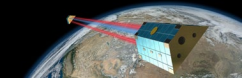


Precise Orbit Determination

Adrian Jäggi

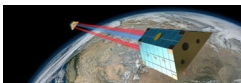
Astronomical Institute
University of Bern



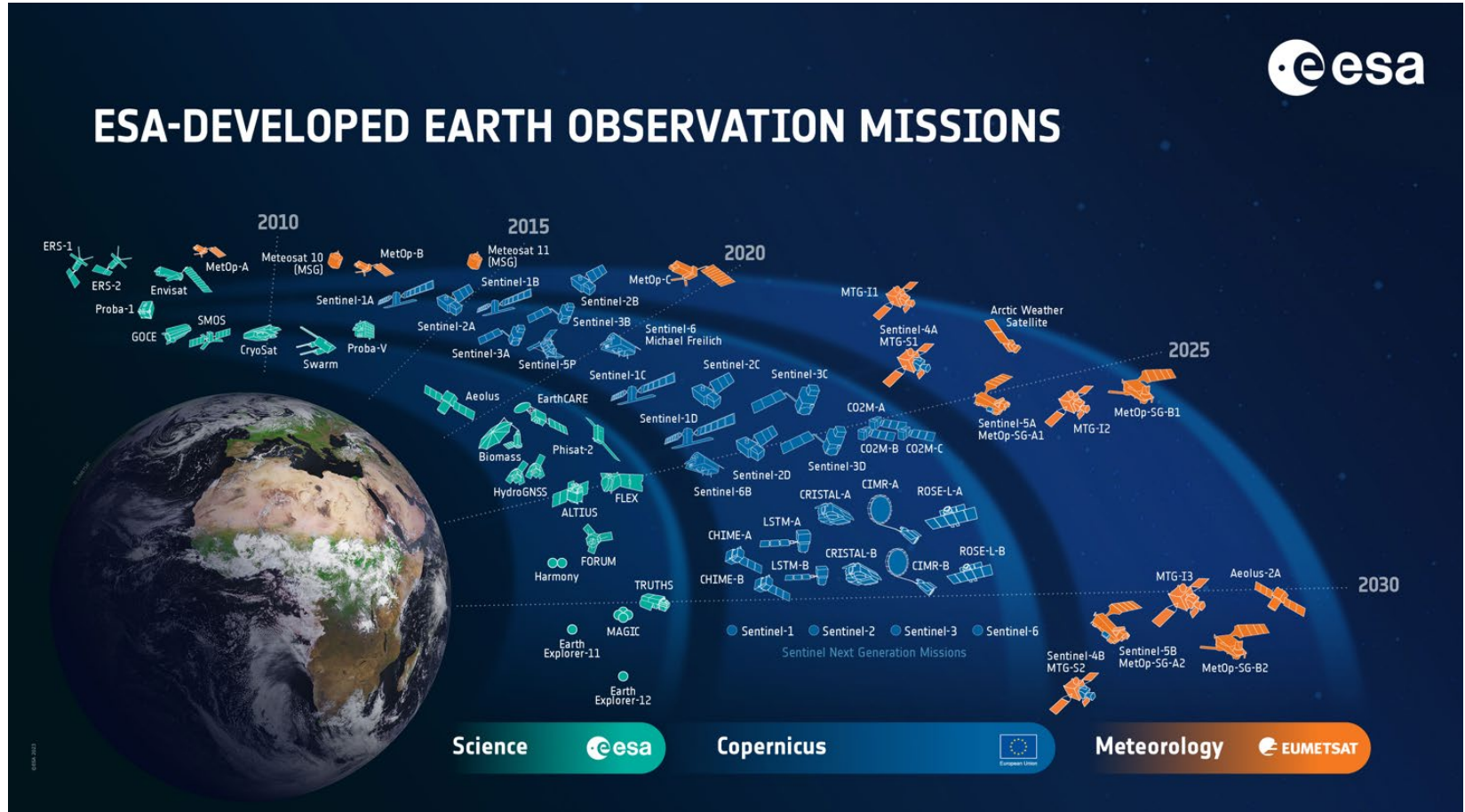


Lecture Contents

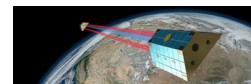
1. Introduction
2. Global Positioning System
3. Different Orbit Representations
4. Principles of Orbit Determination
5. GPS-based LEO POD
6. Orbit Validation



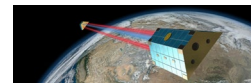
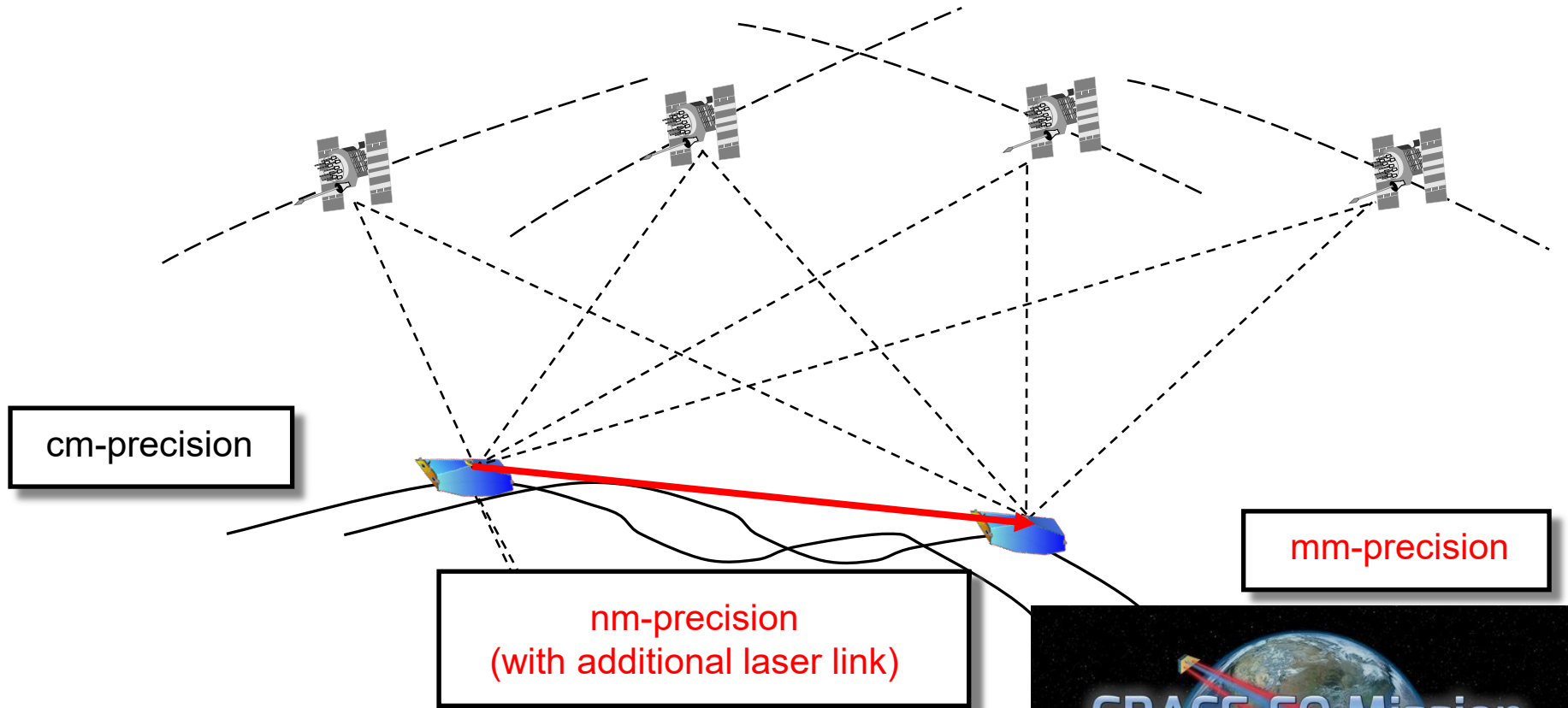
A multitude of Earth Observation Satellites

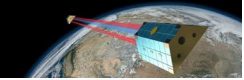


Courtesy: ESA



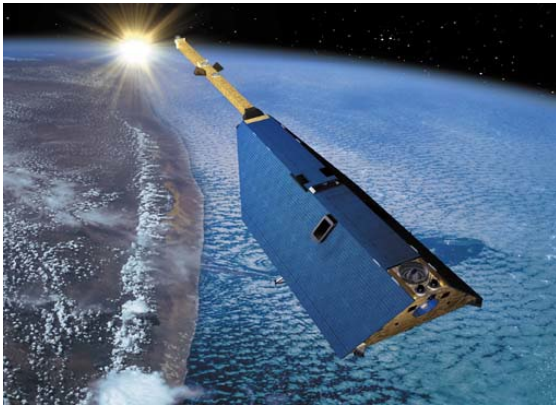
Precise Tracking Data in Near Earth Space





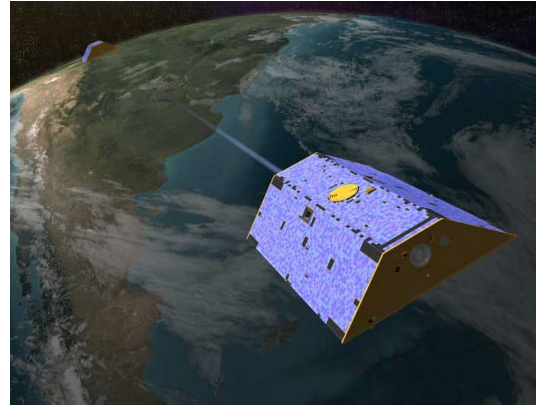
Low Earth Orbiters (LEOs)

CHAMP



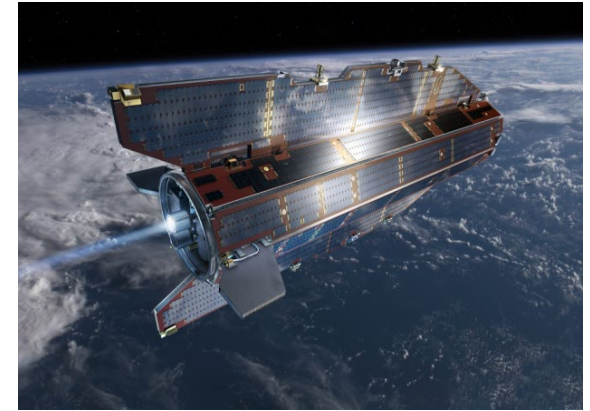
CHallenging
Minisatellite **P**ayload

GRACE



Gravity **R**ecovery **A**nd
Climate **E**xperiment

GOCE



Gravity and
steady-state **O**cean
Circulation **E**xplorer

Of course, there are many more missions equipped with GPS receivers

Jason



Jason-2



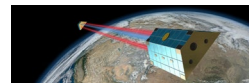
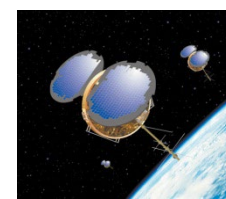
MetOp-A

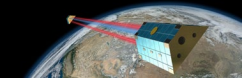


Icesat



COSMIC





LEO Constellations

TanDEM-X



Swarm



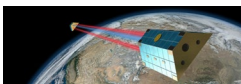
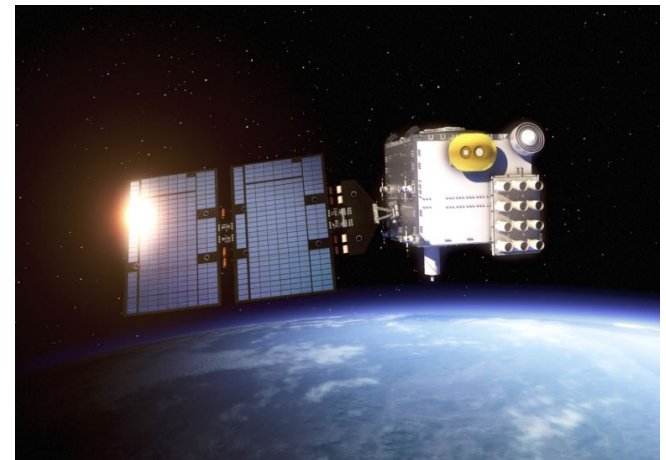
Sentinel

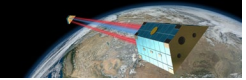


GRACE-FO

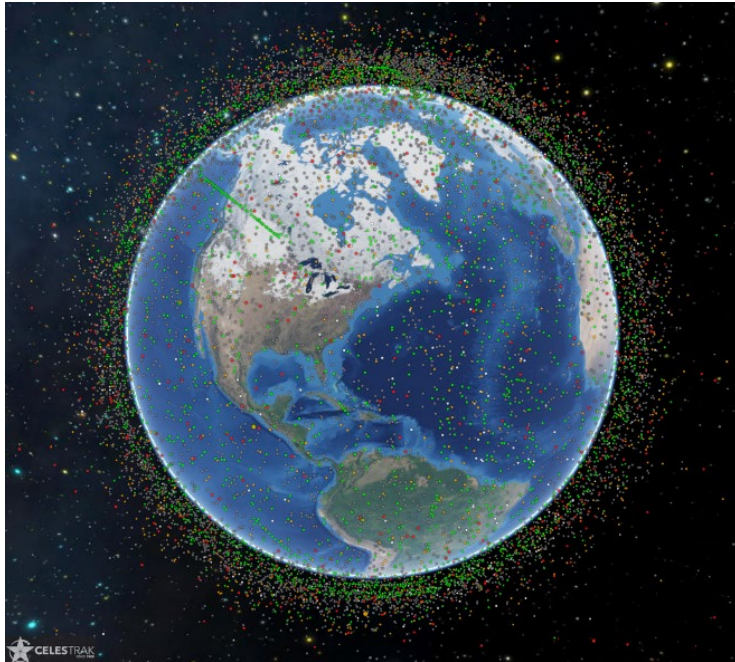


COSMIC-2





LEO CubeSats



A multitude of **CubeSats** already exist or are planned for the future

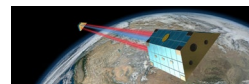
Spire

e.g.



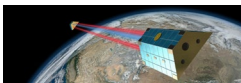
~ 150 Spire satellites are already in different orbits, offering dual-frequency high-quality GPS data

Many more will follow ...



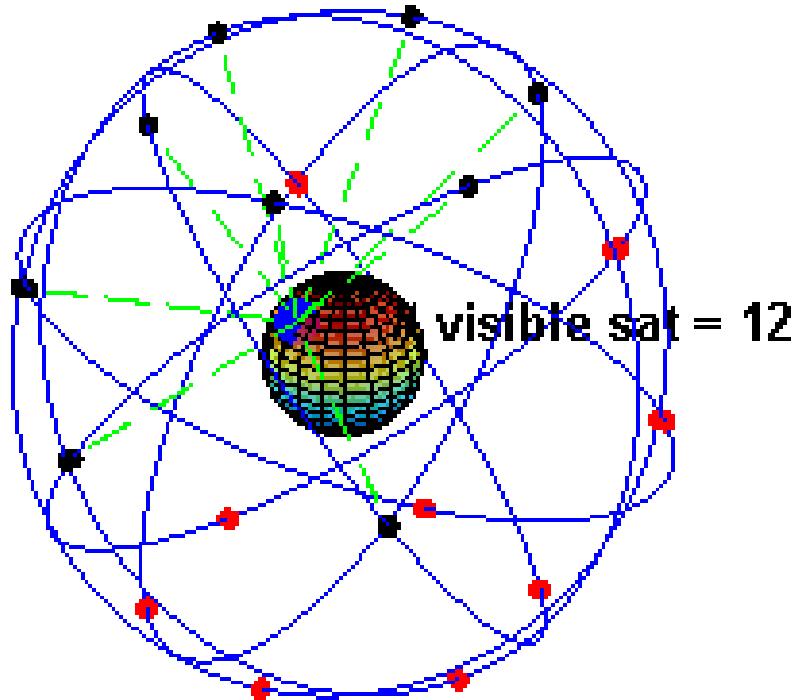


Global Positioning System

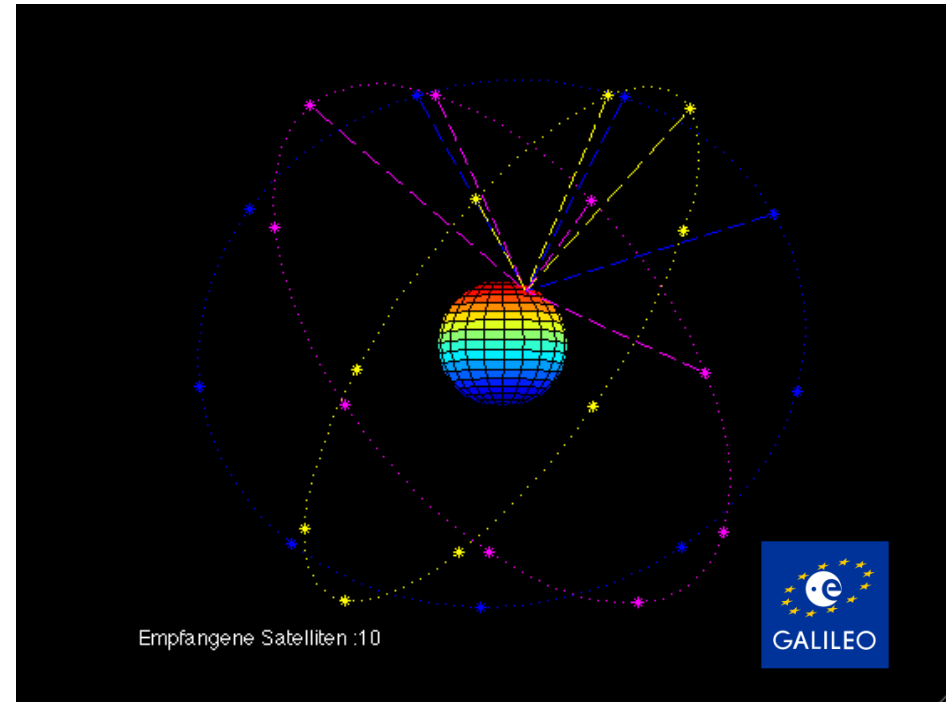


Introduction to GPS

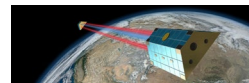
GPS

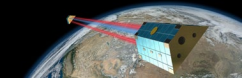


Galileo



Other **Global Navigation Satellite Systems** (GNSS) are also available (GLONASS, Galileo, Beidou), but for a long time no multi-GNSS spaceborne receivers were in orbit. This changed with the launches of Fengyun-3, COSMIC-2, Sentinel-6.



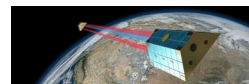


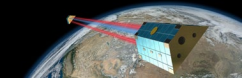
Introduction to GPS

GPS: Global Positioning System

Characteristics:

- Satellite system for (real-time) **Positioning** and **Navigation**
- **Global** (everywhere on Earth, up to altitudes of 5000km) and **at any time**
- **Unlimited** number of users
- **Weather-independent** (radio signals are passing through the atmosphere)
- **3-dimensional position, velocity** and **time** information





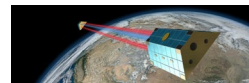
GPS Segments

The GPS consists of **3 main segments**:

- **Space Segment:** the satellites and the constellation of satellites
- **Control Segment:** the ground stations, infrastructure and software for operation and monitoring of the GPS
- **User Segment:** all GPS receivers worldwide and the corresponding processing software

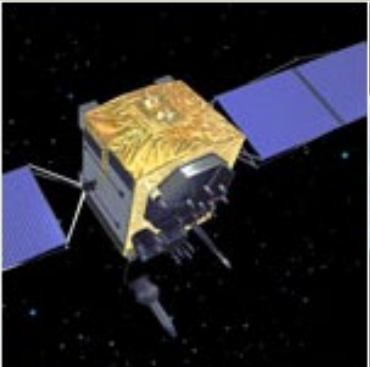
We should add an important **4th segment**:

- **Ground Segment:** all civilian permanent networks of reference sites and the international/regional/local services delivering products for the users

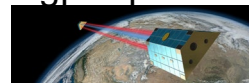


Space Segment

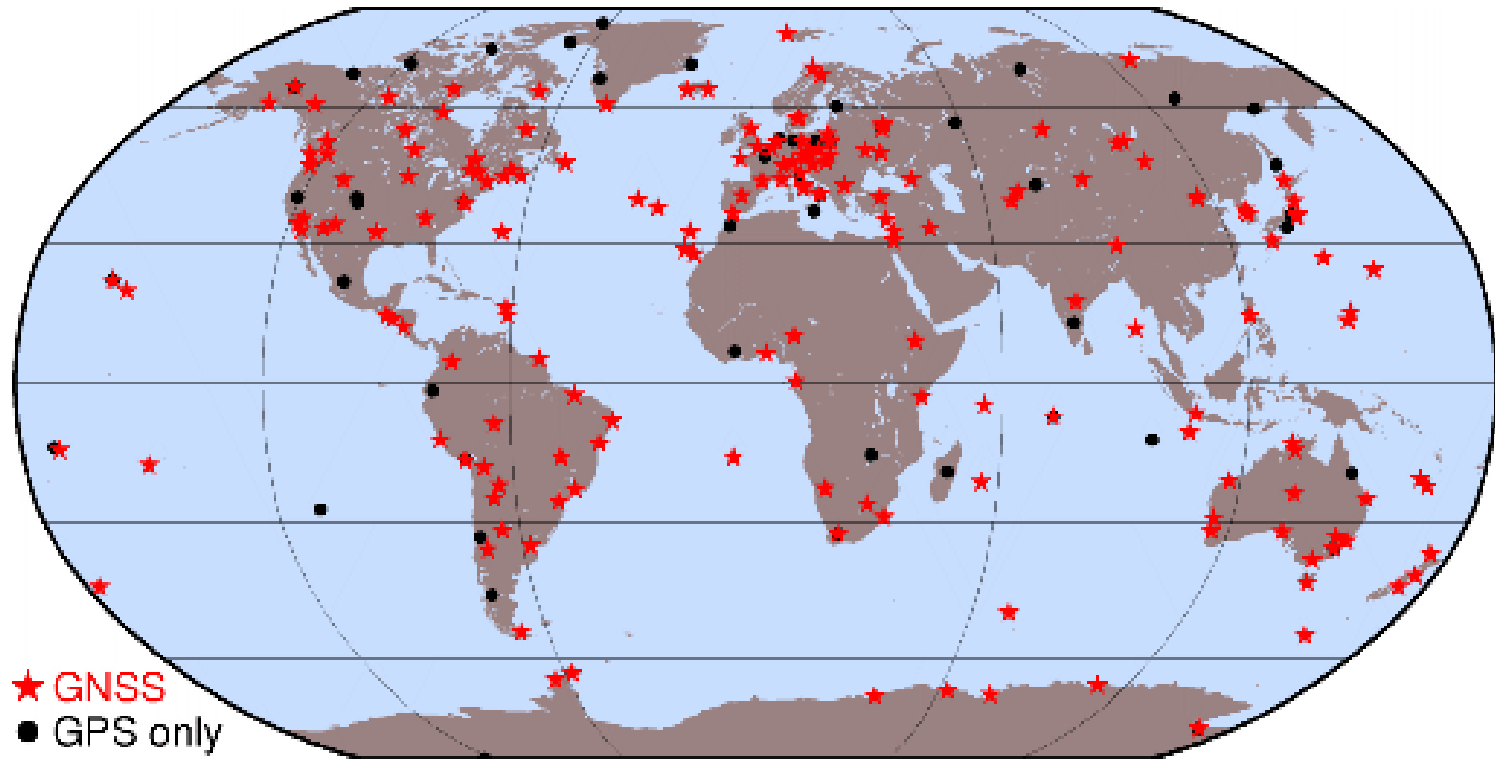
- The space segment nominally consists of **24 satellites**, presently: 31 active GPS satellites
- Constellation design: at least **4 satellites** in view from **any location** on the Earth at **any time**

LEGACY SATELLITES		MODERNIZED SATELLITES		
				
BLOCK IIA	BLOCK IIR	BLOCK IIR-M	BLOCK IIF	GPS III/IIIF
0 operational	6 operational	7 operational	12 operational	6 operational

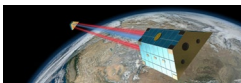
<https://www.gps.gov/systems/gps/space/>



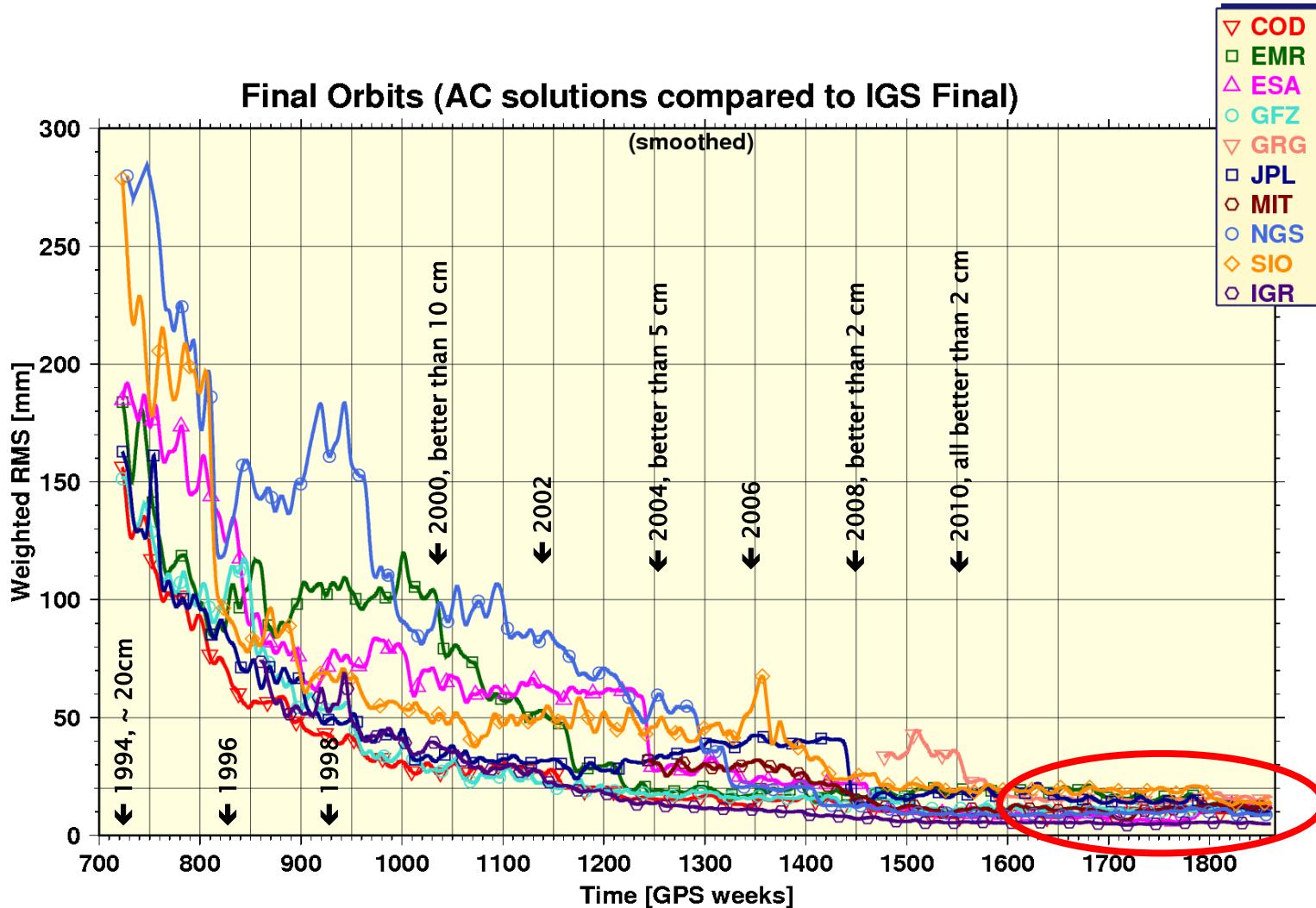
Global Network of the IGS



IGS stations used for computation of final orbits at CODE (Dach et al., 2009)

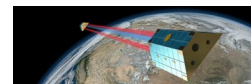


Performance of IGS Final Orbits

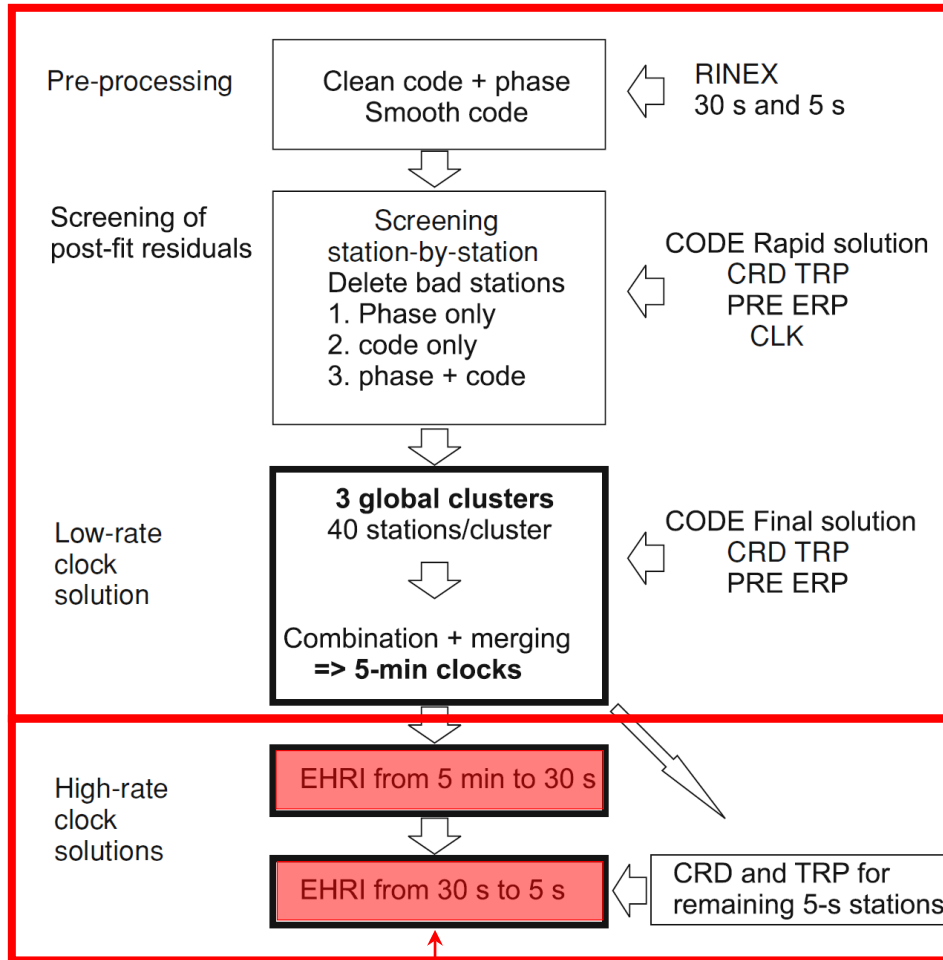


NOAA NGS, 19.09.2015 19:19 (GMT)

<http://acc.igs.org/>



Computation of High-Rate Clocks

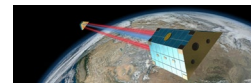


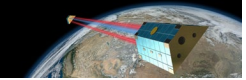
(Bock et al., 2009)

The final clock product with 5 min sampling is based on undifferenced GPS data of typically 120 stations of the IGS network

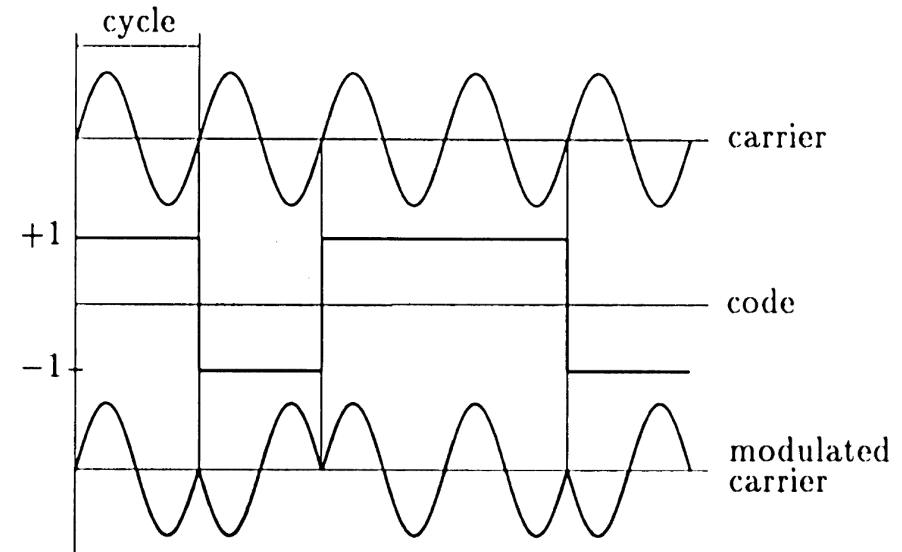
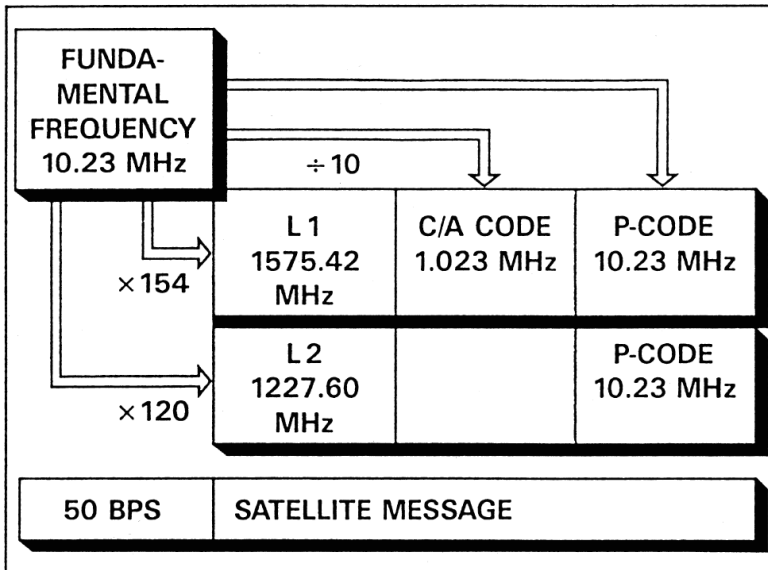
The IGS 1 Hz network is finally used for clock densification to 5 sec

The 5 sec clocks are interpolated to 1 sec as needed for 1 Hz kinematic LEO POD





GPS Signals



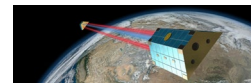
Bits encoded on carrier by phase modulation:

Signals driven by an **atomic clock**

Two **carrier signals** (sine waves):

- **L1**: $f = 1575.43$ MHz, $\lambda = 19$ cm
- **L2**: $f = 1227.60$ MHz, $\lambda = 24$ cm

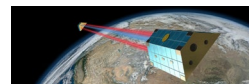
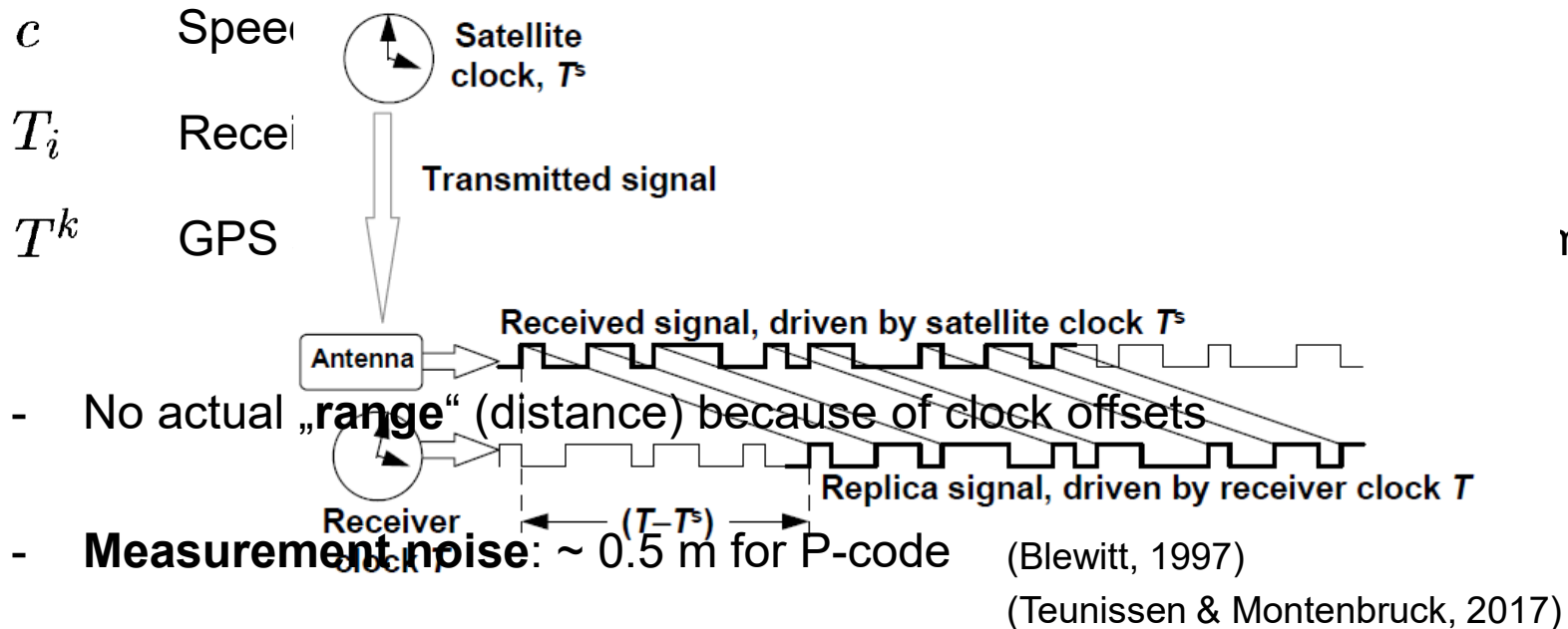
- **C/A-code** (Clear Access / Coarse Acquisition)
- **P-code** (Protected / Precise)
- **Broadcast/Navigation Message**

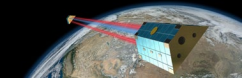


Pseudorange / Code Measurements

Code Observations P_i^k are defined as:

$$P_i^k \doteq c (T_i - T^k)$$





Code Observation Equation

$$P_i^k = \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i$$

t_i, t^k GPS time of reception and emission

Δt^k Satellite clock offset $T^k - t^k$

Δt_i Receiver clock offset $T_i - t_i$

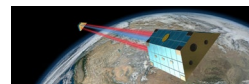
ρ_i^k Distance between receiver and satellite $c (t_i - t^k)$

Known from ACs or IGS:

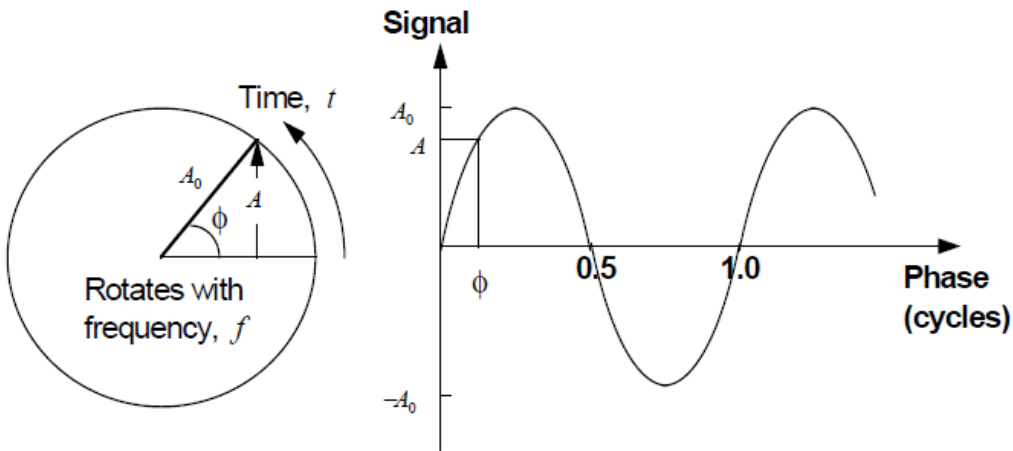
- satellite positions (x^{kj}, y^{kj}, z^{kj})
- satellite clock offsets Δt^{kj}

4 unknown parameters:

- receiver position (x_i, y_i, z_i)
- receiver clock offset Δt_i



Carrier Phase Measurements (1)



Phase ϕ (in cycles) increases linearly with time t :

$$\phi = f \cdot t$$

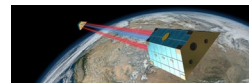
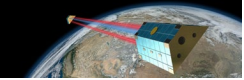
where f is the frequency

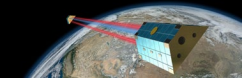
The **satellite** generates with its clock the phase signal ϕ^k . At emission time T^k (in satellite clock time) we have

$$\phi^k = f \cdot T^k$$

The same phase signal, e.g., a wave crest, propagates from the satellite to the receiver, but the receiver measures only the fractional part of the phase and does not know the **integer number of cycles** N_i^k (phase ambiguity):

$$\phi_i^k = \phi^k - N_i^k = f \cdot T^k - N_i^k$$





Carrier Phase Measurements (2)

The **receiver** generates with its clock a **reference phase**. At time of reception T_i of the satellite phase ϕ_i^k (in receiver clock time) we have:

$$\phi_i = f \cdot T_i$$

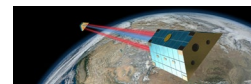
The actual **phase measurement** is the difference between receiver reference phase ϕ_i and satellite phase ϕ_i^k :

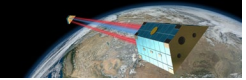
$$\psi_i^k = \phi_i - \phi_i^k = f \cdot T_i - (f \cdot T^k - N_i^k) = f \cdot (T_i - T^k) + N_i^k$$

Multiplication with the wavelength $\lambda = c/f$ leads to the **phase observation equation** in meters:

$$\begin{aligned} L_i^k &= \lambda \cdot \psi_i^k = c \cdot (T_i - T^k) + \lambda \cdot N_i^k \\ &= \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i + \lambda \cdot N_i^k \end{aligned}$$

Difference to the pseudorange observation: **integer ambiguity term** N_i^k



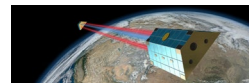


Detailed Observation Equation

$$L_i^k = \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i + \cancel{I_i^k} + \cancel{I_i^k} + \lambda \cdot N_i^k \\ + \Delta_{rel} - c \cdot b^k + c \cdot b_i + m_i^k + \epsilon_i^k$$

ρ_i^k	Distance between satellite and receiver	←	Satellite positions and clocks
Δt^k	Satellite clock offset wrt GPS time	←	are known from the IGS
Δt_i	Receiver clock offset wrt GPS time		
T_i^k	Tropospheric delay	←	Not existent for LEOs
I_i^k	Ionospheric delay	←	Cancels out (first order only)
N_i^k	Phase ambiguity		when forming the ionosphere-free linear combination:
Δ_{rel}	Relativistic corrections		
b^k	Delays in satellite (cables, electronics)		
b_i	Delays in receiver and antenna		
m_i^k	Multipath, scattering, bending effects		
ϵ_i^k	Measurement error		

$$L_c = \frac{f_1^2}{f_1^2 - f_2^2} L_1 - \frac{f_2^2}{f_1^2 - f_2^2} L_2$$





Geometric Distance

Geometric distance ρ_{leo}^k is given by:

$$\rho_{leo}^k = |\mathbf{r}_{leo}(t_{leo}) - \mathbf{r}^k(t_{leo} - \tau_{leo}^k)|$$

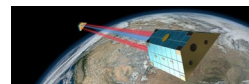
\mathbf{r}_{leo} Inertial position of LEO antenna phase center at reception time

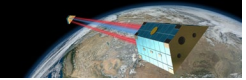
\mathbf{r}^k Inertial position of GPS antenna phase center of satellite k at emission time

τ_{leo}^k Signal traveling time between the two phase center positions

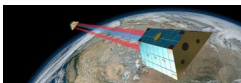
Different ways to represent \mathbf{r}_{leo} :

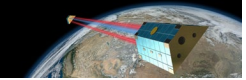
- **Kinematic** orbit representation
- **Dynamic** or **reduced-dynamic** orbit representation





Different Orbit Representations





Kinematic Orbit Representation (1)

Satellite position $\mathbf{r}_{leo}(t_{leo})$ (in inertial frame) is given by:

$$\mathbf{r}_{leo}(t_{leo}) = \mathbf{R}(t_{leo}) \cdot (\mathbf{r}_{leo,e,0}(t_{leo}) + \delta\mathbf{r}_{leo,e,ant}(t_{leo}))$$

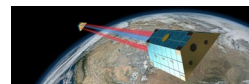
\mathbf{R} Transformation matrix from Earth-fixed to inertial frame

$\mathbf{r}_{leo,e,0}$ LEO center of mass position in Earth-fixed frame

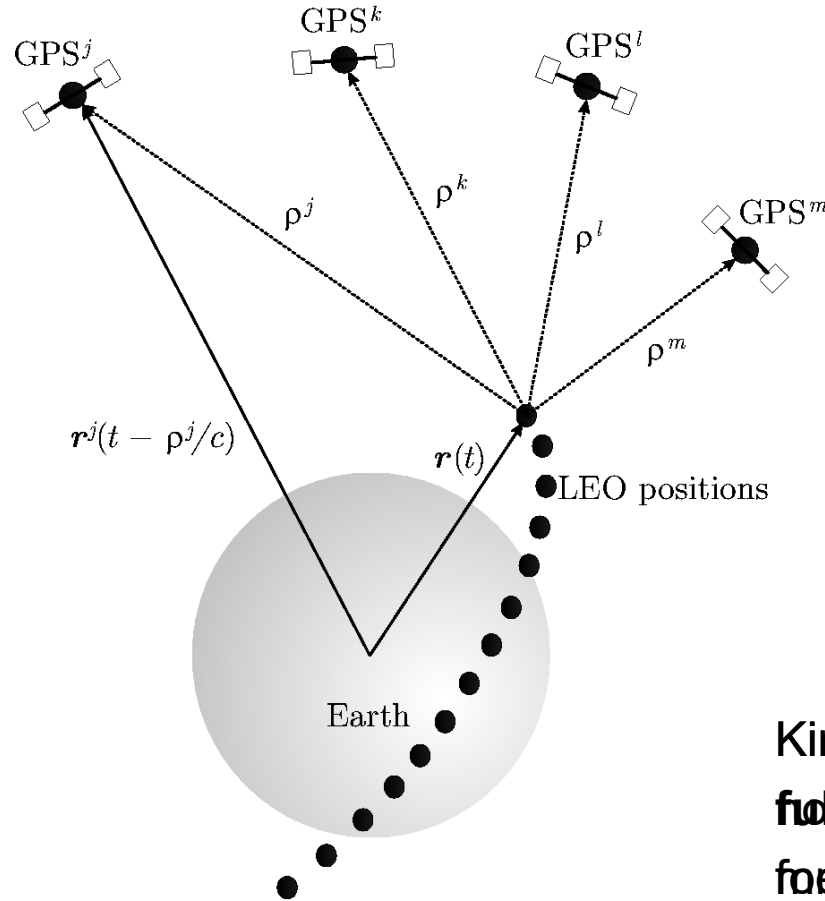
$\delta\mathbf{r}_{leo,e,ant}$ LEO antenna phase center offset in Earth-fixed frame

Kinematic positions $\mathbf{r}_{leo,e,0}$ are estimated for each **measurement epoch**:

- Measurement epochs **need not** to be identical with nominal epochs
- Positions are **independent** of models describing the LEO dynamics. Velocities and accelerations cannot be provided in a "strict" sense.



Kinematic Orbit Representation (2)



A kinematic orbit is an ephemeris at **discrete** measurement epochs

Kinematic positions are **fully independent** of the force models used for (LEO to a high degree) (Saggl et al., 2011) (Schäfer, 2004)



Kinematic Orbit Representation (3)

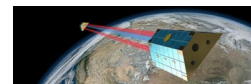
Measurement epochs
(in GPS time)

Positions (km)
(Earth-fixed)

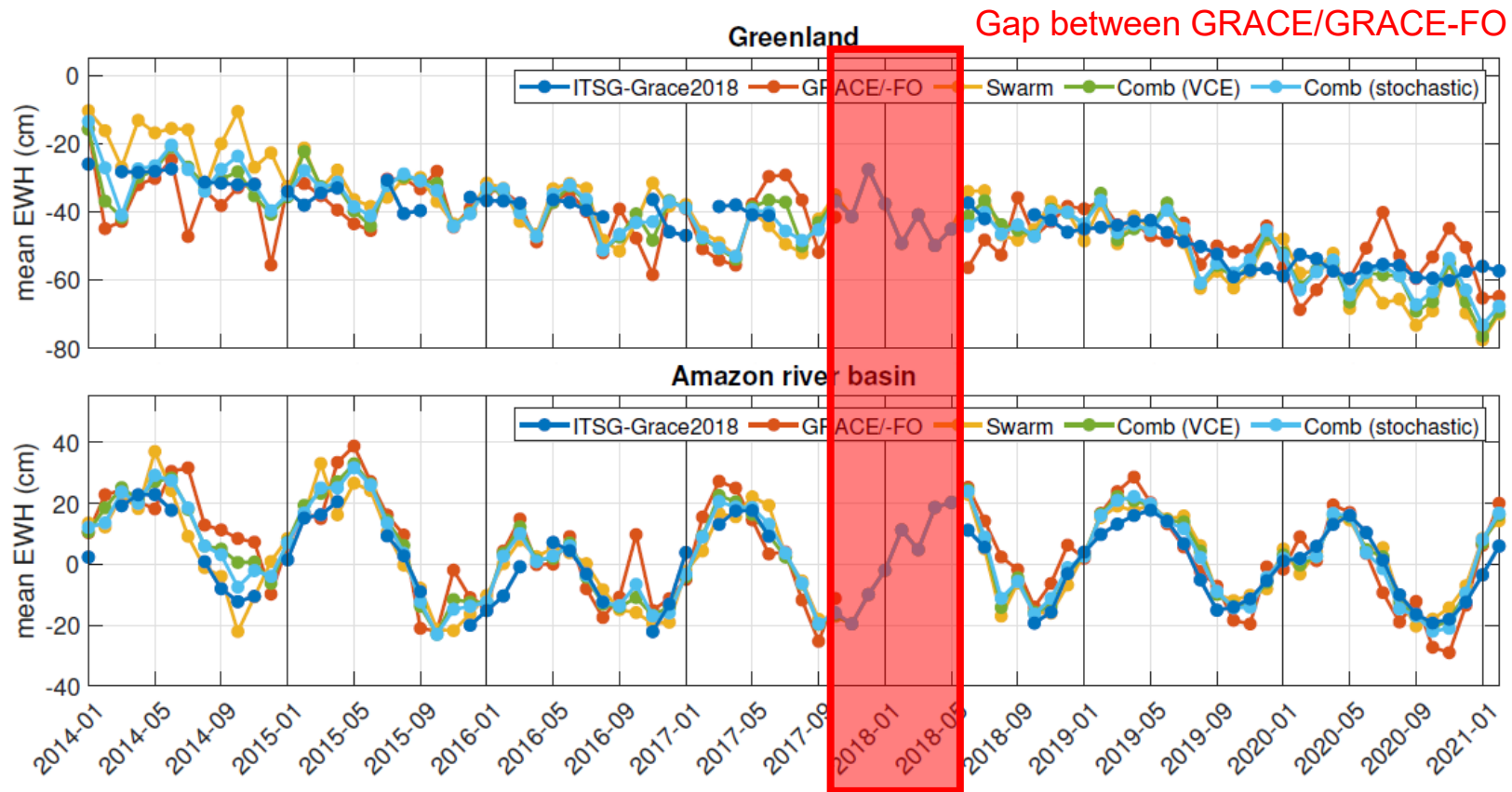
* 2009 11 2	0	0	0.80678020		
PL15	-390.612059	6623.987679	73.104149	193219.797196	
* 2009 11 2	0	0	1.80678020		
PL15	-389.240315	6624.166512	65.402457	193219.799413	
* 2009 11 2	0	0	2.80678020		
PL15	-387.868014	6624.336133	57.700679	193219.801634	
* 2009 11 2	0	0	3.80678020		
PL15	-386.495163	6624.496541	49.998817	193219.803855	
* 2009 11 2	0	0	4.80678019		
PL15	-385.121760	6624.647724	42.296889	193219.806059	
* 2009 11 2	0	0	5.80678019		
PL15	-383.747819	6624.789703	34.594896	193219.808280	
* 2009 11 2	0	0	6.80678019		
PL15	-382.373332	6624.922464	26.892861	193219.810495	
* 2009 11 2	0	0	7.80678019		
PL15	-380.998306	6625.046003	19.190792	193219.812692	
* 2009 11 2	0	0	8.80678019		
PL15	-379.622745	6625.160329	11.488692	193219.814899	
* 2009 11 2	0	0	9.80678018		
PL15	-378.246651	6625.265448	3.786580	193219.817123	

Clock correction to nominal epoch (μ s), e.g., to epoch 00:00:03

Excerpt of kinematic GOCE positions at begin of 2 Nov, 2009

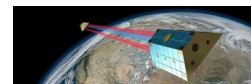


Recovery of Large-Scale Time-VARIABLE Gravity Signals

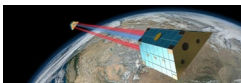
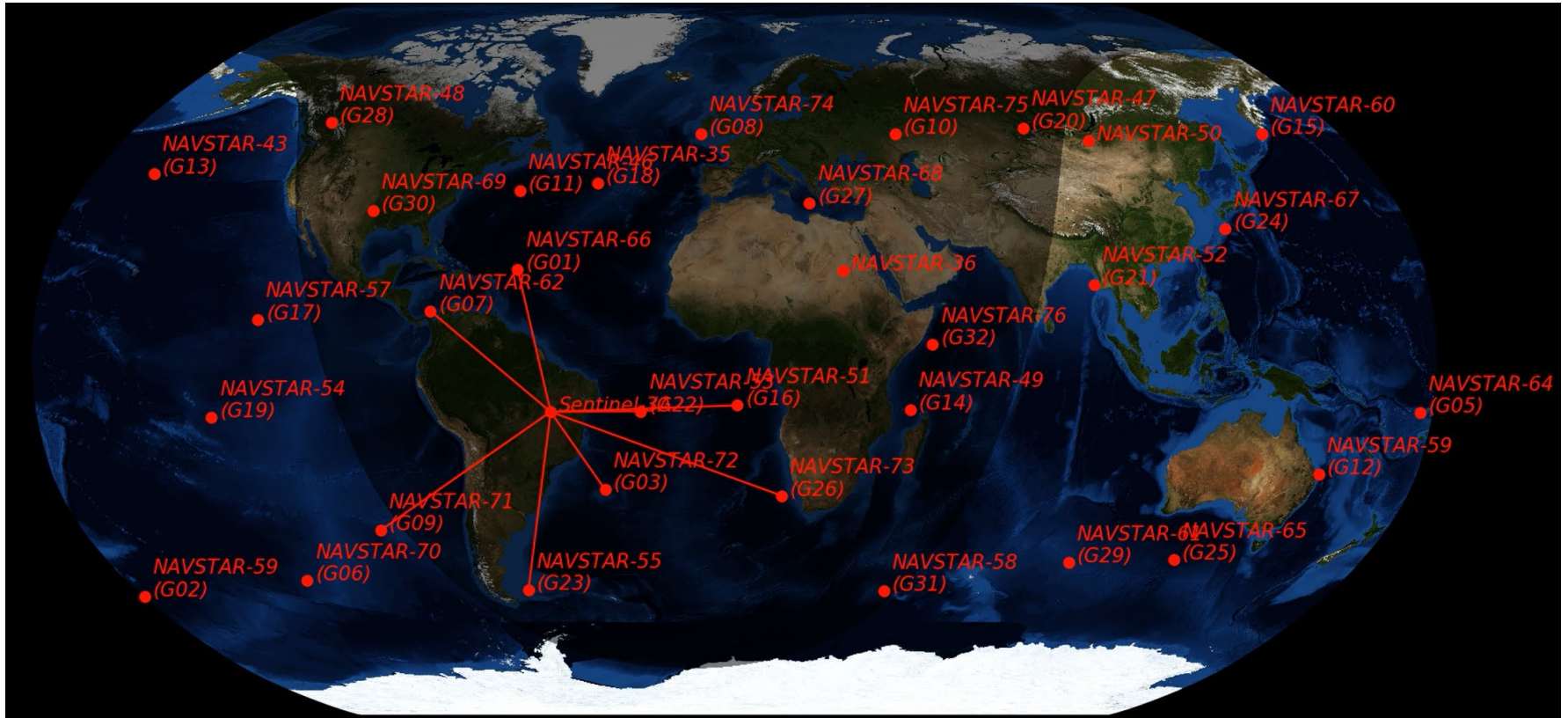


Kinematic positions allow it to recover the **long wavelength** part of the Earth's time-variable gravity field. The scatter of monthly gravity field solutions is larger than from dedicated GRACE/GRACE-FO data, but trends and annual signals may be derived remarkably well.

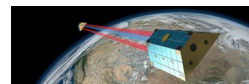
(Grombein et al., 2022)



Example: Sentinel-3A GPS Tracking

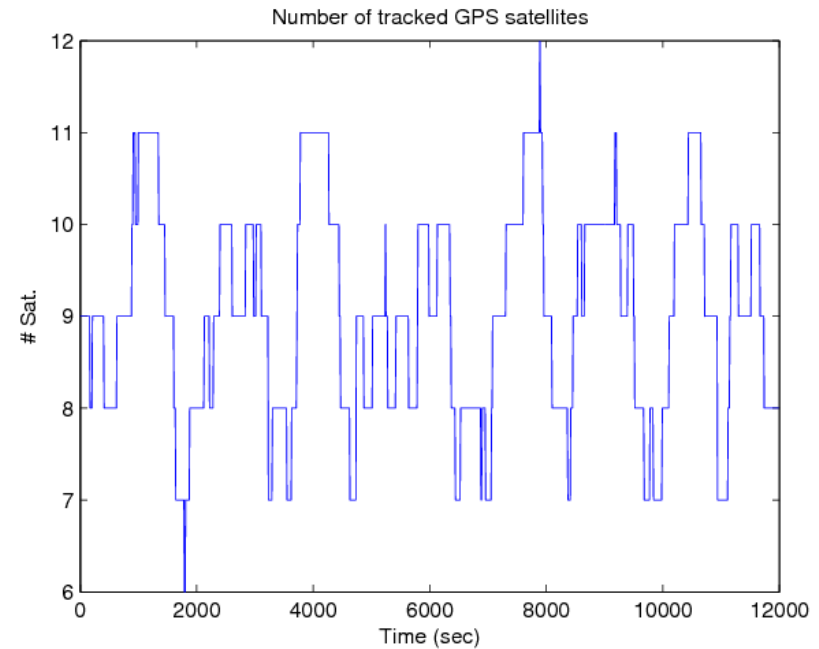
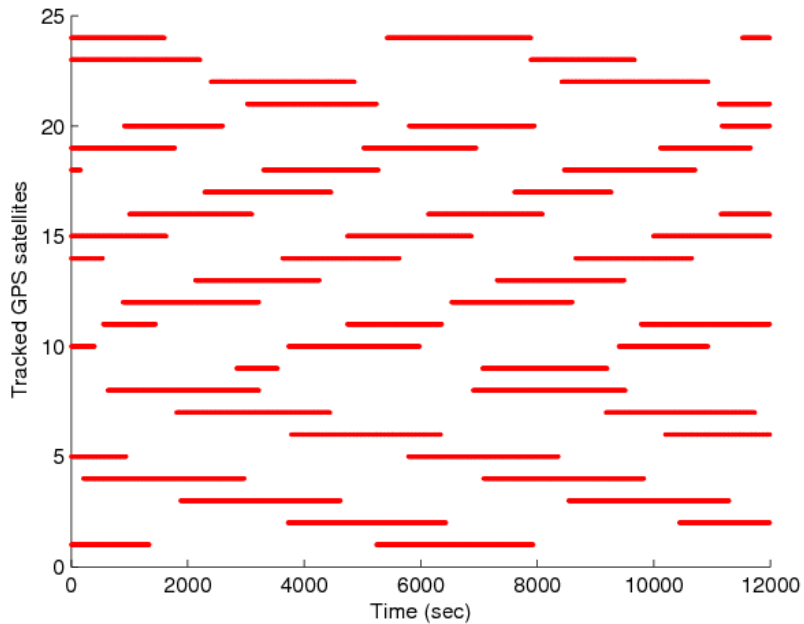


Example: Sentinel-6A multi-GNSS Tracking

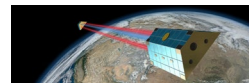


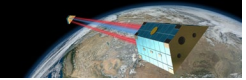


GPS Satellite Visibility



Tracking scenario of a simulated data set (left). Up to 12 GPS satellites are at maximum simultaneously visible from the LEO satellite (right). The viewing geometry is continuously changing due to the orbital motion of all satellites.





Dynamic Orbit Representation (1)

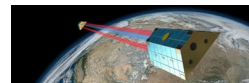
Satellite position $\mathbf{r}_{leo}(t_{leo})$ (in inertial frame) is given by:

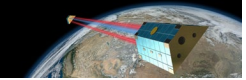
$$\mathbf{r}_{leo}(t_{leo}) = \mathbf{r}_{leo,0}(t_{leo}; a, e, i, \Omega, \omega, u_0; Q_1, \dots, Q_d) + \delta\mathbf{r}_{leo,ant}(t_{leo})$$

$\mathbf{r}_{leo,0}$	LEO center of mass position
$\delta\mathbf{r}_{leo,ant}$	LEO antenna phase center offset
$a, e, i, \Omega, \omega, u_0$	LEO initial osculating orbital elements
Q_1, \dots, Q_d	LEO dynamical parameters

Satellite trajectory $\mathbf{r}_{leo,0}$ is a particular solution of an **equation of motion**

- One set of **initial conditions** (orbital elements) is estimated per arc.
Dynamical parameters of the force model may be estimated on request.





Dynamic Orbit Representation (2)

Equation of motion (in inertial frame) is given by:

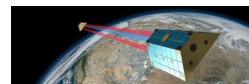
$$\ddot{\mathbf{r}} = -GM \frac{\mathbf{r}}{r^3} + \mathbf{f}_1(t, \mathbf{r}, \dot{\mathbf{r}}, Q_1, \dots, Q_d)$$

with initial conditions

$$\mathbf{r}(t_0) = \mathbf{r}(a, e, i, \Omega, \omega, u_0; t_0)$$

$$\dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}(a, e, i, \Omega, \omega, u_0; t_0)$$

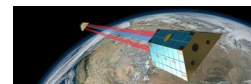
The **acceleration** \mathbf{f}_1 consists of **gravitational** and **non-gravitational** perturbations taken into account to model the satellite trajectory. Unknown parameters Q_1, \dots, Q_d of force models may appear in the equation of motion together with deterministic (known) accelerations given by analytical models.

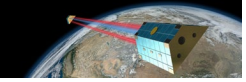




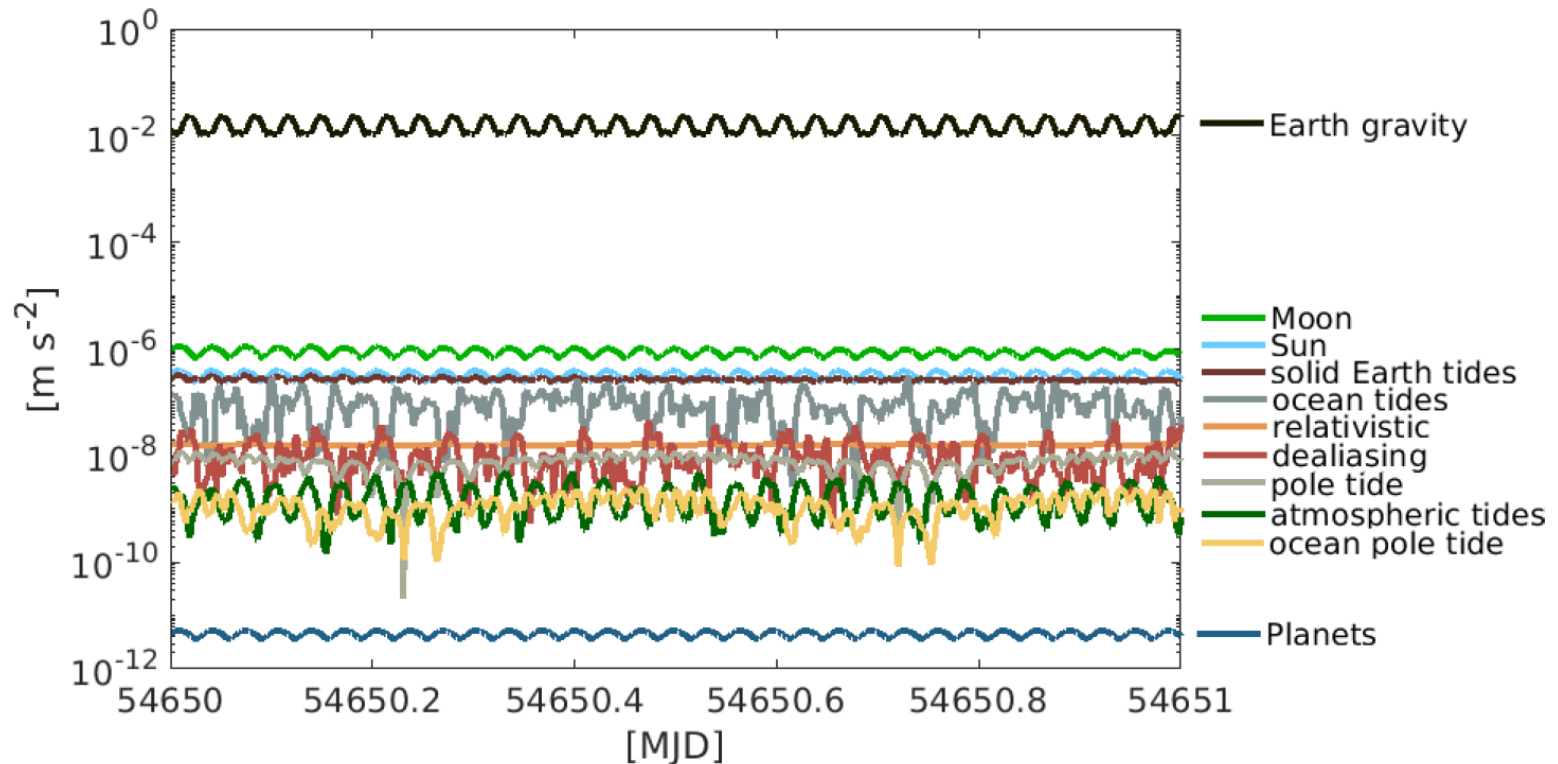
Perturbing Accelerations of a LEO Satellite

Force	Acceleration (m/s ²)
Central term of Earth's gravity field	8.42
Oblateness of Earth's gravity field	0.015
Atmospheric drag	0.00000079
Higher order terms of Earth's gravity field	0.00025
Attraction from the Moon	0.0000054
Attraction from the Sun	0.0000005
Direct solar radiation pressure	0.000000097
...	...



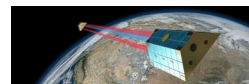


Perturbing Accelerations of a LEO Satellite

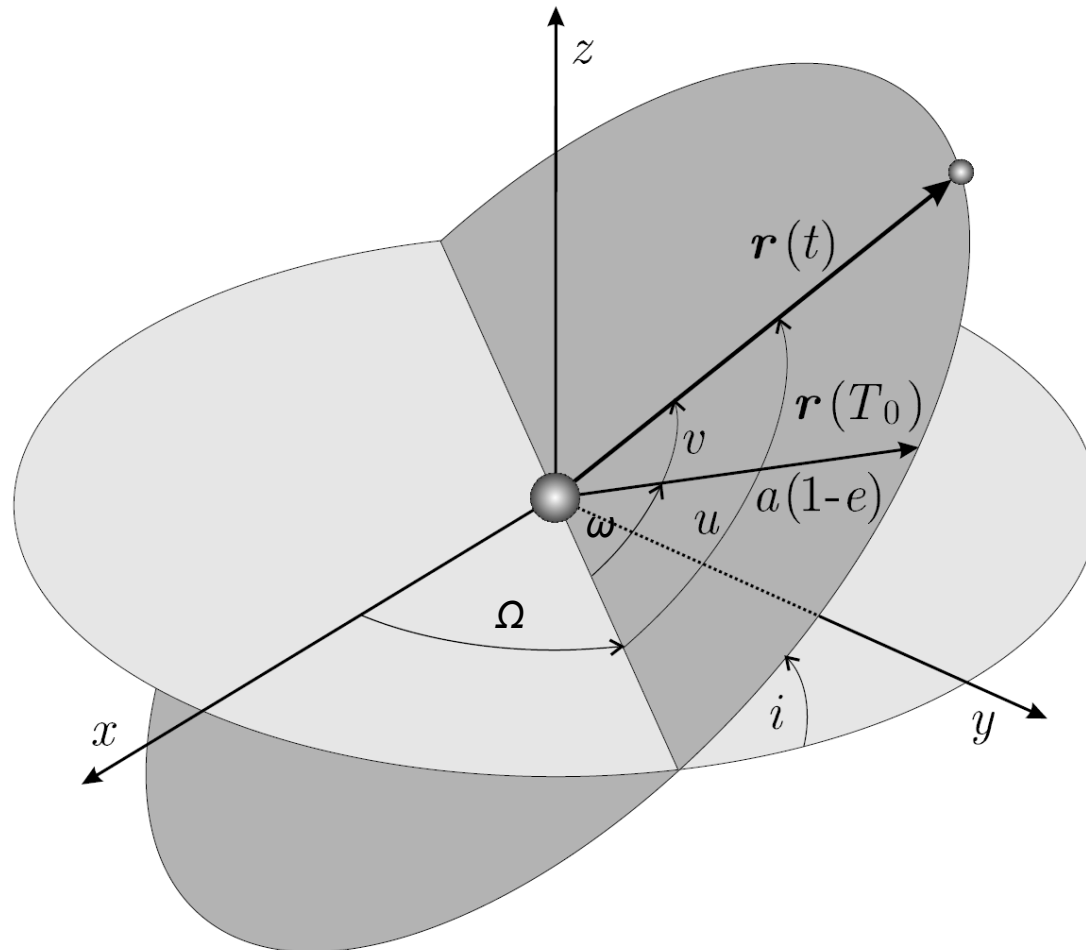


Norm of the **COST-G benchmark accelerations** along a GRACE satellite orbit. The benchmark data set may be used as a reference data set and provides the opportunity to test the implementation of corresponding background models.

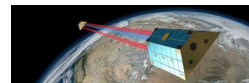
(Mayer-Gürr and Kvas, 2019; Lasser et al., 2020)

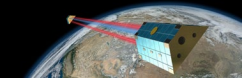


Osculating Orbital Elements

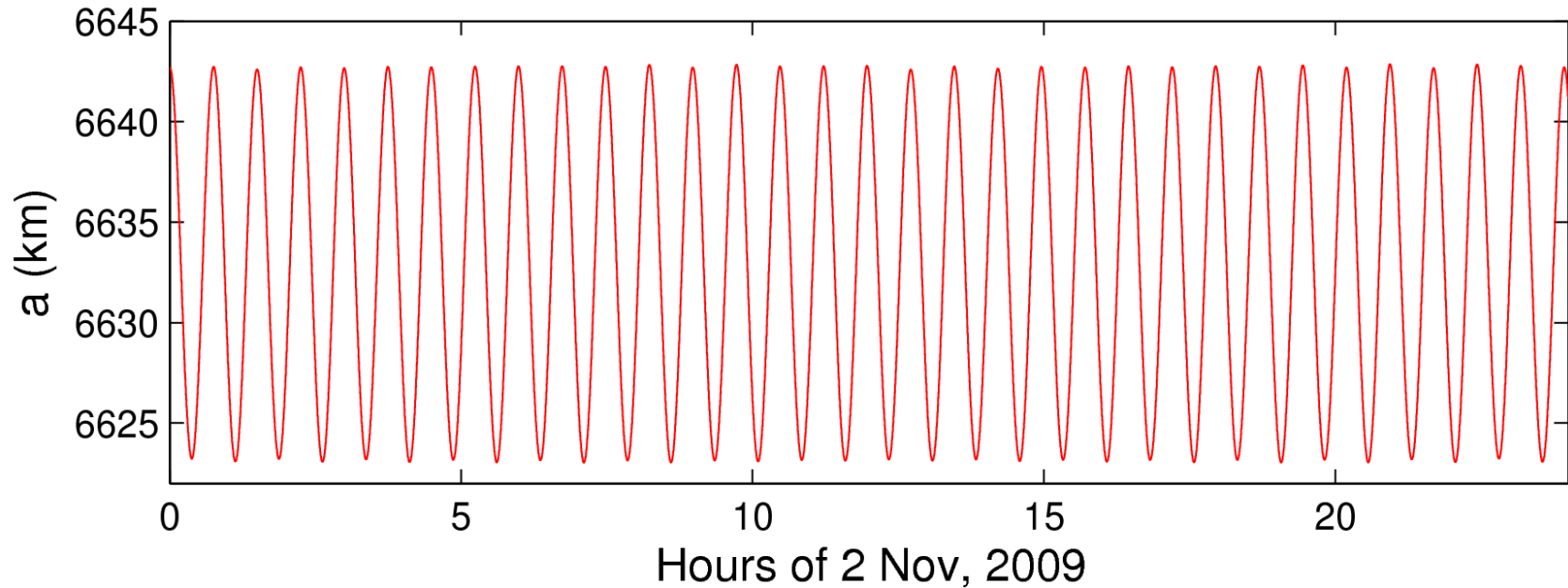


(Beutler, 2005)



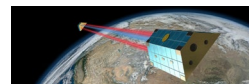


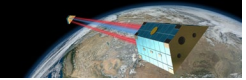
Osculating Orbital Elements of GOCE (1)



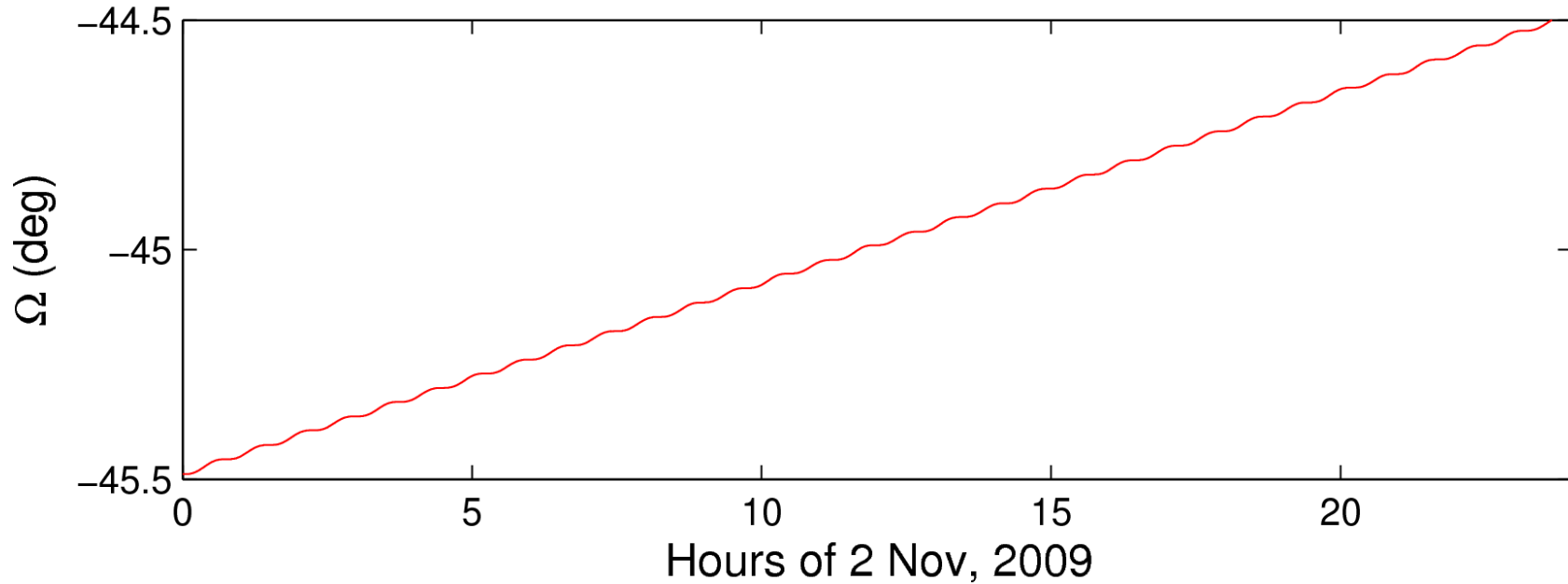
Semi-major axis:

Twice-per-revolution variations of about ± 10 km around the mean semi-major axis of 6632.9 km, which corresponds to a mean altitude of 254.9 km



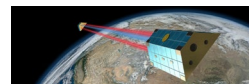


Osculating Orbital Elements of GOCE (2)

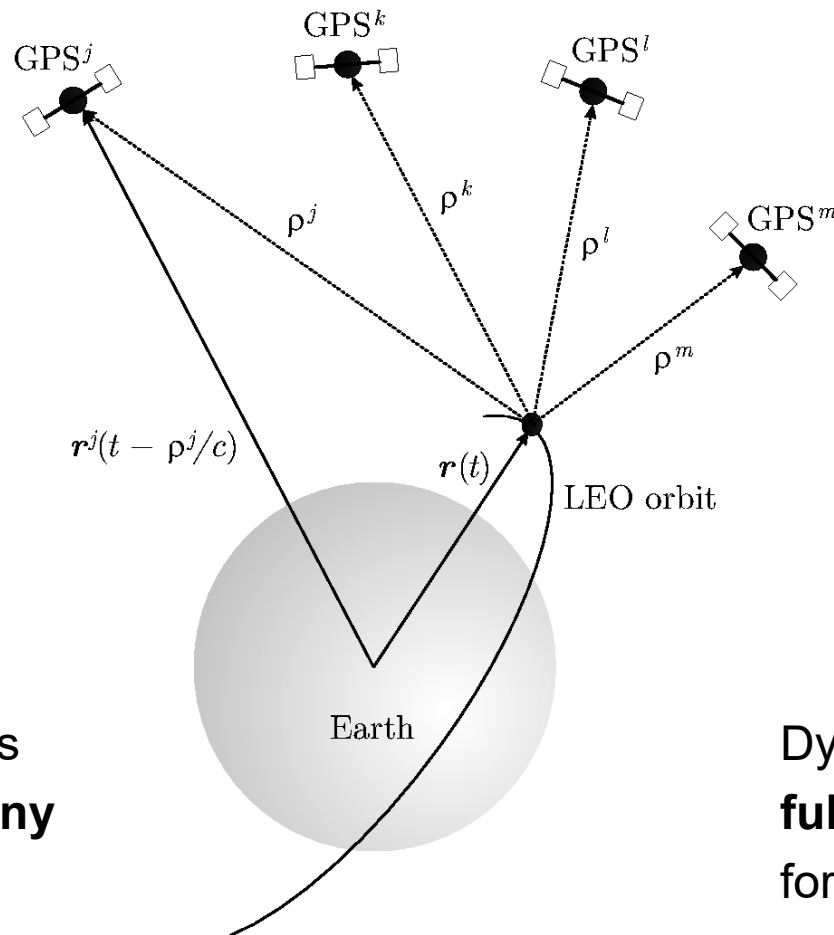


Right ascension of ascending node:

Twice-per-revolution variations and linear drift of about $+1^\circ/\text{day}$ ($360^\circ/365\text{days}$) due to the sun-synchronous orbit

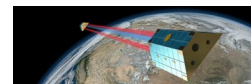


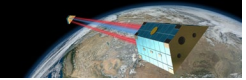
Dynamic Orbit Representation (3)



Dynamic orbit positions may be computed at **any epoch** within the arc

Dynamic positions are **fully dependent** on the force models used, e.g., on the gravity field model





Reduced-Dynamic Orbit Representation (1)

Equation of motion (in inertial frame) is given by:

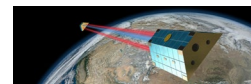
$$\ddot{\mathbf{r}} = -GM \frac{\mathbf{r}}{r^3} + \mathbf{f}_1(t, \mathbf{r}, \dot{\mathbf{r}}, Q_1, \dots, Q_d, P_1, \dots, P_s)$$

P_1, \dots, P_s

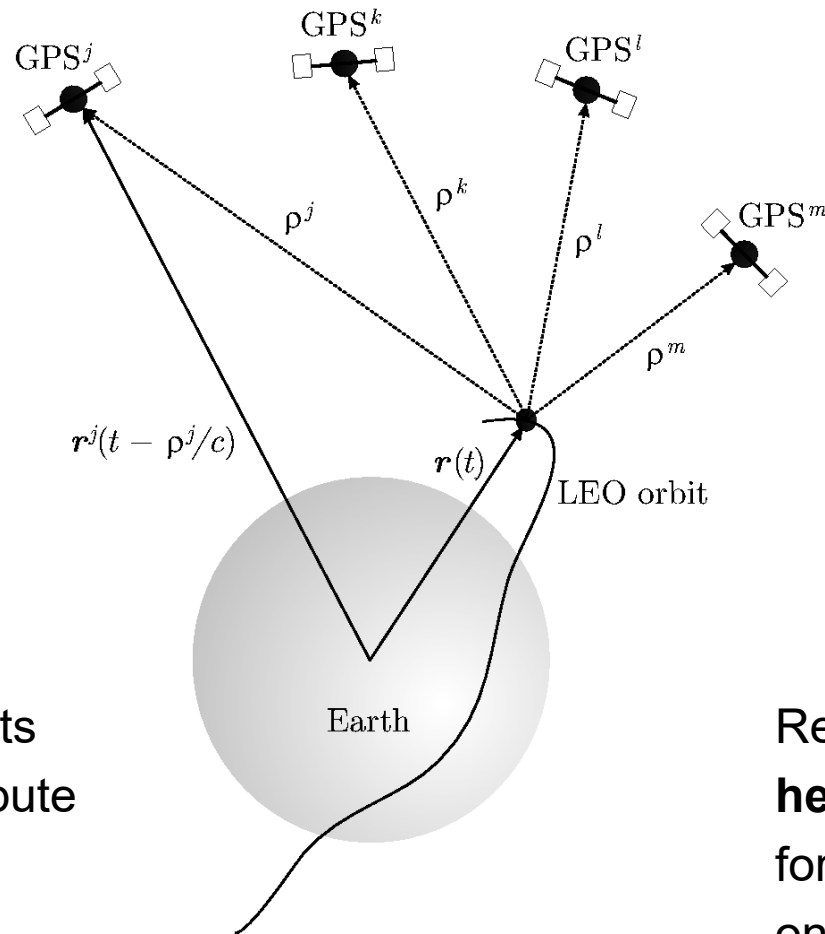
Pseudo-stochastic parameters

Pseudo-stochastic parameters are:

- additional empirical parameters characterized by a priori known **statistical properties**, e.g., by expectation values and a priori variances
- useful to **compensate** for deficiencies in dynamic models, e.g., deficiencies in models describing non-gravitational accelerations
- often set up as **piecewise constant accelerations** to ensure that satellite trajectories are continuous and differentiable at any epoch



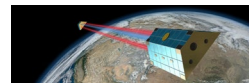
Reduced-Dynamic Orbit Representation (2)



Reduced-dynamic orbits are well suited to compute LEO orbits of **highest quality**

(Jäggi et al., 2006; Jäggi, 2007)

Reduced-dynamic orbits **heavily depend** on the force models used, e.g., on the gravity field model



Reduced-dynamic Orbit Representation (3)

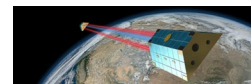
Positions (km) &
Velocities (dm/s)
(Earth-fixed)

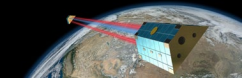
Position epochs
(in GPS time)

* 2009 11 2	0	0	0.00000000		
PL15	-391.718353	6623.836682	79.317661	999999.999999	
VL15	13710.157683	1908.731015	-77015.601314	999999.999999	
* 2009 11 2	0	0	10.00000000		
PL15	-377.980705	6625.284690	2.298385	999999.999999	
VL15	13764.602016	987.250587	-77021.193676	999999.999999	
* 2009 11 2	0	0	20.00000000		
PL15	-364.190222	6625.811136	-74.721213	999999.999999	
VL15	13815.825127	65.631014	-77016.232293	999999.999999	
* 2009 11 2	0	0	30.00000000		
PL15	-350.350131	6625.415949	-151.730567	999999.999999	
VL15	13863.820409	-855.995477	-77000.719734	999999.999999	
* 2009 11 2	0	0	40.00000000		
PL15	-336.463660	6624.099187	-228.719134	999999.999999	
VL15	13908.581905	-1777.497047	-76974.660058	999999.999999	
* 2009 11 2	0	0	50.00000000		
PL15	-322.534047	6621.861041	-305.676371	999999.999999	
VL15	13950.104280	-2698.741871	-76938.058807	999999.999999	
* 2009 11 2	0	1	0.00000000		
PL15	-308.564533	6618.701833	-382.591743	999999.999999	
VL15	13988.382807	-3619.598277	-76890.923043	999999.999999	

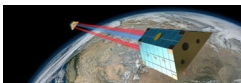
Clock corrections
are not provided

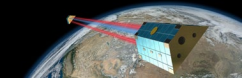
Excerpt of reduced-dynamic GOCE positions at begin of 2 Nov, 2009





Principles of Orbit Determination





Principle of Orbit Determination

The **actual orbit** $\mathbf{r}(t)$ is expressed as a truncated Taylor series:

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \sum_{i=1}^n \frac{\partial \mathbf{r}_0}{\partial P_i}(t) \cdot (P_i - P_{0,i})$$

$\mathbf{r}_0(t)$

A priori orbit

$\frac{\partial \mathbf{r}_0}{\partial P_i}(t)$

Partial derivative of the a priori orbit $\mathbf{r}_0(t)$ w.r.t. parameter P_i

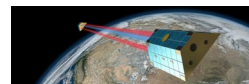
$P_{0,i}$

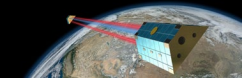
A priori parameter values of the a priori orbit $\mathbf{r}_0(t)$

P_i

Parameter values of the improved orbit $\mathbf{r}(t)$

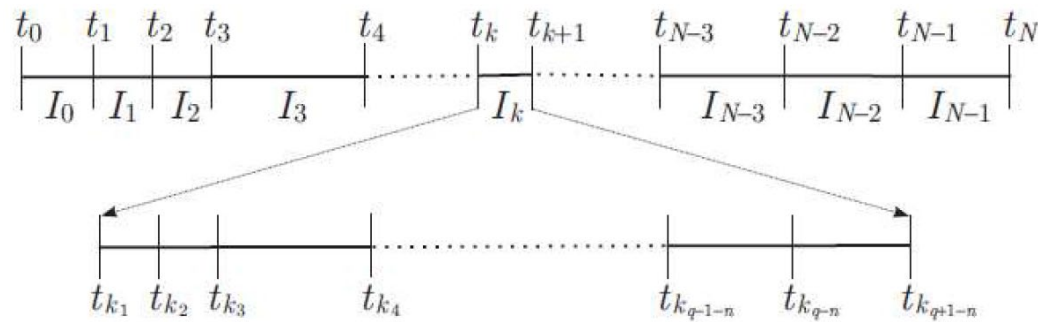
A **least-squares** adjustment of spacecraft tracking data yields **corrections** to the a priori parameter values $P_{0,i}$. Using the above equation, the improved (linearized) orbit $\mathbf{r}(t)$ may be eventually computed.





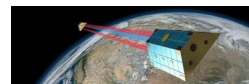
Numerical Integration (1)

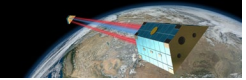
Collocation algorithms (one particular class of numerical integration techniques) are subsequently used to briefly illustrate the principles of numerical integration:



The original interval is divided into N integration intervals. For each interval I_k a further subdivision is performed according to the order q of the adopted method. At these points t_{k_j} the numerical solution is requested to solve the differential equation system of order n .

(Beutler, 2005)





Numerical Integration (2)

Initial value problem in the interval I_k is given by:

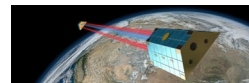
$$\ddot{\mathbf{r}}_k = \mathbf{f}(t, \mathbf{r}_k, \dot{\mathbf{r}}_k)$$

with initial conditions

$$\mathbf{r}_k(t_k) \doteq \mathbf{r}_{k0} \quad \text{and} \quad \dot{\mathbf{r}}_k(t_k) \doteq \dot{\mathbf{r}}_{k0}$$

where the initial values are defined as

$$\mathbf{r}_{k0}^{(i)} = \begin{cases} \mathbf{r}_0^{(i)} & ; k = 0 \\ \mathbf{r}_{k-1}^{(i)}(t_k) & ; k > 0 \end{cases}$$





Numerical Integration (3)

The **collocation method** approximates the solution in the interval I_k by:

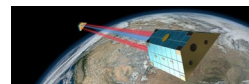
$$\mathbf{r}_k(t) \doteq \sum_{l=0}^q \frac{1}{l!} (t - t_k)^l \mathbf{r}_{k0}^{(l)}$$

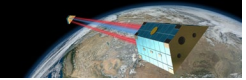
The coefficients $\mathbf{r}_{k0}^{(l)}$, $l = 0, \dots, q$ are obtained by requesting that the numerical solution assumes the initial values and solves the differential equation system at $q - 1$ different epochs t_{kj} , $j = 1, \dots, q - 1$. This leads to the conditions

$$\sum_{l=2}^q \frac{(t_{kj} - t_k)^{l-2}}{(l-2)!} \mathbf{r}_{k0}^{(l)} = \mathbf{f}(t_{kj}, \mathbf{r}_k(t_{kj}), \dot{\mathbf{r}}_k(t_{kj})), \quad j = 1, \dots, q - 1.$$

They are non-linear but can be solved efficiently by an iterative procedure.

(Beutler, 2005)





Partial Derivatives

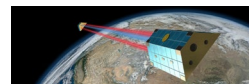
The partial of the r -th observation w.r.t. orbit parameter P_i may be expressed as

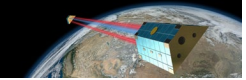
$$\frac{\partial F_r(\mathbf{X})}{\partial P_i} = (\nabla (F_r(\mathbf{X})))^T \cdot \frac{\partial \mathbf{r}_0}{\partial P_i}(t)$$

with the gradient given by

$$(\nabla (F_r(\mathbf{X})))^T = \left(\frac{\partial F_r(\mathbf{X})}{\partial r_{0,1}} \quad \frac{\partial F_r(\mathbf{X})}{\partial r_{0,2}} \quad \frac{\partial F_r(\mathbf{X})}{\partial r_{0,3}} \right)$$

if the observations only depend on the geocentric position vector and are referring to only one epoch. The gradient only depends on the type of observations used, whereas the second term is independent of the observation type and related to the **variational equations**. This separates the observation-specific (**geometric**) part from the **dynamic** part.





Variational Equations

For each orbit parameter P_i the corresponding **variational equation** reads as

$$\ddot{\mathbf{z}}_{P_i} = \mathbf{A}_0 \cdot \mathbf{z}_{P_i} + \mathbf{A}_1 \cdot \dot{\mathbf{z}}_{P_i} + \frac{\partial \mathbf{f}_1}{\partial P_i}$$

with $\mathbf{z}_{P_i}(t) \doteq \frac{\partial \mathbf{r}_0}{\partial P_i}(t)$ and the 3 x 3 matrices defined by

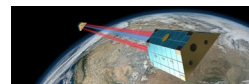
$$A_{0[i;k]} \doteq \frac{\partial f_i}{\partial r_{0,k}} \quad \text{and} \quad A_{1[i;k]} \doteq \frac{\partial f_i}{\partial \dot{r}_{0,k}}$$

f_i i -th component of the total acceleration \mathbf{f}

$r_{0,k}$ k -th component of the geocentric position \mathbf{r}_0

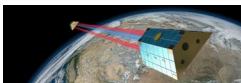
For each orbit parameter P_i the **variational equation** is a linear differential equation system of second order in time. Their solutions are all needed for orbit determination.

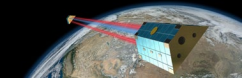
(Jäggi, 2007)





GPS-based LEO POD

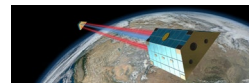




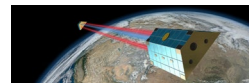
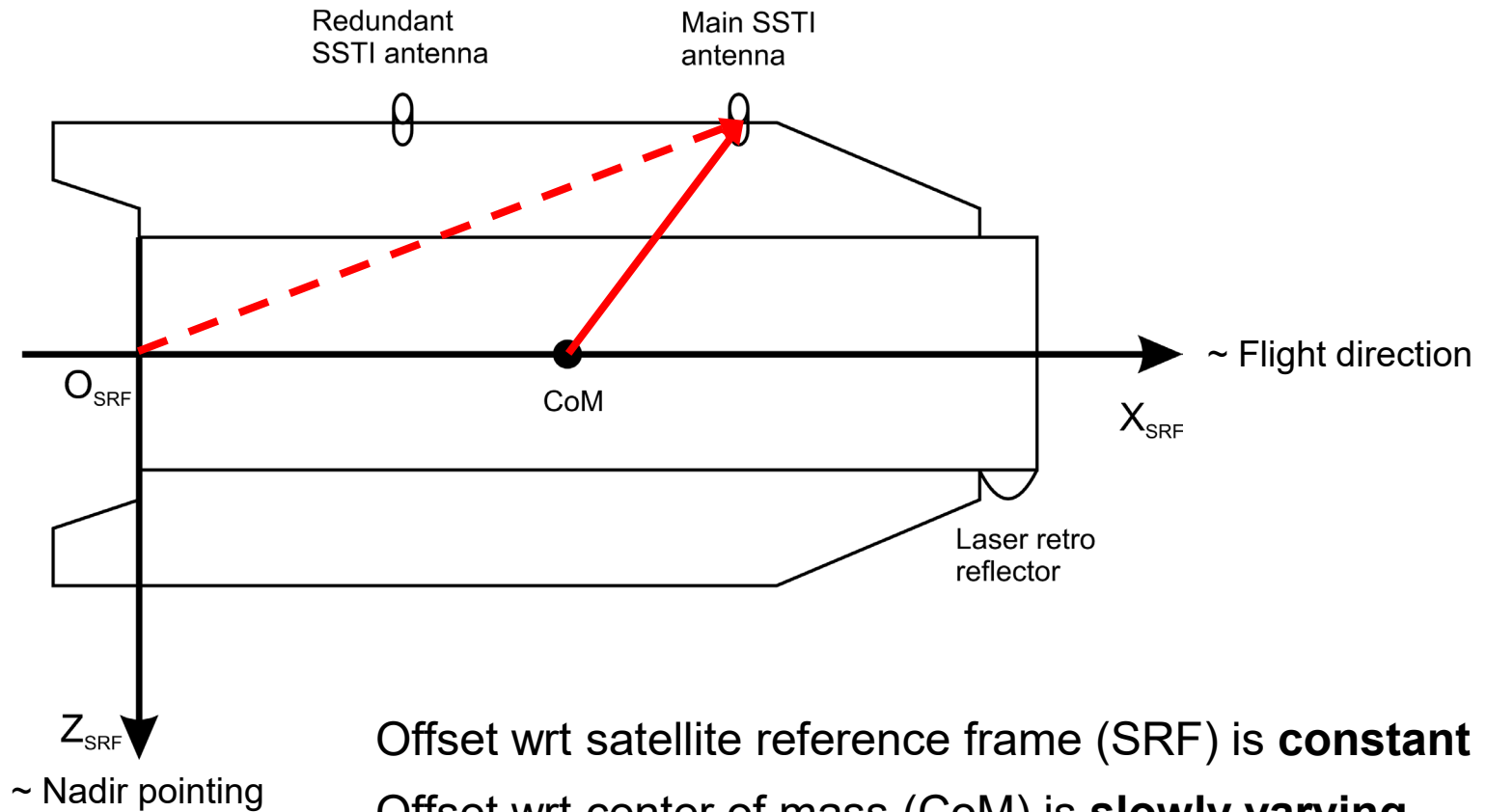
LEO Sensor Offsets

Phase center offsets $\delta r_{leo,ant}$:

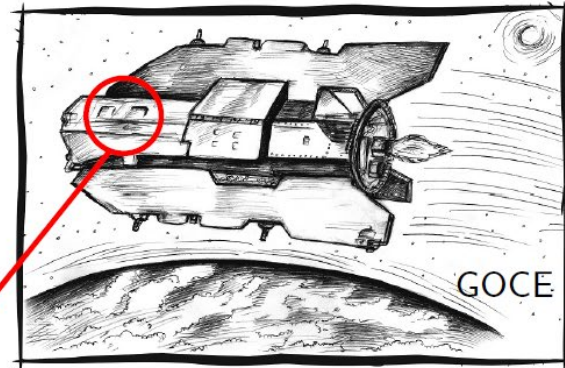
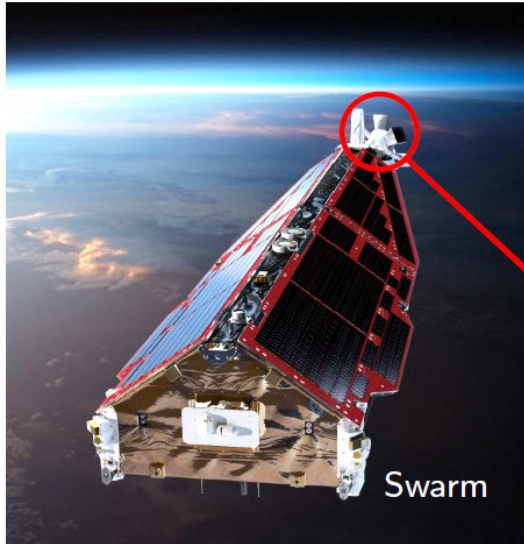
- are needed in the inertial or Earth-fixed frame and have to be transformed from the satellite frame using **attitude data** from the star-trackers
- consist of a frequency-independent **instrument offset**, e.g., defined by the center of the instrument's mounting plane (CMP) in the satellite frame
- consist of frequency-dependent **phase center offsets** (PCOs), e.g., defined wrt the center of the instrument's mounting plane in the antenna frame (ARF)
- consist of frequency-dependent **phase center variations** (PCVs) varying with the direction of the incoming signal, e.g., defined wrt the PCOs in the antenna frame



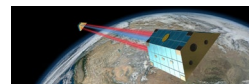
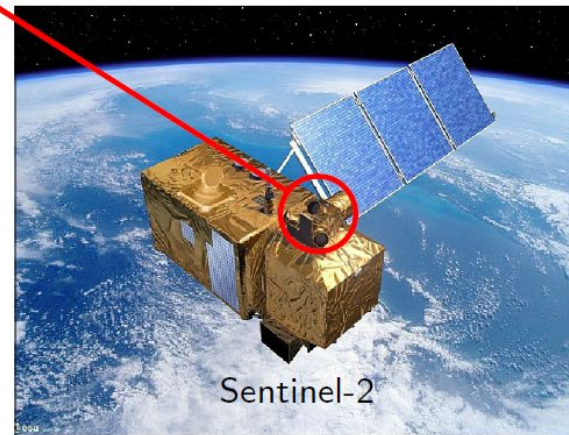
Example: GOCE Sensor Offsets (1)



Example: GOCE Sensor Offsets (2)



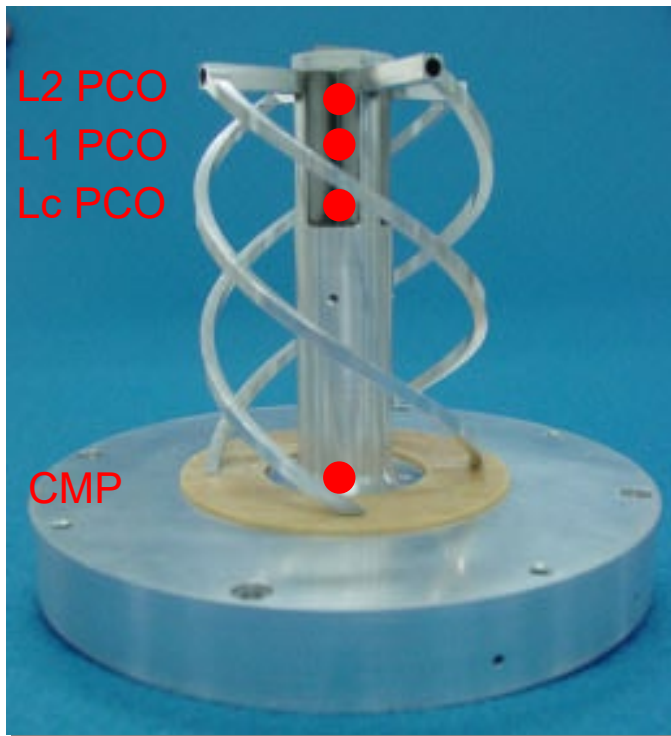
Star tracker





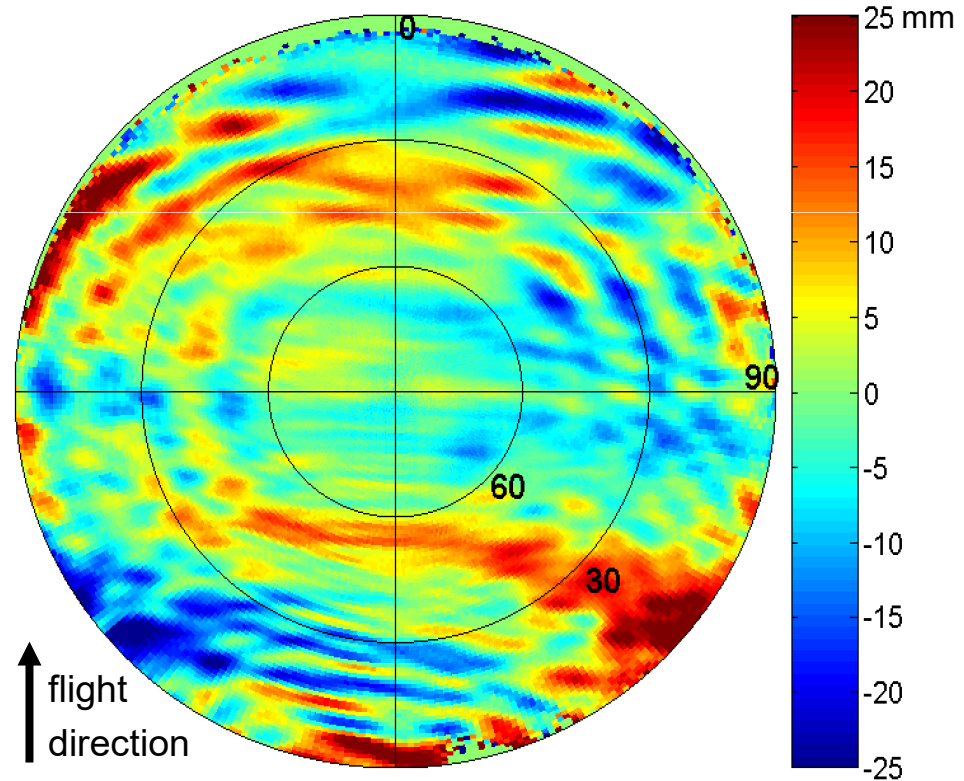
Spaceborne GPS Antennas: GOCE

L1, L2, Lc phase center offsets

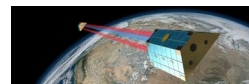


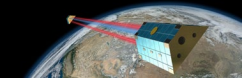
Measured from ground calibration
in anechoic chamber

Lc phase center variations



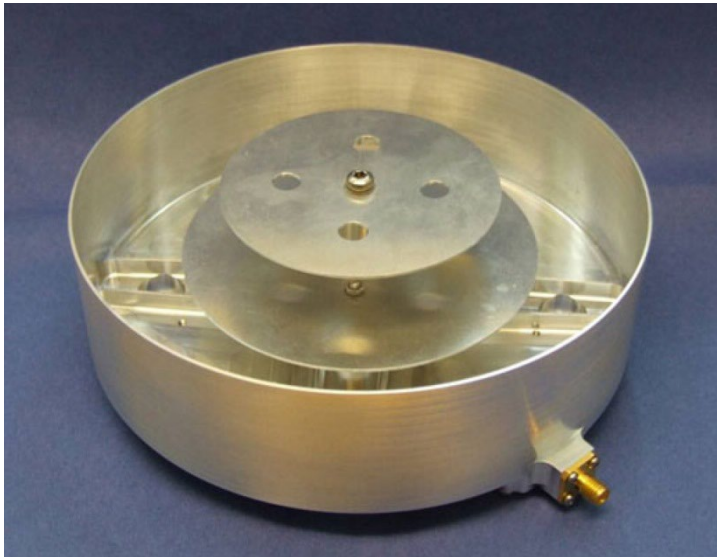
Empirically derived during orbit determination
according to Jäggi et al. (2009)





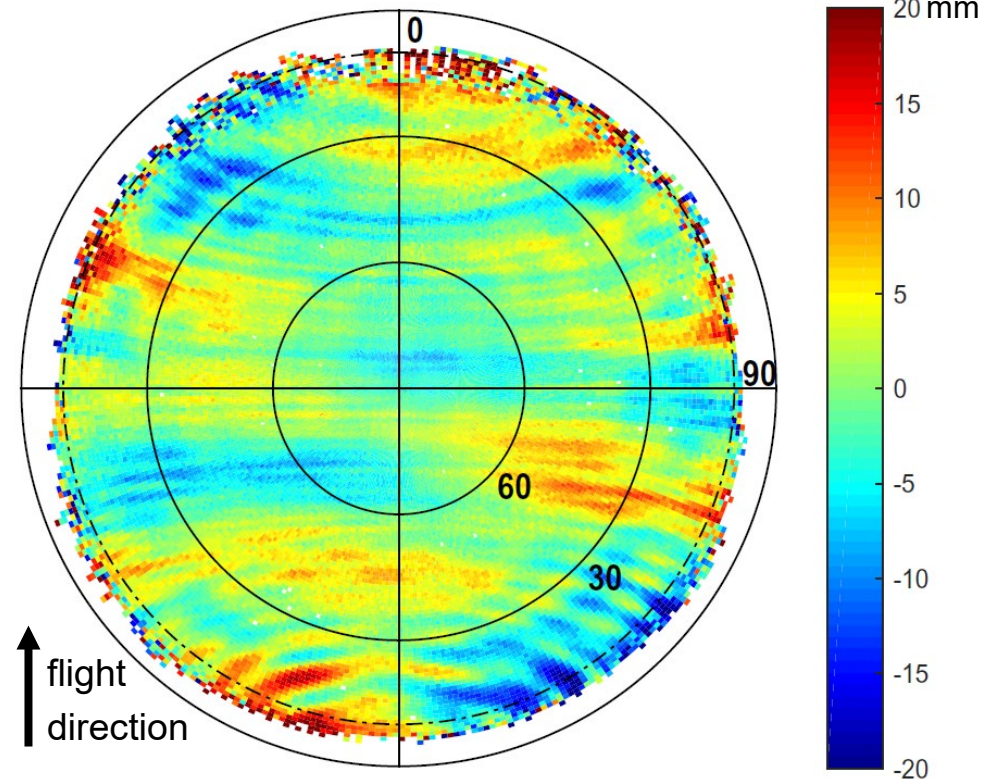
Spaceborne GPS Antennas: Swarm

Swarm GPS antenna

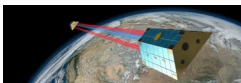


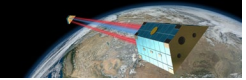
Multipath shall be minimized by chokering

Lc phase center variations

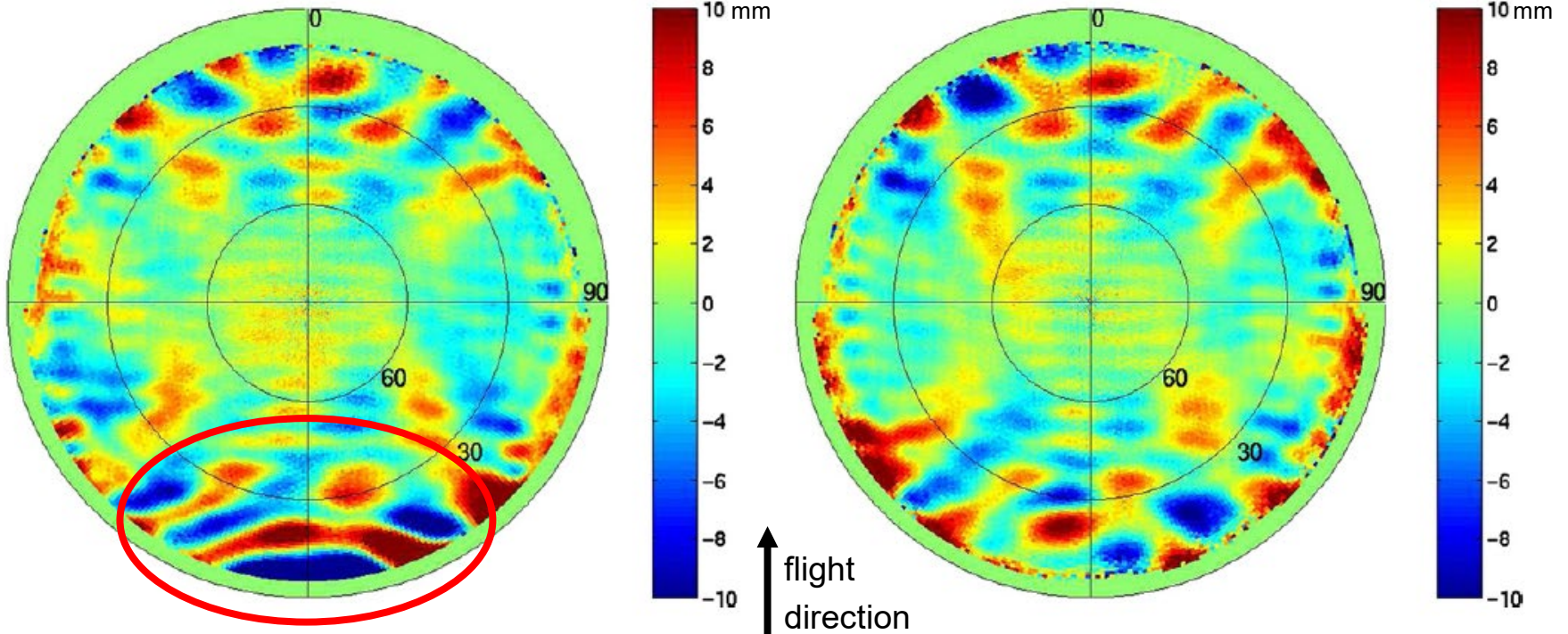


Empirically derived during orbit determination according to Jäggi et al. (2016)





Spaceborne GPS Antennas: GRACE

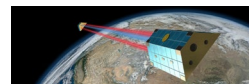


GRACE-A

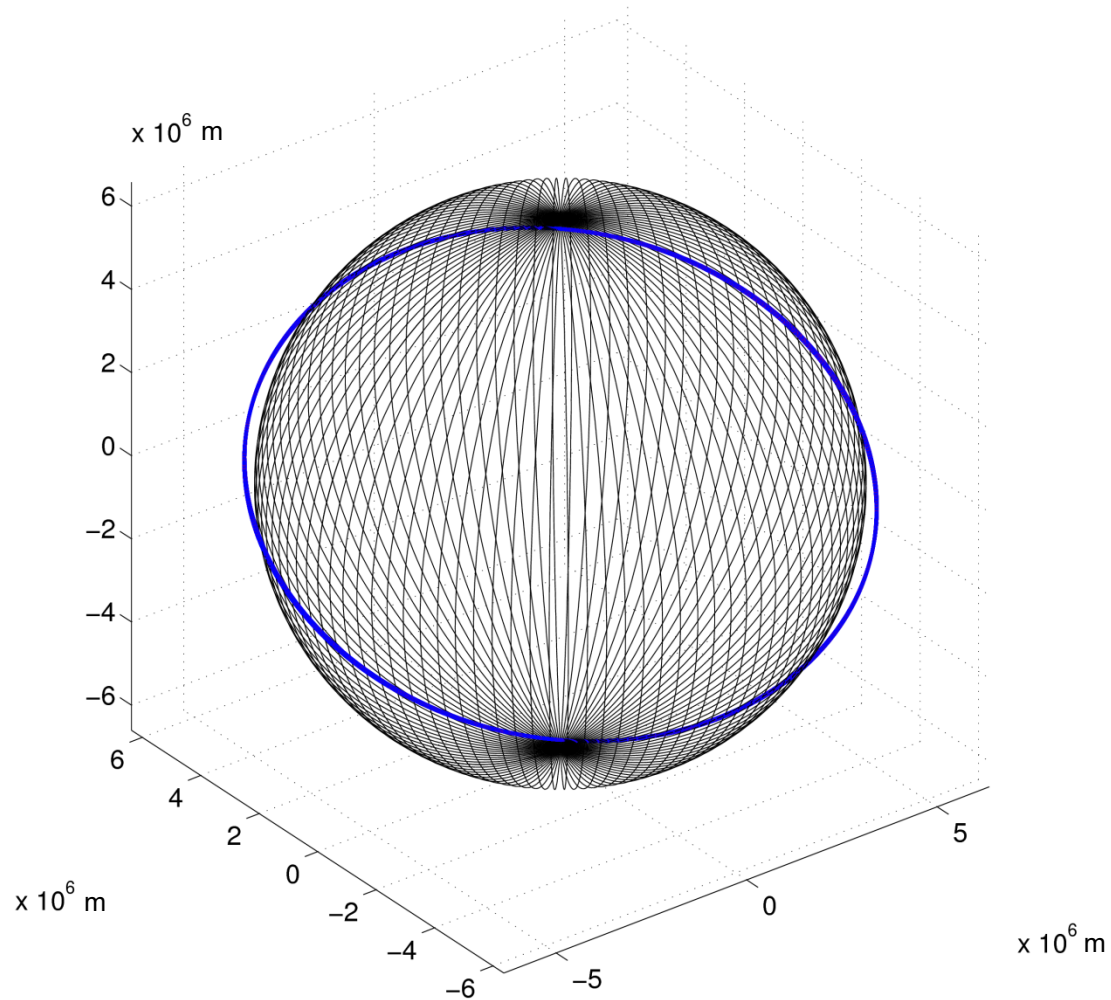
(occultation antenna switched on)

GRACE-B

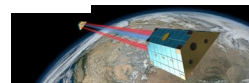
(Jäggi et al., 2009)



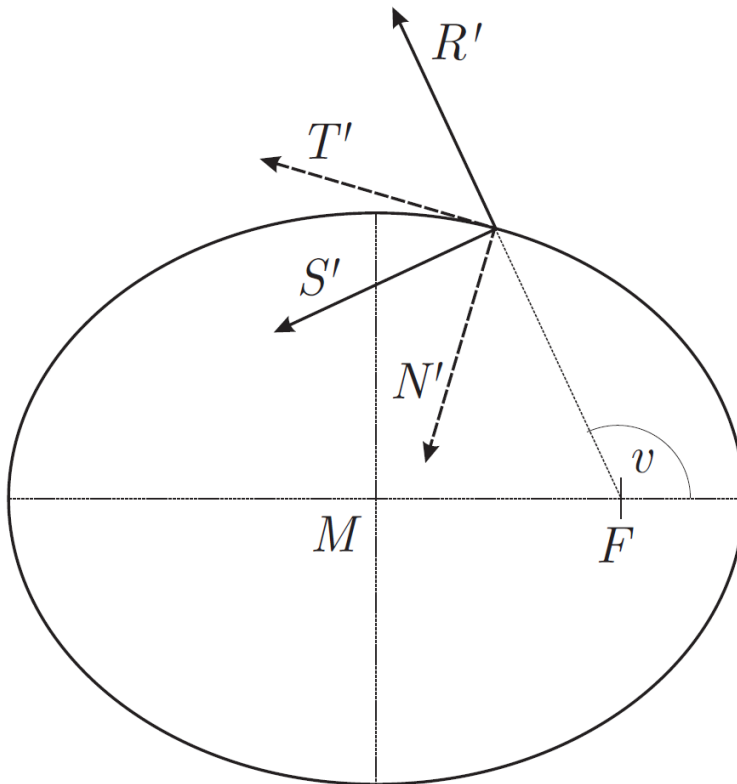
Visualization of Orbit Solutions



It is more instructive to look at differences between orbits in well suited coordinate systems ...



Co-Rotating Orbital Frames



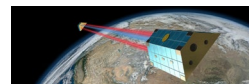
R, S, C unit vectors are pointing:

- into the radial direction
- normal to **R** in the orbital plane
- normal to the orbital plane (cross-track)

T, N, C unit vectors are pointing:

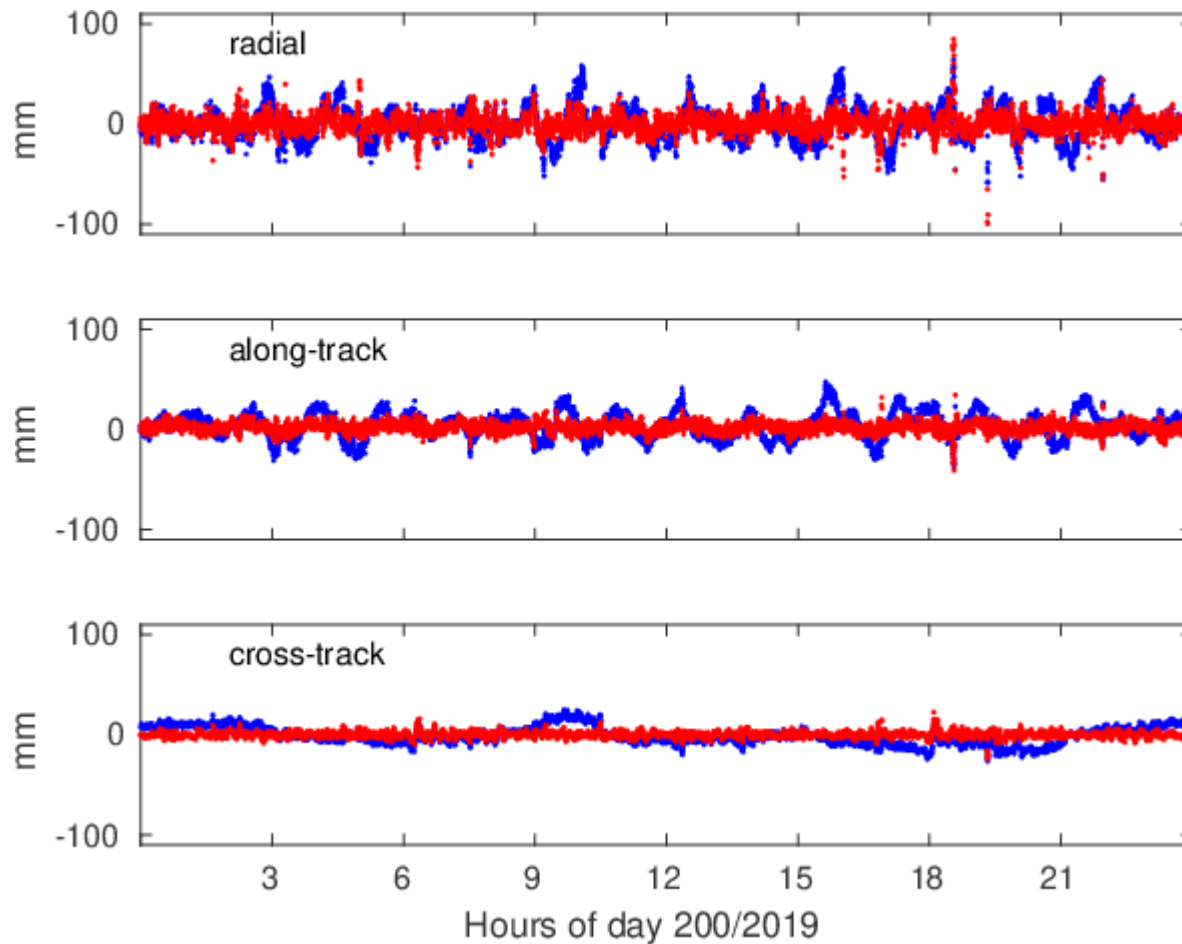
- into the tangential (along-track) direction
- normal to **T** in the orbital plane
- normal to the orbital plane (cross-track)

Small eccentricities: **S**~**T** (velocity direction)

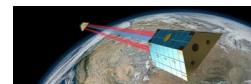


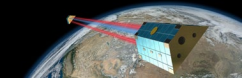
Orbit Differences KIN-RD (Sentinel-3A)

Differences at epochs of kin. positions



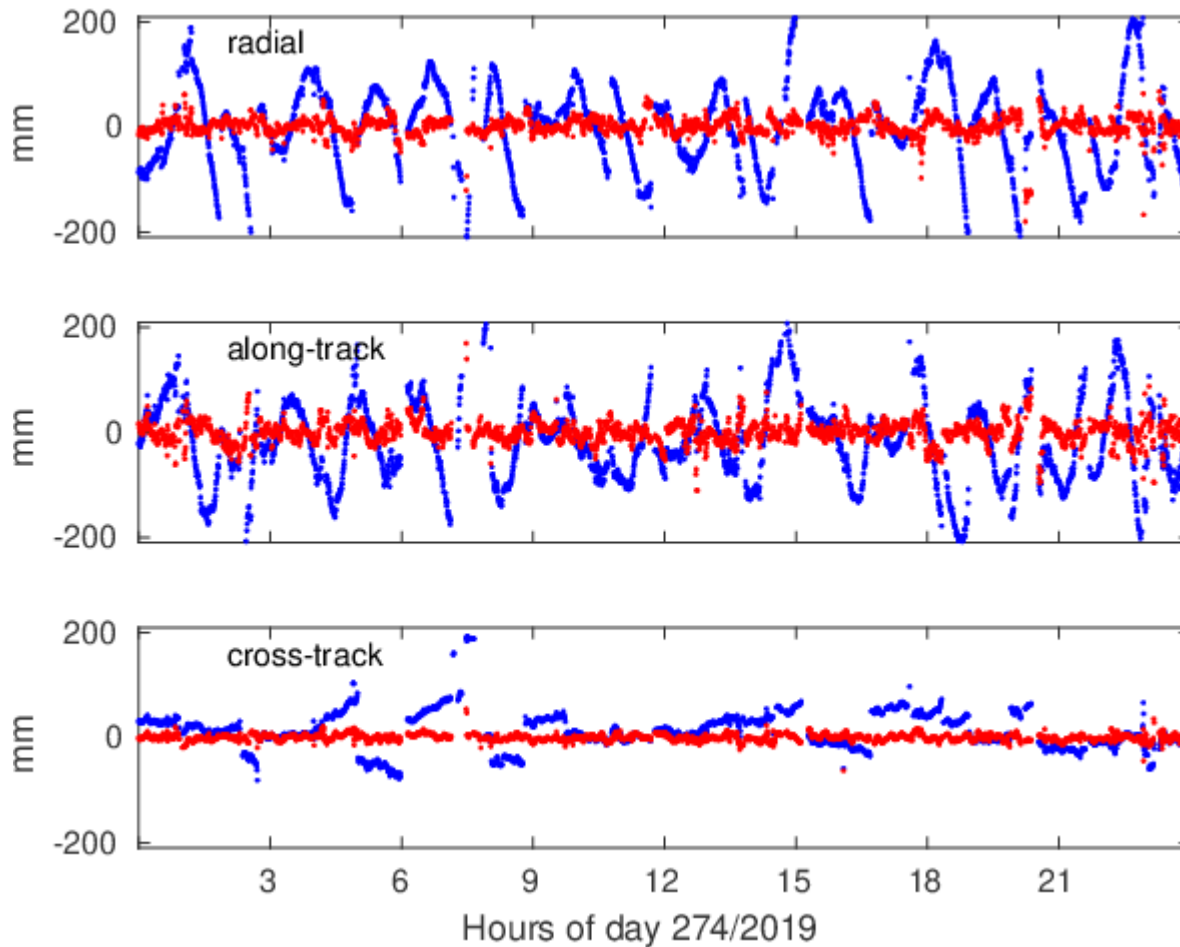
Comparison of **ambiguity-float** solutions and **ambiguity-fixed** solutions.



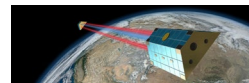


Orbit Differences KIN-RD (COSMIC-2)

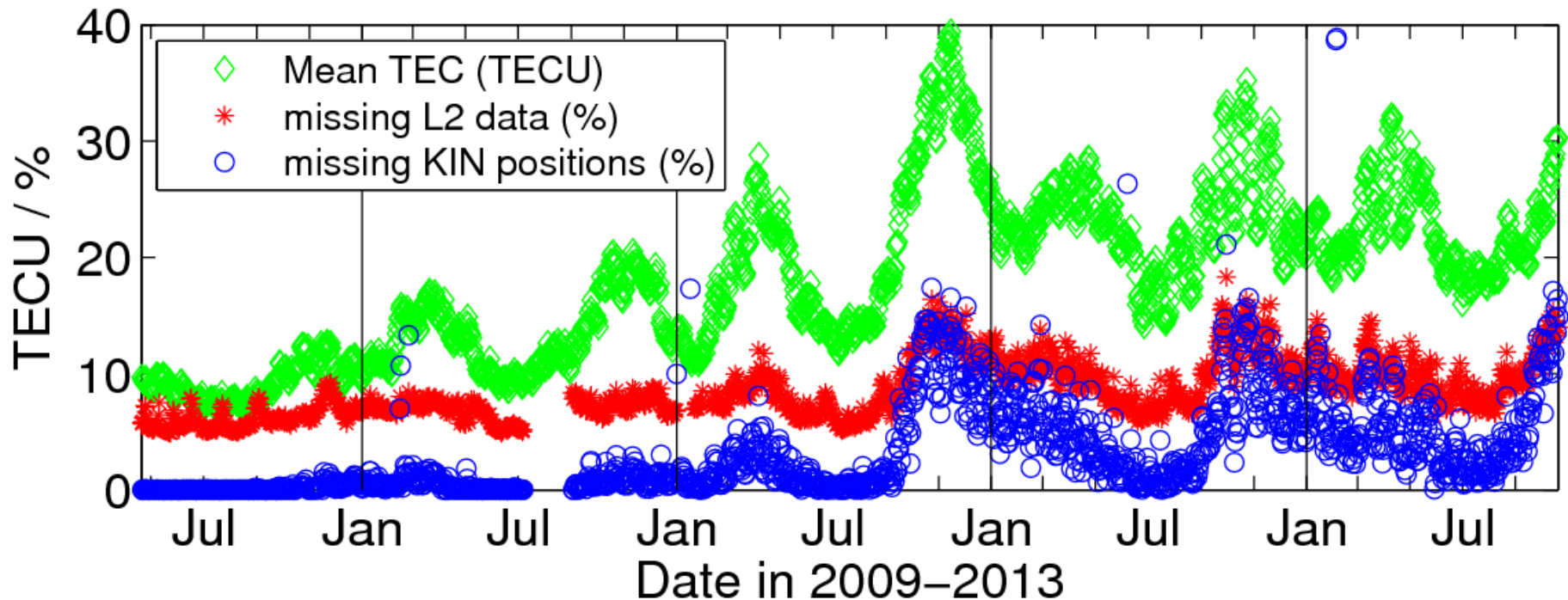
Differences at epochs of kin. positions, FM-1, POD-1



Comparison of **ambiguity-float** solutions and **ambiguity-fixed** solutions. (Jäggi et al. 2021)

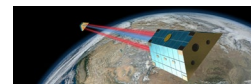


Orbit Differences KIN-RD (GOCE)

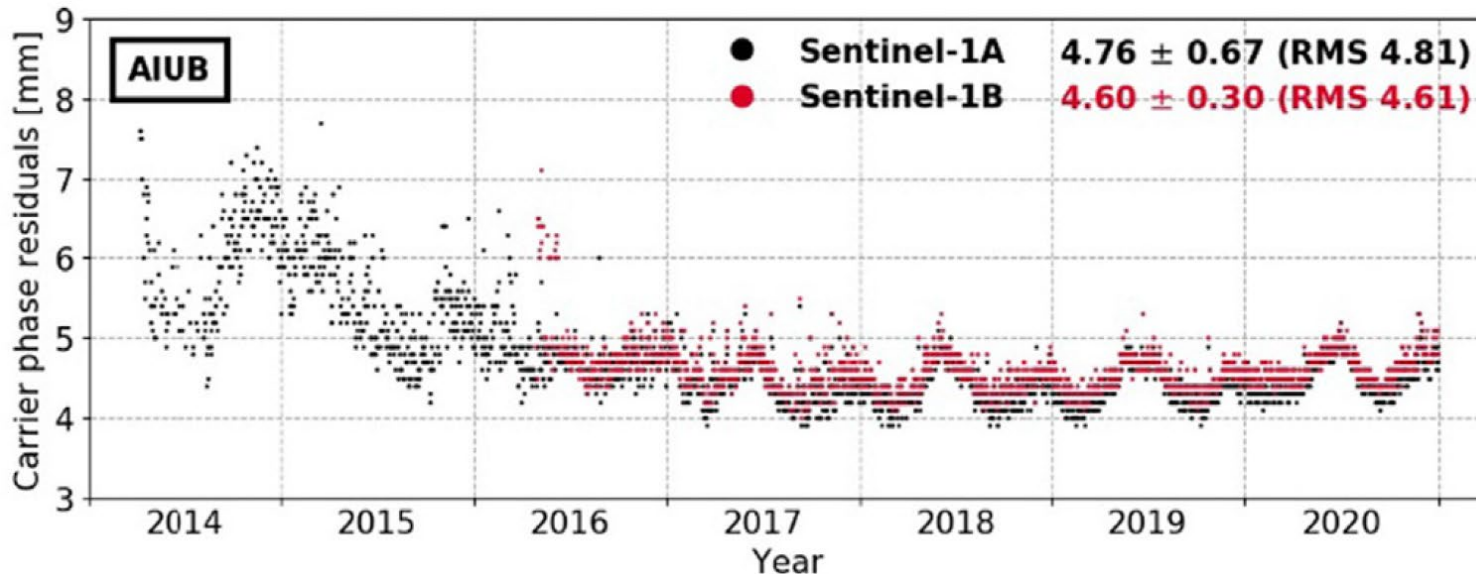


The result illustrates the **consistency** between both orbit-types for the GOCE Precise Science Orbit (PSO) product. The level of the differences is usually given by the quality of the kinematic positions. The differences are highly correlated with the **ionosphere activity** and with **data losses on L2**.

(Bock et al., 2014)



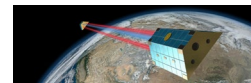
RMS of Carrier Phase Residuals (Sentinel-1)



The plot illustrates the **quality** of the orbit fit. The level is given by the adopted parametrization, depending on how dynamic the orbit parametrization is.

The variations reveal again the impact of the **ionosphere activity** and also further **modeling deficiencies**.

(Fernández et al., 2022)

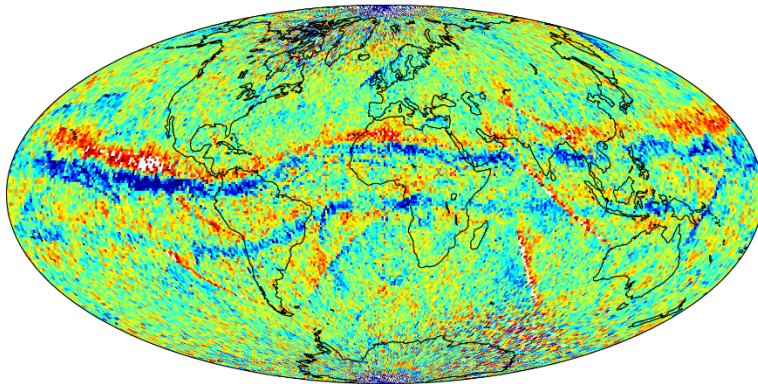


Consequences of Ionospheric Effects in Orbits

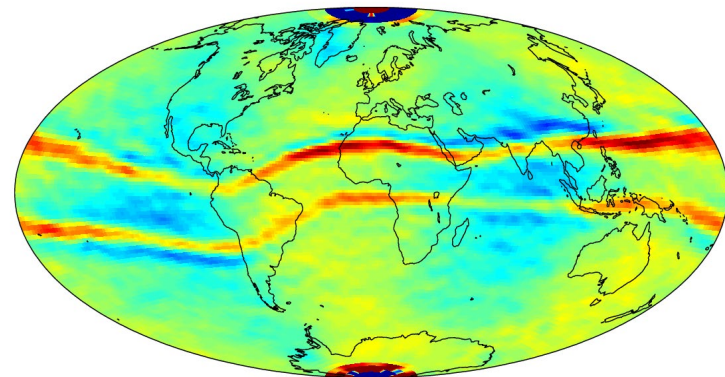
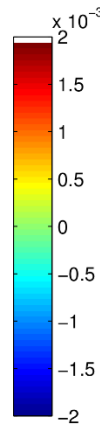
For GOCE systematic effects around the geomagnetic equator were observed in the ionosphere-free GPS phase residuals => **affects kinematic positions**

Degradation of kinematic positions around the geomagnetic equator propagates into gravity field solutions.

mean residuals at ionosphere-crossing: 2011, doys 245–365

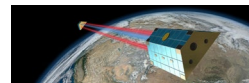


Phase observation residuals
(- 2 mm ... +2 mm) mapped
to the ionosphere piercing
point



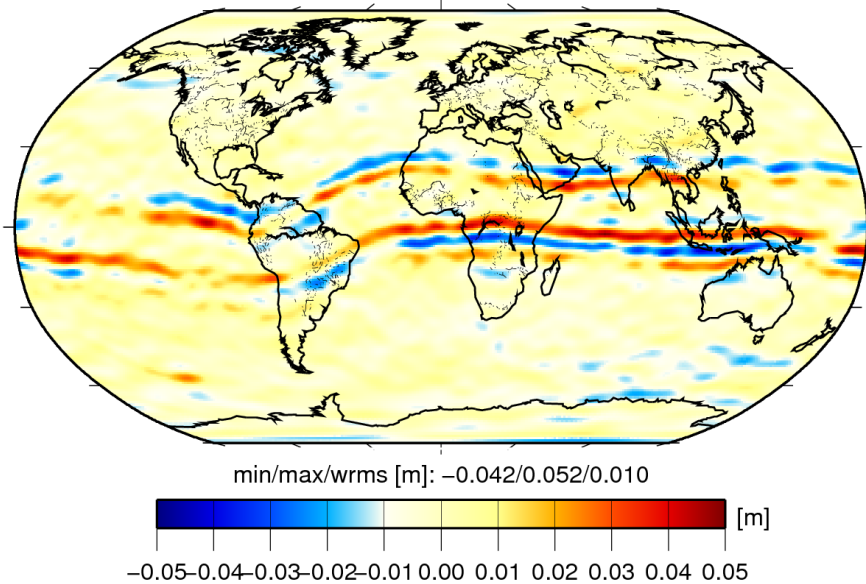
Geoid height differences
(-5 cm ... 5 cm);
R4 period

(Jäggi et al., 2015)

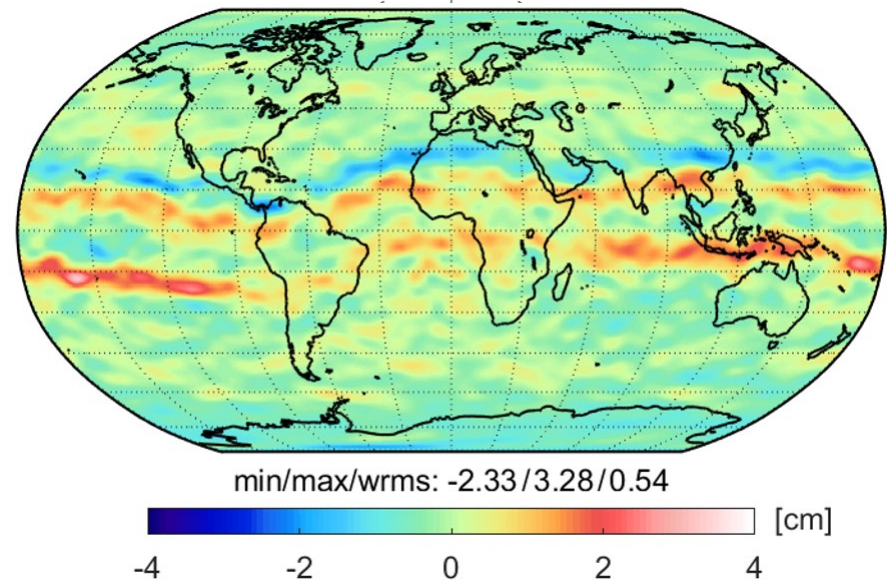


Systematic Errors in GPS Data (1)

Original GPS Data
(Swarm)



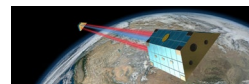
Original GPS Data
(GRACE)



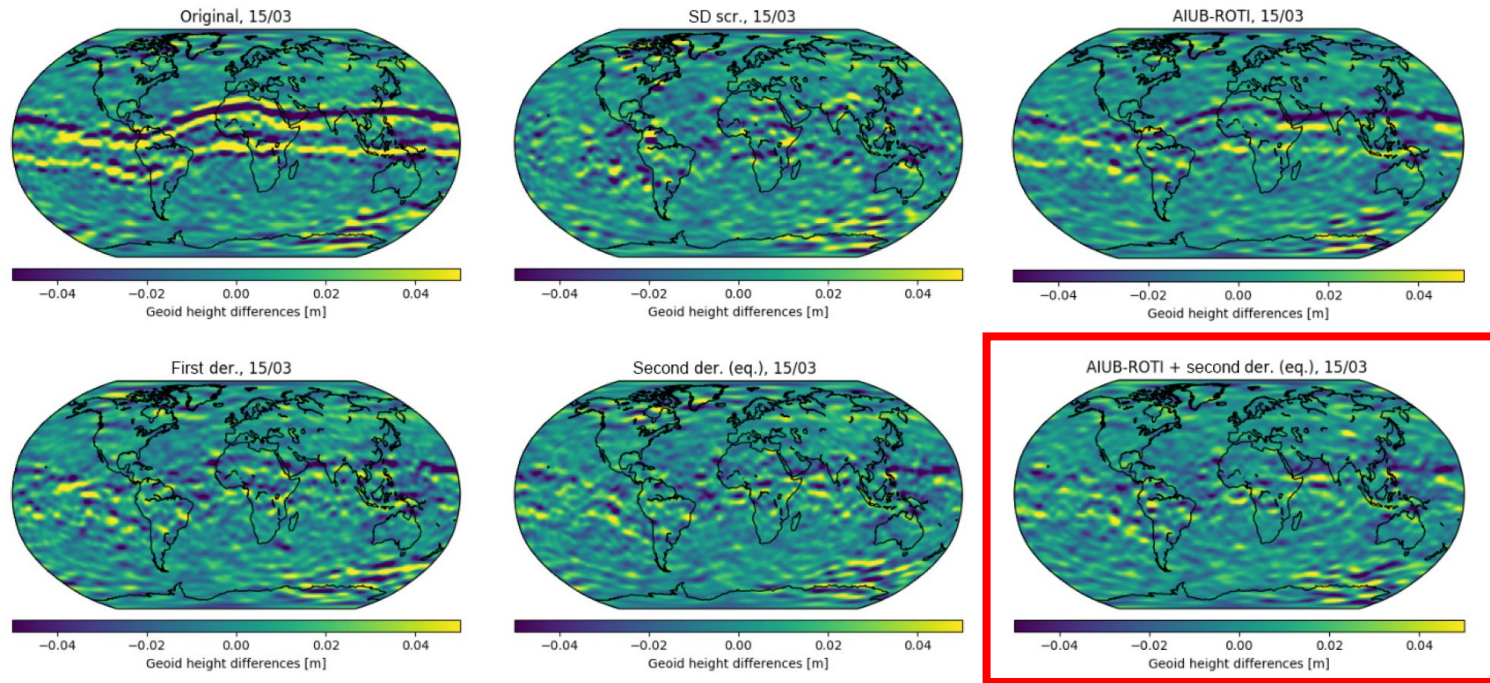
(Differences wrt GOCO05S, 400 km Gauss smoothing adopted)

Systematic signatures along the geomagnetic equator are **"not"** visible when using original L1B RINEX GPS data files from the GRACE mission.

(Jäggi et al., 2016)



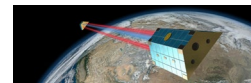
Systematic Errors in GPS Data (2)



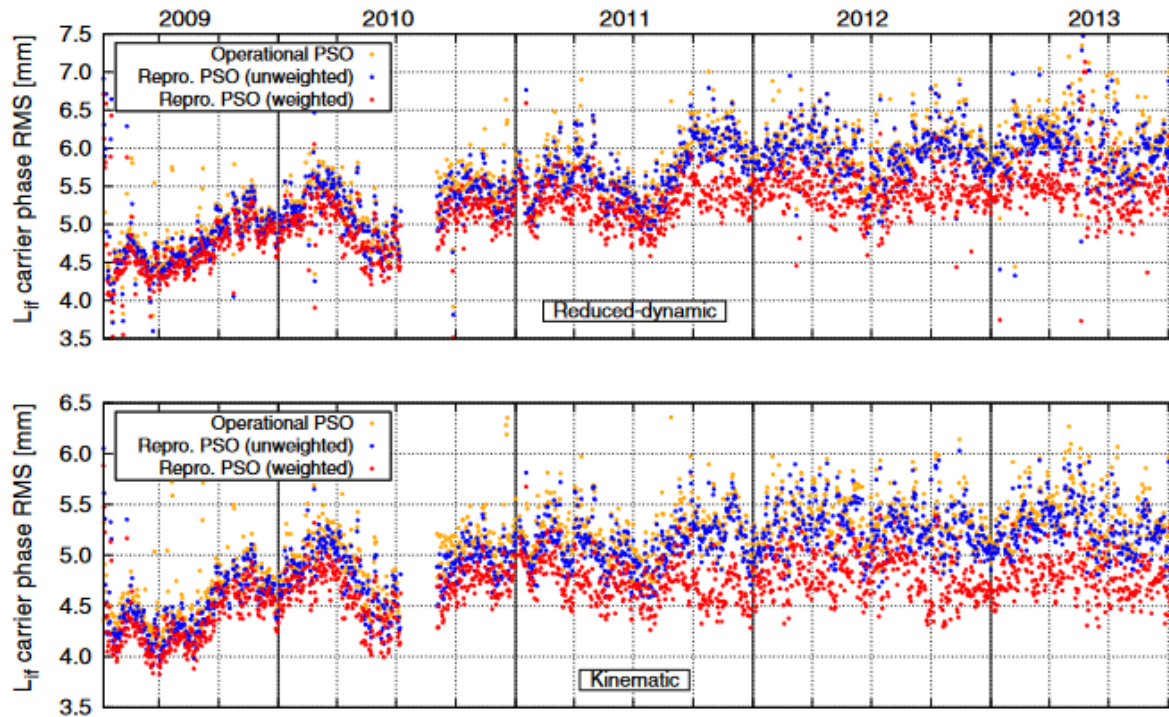
(Differences wrt JPL-GRACE-RL06, 400 km Gauss smoothing adopted)

Systematic signatures along the geomagnetic equator may be efficiently reduced when down-weighting the GPS data using derivatives of the geometry-free linear combination. ROTI-based down-weighting additionally reduces scintillation noise.

(Schreiter et al., 2019)

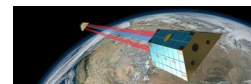


Systematic Errors in GPS Data (3)

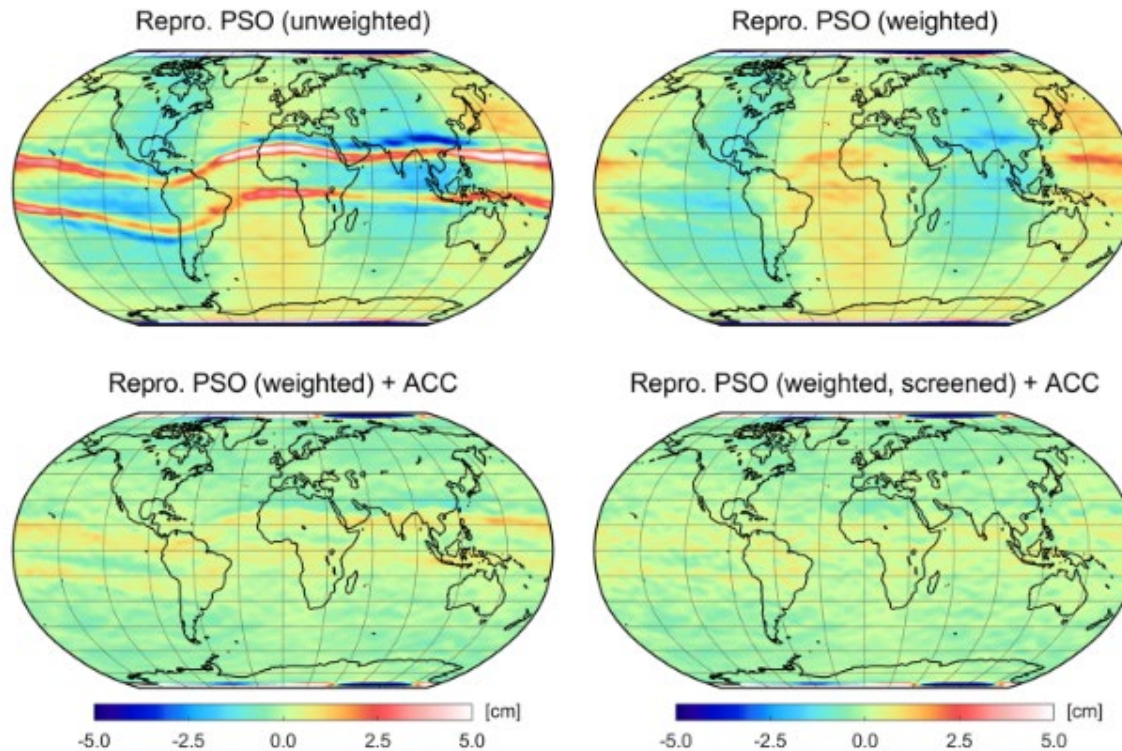


The plot illustrates the importance of adequate weighting strategies, as they were used for the reprocessing of the GOCE Precise Science Orbits, for the quality of the orbit fit when high ionosphere activity is present.

(Arnold et al., 2023)



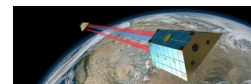
Systematic Errors in GPS Data (4)



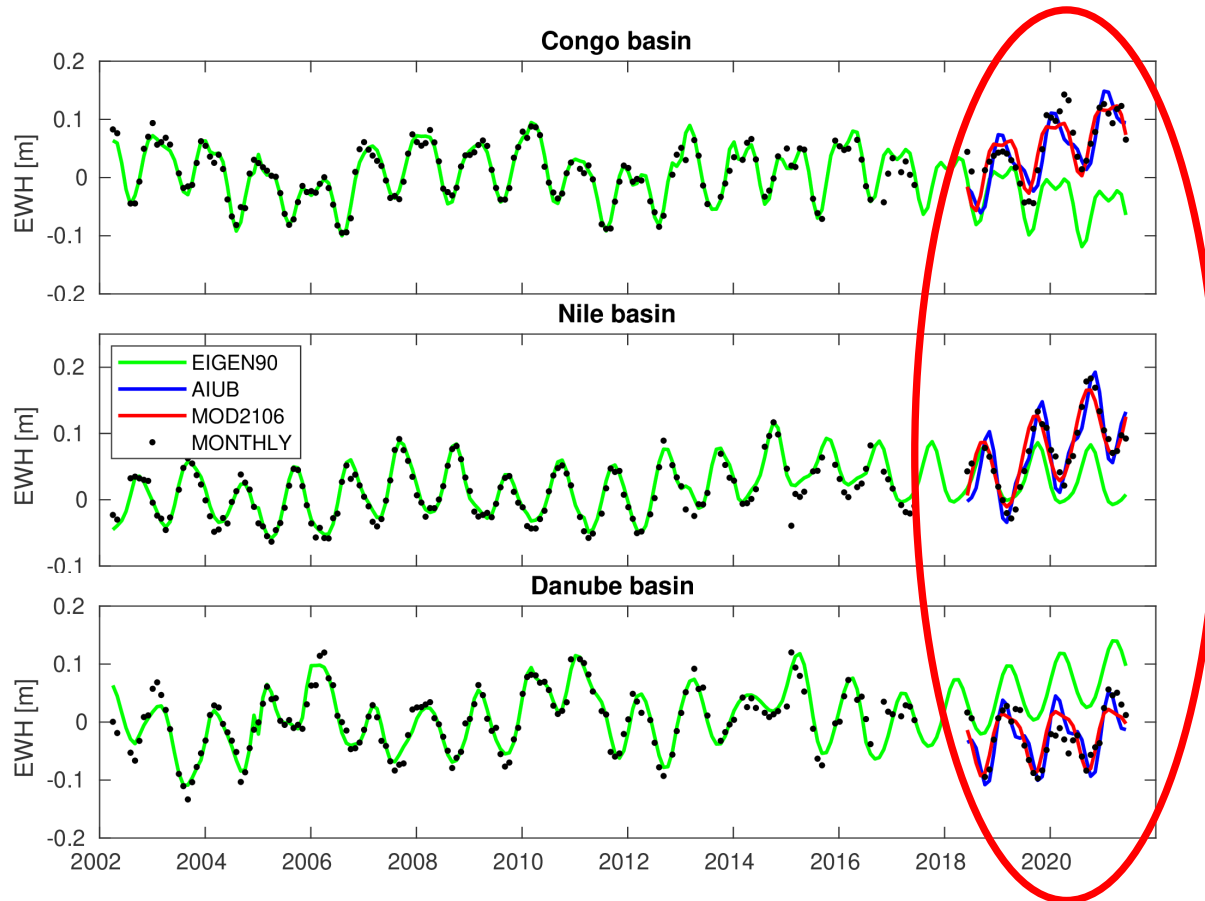
(Differences wrt ITSG-Grace2018, 300 km Gauss smoothing adopted)

Remarkable reduction of ionosphere-induced artifacts along the geomagnetic equator when using the kinematic positions of the reprocessed GOCE Precise Science Orbits for gravity field recovery.

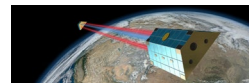
(Arnold et al., 2023)



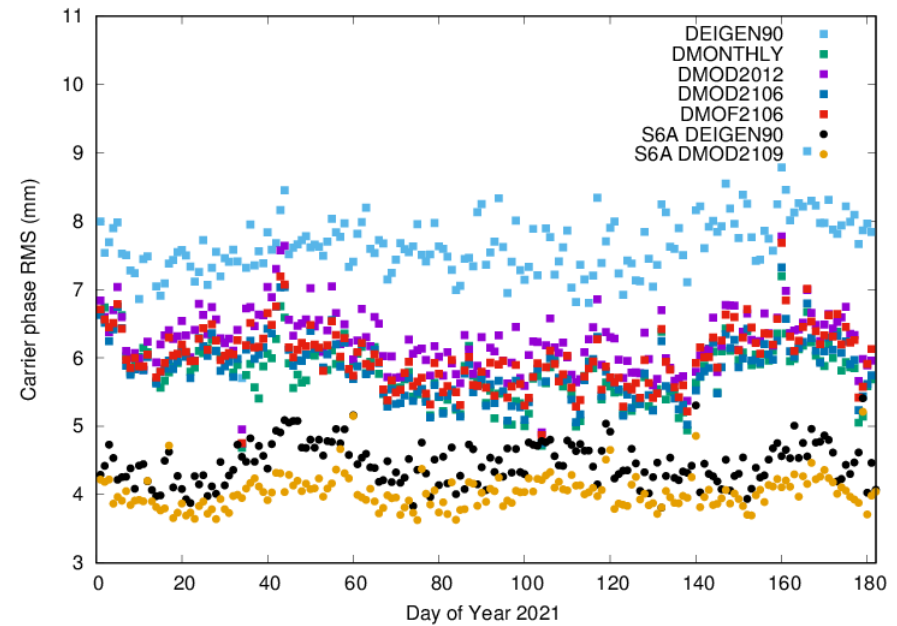
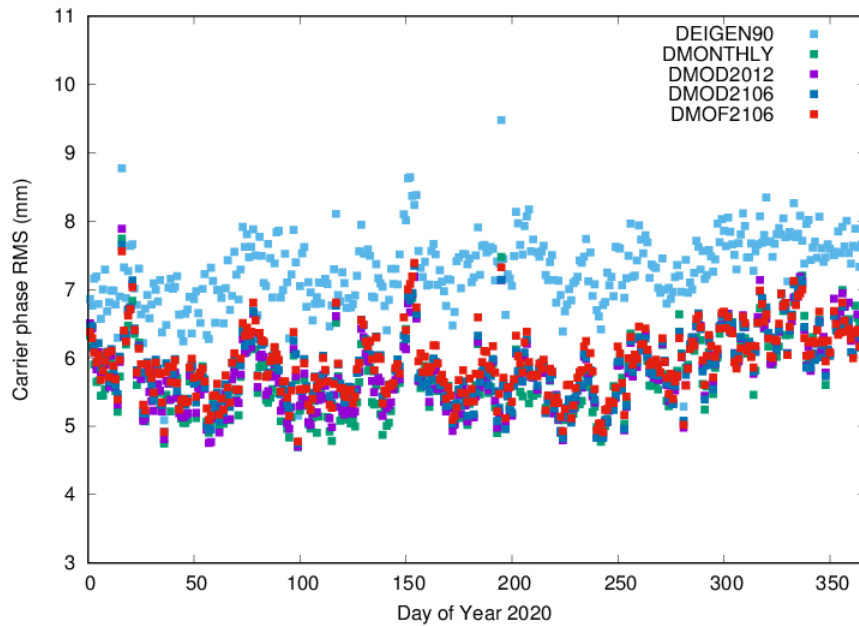
Sensitivity to the Time-Varyable Gravity Field Model (1)



The predictions of the EIGEN-GRGS.RL04 model (containing no data after 2017) are rather poor as shown here for three river basins. The fitted signal model (FSM) to COST-G monthly solutions has clear advantages. (Peter et al., 2022)



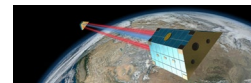
Sensitivity to the Time-Varyable Gravity Field Model (2)



Monthly COST-G GRACE-FO gravity fields (■) outperform GRACE-based static fields (□) with co-estimated time-variations for Sentinel-3B (left and right plot).

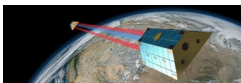
COST-G fitted signal models (FSM) perform comparably good (■ □ ■, left plot), which also holds for Sentinel-6A (●, right plot). In particular the COST-G FSM show also good prediction capabilities (■, right plot).

(Peter et al., 2022)

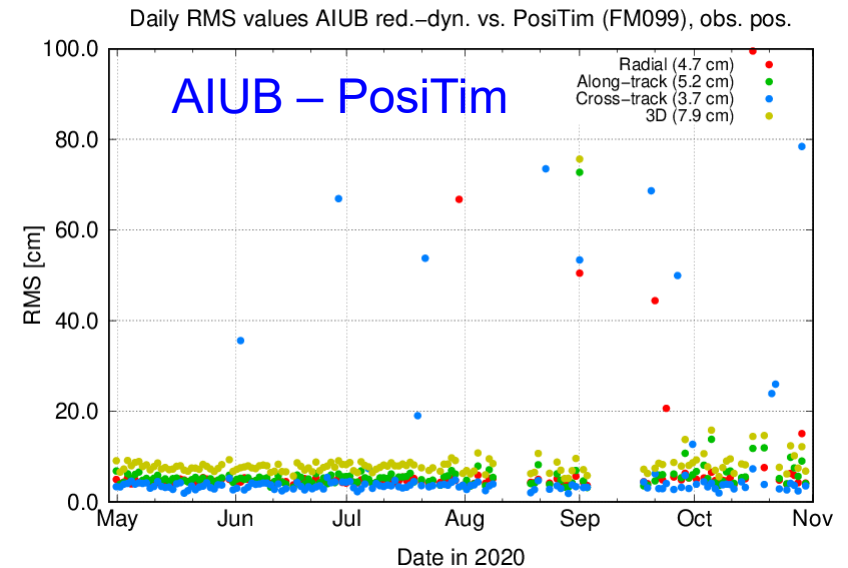
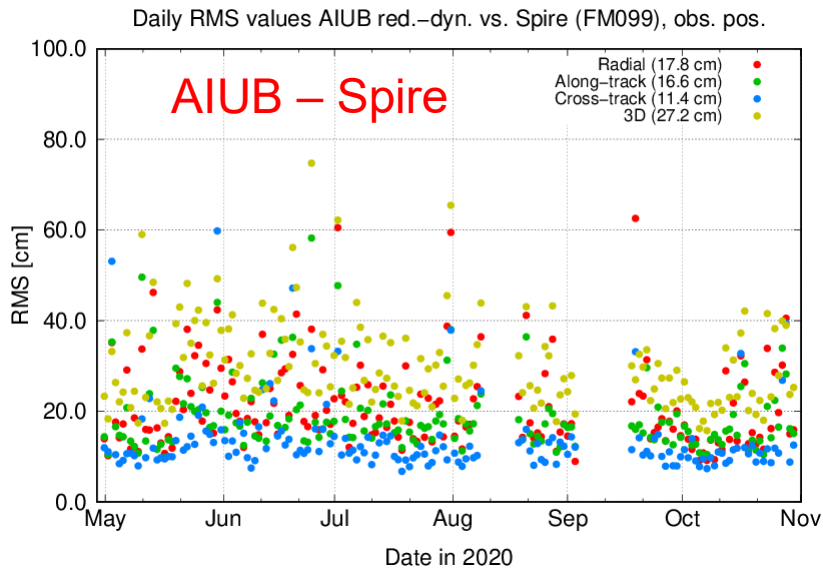




Orbit Validation



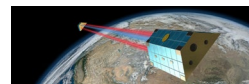
Orbit Comparisons (Spire FM099)



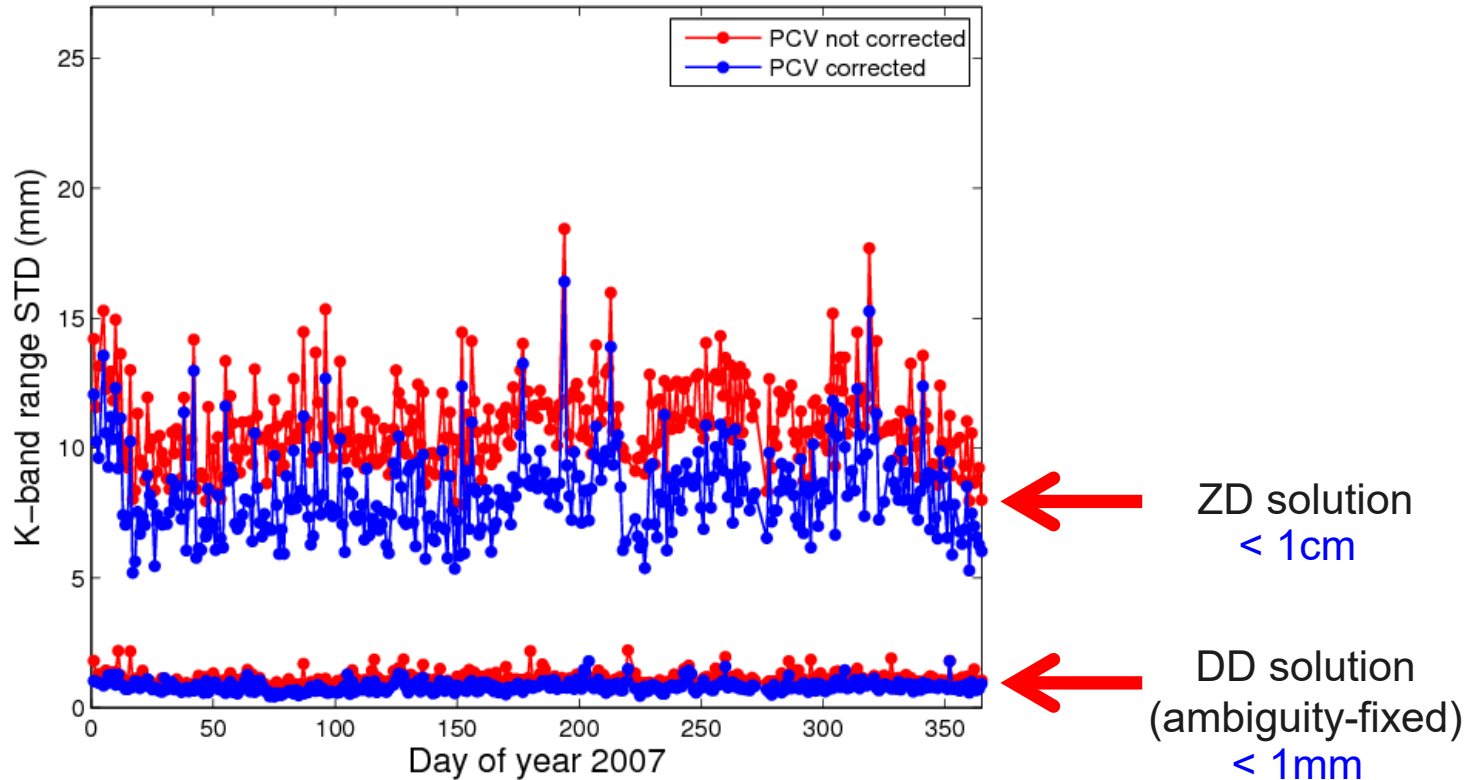
Orbit comparisons between solutions computed **with different software packages** are helpful, especially if no external orbit validations are possible.

The plots show for the example of one Spire satellite that orbit differences **AIUB – PosiTim** are significantly smaller than the orbit differences **AIUB – Spire**.

(Jäggi et al., 2022)

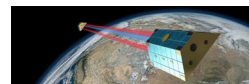


GRACE Orbit Validation with K-Band

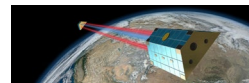
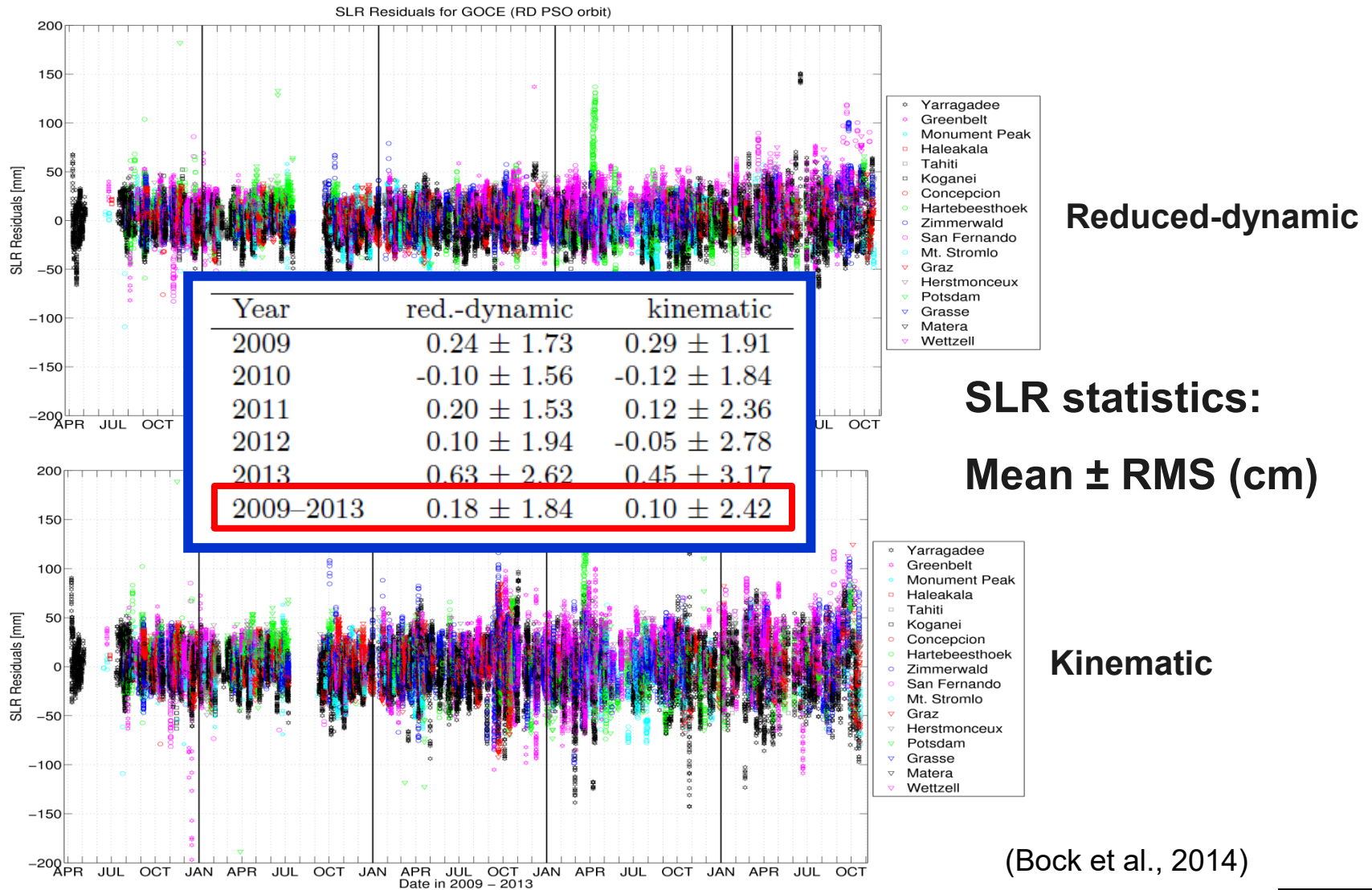


The ultra-precise and continuously available K-Band data allow it to validate the **inter-satellite distances** between the GRACE satellites. Thanks to this validation, e.g., PCV maps were recognized to be crucial for high-quality POD.

(Jäggi et al., 2009)

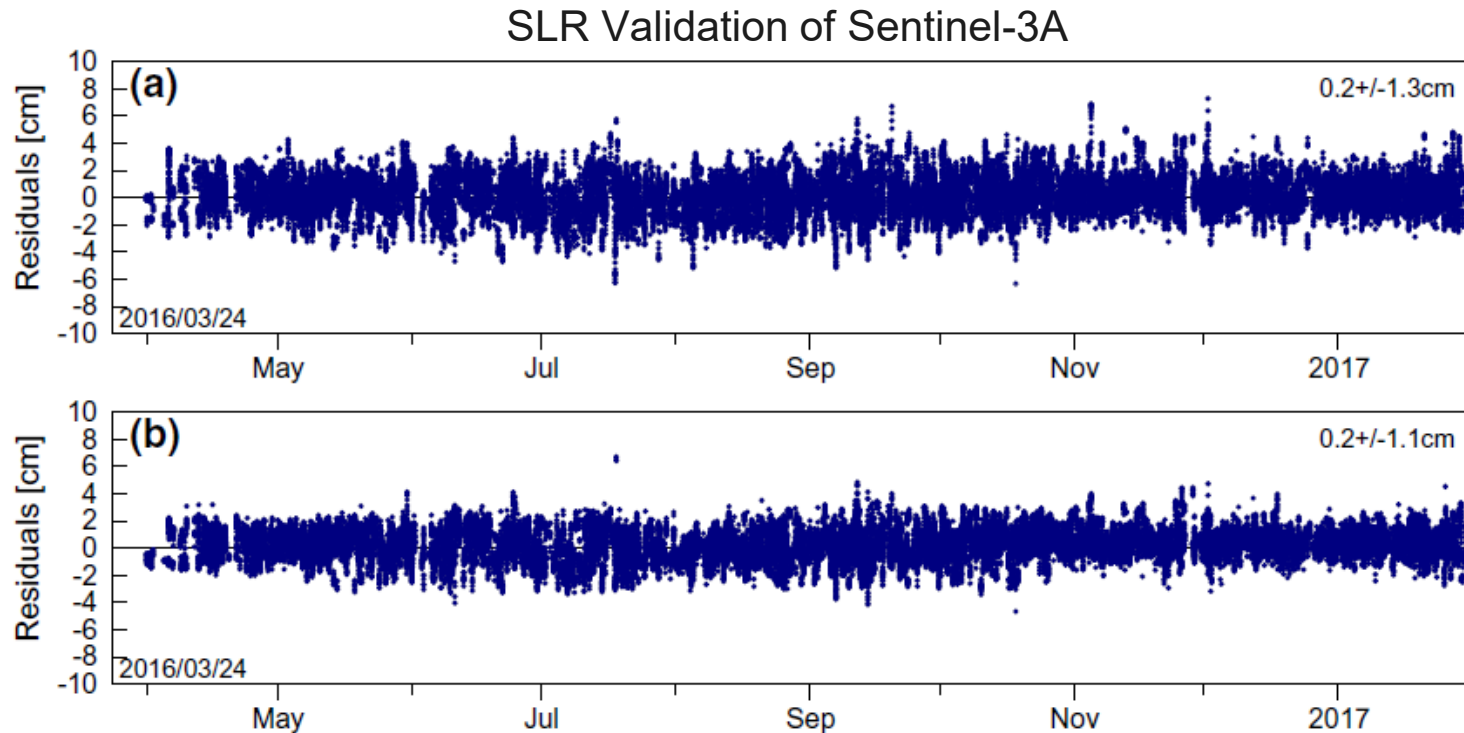


Orbit Validation with SLR (GOCE)



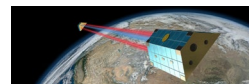


Impact of Undifferenced Ambiguity Resolution (1)

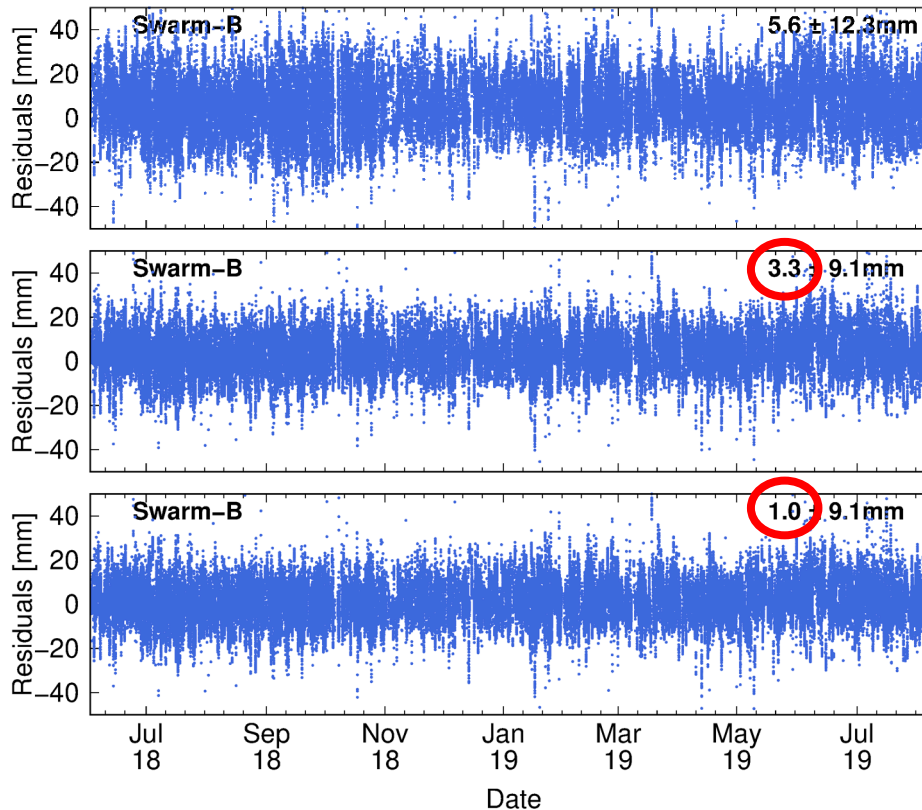
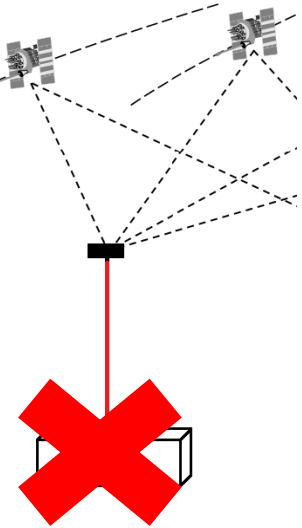
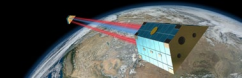


Single-receiver ambiguity fixing may be enabled by using phase bias products and corresponding clock products provided by the IGS analysis centers without the need to form any baselines. It allows to identify lateral offsets in the GPS antenna or center-of-mass location and to significantly stabilize the LEO trajectories.

(Montenbruck et al., 2018)



Impact of Undifferenced Ambiguity Resolution (2)



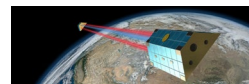
Ambiguity-float, no non-grav. modeling

Ambiguity-fixed, no non-grav. modeling

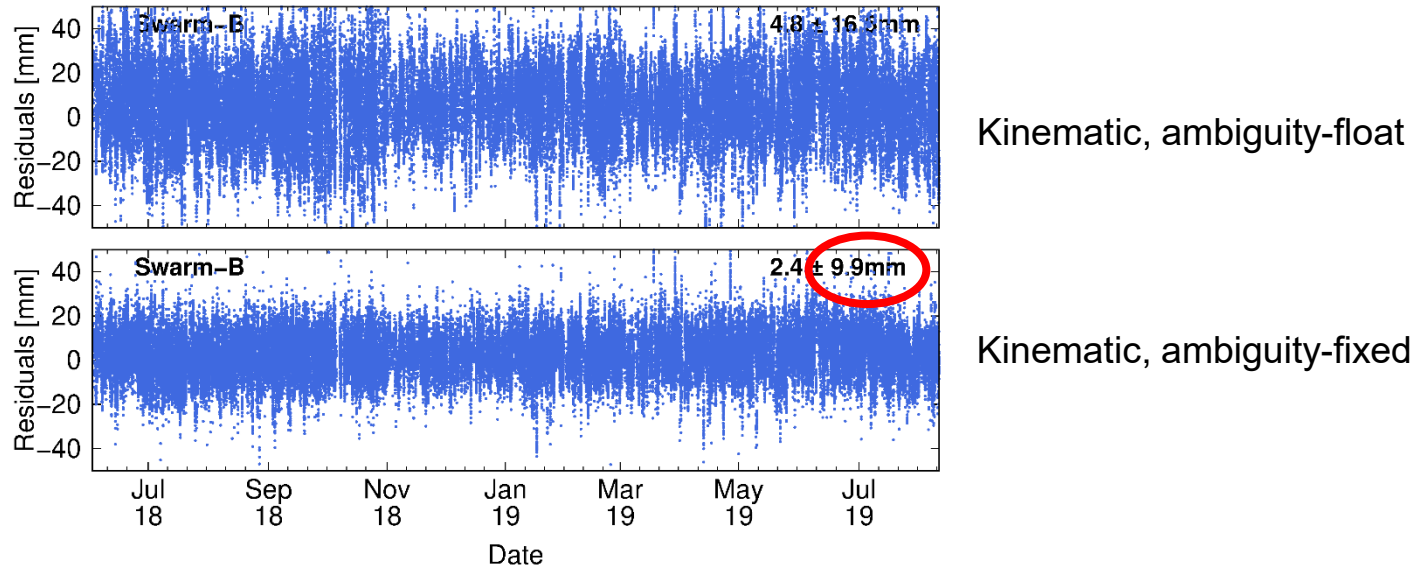
Ambiguity-fixed, with non-grav. modeling

LEO POD significantly profits from single-receiver ambiguity fixing techniques and high-quality signal-specific phase bias products, e.g., by Schaer et al. (2021).

(Arnold et al., 2019; Mao et al., 2021)



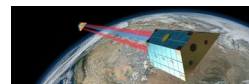
Impact of Undifferenced Ambiguity Resolution (3)



The SLR STD of ambiguity-fixed kinematic orbits (9.9mm) is only marginally worse than for the ambiguity-fixed dynamic orbits (9.1mm, see previous slide).

This nicely illustrates the limitation of SLR to “distinguish” between the orbits.

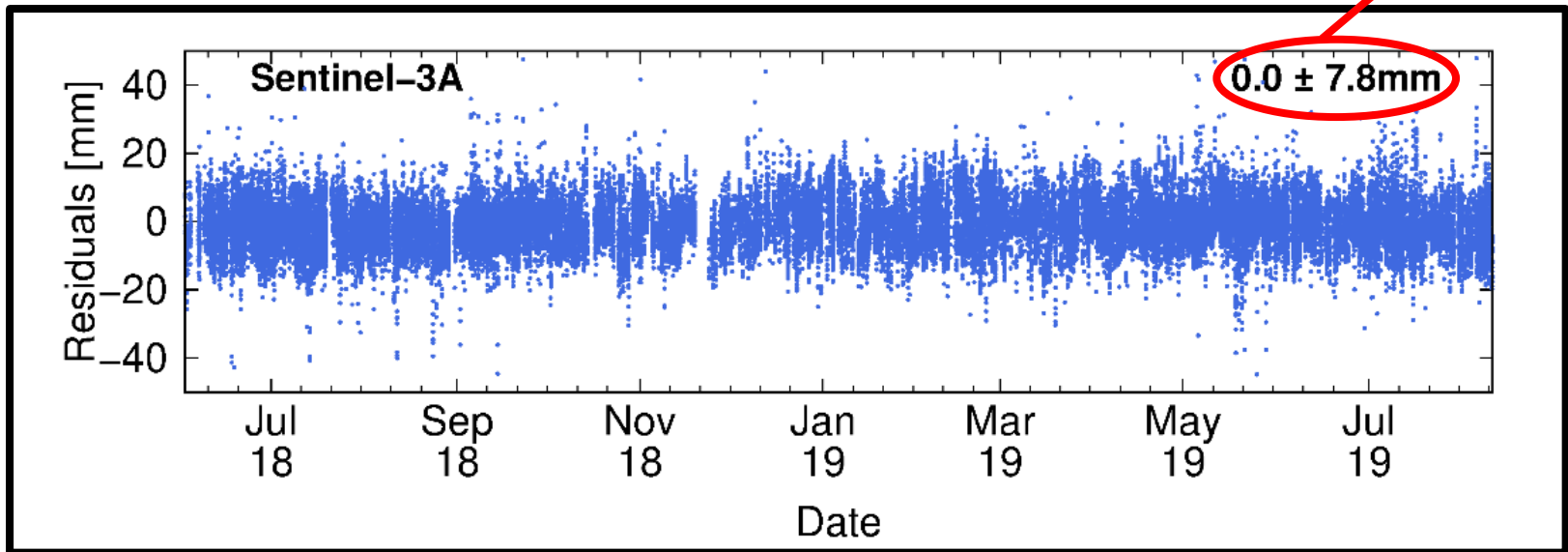
Comparisons to ambiguity-fixed kinematic orbits should be regularly performed to detect inconsistencies, e.g., related to wrong GPS antenna phase center offsets.



Orbit Validation – or SLR Network Validation?

Corrections from 1-year of dynamic, ambiguity-fixed Swarm-A/B/C, Sentinel-3A/B and GRACE-FO-C/D orbits.

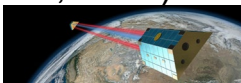
compared to 1.5 ± 10.5 mm

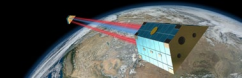


Graz	78393402	2.7 ± 0.2	3.2 ± 0.2	8.7 ± 0.7	11.9 ± 0.4
Herstmonceux	78403501	3.1 ± 0.3	1.5 ± 0.3	-4.1 ± 1.0	-2.5 ± 0.6
Potsdam	78418701	0.9 ± 0.3	3.7 ± 0.3	17.1 ± 0.9	-0.6 ± 0.6
Matera	79417701	1.7 ± 0.4	4.8 ± 0.4	4.2 ± 2.0	-5.3 ± 1.0

Some larger corrections ask for further investigations, e.g. comparisons to LAGEOS-based coordinate solutions. Investigations are on-going ...

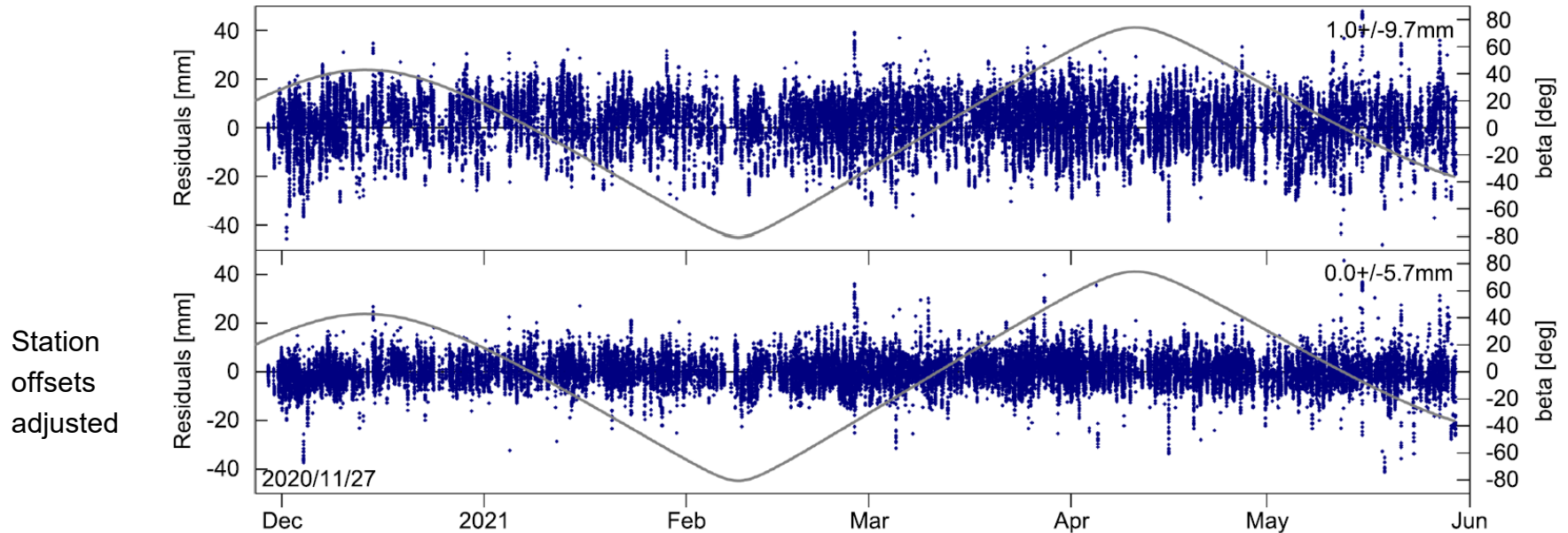
(Arnold et al., 2019)





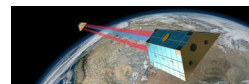
Multi-GNSS LEO POD

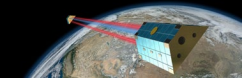
SLR Validation of Sentinel-6A



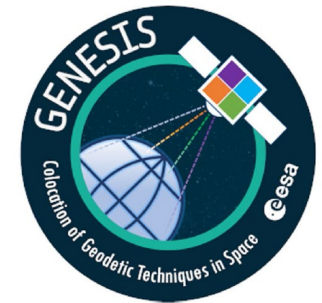
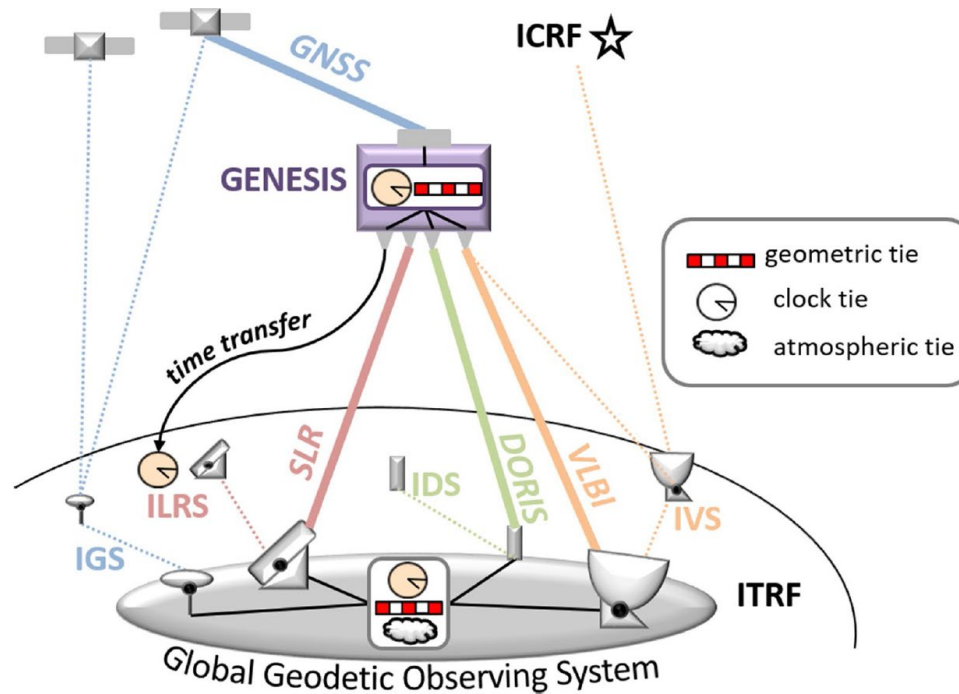
While **Galileo** measurements exhibit **30–50% smaller RMS errors** than those of GPS, the POD benefits most from the availability of an increased number of satellites. For Sentinel-6A a 1-cm consistency of ambiguity-fixed GPS-only and Galileo-only solutions with the dual-constellation orbits can be demonstrated.

(Montenbruck et al., 2021)



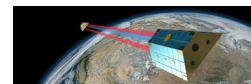


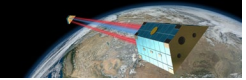
Outlook: GENESIS



GENESIS will be a dynamic space geodetic observatory carrying all the geodetic instruments referenced to one another through carefully calibrated **space ties**. The co-location of the techniques in space will allow it to determine the Terrestrial Reference Frame (TRF) with **1-mm accuracy** and **0.1 mm/year stability**.

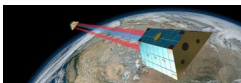
(Delva et al., 2023)





Literature (1)

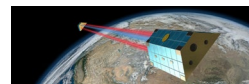
- Arnold, D., A. Jäggi, S. Schaer, C. Kobel, U. Meyer, L. Geisser, O. Montenbruck (2019): Performance of dynamic and ambiguity-fixed LEO orbits in SLR validation and network calibration. OSTST 2019 Meeting, October 21-25, 2019.
- Arnold, D., T. Grombein, L. Schreiter, V. Sterken, A. Jäggi (2023): Reprocessed precise science orbits and gravity field recovery for the entire GOCE mission. *Journal of Geodesy*, 97, 67, doi: 10.1007/s00190-023-01752-y
- Beutler, G. (2005) *Methods of Celestial Mechanics. Vol 1: Physical, Mathematical, and Numerical Principles*. Springer, ISBN 3-540-40749-9
- Blewitt, G. (1997): Basics of the GPS Technique: Observation Equations, in *Geodetic Applications of GPS*, Swedish Land Survey, pp. 10-54, available at http://www.sbg.ac.at/mat/staff/revers/lectures/2006_2007/GPS/GPSBasics.pdf
- Bock, H., R. Dach, A. Jäggi, G. Beutler (2009): High-rate GPS clock corrections from CODE: Support of 1 Hz applications. *Journal of Geodesy*, 83(11), 1083-1094, doi: 10.1007/s00190-009-0326-1





Literature (2)

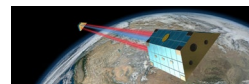
- Bock, H., A. Jäggi, G. Beutler, U. Meyer (2014): GOCE: Precise orbit determination for the entire mission. *Journal of Geodesy*, 88(11), 1047-1060, doi: 10.1007/s00190-014-0742-8
- Dach, R., E. Brockmann, S. Schaer, G. Beutler, M. Meindl, L. Prange, H. Bock, A. Jäggi, L. Ostini (2009): GNSS processing at CODE: status report, *Journal of Geodesy*, 83(3-4), 353-366, doi: 10.1007/s00190-008-0281-2
- Delva, P., and the GENESIS Science Team (2023): GENESIS: co-location of geodetic techniques in space. *Earth, Planets and Space*, 75, 5, doi: 10.1186/s40623-022-01752-w
- Fernández, M., H. Peter, D. Arnold, B. Duan, W. Simons, M. Wermuth, S. Hackel, J. Fernández, A. Jäggi, U. Hugentobler, P. Visser, P. Féménias (2022): Copernicus Sentinel-1 POD Reprocessing Campaign. *Advances in Space Research*, 70(2), 249-267, doi: 10.1016/j.asr.2022.04.036
- Grombein, T., M. Lasser, D. Arnold, U. Meyer, A. Jäggi (2022): Determination and Combination of Monthly Gravity Field Time Series from Kinematic Orbits of GRACE, GRACE-FO and Swarm. *IAG Symposia Series*, Vol. 154, 191-201, doi: 10.1007/1345_2022_163

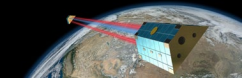




Literature (3)

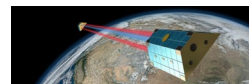
- Jäggi, A., U. Hugentobler, G. Beutler (2006): Pseudo-stochastic orbit modeling techniques for low-Earth satellites. *Journal of Geodesy*, 80(1), 47-60, doi: 10.1007/s00190-006-0029-9
- Jäggi, A. (2007): Pseudo-Stochastic Orbit Modeling of Low Earth Satellites Using the Global Positioning System. *Geodätisch-geophysikalische Arbeiten in der Schweiz*, 73, Schweizerische Geodätische Kommission, available at <http://www.sgc.ethz.ch/sgc-volumes/sgk-73.pdf>
- Jäggi, A., R. Dach, O. Montenbruck, U. Hugentobler, H. Bock, G. Beutler (2009): Phase center modeling for LEO GPS receiver antennas and its impact on precise orbit determination. *Journal of Geodesy*, 83(12), 1145-1162, doi: 10.1007/s00190-009-0333-2
- Jäggi, A., L. Prange, U. Hugentobler (2011): Impact of covariance information of kinematic positions on orbit reconstruction and gravity field recovery. *Advances in Space Research*, 47(9), 1472-1479, doi: 10.1016/j.asr.2010.12.009
- Jäggi, A., H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2015): GOCE: assessment of GPS-only gravity field determination. *Journal of Geodesy*, 89(1), 33-48. doi: 10.1007/s00190-014-0759-z

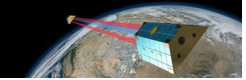




Literature (4)

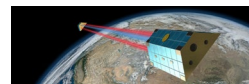
- Jäggi, A., C. Dahle, D. Arnold, H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2016): Swarm kinematic orbits and gravity fields from 18 months of GPS data. *Advances in Space Research*, 57(1), 218-233, doi: 10.1016/j.asr.2015.10.035
- Jäggi, A., C. Dahle, D. Arnold, H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2016): Swarm kinematic orbits and gravity fields from 18 months of GPS data. *Advances in Space Research*, 57(1), 218-233, doi: 10.1016/j.asr.2015.10.035
- Jäggi, A., D. Arnold, J. Weiss, D. Hunt (2021): Assessment of COSMIC-2 reduced-dynamic and kinematic orbit determination, vEGU21: Gather Online, 19-30 April, 2021
- Jäggi, A., H. Peter, D. Arnold, X. Mao (2022): Precise Orbit Determination of the Spire Satellite Constellation for Geodetic, Geophysical, and Ionospheric Applications – Project Overview and First Orbit Determination Results. COSPAR 2022 44th Scientific Assembly, Athens, Greece, 16-24 July, 2022





Literature (5)

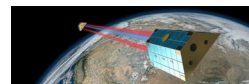
- Lasser, M., U. Meyer, A. Jäggi, T. Mayer-Gürr, A. Kvas, K.H. Neumayer, C. Dahle, F. Flechtner, J.-M. Lemoine, I. Koch, M. Weigelt, J. Flury (2020): Benchmark data for verifying background model implementations in orbit and gravity field determination software. *Advances in Geosciences*, 55, 1-11, doi: 10.5194/adgeo-55-1-2020
- Mao, X., D. Arnold, V. Girardin, A. Villiger, A. Jäggi (2021): Dynamic GPS-based LEO orbit determination with 1 cm precision using the Bernese GNSS Software. *Advances in Space Research*, 67(2), 788-805, doi: 10.1016/j.asr.2020.10.012
- Mayer-Gürr, T., A. Kvas (2019): COST-G software comparison, Graz University of Technology, benchmark data set available at: <ftp://ftp.tugraz.at/outgoing/ITSG/COST-G/softwareComparison/>
- Montenbruck, O., S. Hackel, A. Jäggi (2018): Precise orbit determination of the Sentinel-3A altimetry satellite using ambiguity-fixed GPS carrier phase observations. *Journal of Geodesy*, 92(7), 711-726, doi: 10.1007/s00190-017-1090-2





Literature (6)

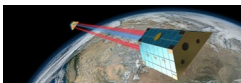
- Montenbruck, O., S. Hackel, M. Wermuth, F. Zangerl (2021): Sentinel-6A precise orbit determination using a combined GPS/Galileo receiver. *Journal of Geodesy*, 95, 109, doi: 10.1007/s00190-021-01563-z
- Peter, H., U. Meyer, M. Lasser, A. Jäggi (2022): COST-G gravity field models for precise orbit determination of Low Earth Orbiting Satellites. *Advances in Space Research*, 69(12), 4155-4168, doi: 10.1016/j.asr.2022.04.005
- Schaer, S., A. Villiger, D. Arnold, R. Dach, L. Prange, A. Jäggi (2021): The CODE ambiguity-fixed clock and phase bias analysis products: generation, properties, and performance. *Journal of Geodesy*, 95, 81, doi: 10.1007/s00190-021-01521-9
- Schreiter, L., D. Arnold, V. Sterken, A. Jäggi (2019): Mitigation of ionospheric signatures in Swarm GPS gravity field estimation using weighting strategies. *Annales Geophysicae*, 37, 111-127, doi: 10.5194/angeo-37-111-2019
- Svehla, D., M. Rothacher (2004): Kinematic Precise Orbit Determination for Gravity Field Determination, in *A Window on the Future of Geodesy*, edited by F. Sanso, pp. 181-188, Springer, doi: 10.1007/b139065

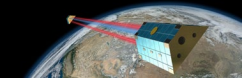




GNSS Background Literature

Teunissen, P.J.G., O. Montenbruck (2017): Springer Handbook of Global Navigation Satellite Systems, Springer, ISBN: 978-3-030-73172-4





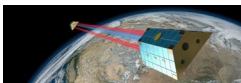
Kinematic LEO Orbit Products (1)

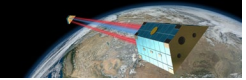
Arnold, D., A. Jäggi (2020): AIUB GRACE kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.
http://www.aiub.unibe.ch/download/LEO_ORBITS/GRACE, doi:
10.48350/158372.

Arnold, D., A. Jäggi (2020): AIUB GRACE-FO kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.
http://www.aiub.unibe.ch/download/LEO_ORBITS/GRACE-FO, doi:
10.7892/boris.147231.

Arnold, D., A. Jäggi (2020): AIUB Sentinel-1A kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.
http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-1A, doi:
10.7892/boris.145822.

Arnold, D., A. Jäggi (2020): AIUB Sentinel-1B kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.
http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-1B, doi:
10.7892/boris.145906.





Kinematic LEO Orbit Products (2)

Arnold, D., A. Jäggi (2020): AIUB Sentinel-2A kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.

http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-2A, doi:
10.7892/boris.146129.

Arnold, D., A. Jäggi (2020): AIUB Sentinel-2B kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.

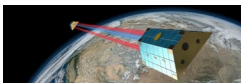
http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-2B, doi:
10.7892/boris.146133.

Arnold, D., A. Jäggi (2020): AIUB Sentinel-3A kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.

http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-3A, doi:
10.7892/boris.147286.

Arnold, D., A. Jäggi (2020): AIUB Sentinel-3A kinematic orbits, release 01.
Published by Astronomical Institute, University of Bern.

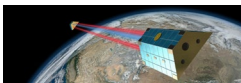
http://www.aiub.unibe.ch/download/LEO_ORBITS/Sentinel-3B, doi:
10.7892/boris.147287.





Kinematic LEO Orbit Products (3)

Arnold, D., A. Jäggi (2020): AIUB Swarm kinematic orbits, release 03.
Published by Astronomical Institute, University of Bern.
http://www.aiub.unibe.ch/download/LEO_ORBITS/SWARM, doi:
10.48350/158373.



Pocket Guide of Least-Squares Adjustment (1)

The system of **Observation Equations** is given by:

$$\mathbf{L}' + \boldsymbol{\epsilon} = \mathbf{F}(\mathbf{X})$$

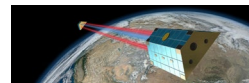
or, if \mathbf{F} is a non-linear function of the parameters, in its **linearized** form:

$$\mathbf{L}' + \boldsymbol{\epsilon} = \mathbf{F}(\mathbf{X}_0) + \mathbf{A} \mathbf{x}$$

\mathbf{L}'	Tracking observations	\mathbf{X}_0	A priori parameter values
$\boldsymbol{\epsilon}$	Observation corrections	\mathbf{x}	Parameter corrections
\mathbf{F}	Functional model	\mathbf{X}	Improved parameter values, i.e., $\mathbf{X} = \mathbf{X}_0 + \mathbf{x}$

$$\mathbf{A} \doteq \left. \frac{\partial \mathbf{F}(\mathbf{X})}{\partial \mathbf{X}} \right|_{\mathbf{X}=\mathbf{X}_0}$$

First design matrix



Pocket Guide of Least-Squares Adjustment (2)

The system of **Normal Equations** is obtained by minimizing $\epsilon^T P \epsilon$:

$$\left(A^T P A \right) x - A^T P l = N x - b = 0$$

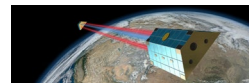
$N \doteq A^T P A$ Normal equation matrix

$b \doteq A^T P l$ Right-hand side with "O-C" term $l \doteq L' - F(X_0)$

$P = \sigma_0^2 C_{ll}^{-1}$ Weight matrix, from covariance matrix C_{ll} of observations

For a **regular** normal equation matrix the parameter corrections follow as:

$$x = \left(A^T P A \right)^{-1} A^T P l = N^{-1} b$$



Pocket Guide of Least-Squares Adjustment (3)

The **a posteriori standard deviation of unit weight** is computed as:

$$m_0 = \sqrt{\frac{\boldsymbol{\epsilon}^T P \boldsymbol{\epsilon}}{f}}$$

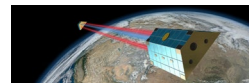
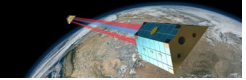
f Degree of freedom (number of observations minus number of parameters)

The **covariance matrix** of the adjusted parameters is given by

$$\mathbf{C}_{xx} = m_0^2 \mathbf{Q}_{xx} = m_0^2 \mathbf{N}^{-1}$$

and their a posteriori standard deviations follow from the diagonal elements:

$$m_x = \sqrt{C_{xx}} = m_0 \sqrt{Q_{xx}}$$



Pocket Guide of Least-Squares Adjustment (4)

Parameter pre-elimination is useful to handle a large number of parameters efficiently. Let us sub-divide the system of normal equations into two parts:

$$\begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$$

We may reduce the normal equation system by pre-eliminating epoch-specific parameters \mathbf{x}_2 , which yields the modified system of normal equations as

$$\mathbf{N}_{11}^* \mathbf{x}_1 = \mathbf{b}_1^*$$

where

$$\mathbf{N}_{11}^* = \mathbf{N}_{11} - \mathbf{N}_{12} \mathbf{N}_{22}^{-1} \mathbf{N}_{21} \quad \text{is the normal equation matrix of } \mathbf{x}_1$$

$$\mathbf{b}_1^* = \mathbf{b}_1 - \mathbf{N}_{12} \mathbf{N}_{22}^{-1} \mathbf{b}_2 \quad \text{is the corresponding right-hand side of the normal equation system}$$

