak{M}^{\perp} = \{0\}$, which coincides with the defect $\dim\operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi})$ for $\phi \in (0, \phi_a)$ by Theorem 4.9 5) a), can be an arbitrary $n \in \mathbb{N} \cup \{\infty\}$.

In fact, by Schmüdgen's theorem, see [36, Thm. 5.1], for an arbitrary dense operator range $\mathcal{R}$ in a separable Hilbert space, there exists a subspace $\mathfrak{N}$ such that $\mathfrak{N} \cap \mathcal{R} = \mathfrak{N}^{\perp} \cap \mathcal{R} = \{0\}$. Clearly, $\dim\mathfrak{N} = \dim\mathfrak{N}^{\perp} = \infty$. It follows that for each $n \in \mathbb{N} \cup \{\infty\}$ any n-dimensional subspace $\mathfrak{M}^{\perp} \subset \mathfrak{N}^{\perp}$ possesses the property $\mathfrak{M}^{\perp} \cap \mathcal{R} = \{0\}$.

Remark 4.11. Since the operator $T_0 = Q + \mathrm{i}(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}$ is sectorial, there is a bounded and nonnegative selfadjoint operator X_0 in $\mathfrak{H}$ such that $(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = Q^{\frac{1}{2}}X_0Q^{\frac{1}{2}}$; in fact, X_0 is a contraction because T_0 is $\pi/4$-sectorial. By (4.24), (4.25), it follows that, for all $a \in \mathbb{R}$,

$$Q^{\frac{1}{2}}(I \pm \mathrm{i}aX_0) = (I \mp aZ_0^*)Q^{\frac{1}{2}}, \quad (I \pm \mathrm{i}aX_0)Q^{\frac{1}{2}} = Q^{\frac{1}{2}}(I \pm aZ_0).$$

If $a \in (0,1)$, then, since $\operatorname{ran}((I \pm \mathrm{i}aX_0) = \mathfrak{H}$, the equalities $((I \mp aZ_0^*)\operatorname{ran} Q^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}$ hold and

$$Q^{-\frac{1}{2}}(I \mp aZ_0^*)^{-1}h = (I \pm \mathrm{i}aX_0)^{-1}Q^{-\frac{1}{2}}h, \quad h \in \operatorname{ran} Q^{\frac{1}{2}}.$$

The operator $\mathcal{A}_{a,\phi}$ defined in (4.28) can be rewritten in the forms

$$\begin{aligned}\mathcal{A}_{a,\phi} &= \mathrm{i}\exp(-\mathrm{i}\phi)Q^{-\frac{1}{2}}\left(I + \mathrm{i}aX_0\right)^{-1}Q^{-\frac{1}{2}}\\ &= Q^{-\frac{1}{2}}\left[(a\cos\phi\, X_0 + \sin\phi\, I) + \mathrm{i}\,(a\sin\phi\, X_0 - \cos\phi\, I)\right]^{-1}Q^{-\frac{1}{2}}.\end{aligned}$$

Then, if $f \in \operatorname{dom}\mathcal{A}_{a,\phi}^* = \operatorname{ran} Q$ for $a \in (0,1)$ and $\phi \in (0, \phi_a)$, using (4.44), (4.45) and (4.25), we obtain

$$\begin{aligned}((\operatorname{Re}\mathcal{A}_{a,\phi})f, f) = \Big(\operatorname{Re}(\mathcal{T}_{a,\phi})\mathcal{A}_{a,\phi}^*f, \mathcal{A}_{a,\phi}^*f\Big) &= \left\|\big(a\cos\phi\, X_0 + \sin\phi\, I\big)^{\frac{1}{2}}Q^{-\frac{1}{2}}(I + aZ_0^*)^{-1}f\right\|^2\\ &= \left\|\big(a\cos\phi\, X_0 + \sin\phi\, I\big)^{\frac{1}{2}}(I - \mathrm{i}aX_0)^{-1}Q^{-\frac{1}{2}}f\right\|^2.\end{aligned}$$

Because the operator $a\cos\phi X_0 + \sin\phi I$ is positive definite, the closure of the above quadratic form is given by

$$\left\|\big(a\cos\phi\, X_0 + \sin\phi\, I\big)^{\frac{1}{2}}(I - \mathrm{i}aX_0)^{-1}Q^{-\frac{1}{2}}h\right\|^2, \quad h \in \operatorname{dom} Q^{-\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}.$$

The associated nonnegative selfadjoint operator is the Friedrichs extension of $\operatorname{Re}\mathcal{A}_{a,\phi}$, it coincides with $(\mathcal{A}_{a,\phi})_R$ and has the form

$$\begin{aligned}(\mathcal{A}_{a,\phi})_R &= Q^{-\frac{1}{2}}(I + a^2X_0^2)^{-1}(a\cos\phi\, X_0 + \sin\phi\, I)Q^{-\frac{1}{2}},\\ \operatorname{dom}(\mathcal{A}_{a,\phi})_R &= Q^{\frac{1}{2}}(I + a^2X_0^2)\big(a\cos\phi\, X_0 + \sin\phi\, I\big)^{-1}\operatorname{ran} Q^{\frac{1}{2}}.\end{aligned}$$

Remark 4.12. Let $\mathcal{A}_{a,\phi} = ((\mathcal{A}_{a,\phi})_R)^{\frac{1}{2}}(I + \mathrm{i}G_{a,\phi})((\mathcal{A}_{a,\phi})_R)^{\frac{1}{2}}$ with a bounded selfadjoint operator $G_{a,\phi}$ be the canonical representation of $\mathcal{A}_{a,\phi}$, see (2.2), and set

$$\widetilde{\mathcal{A}}_{a,\phi} := ((\mathcal{A}_{a,\phi})_R)^{\frac{1}{2}}(I + \mathrm{i}G_{a,\phi})^{-1}((\mathcal{A}_{a,\phi})_R)^{\frac{1}{2}}.$$

According to Corollary 3.6 and its proof, see (3.6), (3.7), the operator $\widetilde{\mathcal{A}}_{a,\phi}$ is m-sectorial and

$$\begin{aligned}&\operatorname{dom}\widetilde{\mathcal{A}}_{a,\phi} = \operatorname{dom}\widetilde{\mathcal{A}}_{a,\phi}^*, \quad \operatorname{Re}\widetilde{\mathcal{A}}_{a,\phi} \subsetneqq (\widetilde{\mathcal{A}}_{a,\phi})_R,\\ &\operatorname{ran}(\operatorname{Re}\widetilde{\mathcal{A}}_{a,\phi}) = \operatorname{ran}(\operatorname{Re}\mathcal{A}_{a,\phi}), \quad \operatorname{ran}(\operatorname{Im}\widetilde{\mathcal{A}}_{a,\phi}) = \operatorname{ran}(\operatorname{Im}\mathcal{A}_{a,\phi}).\end{aligned}$$

Therefore Theorem 4.9 5) a), b) and 4) continue to hold for $\operatorname{Re}\widetilde{\mathcal{A}}_{a,\phi}$ and $\operatorname{Im}\widetilde{\mathcal{A}}_{a,\phi}$, respectively.

Corollary 4.13. *On* $\mathcal{S}_{(0,\phi_a]} = \{\lambda \in \mathbb{C} \setminus \{0\} : 0 < \arg\lambda \le \phi_a\}$ *with* $\phi_a := \arctan a$, $a \in (0,1)$, *define*

$$\mathcal{A}_a(\lambda) = \frac{1}{\lambda}\Big(a\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} - \mathrm{i}Q\Big)^{-1}, \quad \lambda \in \mathcal{S}_{(0,\phi_a]}.$$

Then the operator function $\mathcal{A}_a(\lambda)$, $\lambda \in \mathcal{S}_{(0,\phi_a]}$, *satisfies*

1) *the operator* $\mathcal{A}_a(\lambda)$ *is unbounded, m-sectorial* (*with semi-angle* $\frac{\pi}{2} - \arg\lambda$), *coercive and dissipative for* $\lambda \in \mathcal{S}_{(0,\phi_a]}$;
2) $\operatorname{dom}\mathcal{A}_a(\lambda) = \operatorname{dom}\mathcal{A}_a^*(\lambda) = \operatorname{ran} Q$, $\operatorname{dom}((\mathcal{A}_a(\lambda))_R)^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}}$, $\operatorname{Re}\mathcal{A}_a(\lambda) \subsetneqq (\mathcal{A}_a(\lambda))_R$ *for* $\lambda \in \mathcal{S}_{(0,\phi_a]}$;
3) *the imaginary part* $\operatorname{Im}\mathcal{A}_a(\lambda)$ *is a positive definite selfadjoint operator for* $\lambda \in \mathcal{S}_{(0,\phi_a]}$;
4) $\mathcal{A}_a(\lambda)$, $\lambda \in \mathcal{S}_{(0,\phi_a)}$, *forms a holomorphic family of both type* (A) *and* (B) *inside the open sector* $\mathcal{S}_{(0,\phi_a)} := \{\lambda \in \mathbb{C} \setminus \{0\} : 0 < \arg\lambda < \phi_a\}$ *see [28, Chapt. VII];*
5) *the operator* $\operatorname{Re}\mathcal{A}_a(\lambda)$ *is closed and symmetric with defect index* $n = \dim \mathfrak{M}^\perp$ *for* $\lambda \in \mathcal{S}_{(0,\phi_a)}$;
6) *the operator* $\operatorname{Re}\mathcal{A}_a(r\exp(i\phi_a))$ *is non-closed and essentially selfadjoint for* $r > 0$;
7) *if* $\operatorname{ran} Q \cap \mathfrak{M} = \{0\}$, *then* $\operatorname{dom}\mathcal{A}_a(\lambda)^2 \cap \operatorname{dom}\mathcal{A}_a(\lambda)^{*2} = \{0\}$ *for* $\lambda \in \mathcal{S}_{(0,\phi_a]}$.

Proof. Let $\lambda = |\lambda|\exp(i\phi) \in \mathcal{S}_{(0,\phi_a]}$, i.e. $0 < \phi \le \phi_a$. Then $\mathcal{A}_a(\lambda) = \frac{1}{|\lambda|}\mathcal{A}_{a,\phi}$ where $\mathcal{A}_{a,\phi}$ is the operator in Theorem 4.9 given by (4.28). Now Theorem 4.9 implies all the claims. □

5. Existence of specific operators and invariant operator ranges

In view of the results of the previous sections, in this section we establish the existence of some classes of bounded operators leaving invariant a given dense operator range, see Theorem 5.4. First we need some auxiliary results.

Proposition 5.1. *Let* C *be a bounded selfadjoint operator such that* $C \ge 0$, $\ker C = \{0\}$ *and* $\operatorname{ran} C \ne \mathfrak{H}$. *Assume that* U *is a unitary operator in* $\mathfrak{H}$ *with*

$$U \operatorname{ran} C = \operatorname{ran} C.$$

Then the operator $S := UC$ *has the following properties.*

1) a) *the equalities* $\operatorname{ran} S = \operatorname{ran} S^* = \operatorname{ran} C$ *hold;*
   b) *the following are equivalent:*
      (i) $-1 \in \rho(U)$,
      (ii) $\operatorname{ran}(S^* + |S|) = \operatorname{ran} S^*$;
   c) *the following are equivalent:*
      (i) $(U + I)\operatorname{ran} C = \operatorname{ran} C$,
      (ii) $\operatorname{ran}(S + |S|) = \operatorname{ran} S^*$;
2) *if*

$$\operatorname{ran} S = \operatorname{ran} S^* = \operatorname{ran}(S^* + |S|),$$

then the unbounded operator

$$A := |S|^{-2}(S^* + |S|)|S|^{-1} \tag{5.1}$$

is m-sectorial and
   a) *the equality* $\operatorname{ran}(S + |S|) = \operatorname{ran} S$ *implies*

$$\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R, \quad \operatorname{Re} A = A_R;$$

   b) *the inclusion* $\operatorname{ran}(S + |S|) \subsetneqq \operatorname{ran} S$ *implies*

$$\operatorname{dom} A = \operatorname{dom} A^* \ne \operatorname{dom} A_R, \quad \operatorname{Re} A \subsetneqq A_R.$$

Proof. 1) Since $C \geq 0$ and U is unitary, we have $C = \sqrt{S^*S} = |S|$ and $S^* = CU^{-1}$. This implies $\operatorname{ran}|S| = \operatorname{ran} C = \operatorname{ran} S^*$, $\operatorname{ran} S = U\operatorname{ran} C = \operatorname{ran} C = \operatorname{ran} S^* = \operatorname{ran}|S|$ which proves a). Further,

$$S + |S| = (U+I)|S|, \quad S^* + |S| = |S|(U^{-1} + I).$$

Therefore,

$$\begin{aligned} &\operatorname{ran}(S+|S|) = (U+I)\operatorname{ran}|S| \subseteq \operatorname{ran} C = \operatorname{ran} S, && \operatorname{ran}(S+|S|) = \operatorname{ran} S \Longleftrightarrow (U+I)\operatorname{ran} S = \operatorname{ran} S,\\ &\operatorname{ran}(S^*+|S|) \subseteq \operatorname{ran}|S| = \operatorname{ran} S, && \operatorname{ran}(S^*+|S|) = \operatorname{ran} S \Longleftrightarrow \operatorname{ran}(U^{-1}+I) = \mathfrak{H}, \end{aligned}$$

and $\operatorname{ran}(U^{-1}+I) = \mathfrak{H} \Longleftrightarrow -1 \in \rho(U^{-1}) \Longleftrightarrow -1 \in \rho(U)$ since U is unitary. Together with a), this proves c) and b).

2) If we suppose that $\operatorname{ran}(S^*+|S|) = \operatorname{ran} S$, then 1) b) shows that $U+I$ has bounded inverse. Since $U^{-1} = |S|^{-1}S^*$, we have $U^* + I = |S|^{-1}(S^*+|S|)$. The operator

$$F := \mathrm{i}(I-U)(I+U)^{-1}$$

is bounded, selfadjoint and $I + \mathrm{i}F = 2(U^*+I)^{-1}$, $I - \mathrm{i}F = 2(I+U)^{-1}$. Since F is bounded,

$$T := \frac{1}{2}C(I+\mathrm{i}F)C = C(I+U^*)^{-1}C = |S|(S^*+|S|)^{-1}|S|^2$$

is a bounded sectorial operator with $\ker T = \ker T^* = \{0\}$ since $\ker C = \{0\}$. It follows that the operator A in (5.1) satisfies $A = T^{-1}$ and hence A is an unbounded m-sectorial operator with

$$\operatorname{dom} A^{-1} = \operatorname{ran} T = \operatorname{ran}(C(I+\mathrm{i}F)C), \quad \operatorname{dom} A^{-*} = \operatorname{ran} T^* = \operatorname{ran}(C(I-\mathrm{i}F)C).$$

Because $U = (I+\mathrm{i}F)(I-\mathrm{i}F)^{-1}$ and $U\operatorname{ran} C = \operatorname{ran} C$, we have $(I+\mathrm{i}F)(I-\mathrm{i}F)^{-1}\operatorname{ran} C = \operatorname{ran} C$ or, equivalently, $(I+\mathrm{i}F)\operatorname{ran} C = (I-\mathrm{i}F)\operatorname{ran} C$. Now Theorem 3.11 3) implies that $\operatorname{ran} T = \operatorname{ran} T^*$ and $\operatorname{dom} A = \operatorname{dom} A^*$. If $\operatorname{ran}(S+|S|) = \operatorname{ran} S$, then 1) c) shows that

$$\operatorname{ran} C = (U+I)\operatorname{ran} C = (I-\mathrm{i}F)^{-1}\operatorname{ran} C.$$

Applying Theorem 3.11 4), we obtain that even $\operatorname{dom} A = \operatorname{dom} A^* = \operatorname{dom} A_R$ which proves a). On the other hand, if $\operatorname{ran}(S+|S|) \subsetneqq \operatorname{ran} S$, then $(U+I)\operatorname{ran} C \subsetneqq \operatorname{ran} C$ by 1) b) and hence

$$(I-\mathrm{i}F)^{-1}\operatorname{ran} C \subsetneqq \operatorname{ran} C.$$

According to Theorem 3.11 3) and 4), this yields that $\operatorname{Re} A \subsetneqq A_R$. □

Proposition 5.2. *Let C be a bounded selfadjoint operator such that $C \geq 0$, $\ker C = \{0\}$ and $\operatorname{ran} C \neq \mathfrak{H}$. Assume that U is a unitary operator in $\mathfrak{H}$ such that*

$$U\operatorname{ran} C = \operatorname{ran} C.$$

Then the operator $W := C^{-1}UC$ has the following properties.

1) *W is bounded and boundedly invertible in $\mathfrak{H}$;*
2) *the equality $W^*C^2W = C^2$ holds;*

3) *the linear manifolds* $\operatorname{ran} C$ *and* $\operatorname{ran} C^2$ *are invariant for the operators* W^* *and* W^{-*}*, i.e.*

$$W^* \operatorname{ran} C = \operatorname{ran} C, \quad W^* \operatorname{ran} C^2 = \operatorname{ran} C^2;$$

4) *the operator* $Z := W^* \upharpoonright \mathfrak{H}_C$ *is unitary in the Hilbert space* $\mathfrak{H}_C$*, see* (2.21)*, and unitarily equivalent to* U^{-1}*;*
5) *the following are equivalent:*
   (i) $-1 \in \rho(U)$,
   (ii) $(W^* + I) \operatorname{ran} C = \operatorname{ran} C$;
6) *the following are equivalent:*
   (i) $(U + I) \operatorname{ran} C = \operatorname{ran} C$,
   (ii) $-1 \in \rho(W)$,
   (iii) $(W^* + I) \operatorname{ran} C^2 = \operatorname{ran} C^2$.

Proof. 1) The operator W is well-defined and bounded in $\mathfrak{H}$ by Douglas' Lemma, see Theorem 2.1, or the Closed Graph Theorem. Clearly, W is injective. Because $UC = CW$ and $U \operatorname{ran} C = \operatorname{ran} C$, it follows that $\operatorname{ran} W = \mathfrak{H}$ and hence $0 \in \rho(W)$.

2), 3) Since $CU^* = W^*C$, the equalities $C^2 = W^*C^2W$ and $C^2 = W^{*n}C^2W^n$ hold for all $n \in \mathbb{N}$. Because $\operatorname{ran} U^* = \operatorname{ran} W = \mathfrak{H}$, it follows that

$$W^* \operatorname{ran} C = \operatorname{ran} C, \quad W^* \operatorname{ran} C^2 = \operatorname{ran} C^2.$$

5) Furthermore, the equality $C(U^* + I) = (W^* + I)C$ yields the equivalence $\operatorname{ran}(U^* + I) = \mathfrak{H} \iff (W^* + I)\operatorname{ran} C = \operatorname{ran} C$. Since U is unitary, we have $(U^* + I) = (U + I)U^{-1}$ and thus $\operatorname{ran}(U^* + I) = \mathfrak{H} \iff \operatorname{ran}(U + I) = \mathfrak{H}$. Because $\ker(U + I) = (\operatorname{ran}(U^* + I))^{\perp}$, the claimed equivalence follows.

4) Because U is unitary in $\mathfrak{H}$ and $U^{-1} = C^{-1}W^*C$, we deduce that the operator $Z = W^* \upharpoonright \mathfrak{H}_C$ is unitary in the Hilbert space $\mathfrak{H}_C$ and

$$U^{-1} = C^{-1}ZC, \quad U = C^{-1}Z^{-1}C = C^{-1}W^{-*}C.$$

Since C maps $\mathfrak{H}_C$ unitarily on $\mathfrak{H}$, this shows that Z in $\mathfrak{H}_C$ is unitarily equivalent to U^{-1} in $\mathfrak{H}$.

6) The equality $(U + I)C = C(W + I)$ yields that

$$(U + I)\operatorname{ran} C = \operatorname{ran} C \iff \operatorname{ran}(W + I) = \mathfrak{H} \iff -1 \in \rho(W).$$

If $(U + I)\operatorname{ran} C = \operatorname{ran} C$ and $x \in \ker(W + I)$, there exists $y \in \mathfrak{H}$ with $Cy = (U + I)Cx = C(W + I)x = 0$ which shows that $y = 0$. If we note that $U \operatorname{ran} C = \operatorname{ran} C$ implies that $(U^* + I)\operatorname{ran} C = (U^* + I)U \operatorname{ran} C = (U + I)\operatorname{ran} C = \operatorname{ran} C$ and $\overline{\operatorname{ran}}\, C = \mathfrak{H}$, it follows that $\ker(U + I) = (\operatorname{ran}(U^* + I))^{\perp} = \{0\}$. This and $\ker C = \{0\}$ yield that $y = 0$ implies $x = 0$. This proves the equivalence of (i) and (ii). Since

$$(U + I)C = U(U^{-1} + I)C = UC^{-1}(W^* + I)C^2 = UCC^{-2}(W^* + I)C^2$$

and $U \operatorname{ran} C = \operatorname{ran} C$ by assumption, we conclude that

$$(U + I)\operatorname{ran} C = \operatorname{ran} C \iff (W^* + I)\operatorname{ran} C^2 = \operatorname{ran} C^2,$$
$$(U + I)\operatorname{ran} C \subsetneqq \operatorname{ran} C \iff (W^* + I)\operatorname{ran} C^2 \subsetneqq \operatorname{ran} C^2. \quad \square$$

Remark 5.3. The condition $W^* \operatorname{ran} C = \operatorname{ran} C$ follows from $W^*C^2W = C^2$ and $0 \in \rho(W)$. In fact, the latter yield $W^*C^2 = C^2W^{-1}$, $W^{-*}C^2 = C^2W$ which imply $W^* \operatorname{ran} C^2 = \operatorname{ran} C^2$ and $W^{-*} \operatorname{ran} C^2 = \operatorname{ran} C^2$. Hence, by interpolation [18], we conclude that $\operatorname{ran} C$ is an invariant operator range for W^* and W^{-*}.

The next theorem on operator ranges follows from Proposition 4.2 and Theorem 4.9 combined with Theorem 3.11, Remark 3.12, Proposition 5.1 and Proposition 5.2.

Theorem 5.4. *Let $\mathcal{R}$ be a dense operator range. Then the following hold.*

a) *there exists a bounded selfadjoint operator F such that*

$$(I - \mathrm{i}F)^{-1}\mathcal{R} = (I + \mathrm{i}F)^{-1}\mathcal{R} \subsetneqq \mathcal{R};$$

b) *there exists a bounded selfadjoint operator F such that*

$$(I - \mathrm{i}F)\mathcal{R} = (I + \mathrm{i}F)\mathcal{R} = \mathcal{R};$$

c) *there exists a unitary operator U such that*

$$-1 \in \rho(U), \quad U\mathcal{R} = \mathcal{R}, \quad (U + I)\mathcal{R} \subsetneqq \mathcal{R};$$

d) *there exists a unitary operator U such that*

$$-1 \in \rho(U), \quad U\mathcal{R} = (U + I)\mathcal{R} = \mathcal{R},$$

i.e. $\mathcal{R}$ is an invariant operator range for U, $U^*(= U^{-1})$ and $(U + I)^{-1}$;

e) *there exists a nonselfadjoint operator $S \in \mathbf{B}(\mathfrak{H})$ such that*

$$\ker S = \{0\}, \quad \mathrm{ran}\, S = \mathrm{ran}\, S^* = \mathrm{ran}\,(S^* + |S|) = \mathcal{R}, \quad \mathrm{ran}\,(S + |S|) \subsetneqq \mathcal{R};$$

f) *there exists a nonselfadjoint operator $S \in \mathbf{B}(\mathfrak{H})$ such that*

$$\ker S = \{0\}, \quad \mathrm{ran}\, S = \mathrm{ran}\, S^* = \mathrm{ran}\,(S^* + |S|) = \mathcal{R}, \quad \mathrm{ran}\,(S + |S|) = \mathcal{R}.$$

g) *there exists $Y \in \mathbf{B}(\mathfrak{H})$ with the properties*

$$0 \in \rho(Y), \quad -1 \in \rho(Y), \quad Y\mathcal{R} = \mathcal{R}, \quad (Y + I)\mathcal{R} = \mathcal{R};$$

h) *there exists $Y \in \mathbf{B}(\mathfrak{H})$ with the properties*

$$0 \in \rho(Y), \quad -1 \notin \rho(Y), \quad Y\mathcal{R} = \mathcal{R}, \quad (Y + I)\mathcal{R} = \mathcal{R}.$$

Proof. Let C be a bounded nonnegative selfadjoint operator such that $\mathrm{ran}\, C = \mathcal{R}$. By Proposition 4.2 there exists an unbounded m-sectorial coercive operator A in a Hilbert space $\mathfrak{H}$ with $\mathrm{dom}\, A = \mathrm{dom}\, A^* = \mathrm{dom}\, A_R = \mathrm{ran}\, C^2$ and hence $\mathrm{Re}\, A = A_R$. The operator A^{-1} is of the form $A^{-1} = C(I + \mathrm{i}F)C$. By Theorem 3.11 4), Remark 3.12 1), and Proposition 5.1, we have

$$\begin{aligned}
&(I + \mathrm{i}F)\mathcal{R} = (I - \mathrm{i}F)\mathcal{R} = \mathcal{R},\\
&U\mathcal{R} = \mathcal{R},\ (U + I)\mathcal{R} = \mathcal{R},\\
&\mathrm{ran}\, S = \mathrm{ran}\, S^* = \mathrm{ran}\,(S^* + |S|) = \mathrm{ran}\,(S + |S|) = \mathcal{R}
\end{aligned}$$

with $U := (I - \mathrm{i}F)^{-1}(I + \mathrm{i}F)$ and $S := UC$. This proves claims b), d) and f).

Set $Q := C^2$ and let $\mathfrak{M}$ be a subspace in $\mathfrak{H}$ such that $\operatorname{ran} C^2 \cap \mathfrak{M}^\perp = \{0\}$. Following Theorem 4.9, we define the bounded sectorial operator

$$T_a := Q + \mathrm{i}a(QP_{\mathfrak{M}}Q)^{\frac{1}{2}} = C^2 + a(C^2P_{\mathfrak{M}}C^2)^{\frac{1}{2}}, \quad a \in (0,1),$$

and, for $\phi \in (0, \arctan a]$, we set

$$\mathcal{T}_{a,\phi} := -\mathrm{i}\exp(\mathrm{i}\phi)\left(Q + \mathrm{i}a\,(QP_{\mathfrak{M}}Q)^{\frac{1}{2}}\right).$$

Then the operator $\mathcal{A}_{a,\phi} := \mathcal{T}_{a,\phi}^{-1}$ is m-sectorial. Due to [26, Thm. VI.3.2] there exist a bounded selfadjoint operator $C_{a,\phi} \geq 0$ with $\ker C_{a,\phi} = \{0\}$ and a bounded selfadjoint operator $F_{a,\phi}$ in $\mathfrak{H}$ such that

$$\mathcal{T}_{a,\phi} = C_{a,\phi}(I + \mathrm{i}F_{a,\phi})C_{a,\phi}.$$

By Theorem 4.9 3) and 5), we have $\operatorname{dom}\mathcal{A}_{a,\phi}^{\frac{1}{2}} = \operatorname{ran} Q^{\frac{1}{2}} = \operatorname{ran} C = \mathcal{R}$ and $\operatorname{Re}\mathcal{A}_{a,\phi} \subsetneqq (\mathcal{A}_{a,\phi})_R$. Then (2.14) implies that

$$\operatorname{ran} C_{a,\phi} = \operatorname{ran}\mathcal{T}_{a,\phi}^{\frac{1}{2}} = \operatorname{dom}\mathcal{A}_{a,\phi}^{\frac{1}{2}} = \mathcal{R}.$$

If we define

$$U_{a,\phi} := (I - \mathrm{i}F_{a,\phi})^{-1}(I + \mathrm{i}F_{a,\phi}) = -I + 2(I - \mathrm{i}F_{a,\phi})^{-1}, \quad S_{a,\phi} := U_{a,\phi}C_{a,\phi},$$

then $-1 \in \rho(U_{a,\phi})$ since $F_{a,\phi}$ is bounded. Now Theorem 3.11 3) and Proposition 5.1 yield that

$$\begin{aligned}&(I - \mathrm{i}F_{a,\phi})\operatorname{ran} C_{a,\phi} = (I + \mathrm{i}F_{a,\phi})\operatorname{ran} C_{a,\phi}, \quad U_{a,\phi}\operatorname{ran} C_{a,\phi} = \operatorname{ran} C_{a,\phi},\\ &\operatorname{ran} S_{a,\phi} = \operatorname{ran} S_{a,\phi}^* = \operatorname{ran}(S_{a,\phi}^* + |S_{a,\phi}|) = \operatorname{ran} C_{a,\phi}.\end{aligned}$$

This implies that

$$(U_{a,\phi} + I)\operatorname{ran} C_{a,\phi} = (I - \mathrm{i}F_{a,\phi})^{-1}\operatorname{ran} C_{a,\phi} \subsetneq \operatorname{ran} C_{a,\phi}, \quad \operatorname{ran}(S_{a,\phi} + |S_{a,\phi}|) \subsetneq \operatorname{ran} C_{a,\phi}.$$

Because $\operatorname{ran} C_{a,\phi} = \mathcal{R}$, and $\operatorname{Re}\mathcal{A}_{a,\phi} \subsetneqq (\mathcal{A}_{a,\phi})_R$, Theorem 3.11 3) and 4) show that

$$\begin{aligned}&(U_{a,\phi} + I)\mathcal{R} \subsetneqq \mathcal{R}, \quad (I - \mathrm{i}F_{a,\phi})^{-1}\mathcal{R} = (I + \mathrm{i}F_{a,\phi})^{-1}\mathcal{R} \subsetneqq \mathcal{R},\\ &\ker S_{a,\phi} = \{0\}, \quad \operatorname{ran} S_{a,\phi} = \operatorname{ran} S_{a,\phi} = \operatorname{ran}(S_{a,\phi}^* + |S_{a,\phi}|) = \mathcal{R}, \quad \operatorname{ran}(S_{a,\phi} + |S_{a,\phi}|) \subsetneqq \mathcal{R}.\end{aligned}$$

This proves claims a), c) and e).

Due to Proposition 5.2, the operators

$$W := C^{-1}UC, \quad W_{a,\phi} := C_{a,\phi}^{-1}U_{a,\phi}C_{a,\phi},$$

are bounded and satisfy $W^*C = CU^*$, $W_{a,\phi}^*C_{a,\phi} = C_{a,\phi}U_{a,\phi}^*$ and $0 \in \rho(W)$, $0 \in \rho(W_{a,\phi})$ as well as

$$(W^* + I)\mathcal{R} = (W_{a,\phi}^* + I)\mathcal{R} = \mathcal{R}.$$

Further, the equality $(U + I)\operatorname{ran} C = \operatorname{ran} C$ is equivalent to $-1 \in \rho(W)$ and $(U_{a,\phi} + I)\operatorname{ran} C_{a,\phi} \subsetneqq \operatorname{ran} C_{a,\phi}$ implies $-1 \notin \rho(W_{a,\phi})$. Thus for $Y = W^*$ claim (g) holds, while for $Y = W_{a,\phi}^*$ claim (h) holds. □

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