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Spectral Theory of the Klein–Gordon Equation in Pontryagin Spaces

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Abstract: In this paper we investigate an abstract Klein–Gordon equation by means of indefinite inner product methods. We show that, under certain assumptions on the potential which are more general than in previous works, the corresponding linear operator A is self-adjoint in the Pontryagin space $\mathcal{K}$ induced by the so-called energy inner product. The operator A possesses a spectral function with critical points, the essential spectrum of A is real with a gap around 0, and the non-real spectrum consists of at most finitely many pairs of complex conjugate eigenvalues of finite algebraic multiplicity; the number of these pairs is related to the 'size' of the potential. Moreover, A generates a group of bounded unitary operators in the Pontryagin space $\mathcal{K}$. Finally, the conditions on the potential required in the paper are illustrated for the Klein–Gordon equation in $\mathbb{R}^n$; they include potentials consisting of a Coulomb part and an L_p-part with $n \le p < \infty$.

1. Introduction

The motion of a relativistic spinless particle of mass m and charge e in an electrostatic field with potential q is described by the Klein–Gordon equation

$$\left(\left(\frac{\partial}{\partial t} - \mathrm{i}\, eq\right)^2 - \Delta + m^2\right)\psi = 0, \tag{1.1}$$

where the velocity of light has been normalized to 1; here ψ is a complex-valued function of $t \in \mathbb{R}$ and of $x \in \mathbb{R}^n$. An abstract model for this equation is obtained if we replace the strictly positive self-adjoint operator generated by the differential expression $-\Delta + m^2$ in the function space $L_2(\mathbb{R}^n)$ by a strictly positive self-adjoint operator H_0 in a Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$ and the operator of multiplication by the function eq in $L_2(\mathbb{R}^n)$ by a symmetric operator V in $\mathcal{H}$:

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\, V\right)^2 + H_0\right)u = 0; \tag{1.2}$$

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here u is a function of t with values in $\mathcal{H}$. The abstract Klein–Gordon equation (1.2) can be transformed into a first order differential equation for a vector function $\mathbf{x}$ with two components in an appropriate product Hilbert space $\mathcal{G}$ and a linear operator A in $\mathcal{G}$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}\, A\mathbf{x}. \tag{1.3}$$

This can be achieved by different substitutions leading to different operators A; however, in general this is not possible with a self-adjoint operator A in a Hilbert space $\mathcal{G}$.

The operator considered in the present paper arises from the abstract Klein–Gordon equation (1.2) by means of the substitution

$$x = u, \quad y = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}u, \tag{1.4}$$

which leads to a first order differential equation for $\mathbf{x} = (x\ y)^{\mathrm{t}}$ of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}\, \widehat{A}\mathbf{x}, \quad \widehat{A} = \begin{pmatrix} 0 & I \\ H_0 - V^2 & 2V \end{pmatrix}. \tag{1.5}$$

Since both operators H_0 and V are in general unbounded, the block operator matrix $\widehat{A}$ in (1.5) may not even be densely defined nor closed. To this end, suitable assumptions have to be imposed on the potential V so that we can associate a closed operator A with the block operator matrix $\widehat{A}$. If the potential V is not small, $\widehat{A}$ does not exhibit symmetry in any Hilbert space. However, formally, if we introduce the so-called energy inner product $\langle\cdot,\cdot\rangle$ which, for suitable elements $\mathbf{x} = (x\ y)^{\mathrm{t}}$, $\mathbf{x}' = (x'\ y')^{\mathrm{t}}$ of $\mathcal{H} \oplus \mathcal{H}$, is given by

$$\langle \mathbf{x}, \mathbf{x}' \rangle = \left(\begin{pmatrix} H_0 - V^2 & 0 \\ 0 & I \end{pmatrix} \mathbf{x}, \mathbf{x}' \right) = \left((H_0 - V^2)x, x' \right) + (y, y'), \tag{1.6}$$

then it is not difficult to see that $\widehat{A}$ is symmetric with respect to $\langle\cdot,\cdot\rangle$:

$$\langle \widehat{A}\mathbf{x}, \mathbf{x}' \rangle = \left(\begin{pmatrix} 0 & H_0 - V^2 \\ H_0 - V^2 & 2V \end{pmatrix} \mathbf{x}, \mathbf{x}' \right).$$

The inner product $\langle\cdot,\cdot\rangle$ is in general indefinite; under our assumptions on the potential V, it is negative definite on a subspace of finite dimension so that the space $\mathcal{G}$ equipped with $\langle\cdot,\cdot\rangle$ becomes a so-called Pontryagin space.

For the Klein–Gordon equation in $\mathbb{R}^n$, the operator $\widehat{A}$ in the energy inner product $\langle\cdot,\cdot\rangle$ has been studied in a number of papers, see, e.g., [SSW40, Lun73a, Lun73b, Eck76, Eck80, Sch76, Kak76, Wed77, Wed78, Jon79, Naj80a, Naj80b, Naj83, Bac04], and the unpublished manuscript [LN96][1]; some of these works also consider the corresponding abstract operator $\widehat{A}$ in a Pontryagin space, but under more restrictive assumptions on the potential V.

The operator A and the energy inner product $\langle\cdot,\cdot\rangle$ studied in this paper are related to other operators associated with the abstract Klein–Gordon equation (see [LNT06]). They arise from the second order differential equation (1.2) by means of the substitution

$$x = u, \quad y = \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V \right) u, \tag{1.7}$$

[1] This manuscript was the starting point for the present paper; unfortunately, Professor Branko Najman died in August 1996.

which leads to a first order differential equation (1.3) for $\mathbf{x} = (x\ y)^{\mathrm{t}}$ of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}\begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}\mathbf{x}. \tag{1.8}$$

The operator A_1, for example, is obtained from (1.8) as the closure of the block operator matrix therein in the Hilbert space $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$; it turns out to be symmetric with respect to the so-called charge inner product $[\cdot, \cdot]$, which is defined on elements $\mathbf{x} = (x\ y)^{\mathrm{t}}$, $\mathbf{x}' = (x'\ y')^{\mathrm{t}}$ of $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ by a relation of the form

$$[\mathbf{x}, \mathbf{x}'] = \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}' \right) = (x, y') + (y, x'). \tag{1.9}$$

Independently of the potential V, the charge inner product is in general negative on an infinite dimensional subspace and hence leads to a so-called Krein space. The energy inner product is related to the charge inner product as follows:

$$\langle \mathbf{x}, \mathbf{x}' \rangle = [A_1 \widetilde{W}\mathbf{x}, \widetilde{W}\mathbf{x}'], \quad \widetilde{W} = \begin{pmatrix} I & 0 \\ -V & I \end{pmatrix}, \tag{1.10}$$

for suitable elements $\mathbf{x}$, $\mathbf{x}'$ of the Pontryagin space $\big(\mathcal{G}, \langle \cdot, \cdot \rangle\big)$; under our assumptions on V, the operator $\widetilde{W} : \mathcal{G} \to \mathcal{G}_1$ is bounded. The spectral properties of the operator A_1 and of another operator A_2 associated with (1.8) in the charge inner product and their relations to the operator A are investigated in a separate paper (see [LNT06]).

The present paper is organized as follows: In the next Sect. 2 we briefly review results from the theory of self-adjoint operators in Pontryagin spaces. In Sect. 3 we associate the operator A with (1.5); it acts in the space $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$, where $\mathcal{H}_{1/2}$ is the Hilbert space given by $\mathcal{D}\big(H_0^{1/2}\big)$ with norm $\|H_0^{1/2} \cdot \|$. We show that if

(i) $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ (i.e., $S = VH_0^{-1/2}$ is bounded) and
(ii) $I - S^*S$ is boundedly invertible,

then the operator

$$A = \begin{pmatrix} 0 & I \\ H & 2V \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{H}_{1/2},$$

is closed and boundedly invertible in $\mathcal{G}$; here H is the self-adjoint operator in $\mathcal{H}$ given by $H = H_0^{1/2}(I - S^*S)H_0^{1/2}$. In Sect. 4 we introduce the indefinite inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$ and we prove that under the above assumptions (i) and (ii) the space $\mathcal{G}$ equipped with this inner product is a Krein space $\mathcal{K}$ and A is a self-adjoint operator in $\mathcal{K}$ with non-empty resolvent set. In addition, we study the relation (1.10) of the energy inner product $\langle \cdot, \cdot \rangle$ with the operator A_1 and the corresponding charge inner product $[\cdot, \cdot]$. Section 5 contains the main result about the spectral properties of A. Under the additional assumption

(iii) $S = VH_0^{-1/2} = S_0 + S_1$ with $\|S_0\| < 1$ and a compact operator S_1,

we show that $\mathcal{K}$ is a Pontryagin space of index κ, where κ is the number of negative eigenvalues of $I - S^*S$, the operator A possesses a spectral function with at most finitely many critical points, the non-real spectrum of A consists of at most κ pairs of complex conjugate eigenvalues, and the essential spectrum of A is real and has a gap of size at least $2(1-\|S_0\|)m$ around 0. Moreover, the operator A generates a strongly continuous group $\big(\exp(\mathrm{i}At)\big)_{t\in\mathbb{R}}$ of unitary operators in the Pontryagin space $\mathcal{K}$ and hence the Cauchy problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution for all initial values $\mathbf{x}_0 \in \mathcal{H}_{1/2} \oplus \mathcal{H}$. Since ∞ is not a critical point for a self-adjoint operator in a Pontryagin space, the group $\big(\exp(\mathrm{i}At)\big)_{t\in\mathbb{R}}$ is uniformly bounded in $\mathcal{K}$; therefore the time-asymptotic behaviour of the solution $\mathbf{x}$ and hence of the solution of the abstract Klein–Gordon equation (1.2) is the same as in a Hilbert space. This is not the case for the self-adjoint operator A_1 in the Krein space $\mathcal{G}_1$ since there ∞ is a critical point (see [LNT06]).

Finally, in Sect. 6, we consider the Klein–Gordon equation in $\mathbb{R}^n$ and present sufficient conditions for the above assumptions. In particular, we show that our results apply to potentials V of the form $V = V_0 + V_1$ with a Coulomb part $V_0(x) = \gamma/|x|$, $x \in \mathbb{R}^n \setminus \{0\}$, with $\gamma < (n-2)/2$ and $V_1 \in L_p(\mathbb{R}^n)$ with $n \le p < \infty$.

2. Preliminaries

1. *Notations and definitions from spectral theory.* For a closed linear operator A in a Hilbert space $\mathcal{G}$ with domain $\mathcal{D}(A)$ we denote by $\rho(A)$, $\sigma(A)$, and $\sigma_{\mathrm{p}}(A)$ its resolvent set, spectrum, and point spectrum (or set of eigenvalues), respectively. For $\lambda \in \sigma_{\mathrm{p}}(A)$ the algebraic eigenspace of A at λ is denoted by $\mathcal{L}_\lambda(A)$. The operator A is called *Fredholm* if its kernel is finite dimensional and its range is finite codimensional (and hence closed), see, e.g., [GGK90, Chapter IV, §5.1]. The *essential spectrum* of A is defined by

$$\sigma_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}.$$

An eigenvalue $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ is called *of finite type* if λ_0 is isolated (i.e., a punctured neighbourhood of λ_0 belongs to $\rho(A)$) and $A - \lambda_0$ is Fredholm or, equivalently, the corresponding Riesz projection is finite dimensional.

2. *Linear spaces with inner products.* A *Krein space* $(\mathcal{K}, [\cdot,\cdot])$ is a linear space $\mathcal{K}$ which is equipped with an (indefinite) inner product (i.e., a hermitian sesquilinear form) $[\cdot,\cdot]$ such that $\mathcal{K}$ can be written as

$$\mathcal{K} = \mathcal{G}_+ [\dot{+}] \mathcal{G}_-, \tag{2.1}$$

where $(\mathcal{G}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and $[\dot{+}]$ means that the sum of $\mathcal{G}_+$ and $\mathcal{G}_-$ is direct and $[\mathcal{G}_+, \mathcal{G}_-] = \{0\}$. The norm topology on a Krein space $\mathcal{K}$ is the norm topology of the orthogonal sum of the Hilbert spaces $\mathcal{G}_\pm$ in (2.1). It can be shown that this norm topology is independent of the particular decomposition (2.1); all topological notions in $\mathcal{K}$ refer to this norm topology and $\|\cdot\|$ denotes any of the equivalent norms.

Krein spaces often arise as follows: In a given Hilbert space $(\mathcal{G}, (\cdot,\cdot))$, every bounded self-adjoint operator G in $\mathcal{G}$ with $0 \in \rho(G)$ induces an inner product

$$[x, y] := (Gx, y), \quad x, y \in \mathcal{G}, \tag{2.2}$$

such that $(\mathcal{G}, [\cdot,\cdot])$ becomes a Krein space; here, in the decomposition (2.1), we can choose $\mathcal{G}_+$ as the spectral subspace of G corresponding to the positive spectrum of

G and $\mathcal{G}_-$ as the spectral subspace of G corresponding to the negative spectrum of G. A subspace $\mathcal{L}$ of a linear space $\mathcal{K}$ with inner product $[\cdot,\cdot]$ is called *non-degenerated* if there exists no $x \in \mathcal{L}$, $x \neq 0$, such that $[x, \mathcal{L}] = 0$, otherwise $\mathcal{L}$ is called *degenerated*; note that a Krein space $\mathcal{K}$ is always non-degenerated, but it may have degenerated subspaces. An element $x \in \mathcal{K}$ is called *positive* (*non-negative, negative, non-positive, neutral*, respectively) if $[x, x] > 0$ (≥ 0, < 0, ≤ 0, $= 0$, respectively); a subspace of $\mathcal{K}$ is called *positive* (*non-negative*, etc., respectively), if all its nonzero elements are positive (non-negative, etc., respectively). For the definition and simple properties of Krein spaces and linear operators therein we refer to [Bog74, Lan82, AI89].

3. *Self-adjoint operators in Krein spaces.* For a closed linear operator A in a Krein space $\mathcal{K}$ with dense domain $\mathcal{D}(A)$, the (*Krein space*) *adjoint* A^+ of A is the densely defined operator in $\mathcal{K}$ given by

$$\mathcal{D}(A^+) := \{y \in \mathcal{K} : [A\,\cdot\,, y] \text{ is a continuous linear functional on } \mathcal{D}(A)\}$$

and the relation

$$[Ax, y] = [x, A^+y], \quad x \in \mathcal{D}(A), \ \ y \in \mathcal{D}(A^+).$$

The operator A is called *symmetric* if $A \subset A^+$ and *self-adjoint* if $A = A^+$. The spectrum of a self-adjoint operator A in a Krein space $\mathcal{K}$ is always symmetric to the real axis; note that both the spectrum $\sigma(A)$ or the resolvent set $\rho(A)$ may be empty. An *orthogonal projection* P in a Krein space $\mathcal{K}$ is a self-adjoint projection in $\mathcal{K}$; note that orthogonal projections in a Krein space may have norm > 1.

If for a self-adjoint operator A in a Krein space $\mathcal{K}$ with $\lambda_0 \in \sigma_p(A)$ all the eigenvectors at λ_0 are positive (negative, respectively), then λ_0 is called an *eigenvalue of positive* (*negative*, respectively) *type*. A positive or negative eigenvector x_0 of A at λ_0 does not have any associated vectors. Consequently, if for an eigenvector x_0 at λ_0 there exists an element x_1 such that $(A - \lambda_0)x_1 = x_0$, then x_0 is neutral.

4. *Self-adjoint operators in Pontryagin spaces.* If in some decomposition (2.1) one of the components $\mathcal{G}_\pm$ is of finite dimension, it is of the same dimension in all such decompositions, and the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called a *Pontryagin space*. For the Pontryagin spaces $\mathcal{K}$ occurring in this paper, the negative component $\mathcal{G}_-$ is of finite dimension, say κ; in this case, $\mathcal{K}$ is called a *Pontryagin space with negative index* say κ. If $\mathcal{K}$ arises from a Hilbert space $\mathcal{G}$ by means of a self-adjoint operator G with inner product (2.2), then $\mathcal{K}$ is a Pontryagin space with negative index κ if and only if the negative spectrum of the invertible operator G consists of exactly κ eigenvalues, counted according to their multiplicities. In a Pontryagin space $\mathcal{K}$ with negative index κ each non-positive subspace is of dimension $\leq \kappa$, and a non-positive subspace is maximal non-positive (that is, it is not properly contained in another non-positive subspace) if and only if it is of dimension say κ.

If $\mathcal{L}$ is a non-degenerated linear space with inner product $[\cdot,\cdot]$ such that for a κ-dimensional subspace $\mathcal{L}_-$ we have

$$[x, x] < 0, \quad x \in \mathcal{L}_-, \ \ x \neq 0,$$

but there is no $(\kappa + 1)$-dimensional subspace with this property, then there exists a Pontryagin space $\mathcal{K}$ with negative index κ such that $\mathcal{L}$ is a dense subset of $\mathcal{K}$. This means that $\mathcal{L}$ can be completed to a Pontryagin space in a similar way as a pre-Hilbert space can be completed to a Hilbert space.

The spectrum of a self-adjoint operator in a Pontryagin space is real with the possible exception of at most κ non-real pairs of eigenvalues $\lambda, \overline{\lambda}$ of finite type; this estimate can be improved by taking multiplicities into account (see (2.3) below). According to a theorem of Pontryagin, a self-adjoint operator A in a Pontryagin space with negative index κ has a κ-dimensional invariant non-positive subspace $\mathcal{L}_-^{\max}$:

$$\mathcal{L}_-^{\max} \subset \mathcal{D}(A), \quad A\mathcal{L}_-^{\max} \subset \mathcal{L}_-^{\max};$$

the subspace $\mathcal{L}_-^{\max}$ can be chosen such that $\operatorname{Im}\big(\sigma(A|\mathcal{L}_-^{\max})\big) \geq 0$. Then the points of $\sigma(A|\mathcal{L}_-^{\max})$ are the eigenvalues of A in the closed upper half plane with a non-positive eigenvector. We denote the set of all eigenvalues of A with a non-positive eigenvector by $\sigma_0(A)$; for a point $\lambda \in \sigma_0(A)$, the maximal dimension of a non-positive subspace of $\mathcal{L}_\lambda(A)$ is denoted by $\kappa_\lambda^-(A)$. Concerning the non-real spectrum of A, the closed linear span of all the algebraic eigenspaces $\mathcal{L}_\lambda(A)$ corresponding to the eigenvalues λ of A in the open upper (or lower) half plane is a neutral subspace of $\mathcal{K}$; for all such points λ the algebraic eigenspaces $\mathcal{L}_\lambda(A)$, $\mathcal{L}_{\overline{\lambda}}(A)$ are skewly linked, that is, to each nonzero $x \in \mathcal{L}_\lambda(A)$ there exists a $y \in \mathcal{L}_{\overline{\lambda}}(A)$ such that $[x, y] \neq 0$ and to each nonzero $y \in \mathcal{L}_{\overline{\lambda}}(A)$ there exists an $x \in \mathcal{L}_\lambda(A)$ such that $[x, y] \neq 0$. In particular, $\dim \mathcal{L}_{\overline{\lambda}}(A) = \dim \mathcal{L}_\lambda(A)$ and the Jordan structure of A in $\mathcal{L}_\lambda(A)$ and in $\mathcal{L}_{\overline{\lambda}}(A)$ is the same. Further, the relation

$$\kappa = \sum_{\lambda\in\sigma_0(A)\cap\mathbb{R}} \kappa_\lambda^-(A) + \sum_{\lambda\in\sigma(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A) \tag{2.3}$$

holds, which yields estimates for the number of points of $\sigma_0(A)$. All (real) points $\lambda \in \sigma(A) \setminus \sigma_0(A)$ are *spectral points of positive type*, by which we mean that they are either eigenvalues of positive type or, if they belong to the continuous spectrum, that for each sequence $(x_n) \subset \mathcal{D}(A)$,

$$\|x_n\| = 1, \quad (A - \lambda)x_n \to 0 \quad \Longrightarrow \quad \liminf_{n\to\infty} [x_n, x_n] > 0.$$

5. *Spectral functions of self-adjoint operators in Pontryagin spaces.* If q denotes the minimal polynomial or the characteristic polynomial of the restriction $A|\mathcal{L}_-^{\max}$, and q^* is the polynomial given by $q^*(z) = \overline{q(\overline{z})}$, $z \in \mathbb{C}$, then the polynomial q^*q is independent of the particular choice of the invariant subspace $\mathcal{L}_-^{\max}$, and it is not hard to show that $[q^*(A)q(A)x, x] \geq 0$, $x \in \mathcal{D}(A^{2\kappa})$. As a consequence, a self-adjoint operator A in a Pontryagin space possesses a *spectral function with possible critical points* (see [KL63] and also [Lan82]). In order to introduce it, we call a bounded or unbounded real interval $\Gamma \subset \mathbb{R}$ *admissible* for the operator A if the end points of Γ do not belong to $\sigma_0(A)$. Then, for every admissible interval Γ, there exists an orthogonal projection $E(\Gamma)$ in $\mathcal{K}$ such that the range $E(\Gamma)\mathcal{K}$ is invariant under A and

$$\sigma\big(A\big|E(\Gamma)\mathcal{K}\big) \subset \overline{\Gamma}, \quad \sigma\big(A\big|(I - E(\Gamma))\mathcal{K}\big) \cap \mathbb{R} \subset \overline{\mathbb{R} \setminus \Gamma}.$$

Moreover, the mapping $\Gamma \mapsto E(\Gamma)$ from the semiring $\mathcal{R}_A$ of all admissible intervals into the space of all bounded linear operators in $\mathcal{K}$ is a homomorphism, that is, for $\Gamma_1, \Gamma_2 \in \mathcal{R}_A$,

$$E(\Gamma_1 \cap \Gamma_2) = E(\Gamma_1)E(\Gamma_2), \quad E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) + E(\Gamma_2) - E(\Gamma_1 \cap \Gamma_2),$$

and

$$E(\emptyset) = 0, \quad E(\mathbb{R})\mathcal{K} = \Big(\sum_{\lambda \in \sigma(A)\setminus\mathbb{R}} \mathcal{L}_\lambda(A) \Big)^{[\perp]};$$

here ${}^{[\perp]}$ denotes the orthogonal complement with respect to the indefinite inner product. The *critical points* of the spectral function E are those points $\lambda \in \mathbb{R}$ for which the inner product $[\cdot, \cdot]$ is indefinite on $E(\Gamma)\mathcal{K}$ for each $\Gamma \in \mathcal{R}_A$ containing λ; all critical points of E belong to $\sigma_0(A)$.

If an interval $\Gamma \in \mathcal{R}_A$ does not contain points of $\sigma_0(A)$, then the range $E(\Gamma)\mathcal{K}$ is a positive subspace of $\mathcal{K}$ and hence a Hilbert space. Therefore, with the exception of the points of $\sigma_0(A) \cap \mathbb{R}$, the spectral behaviour of A is that of a self-adjoint operator in a Hilbert space. In particular, for an admissible interval Γ with $\Gamma \cap \sigma_0(A) = \emptyset$,

$$AE(\Gamma) = \int_\Gamma \lambda E(d\lambda);$$

here, if A is an unbounded operator and Γ is an unbounded interval, the expressions on either side coincide as unbounded operators.

Given a point $\lambda_0 \in \sigma_0(A) \cap \mathbb{R}$, we choose an admissible interval $\Gamma = [\alpha, \beta]$ such that $[\alpha, \beta] \cap \sigma_0(A) = \{\lambda_0\}$. If $\mathcal{L}_{\lambda_0}(A)$ is non-degenerated (e.g., if λ_0 is an eigenvalue of negative type), then the strong limits

$$\lim_{\mu \nearrow \lambda_0} E([\alpha, \mu]), \qquad \lim_{\mu \searrow \lambda_0} E([\mu, \beta])$$

exist. They can be considered as spectral projections $E([\alpha, \lambda_0))$ and $E((\lambda_0, \beta])$ of A corresponding to the intervals $[\alpha, \lambda_0)$ and $(\lambda_0, \beta]$, respectively, and the decomposition

$$E(\Gamma)\mathcal{K} = E([\alpha, \lambda_0))\mathcal{K} \,[\dot{+}]\, \mathcal{L}_{\lambda_0}(A) \,[\dot{+}]\, E((\lambda_0, \beta])\mathcal{K} \tag{2.4}$$

holds. If, however, $\mathcal{L}_{\lambda_0}$ is degenerated, then at least one of the quantities

$$\limsup_{\mu \nearrow \lambda_0} \|E([\alpha, \mu])\| \quad \text{or} \quad \limsup_{\mu \searrow \lambda_0} \|E([\mu, \beta])\|$$

is infinite and the subspace $\mathcal{L}_{\lambda_0}(A)$ cannot be split off as in (2.4).

If A is an unbounded self-adjoint operator in a Pontryagin space $\mathcal{K}$, we choose a bounded admissible interval Γ which contains all the real points of $\sigma_0(A)$ and we consider the space

$$\mathcal{L}^1 := E(\Gamma)\mathcal{K} \,[\dot{+}] \sum_{\lambda \in \sigma(A)\setminus\mathbb{R}} \mathcal{L}_\lambda(A).$$

It is a Pontryagin space with negative index κ that reduces A and the restriction $A_1 := A|\mathcal{L}^1$ is a bounded operator. The orthogonal complement $\mathcal{L}^0$ of $\mathcal{L}^1$ in $\mathcal{K}$ is a Hilbert space with respect to the inner product $[\cdot, \cdot]$ and the decomposition

$$\mathcal{K} = \mathcal{L}^1 \,[\dot{+}]\, \mathcal{L}^0$$

yields a corresponding orthogonal decomposition of the operator A:

$$A = A_1 \,[\dot{+}]\, A_0. \tag{2.5}$$

Here A_1 is a bounded self-adjoint operator in the Pontryagin space $\mathcal{L}^1$ with negative index κ and A_0 is a self-adjoint operator in the Hilbert space $\mathcal{L}^0$. Thus, the study of an unbounded self-adjoint operator in a Pontryagin space can always be reduced to the study of a bounded self-adjoint operator in a Pontryagin space and of an unbounded self-adjoint operator in a Hilbert space.

A bounded operator in a Pontryagin space $\mathcal{K}$ is called *unitary* if it maps $\mathcal{K}$ onto itself and

$$[Ux, Uy] = [x, y], \quad x, y \in \mathcal{K}.$$

Using the decomposition (2.5), it readily follows that a self-adjoint operator A in a Pontryagin space generates a group $(\exp(\mathrm{i}tA))_{t\in\mathbb{R}}$ of unitary operators in $\mathcal{K}$ and that this group is exponentially bounded, that is,

$$\|\exp(\mathrm{i}tA)\| \leq C\, \mathrm{e}^{\gamma|t|}, \quad t \in \mathbb{R},$$

with positive constants C and γ. This was first proved by M.A. Naĭmark in [Naĭ66].

3. An Operator Associated with the Abstract Klein–Gordon Equation

Let $\big(\mathcal{H}, (\cdot,\cdot)\big)$ be a Hilbert space with corresponding norm $\|\cdot\|$, H_0 a strictly positive self-adjoint operator in $\mathcal{H}$, $H_0 \geq m^2 > 0$, and V a symmetric operator in $\mathcal{H}$. By means of the operator H_0 we introduce the Hilbert space $\big(\mathcal{H}_{1/2}, (\cdot,\cdot)_{1/2}\big)$ as

$$\mathcal{H}_{1/2} := \mathcal{D}\big(H_0^{1/2}\big), \quad (x, y)_{1/2} := \big(H_0^{1/2}x, H_0^{1/2}y\big), \quad x, y \in \mathcal{H}_{1/2}. \tag{3.1}$$

In the orthogonal sum $\mathcal{G} := \mathcal{H}_{1/2} \oplus \mathcal{H}$ with norm

$$\|\mathbf{x}\|_{\mathcal{G}} = \Big(\|H_0^{1/2}x\|^2 + \|y\|^2\Big)^{1/2}, \quad \mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{G},$$

we consider the block operator matrix $\widehat{A}$, formally given by

$$\widehat{A} := \begin{pmatrix} 0 & I \\ H_0 - V^2 & 2V \end{pmatrix}, \tag{3.2}$$

which arises from the differential equation (1.2) by means of the substitution (1.4) (see (1.5)).

In order to associate a well-defined operator with the entry $H_0 - V^2$ in $\widehat{A}$, we make the following assumption:

Ass. (i) $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$;
this condition implies that the operator

$$S := VH_0^{-1/2} \tag{3.3}$$

is everywhere defined and bounded on $\mathcal{H}$.

In the next section we need that the operator associated with the formal expression $H_0 - V^2$ is boundedly invertible. In order to assure this we also assume

Ass. (ii) $1 \in \rho(S^*S)$, that is, the operator $I - S^*S$ is boundedly invertible.

In this case the operator $H_0^{-1/2}(I - S^*S)^{-1}H_0^{-1/2}$ is everywhere defined, injective, bounded and self-adjoint in $\mathcal{H}$; therefore the operator

$$H := H_0^{1/2}(I - S^*S)H_0^{1/2}, \quad \mathcal{D}(H) = \{x \in \mathcal{H}_{1/2} : (I - S^*S)H_0^{1/2}x \in \mathcal{H}_{1/2}\} \tag{3.4}$$

is self-adjoint and boundedly invertible in $\mathcal{H}$. The operator H can also be considered as a densely defined closed operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$, for which we use the same symbol H: it is densely defined because $\mathcal{D}(H)$ is dense in $\mathcal{H}$ and the inclusion $\mathcal{H}_{1/2} \hookrightarrow \mathcal{H}$ is continuous; it is closed since the middle factor is closed in $\mathcal{H}$, the left factor is boundedly invertible in $\mathcal{H}$ and the right factor is boundedly invertible as an operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$ (see [Kat66, Sect. III.5.2]).

Remark 3.1. The operator H in $\mathcal{H}$ (from $\mathcal{H}_{1/2}$ to $\mathcal{H}$, respectively) can also be defined by means of quadratic forms if we replace the conditions (i) and (ii) by
Ass. (i′) V is $H_0^{1/2}$-bounded with relative bound less than 1.

In fact, Assumption (i) is equivalent to the fact that V is $H_0^{1/2}$-bounded, that is, $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ and there exist constants $a,\ b \geq 0$, such that

$$\|Vx\| \leq a\,\|x\| + b\,\|H_0^{1/2}x\|, \quad x \in \mathcal{D}(H_0^{1/2}). \tag{3.5}$$

In Assumption (i′) it is required, in addition, that (3.5) holds with $b < 1$, or, equivalently (see [Kat82, Sect. V.4.1]), there exist constants $a',\ b' \geq 0$, $b' < 1$, such that

$$\|Vx\|^2 \leq a'^2\,\|x\|^2 + b'^2\,\|H_0^{1/2}x\|^2, \quad x \in \mathcal{D}(H_0^{1/2}). \tag{3.6}$$

If we introduce the forms

$$\begin{aligned} \mathbf{h}[x,y] &:= \big(H_0^{1/2}x, H_0^{1/2}y\big), \quad x, y \in \mathcal{D}\big(H_0^{1/2}\big), \\ \mathbf{v}_2[x,y] &:= (Vx, Vy), \quad x, y \in \mathcal{D}(V), \end{aligned}$$

then (3.6) (and hence (i′)) implies that the form $\mathbf{v}_2$ is $\mathbf{h}$–bounded with relative form-bound less than 1. Then, according to [Kat82, Theorem VI.3.9], the form sum $\mathbf{h} + \mathbf{v}_2$ is closed and symmetric, and the entry $H_0 - V^2$ in (3.7) can be defined by means of the self-adjoint operator in $\mathcal{H}$ induced by the form sum $\mathbf{h} + \mathbf{v}_2$. Our choice of the conditions (i) and (ii) rather than (i′) is due to the fact that Assumption (ii) is needed in the next section for other reasons.

With the formal matrix $\widehat{A}$ in (3.2) we now associate the block operator matrix A in $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$ defined by

$$A = \begin{pmatrix} 0 & I \\ H & 2V \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{D}\big(H_0^{1/2}\big). \tag{3.7}$$

Lemma 3.2. *If $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ and $1 \in \rho(S^*S)$, then the operator A from* (3.7) *is boundedly invertible, and hence closed in $\mathcal{G}$, with*

$$A^{-1} = \begin{pmatrix} -2H^{-1}V & H^{-1} \\ I & 0 \end{pmatrix}. \tag{3.8}$$

Proof. By the assumptions, H is a boundedly invertible operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$. Hence formally the inverse of A is given by (3.8). It remains to be shown that A^{-1} is a bounded operator in $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$.

This follows from the facts that H^{-1} is a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{1/2}$, the identity I is bounded as an operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$ since the inclusion $\mathcal{H}_{1/2} \hookrightarrow \mathcal{H}$ is continuous, and $H^{-1}V = H_0^{-1/2}(I - S^*S)^{-1}H_0^{-1/2}V$ is a bounded operator in $\mathcal{H}_{1/2}$. For the latter, we observe that V is bounded from $\mathcal{H}_{1/2}$ to $\mathcal{H}$ by the first assumption, $(I - S^*S)^{-1}H_0^{-1/2}$ is bounded in $\mathcal{H}$ by the second assumption, and $H_0^{-1/2}$ is bounded from $\mathcal{H}$ to $\mathcal{H}_{1/2}$. □

The operator A is related to another operator associated with the Klein–Gordon equation (1.2) which is formally given by (1.8) and arises from the substitution (1.7): In the orthogonal sum $\mathcal{G}_1 := \mathcal{H} \oplus \mathcal{H}$ we consider the operator

$$\widehat{A}_1 := \begin{pmatrix} V & I \\ H_0 & V \end{pmatrix} \tag{3.9}$$

with domain

$$\mathcal{D}(\widehat{A}_1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}(V) \cap \mathcal{D}(H_0),\ y \in \mathcal{D}(V) \right\}.$$

It has been shown in [LNT06, Thm. 3.1] that Assumption (i) implies that $\widehat{A}_1$ is closable with closure A_1 given by

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}\big(H_0^{1/2}\big),\ H_0^{1/2}x + S^*y \in \mathcal{D}\big(H_0^{1/2}\big) \right\}, \tag{3.10}$$

$$A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Vx + y \\ H_0^{1/2}(H_0^{1/2}x + S^*y) \end{pmatrix}. \tag{3.11}$$

In order to establish the relation between A and A_1, we introduce the unbounded operator W from $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$ as

$$W := \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}, \quad \mathcal{D}(W) := \mathcal{H}_{1/2} \oplus \mathcal{H}.$$

Its inverse

$$W^{-1} = \begin{pmatrix} I & 0 \\ -V & I \end{pmatrix}$$

(denoted by $\widetilde{W}$ in (1.10)) is a bounded operator from $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$ to $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ since V is a bounded operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$ by Assumption (i).

Lemma 3.3. *If Assumptions* (i) *and* (ii) *are satisfied, then*

$$A = WA_1W^{-1}. \tag{3.12}$$

Proof. Using the description of the domain of A_1 from (3.10) and the fact that for $y \in \mathcal{H}_{1/2} \subset \mathcal{D}(V)$ we have $S^* y = \overline{H_0^{-1/2} V} y = H_0^{-1/2} V y \in \mathcal{H}_{1/2}$, we find

$$\begin{aligned}
&\mathcal{D}(W A_1 W^{-1})\\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H} : \begin{pmatrix} x \\ -Vx + y \end{pmatrix} \in \mathcal{D}(A_1),\ A_1 \begin{pmatrix} x \\ -Vx + y \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H} \right\}\\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H} : H_0^{1/2} x + S^*(-Vx+y) \in \mathcal{D}\big(H_0^{1/2}\big),\ Vx - Vx + y \in \mathcal{H}_{1/2} \right\}\\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H} : H_0^{1/2} x - S^* V H_0^{-1/2} H_0^{1/2} x \in \mathcal{D}\big(H_0^{1/2}\big),\ y \in \mathcal{H}_{1/2} \right\}\\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H} : (I - S^* S) H_0^{1/2} x \in \mathcal{D}\big(H_0^{1/2}\big),\ y \in \mathcal{H}_{1/2} \right\}\\
&= \mathcal{D}(H) \oplus \mathcal{H}_{1/2} = \mathcal{D}(A).
\end{aligned}$$

That the operators A and $W A_1 W^{-1}$ coincide is seen as follows: for $x \in \mathcal{D}(H)$ and $y \in \mathcal{H}_{1/2} = \mathcal{D}\big(H_0^{1/2}\big)$ we have, observing (3.11) and (3.4),

$$\begin{aligned}
\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} A_1 \begin{pmatrix} x \\ -Vx + y \end{pmatrix} &= \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} Vx - Vx + y \\ H_0^{1/2}\big(H_0^{1/2} x + S^*(-Vx + y)\big) \end{pmatrix}\\
&= \begin{pmatrix} y \\ H_0^{1/2}\big(H_0^{1/2} x + S^*(-Vx + y)\big) + Vy \end{pmatrix}\\
&= \begin{pmatrix} y \\ Hx + H_0^{1/2} S^* y + Vy \end{pmatrix} = \begin{pmatrix} y \\ Hx + 2Vy \end{pmatrix}\\
&= A \begin{pmatrix} x \\ y \end{pmatrix},
\end{aligned}$$

where we have used that $y \in \mathcal{H}_{1/2} \subset \mathcal{D}(V)$ and $H_0^{1/2} S^* = (S H_0^{1/2})^* = V^* \supset V$. $\square$

4. Indefinite Inner Products

In this section we always suppose that Assumptions (i) and (ii) are satisfied and we consider the operator A from (3.7). Obviously, A is not symmetric with respect to the Hilbert space inner product of $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$. However, it exhibits symmetry with respect to another inner product which is, in general, indefinite. This so-called energy inner product on $\mathcal{G}$ is defined as

$$\langle \mathbf{x}, \mathbf{x}' \rangle := \big(H_0^{1/2} x, H_0^{1/2} x'\big) - (Vx, Vx') + (y, y')$$

for $\mathbf{x} = (x\ y)^{\mathrm{t}},\ \mathbf{x}' = (x'\ y')^{\mathrm{t}} \in \mathcal{G}$, which, using $S = V H_0^{-1/2}$, can also be written as

$$\langle \mathbf{x}, \mathbf{x}' \rangle = \big((I - S^* S) H_0^{1/2} x, H_0^{1/2} x'\big) + (y, y'). \tag{4.1}$$

Lemma 4.1. *Under Assumptions* (i) *and* (ii)*, the space $\mathcal{K} := (\mathcal{G}, \langle \cdot, \cdot \rangle)$ is a Krein space. If, additionally, the number of negative eigenvalues of the operator $I - S^* S$ in $\mathcal{H}$ is finite, say κ, then $\mathcal{K}$ is a Pontryagin space with negative index κ.*

Proof. Due to Assumptions (i) and (ii), the operator $I - S^*S$ is bounded and self-adjoint in $\mathcal{H}$ with $0 \in \rho(I - S^*S)$. Hence $\mathcal{H}$ equipped with the inner product $\big((I - S^*S)\,\cdot, \cdot\big)$ is a Krein space (see Sect. 2.2); if, in addition, the number of negative eigenvalues of $I - S^*S$ in $\mathcal{H}$ is finite, say κ, it is a Pontryagin space with negative index κ. Now the claim follows since $H_0^{1/2} : \mathcal{H}_{1/2} \to \mathcal{H}$ is an isomorphism. □

Remark 4.2. If $\|S\| < 1$, that is, $\kappa = 0$, then $\mathcal{K}$ is a Hilbert space.

Theorem 4.3. *Suppose that $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ and that $1 \in \rho(S^*S)$. Then A is a self-adjoint operator in the Krein space $\mathcal{K}$ with $\rho(A) \neq \emptyset$.*

Proof. For $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{D}\big(H_0^{1/2}\big)$, we obtain, using (4.1),

$$\begin{aligned}
\langle A\mathbf{x}, \mathbf{x}\rangle &= \left\langle \begin{pmatrix} y \\ Hx + 2Vy \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\
&= \Big((I - S^*S)H_0^{1/2}y,\, H_0^{1/2}x\Big) + (Hx + 2Vy, y) \\
&= (y, Hx) + (Hx, y) + 2(Vy, y),
\end{aligned}$$

which is real. Thus the operator A is symmetric in $\mathcal{K}$. In order to prove that A is self-adjoint in $\mathcal{K}$, it remains to be shown that $\rho(A)$ contains a real point μ (then $(A - \mu)^{-1}$ is bounded and symmetric in $\mathcal{K}$ and hence self-adjoint in $\mathcal{K}$). Hence, to complete the proof, it suffices to show that $0 \in \rho(A)$. For $\mathbf{f} = (f\ g)^{\mathrm{t}} \in \mathcal{H}_{1/2} \oplus \mathcal{H}$, the equation $A\mathbf{x} = \mathbf{f}$ with $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{H}_{1/2}$ is equivalent to

$$\begin{aligned}
y &= f, \\
Hx + 2Vy &= g.
\end{aligned}$$

Since $1 \in \rho(S^*S)$, H is boundedly invertible (see (3.4)) and so the second equation with $y = f$ has a unique solution $x \in \mathcal{D}(H)$, whence $A\mathbf{x} = \mathbf{f}$ has the unique solution

$$\mathbf{x} = \begin{pmatrix} H^{-1}(-2Vf + g) \\ f \end{pmatrix} \in \mathcal{D}(H) \oplus \mathcal{H}_{1/2} = \mathcal{D}(A),$$

which proves that $0 \in \rho(A)$. □

The energy inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{G}$ is related to the so-called charge inner product $[\cdot,\cdot]$ on $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ which is given by

$$[\mathbf{x}, \mathbf{x}'] := (x, y') + (y, x') = \big(G\mathbf{x}, \mathbf{x}'\big) \tag{4.2}$$

with

$$G := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Obviously, the space $\mathcal{K}_1 := (\mathcal{G}_1, [\cdot,\cdot])$ is a Krein space for which the positive and negative components in the decomposition (2.1) have the same dimension; in particular, if $\mathcal{H}$ is infinite dimensional (as it is the case for the Klein–Gordon equation), then both components are infinite dimensional.

The operator $\widehat{A}_1$ given by (3.9) is symmetric with respect to the charge inner product in $\mathcal{K}_1$ since, for $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{D}(\hat{A}_1) = \mathcal{D}(H_0) \oplus \mathcal{D}(V)$,

$$[\widehat{A}_1\mathbf{x}, \mathbf{x}] = \big(G\widehat{A}_1\mathbf{x}, \mathbf{x}\big) = (H_0 x, x) + (Vy, x) + (x, Vy) + (y, y),$$

which is real. Moreover, it has been shown in [LNT06] that, under Assumption (i), the closure A_1 of $\widehat{A}_1$ given by (3.10), (3.11) is self-adjoint in $\mathcal{K}_1$ with $\rho(A_1) \neq \emptyset$.

Proposition 4.4. *Between the indefinite inner products* $\langle\cdot,\cdot\rangle$ *of* $\mathcal{G}$ *and* $[\cdot,\cdot]$ *of* $\mathcal{G}_1$ *the following relations hold:*

i) $\langle \mathbf{x}, \mathbf{x}'\rangle = [A_1 W^{-1}\mathbf{x}, W^{-1}\mathbf{x}'],\ \ \mathbf{x} \in W\mathcal{D}(A_1),\ \mathbf{x}' \in \mathcal{G},$

ii) $\langle A\mathbf{x}, \mathbf{x}'\rangle = [A_1^2 W^{-1}\mathbf{x}, W^{-1}\mathbf{x}'],\ \ \mathbf{x} \in \mathcal{D}(A),\ A\mathbf{x} \in W\mathcal{D}(A_1),\ \mathbf{x}' \in \mathcal{G}.$

Proof. i) Let $\mathbf{x} = (x\ y)^{\mathrm{t}} \in W\mathcal{D}(A_1)$, $\mathbf{x}' = (x'\ y')^{\mathrm{t}} \in \mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$, and set

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} := W^{-1}\mathbf{x} = \begin{pmatrix} x \\ -Vx + y \end{pmatrix} \in \mathcal{D}(A_1).$$

Then we have

$$\mathbf{x} = W\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ Vu + v \end{pmatrix},$$

and the left-hand side of i) becomes

$$\begin{aligned}\langle \mathbf{x}, \mathbf{x}'\rangle &= (H_0^{1/2}x, H_0^{1/2}x') - (Vx, Vx') + (y, y') \\ &= (H_0^{1/2}u, H_0^{1/2}x') - (Vu, Vx') + (Vu + v, y').\end{aligned}$$

Using (3.11) and the fact that $x' \in \mathcal{H}_{1/2}$, we can rewrite the right hand side of i) as

$$\begin{aligned}\Big[A_1 W^{-1}\mathbf{x}, W^{-1}\mathbf{x}'\Big] &= \left[A_1\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x' \\ -Vx' + y' \end{pmatrix}\right] = \left[\begin{pmatrix} Vu + v \\ H_0(u + T^*v) \end{pmatrix}, \begin{pmatrix} x' \\ -Vx' + y' \end{pmatrix}\right] \\ &= \big(Vu + v, -Vx' + y'\big) + \big(H_0^{1/2}(u + T^*v), H_0^{1/2}x'\big) \\ &= (Vu + v, -Vx' + y') + \big(H_0^{1/2}u, H_0^{1/2}x'\big) + \big(H_0^{1/2}T^*v, H_0^{1/2}x'\big).\end{aligned}$$

Since $T = VH_0^{-1}$, the last summand equals $(v, TH_0x') = (v, Vx')$ and i) follows.

ii) Let $\mathbf{x} \in \mathcal{D}(A)$ be such that $A\mathbf{x} \in W\mathcal{D}(A_1)$ and hence $W^{-1}A\mathbf{x} \in \mathcal{D}(A_1)$. Since, by the operator equality (3.12), we have $W^{-1}Ax = A_1W^{-1}x$, it follows that $A_1W^{-1}\mathbf{x} \in \mathcal{D}(A_1)$ and further, by i),

$$\langle A\mathbf{x}, \mathbf{x}'\rangle = [A_1W^{-1}A\mathbf{x}, W^{-1}\mathbf{x}'] = [A_1^2W^{-1}\mathbf{x}, W^{-1}\mathbf{x}']$$

for arbitrary $\mathbf{x}' \in \mathcal{G}$. $\square$

Lemma 4.5. *Let* $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$. *Then the set* $W\mathcal{D}(A_1)$ *is dense in* $\mathcal{G}$.

Proof. By (3.10), we have

$$W\mathcal{D}(A_1) = \left\{ \begin{pmatrix} x \\ Vx + y \end{pmatrix} : x \in \mathcal{D}(\overline{V}),\ x + T^* y \in \mathcal{D}(H_0) \right\}.$$

Hence if $(x_0\ y_0)^{\mathrm{t}} \in \mathcal{G}$ is orthogonal to $W\mathcal{D}(A_1)$ with respect to the Hilbert space inner product in $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$, then

$$\left(\begin{pmatrix} x \\ Vx + y \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)_{\mathcal{G}} = (H_0^{1/2} x, H_0^{1/2} x_0) + (Vx + y, y_0) = 0 \tag{4.3}$$

for all $(x\ y)^{\mathrm{t}} \in W\mathcal{D}(A_1)$. If we choose $x = 0$ and $y \in \mathcal{D}(V)$, then $T^* y = \overline{H_0^{-1} V} y = H_0^{-1} V y \in \mathcal{D}(H_0)$ and hence $(0\ y)^{\mathrm{t}} \in W\mathcal{D}(A_1)$. Now (4.3) shows that y_0 is orthogonal in $\mathcal{H}$ to the dense subset $\mathcal{D}(V)$ and thus $y_0 = 0$. If we choose $x \in \mathcal{D}(H_0)$ and $y = 0$, then $(x\ 0)^{\mathrm{t}} \in W\mathcal{D}(A_1)$ because $\mathcal{D}(H_0) \subset \mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ by assumption. Since H_0 is bijective, (4.3) implies that x_0 is orthogonal to $\mathcal{H}$ and hence $x_0 = 0$. □

Remark 4.6. If the operator $I - S^*S$ has only finitely many, say κ, negative eigenvalues, the Pontryagin space $\mathcal{K}$ and the operator A can also be introduced by means of the operator A_1 in the space $\mathcal{G}_1$ as follows. By Proposition 4.4 i), the indefinite inner product $\langle \cdot, \cdot \rangle$ is defined on the dense subset $W\mathcal{D}(A_1)$ of $\mathcal{G}$. Since $I - S^*S$ has only κ negative eigenvalues, the form $[A_1 \cdot, \cdot]$ on $\mathcal{D}(A_1) \subset \mathcal{G}_1$, and hence $\langle \cdot, \cdot \rangle$ on $W\mathcal{D}(A_1)$, has κ negative squares (see [LNT06]). Therefore $\mathcal{K}$ is the Pontryagin space completion of $W\mathcal{D}(A_1) \subset \mathcal{G}$ with respect to the inner product $\langle \cdot, \cdot \rangle$; the operator A can now be defined by the relation in Proposition 4.4 ii).

5. Spectral Properties of the Operator A

In this section we exploit the self-adjointness of the operator A with respect to the indefinite inner product $\langle \cdot, \cdot \rangle$. We show that A possesses a spectral function with at most finitely many critical points, we investigate the structure of the spectrum of A and consider the solvability of an abstract Cauchy problem for A (and hence for the Klein–Gordon equation).

In order to guarantee that $\mathcal{K}$ is a Pontryagin space, in addition to the Assumptions (i) and (ii), we suppose that

Ass. (iii) $S = V H_0^{-1/2} = S_0 + S_1$ with $\|S_0\| < 1$ and a compact operator S_1.

To study the spectrum and essential spectrum of A under Assumption (iii), the following lemma for the particular case $\|S\| < 1$ is useful.

Lemma 5.1. *Suppose that $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ and $1 \in \rho(S^*S)$. Define the quadratic pencil L of bounded operators in $\mathcal{H}$ by*

$$L(\lambda) := I - S^*S + \lambda\big(S^* H_0^{-1/2} + H_0^{-1/2} S\big) - \lambda^2 H_0^{-1}, \quad \lambda \in \mathbb{C}. \tag{5.1}$$

If $\|S\| < 1$, then $\rho(A) = \rho(L)$ and, with $\alpha := (1 - \|S\|)m$,

$$(-\alpha, \alpha) \subset \rho(A).$$

Proof. By Lemma 3.2, the operator A has a bounded inverse A^{-1}. By the spectral mapping theorem (see [EE87, Thm. IX.2.3]), we have

$$\lambda \in \rho(A) \iff \mu := \lambda^{-1} \in \rho\big(A^{-1}\big).$$

If $\|S\| < 1$, then the operator $\Gamma := I - S^*S$ is uniformly positive. Hence, by (3.4), we can write $H = H_0^{1/2}\Gamma H_0^{1/2}$, $H^{-1} = H_0^{-1/2}\Gamma^{-1/2}\Gamma^{-1/2}H_0^{-1/2}$, and thus we can factorize the inverse A^{-1} given by (3.8) as

$$\begin{aligned} A^{-1} &= \begin{pmatrix} -2H^{-1}V & H^{-1} \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} H_0^{-1/2}\Gamma^{-1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -2\,\Gamma^{-1/2}H_0^{-1/2}V & \Gamma^{-1/2}H_0^{-1/2} \\ I & 0 \end{pmatrix}; \end{aligned}$$

here the right factor is an operator from $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$ and the left factor is an operator from $\mathcal{H} \oplus \mathcal{H}$ back to $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$. If we exchange the order of the factors and define the auxiliary operator B in $\mathcal{H} \oplus \mathcal{H}$ by

$$\begin{aligned} B :&= \begin{pmatrix} -2\,\Gamma^{-1/2}H_0^{-1/2}V & \Gamma^{-1/2}H_0^{-1/2} \\ I & 0 \end{pmatrix} \begin{pmatrix} H_0^{-1/2}\Gamma^{-1/2} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} -\Gamma^{-1/2}S^*H_0^{-1/2}\Gamma^{-1/2} - \Gamma^{-1/2}H_0^{-1/2}S\,\Gamma^{-1/2} & \Gamma^{-1/2}H_0^{-1/2} \\ H_0^{-1/2}\Gamma^{-1/2} & 0 \end{pmatrix}, \end{aligned}$$

then $\rho\big(A^{-1}\big) \setminus \{0\} = \rho(B) \setminus \{0\}$; here we have used that $H_0^{-1/2}VH_0^{-1/2} = S^*H_0^{-1/2} = H_0^{-1/2}S$. For $\mu \in \mathbb{C}$, $\mu \neq 0$, we have $\mu \in \rho(B)$ if and only if for every $(f\ g)^{\mathrm{t}} \in \mathcal{H} \oplus \mathcal{H}$ there exists a unique $(x\ y)^{\mathrm{t}} \in \mathcal{H} \oplus \mathcal{H}$ such that

$$\begin{aligned} \big(-\Gamma^{-1/2}S^*H_0^{-1/2}\Gamma^{-1/2} - \Gamma^{-1/2}H_0^{-1/2}S\,\Gamma^{-1/2} - \mu\big)x + \Gamma^{-1/2}H_0^{-1/2}y &= f, \\ H_0^{-1/2}\Gamma^{-1/2}x - \mu\, y &= g. \end{aligned}$$

If we divide both equations by μ ($\neq 0$) and insert the second into the first, we see that this is equivalent to

$$\begin{aligned} -\Gamma^{-1/2}\Big(\frac{1}{\mu}\big(S^*H_0^{-1/2} + H_0^{-1/2}S\big) + \Gamma - \frac{1}{\mu^2}H_0^{-1}\Big)\Gamma^{-1/2}x &= \frac{1}{\mu}f + \frac{1}{\mu^2}\Gamma^{-1/2}H_0^{-1/2}g, \\ y &= \frac{1}{\mu}\big(H_0^{-1/2}\Gamma^{-1/2}x - g\big). \end{aligned}$$

Since $\Gamma^{-1/2}$ is bounded and boundedly invertible, the latter is equivalent to the fact that μ belongs to the resolvent set of the operator pencil given by

$$\frac{1}{\mu}\big(S^*H_0^{-1/2} + H_0^{-1/2}S\big) + \Gamma - \frac{1}{\mu^2}H_0^{-1}$$

or, equivalently, $\lambda = \mu^{-1} \in \rho(L)$ with L given by (5.1). This completes the proof of $\rho(A) = \rho(L)$. Finally, for $\lambda \in \mathbb{R}$, $|\lambda| < (1 - \|S\|)m$, the estimate

$$\begin{aligned} &\|S^*S - \lambda(S^*H_0^{-1/2} + H_0^{-1/2}S) + \lambda^2 H_0^{-1}\| \\ &\quad < \|S\|^2 + 2\big(1 - \|S\|\big)m\frac{1}{m}\|S\| + \big(1 - \|S\|\big)^2 m^2 \frac{1}{m^2} = 1 \end{aligned}$$

and (5.1) show that $\lambda \in \rho(L) = \rho(A)$. □

Theorem 5.2. *Suppose that* $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$*, that* $S = VH_0^{-1/2} = S_0 + S_1$ *with* $\|S_0\| < 1$ *and a compact operator* S_1*, and that* $1 \in \rho(S^*S)$*. Then we have:*

i) $\mathcal{K}$ *is a Pontryagin space with finite negative index* κ*, where* κ *is the number of negative eigenvalues of the operator* $I - S^*S$*.*

ii) *The self-adjoint operator* A *in* $\mathcal{K}$ *has a spectral function with at most finitely many critical points.*

iii) *The non-real spectrum of* A *is symmetric with respect to the real axis and consists of at most* κ *pairs of eigenvalues* λ, $\overline{\lambda}$ *of finite type; the algebraic eigenspaces corresponding to* λ *and* $\overline{\lambda}$ *are isomorphic.*

iv) *The linear span of all the algebraic eigenspaces corresponding to the eigenvalues of* A *in the upper* (*or lower*) *half plane is a neutral subspace of* $\mathcal{K}$ *and*

$$\kappa = \sum_{\lambda\in\sigma_0(A)\cap\mathbb{R}} \kappa_\lambda^-(A) \ + \sum_{\lambda\in\sigma(A)\cap\mathbb{C}^+} \dim \mathcal{L}_\lambda(A);$$

here $\sigma_0(A)$ *denotes the set of all eigenvalues of* A *with non-positive eigenvector.*

v) *The points of* $\sigma(A)\setminus\sigma_0(A)$ (*which are all real*) *are spectral points of positive type.*

vi) *The essential spectrum* $\sigma_{\mathrm{ess}}(A)$ *is real and*

$$\sigma_{\mathrm{ess}}(A) \cap (-\alpha, \alpha) = \emptyset,$$

where $\alpha := (1 - \|S_0\|)\, m$*.*

vii) *The operator* A *generates a strongly continuous group* $\big(\exp(\mathrm{i}At)\big)_{t\in\mathbb{R}}$ *of unitary operators in* $\mathcal{K}$ *and hence the Cauchy problem*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

has the unique solution $\mathbf{x}(t) = \exp\big(\mathrm{i}At\big)\mathbf{x}_0,\ t \in \mathbb{R}$*, for all initial values* $\mathbf{x}_0 \in \mathcal{K}$*.*

Proof. i) Assumption (iii) on S implies that

$$I - S^*S = I - S_0^*S_0 + K,$$

where $I - S_0^*S_0$ is uniformly positive and K is a compact operator in $\mathcal{H}$. Thus $I - S^*S$ has only a finite number κ of negative eigenvalues and hence $\mathcal{K}$ is a Pontryagin space of finite negative index κ by Lemma 4.1.

ii), iii), iv), v), and vii) are immediate consequences of Theorem 4.3 and of i) by [Lan82] (see also Sects. 2.4 and 2.5).

vi) We define an operator A_0 in $\mathcal{G}$ by

$$A_0 := \begin{pmatrix} 0 & I \\ H_0^{1/2}(I - S_0^*S_0)H_0^{1/2} & 2V \end{pmatrix}. \tag{5.2}$$

By the spectral mapping theorem (see [EE87, Thm. IX.2.3]), we have

$$\begin{aligned} \lambda \in \sigma_{\mathrm{ess}}(A) \ &\Longleftrightarrow\ \mu := \lambda^{-1} \in \sigma_{\mathrm{ess}}\big(A^{-1}\big), \\ \lambda \in \sigma_{\mathrm{ess}}(A_0) &\Longleftrightarrow\ \mu := \lambda^{-1} \in \sigma_{\mathrm{ess}}\big(A_0^{-1}\big). \end{aligned} \tag{5.3}$$

The difference $A^{-1} - A_0^{-1}$ is compact since S_1 is compact by assumption; in fact,

$$A^{-1} - A_0^{-1} = A^{-1}\big(A_0 - A\big)A_0^{-1} = A^{-1}\begin{pmatrix} 0 & 0 \\ H_0^{1/2}\,\big(S_1^* S_0 + S_0^* S_1\big)\,H_0^{1/2} & 0 \end{pmatrix} A_0^{-1},$$

which is compact since $S_1^* S_0 + S_0^* S_1$ is compact and

$$A^{-1}\begin{pmatrix} 0 & 0 \\ 0 & H_0^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & H^{-1}H_0^{1/2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_0^{-1/2}(I - S^*S) \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} H_0^{1/2} & 0 \\ 0 & 0 \end{pmatrix} A_0^{-1} = \begin{pmatrix} -2(I - S_0^* S_0)^{-1} H_0^{-1/2} V & (I - S_0^* S_0)^{-1} H_0^{-1/2} \\ 0 & 0 \end{pmatrix}$$

are bounded. By iii), $\sigma(A)$ has empty interior as a subset of $\mathbb{C}$. In addition, part iii) applied to A and A_0 shows that each of the at most two components of $\mathbb{C} \setminus \sigma(A)$ contains a point in $\rho(A_0)$. Hence, by [RS78, Lemma XIII.4], $\sigma_{\mathrm{ess}}\big(A^{-1}\big) = \sigma_{\mathrm{ess}}\big(A_0^{-1}\big)$ and thus, by (5.3),

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_0). \tag{5.4}$$

Now Lemma 5.1 applied to A_0 shows that $(-\alpha, \alpha) \subset \rho(A_0)$ and, consequently, $\sigma_{\mathrm{ess}}(A) \cap (-\alpha, \alpha) = \sigma_{\mathrm{ess}}(A_0) \cap (-\alpha, \alpha) = \emptyset$. □

The special cases that S is compact or that $\|S\| < 1$ in Theorem 5.2 have been considered before (see, e.g., [LN96] and [Naj79], respectively):

Remark 5.3. Suppose that $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ and $1 \in \rho(S^*S)$.

i) If V is $H_0^{1/2}$-compact, then

$$\sigma_{\mathrm{ess}}(A) = \big\{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{\mathrm{ess}}(H_0)\big\}.$$

ii) If $\big\|V H_0^{-1/2}\big\| < 1$, then $\kappa = 0$, $\mathcal{K}$ is a Hilbert space, A is self-adjoint in this Hilbert space, and

$$\sigma(A) \cap (-\alpha, \alpha) = \emptyset$$

with $\alpha = \big(1 - \big\|V H_0^{-1/2}\big\|\big)m$.

Proof. i) If V is $H_0^{1/2}$-compact, we can choose $S_0 = 0$ and $S_1 = V H_0^{-1/2}$ in Assumption (iii). Then (5.4), (5.3), and (5.2) show that

$$\lambda \in \sigma_{\mathrm{ess}}(A) \iff \lambda^{-1} \in \sigma_{\mathrm{ess}}\big(A_0^{-1}\big) = \sigma_{\mathrm{ess}}\left(\begin{pmatrix} -2H_0^{-1}V & H_0^{-1} \\ I & 0 \end{pmatrix}\right).$$

If we define

$$D := \begin{pmatrix} 0 & H_0^{-1} \\ I & 0 \end{pmatrix},$$

then

$$A_0^{-1} - D = \begin{pmatrix} -2H_0^{-1}V & H_0^{-1} \\ I & 0 \end{pmatrix} - \begin{pmatrix} 0 & H_0^{-1} \\ I & 0 \end{pmatrix} = \begin{pmatrix} -2H_0^{-1}V & 0 \\ 0 & 0 \end{pmatrix}$$

is compact by assumption. Moreover, by Theorem 5.2 iii) and v), $\sigma(A_0^{-1})$ has empty interior as a subset of $\mathbb{C}$ and $\mathbb{C} \setminus \sigma(A_0^{-1})$ consists of only one component containing points in $\rho(D)$ (e.g., all non-real points of $\mathbb{C} \setminus \sigma(A_0^{-1})$). Hence [RS78, Lemma XIII.4] shows that $\sigma_{\text{ess}}(A_0^{-1}) = \sigma_{\text{ess}}(D)$. Now the fact that

$$\lambda^{-1} \in \sigma_{\text{ess}}(D) \iff \lambda^{-2} \in \sigma_{\text{ess}}\big(H_0^{-1}\big),$$

which is not difficult to check (see, e.g., [HM01]), completes the proof.

ii) is immediate from Theorem 5.2 v) if we choose $S_0 = VH_0^{-1/2}$ and $S_1 = 0$ in Assumption (iii). □

Remark 5.4. If, under Assumption (iii), Assumption (ii) is not satisfied, that is, $1 \in \sigma_{\text{p}}(S^*S)$, then 0 is an isolated eigenvalue of finite multiplicity of the self-adjoint operator H and, with $\mathcal{N}_0 := \ker H = \ker(I - S^*S)H_0^{1/2}$, the subspace $\mathcal{N}_0 \oplus \{0\}$ is the isotropic subspace of the inner product space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$. Then the factor space $\widetilde{\mathcal{K}} = \mathcal{K}/\mathcal{N}_0$ is a Pontryagin space with negative index again given by the number of negative eigenvalues of $I - S^*S$. Since $\ker A = \mathcal{N}_0 \oplus \{0\}$, the operator A induces a self-adjoint operator $\widetilde{A}$ in this Pontryagin space $\widetilde{\mathcal{K}}$. Then all claims of Theorem 5.2 remain true for $\widetilde{A}$.

6. Assumptions for the Klein–Gordon Equation in $\mathbb{R}^n$

In this section we consider the example of the Klein–Gordon equation in $\mathbb{R}^n$ for which $\mathcal{H} = L_2(\mathbb{R}^n)$ with norm $\|\cdot\|_2$ and scalar product $(\cdot,\cdot)_2$, $H_0 = -\Delta + m^2$, and V stands for the operator of multiplication by a function $V : \mathbb{R}^n \to \mathbb{R}$. In this case, sufficient conditions on the potential V will be established that guarantee the assumptions

Ass. (i) $\mathcal{D}\big(H_0^{1/2}\big) \subset \mathcal{D}(V)$ or, equivalently, $V : \mathcal{H}_{1/2} \to \mathcal{H}$ is bounded,

Ass. (iii) $S = VH_0^{-1/2} = S_0 + S_1$ with $\|S_0\| < 1$ and a compact operator S_1,

which were used in the previous sections. Obviously, (iii) is stronger than (i). Note that, according to Remark 5.4, Assumption (ii), that is, $1 \in \rho(S^*S)$, is not an essential restriction and thus will not be considered here.

It is well-known (see [Tri92, Sects. 1.3.1, 1.3.2]) that for $H_0 = -\Delta + m^2$ the space $\mathcal{H}_{1/2} = \mathcal{D}\big(H_0^{1/2}\big)$ is the Sobolev space of order 1 associated with $L_2(\mathbb{R}^n)$:

$$\mathcal{H}_{1/2} = W_2^1(\mathbb{R}^n).$$

Hence Assumption (i) holds if and only if $W_2^1(\mathbb{R}^n) \subset \mathcal{D}(V)$ or, equivalently, if there exist constants $a, b \geq 0$ such that

$$\|Vu\|_2 \leq a\|u\|_2 + b\|(-\Delta + m^2)^{1/2}u\|_2, \quad u \in W_2^1(\mathbb{R}^n); \tag{6.1}$$

this is equivalent to the $(-\Delta + m^2)$-form boundedness of V^2, that is,

$$(V^2u, u)_2 \leq a(u,u)_2 + b((-\Delta + m^2)u, u)_2, \quad u \in W_2^1(\mathbb{R}^n).$$

Assumption (iii) holds if $V = V_0 + V_1$, where $W_2^1(\mathbb{R}^n) \subset \mathcal{D}(V_i)$ for $i = 0, 1$, V_0 satisfies (6.1) with $a,\ b \geq 0$ such that

$$\frac{a}{m} + b < 1, \tag{6.2}$$

that is, $S_0 = V_0(-\Delta + m^2)^{-1/2}$ is a strict contraction, and $S_1 = V_1(-\Delta + m^2)^{-1/2}$ is compact.

Many different sufficient conditions for the relative boundedness as well as for the relative compactness of a multiplication operator with respect to $(-\Delta + m^2)^{1/2}$ have been established (see, e.g., [Kat82, RS75, Sim71] and the more specialized references therein). In the following we formulate two well-known sufficient conditions in terms of L_p-spaces and Rollnik classes. We start with a well-known relative compactness result, which, for $p = 3$, goes back to Brezis and Kato (see [BK79]).

Theorem 6.1. *If $n \geq 3$ and $V \in L_p(\mathbb{R}^n)$ with $n \leq p < \infty$, then V is $(-\Delta + m^2)^{1/2}$-compact.*[2]

Proof. The operator of multiplication with V in $L_2(\mathbb{R}^n)$ is $(-\Delta+m^2)^{1/2}$-compact if and only if $V(-\Delta + m^2)^{-1/2}$ is compact, that is, if (u_m) is a sequence in the form domain $W_2^1(\mathbb{R}^n)$ of $-\Delta + m^2$ that converges weakly to 0, then (Vu_m) converges strongly to 0 in $L_2(\mathbb{R}^n)$. Now let $n \leq p < \infty$ and $W \in L_p(\mathbb{R}^n)$, and set $q := p/(p-2)$. By Hölder's inequality and the boundedness of the embedding of the Sobolev space $W_2^1(\mathbb{R}^n)$ into $L_{2q}(\mathbb{R}^n)$ (which holds since $p \geq n$, see [EE87, Theorem V.3.7]), we have

$$\|Wu\|_2^2 \leq \|W\|_p^2 \|u\|_{2q}^2 \leq c^2 \|W\|_p^2 \|u\|_{2,1}^2, \quad u \in W_2^1(\mathbb{R}^n); \tag{6.3}$$

here c is the norm of the embedding of $W_2^1(\mathbb{R}^n)$ into $L_{2q}(\mathbb{R}^n)$. Assume now that $(u_m) \subset W_2^1(\mathbb{R}^n)$ converges weakly to 0 and let $\varepsilon > 0$. Since $C_0^\infty(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$ is dense, there exists a function $V_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\|V - V_\varepsilon\|_p < \varepsilon$. Let $\Omega_\varepsilon := \operatorname{supp} V_\varepsilon$ and choose $C_\varepsilon \geq 0$ such that $|V_\varepsilon| \leq C_\varepsilon$. Then we obtain, using (6.3) with $W = V - V_\varepsilon$,

$$\|Vu_m\|_2 \leq \|(V - V_\varepsilon)u_m\|_2 + \|V_\varepsilon u_m\|_2 \leq c\,\varepsilon \|u_m\|_{2,1} + C_\varepsilon \|u_m|_{\Omega_\varepsilon}\|_2. \tag{6.4}$$

Since (u_m) converges weakly in $W_2^1(\mathbb{R}^n)$, it is a bounded sequence in $W_2^1(\mathbb{R}^n)$. Hence, choosing ε sufficiently small, the first term can be made arbitrarily small. The second term becomes arbitrarily small for sufficiently large m: in fact, $W_2^1(\Omega_\varepsilon)$ is compactly embedded in $L_2(\Omega_\varepsilon)$ since Ω_ε is bounded (see [EE87, Theorem V.3.7]) and thus $(u_m|_{\Omega_\varepsilon})$ converges to 0 strongly in $L_2(\Omega_\varepsilon)$. □

For $n = 3$, a criterion for the relative form-boundedness of V^2 with respect to $-\Delta+m^2$ can be formulated in terms of Rollnik potentials, see [RS75], [Sim71]: A measurable function $W : \mathbb{R}^3 \to \mathbb{R}$ is said to belong to the class R of *Rollnik potentials* if

$$\|W\|_R^2 := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|W(x)||W(y)|}{|x-y|^2} \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Theorem 6.2. *If $n = 3$ and $V : \mathbb{R}^3 \to \mathbb{R}$ is a measurable function such that $V^2 \in R + L_\infty(\mathbb{R}^3)$, then V is $(-\Delta + m^2)^{1/2}$-bounded (with relative bound 0). In particular, if $V^2 \in R$, we have*

$$\left\| V(-\Delta + m^2)^{-1/2} \right\| \leq \sqrt{\frac{\|V^2\|_R}{4\pi}}.$$

[2] We thank W.D. Evans for communicating this result to us.

Proof. The first statement may be found in [RS75, Theorem X.19], for the proof see [Sim71, Theorem I.21]. For the second claim, we note that, by [Sim71, (I.13)],

$$\begin{aligned}\big(|V|(-\Delta+m^2)^{-1}|V|u,u\big)_2 &\le \frac{1}{4\pi}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V^2(x)|\,\mathrm{e}^{-m|x-y|}|V^2(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\|u\|_2^2\\ &\le \frac{1}{4\pi}\|V^2\|_R\,\|u\|_2^2,\quad u\in\mathcal{D}(|V|)=\mathcal{D}(V),\end{aligned}$$

which implies that $\|(-\Delta+m^2)^{-1/2}|V|\,\| \le \sqrt{\|V^2\|_R/(4\pi)}$. Hence the densely defined operator $(-\Delta+m^2)^{-1/2}|V|$ is bounded and

$$\begin{aligned}\big\|V(-\Delta+m^2)^{-1/2}\big\| &= \big\|\,|V|(-\Delta+m^2)^{-1/2}\big\| = \big\|(|V|(-\Delta+m^2)^{-1/2})^*\big\|\\ &= \big\|\overline{(-\Delta+m^2)^{-1/2}|V|}\,\big\| \le \sqrt{\frac{\|V^2\|_R}{4\pi}},\end{aligned}$$

follows. □

Remark 6.3. For $n=3$, Theorem 6.1 shows that every $V\in L_p(\mathbb{R}^3)+L_\infty(\mathbb{R}^3)$ with $3\le p<\infty$ is $(-\Delta+m^2)^{1/2}$-bounded (with relative bound 0). This condition is more restrictive than the condition $V^2\in R+L_\infty(\mathbb{R}^3)$ in Theorem 6.2. Indeed, $V\in L_p(\mathbb{R}^3)+L_\infty(\mathbb{R}^3)$ with $p\ge 3$ implies that $V^2\in L_{p/2}(\mathbb{R}^3)+L_p(\mathbb{R}^3)+L_\infty(\mathbb{R}^3)\subset R+L_\infty(\mathbb{R}^3)$ since $L_q(\mathbb{R}^3)+L_\infty(\mathbb{R}^3)\subset R+L_\infty(\mathbb{R}^3)$ for $q\ge 3/2$ (see [Sim71, Corollary I.2]).

The Coulomb potential $V(x)=\gamma/|x|$, $x\in\mathbb{R}^n\setminus\{0\}$, does not have relative bound 0 with respect to $(-\Delta+m^2)^{1/2}$; therefore neither Theorem 6.1 nor Theorem 6.2 apply to it. In this case, however, Assumption (i) is an immediate consequence of the Hardy inequality.

Proposition 6.4. *The Coulomb potential* $V(x)=\gamma/|x|$, $x\in\mathbb{R}^n\setminus\{0\}$, *with* $\gamma\in\mathbb{R}$ *satisfies Assumption* (i) *for* $n\ge 3$*; in fact,*

$$\|V(-\Delta+m^2)^{-1/2}\| \le \frac{2|\gamma|}{n-2}.$$

Proof. The classical Hardy inequality (see [HLP88, Theorem 330]) shows that, for $u\in W_2^1(\mathbb{R}^n)$,

$$\|Vu\|_2^2 \le \frac{4\gamma^2}{(n-2)^2}\|\nabla u\|_2^2 \le \frac{4\gamma^2}{(n-2)^2}\|(-\Delta+m^2)^{1/2}u\|_2^2,$$

which yields the desired estimate. □

As a consequence of Theorems 6.1, 6.2, and Proposition 6.4, we obtain:

Example 6.5. Let $n\ge 3$. Assumption (iii) (and hence (i)) is satisfied if

$$V=V_0+V_1,$$

where $V_1\in L_p(\mathbb{R}^n)$ with $n\le p<\infty$, and for V_0 one of the following holds:

i) $V_0\in L_\infty(\mathbb{R}^n)$ with $\|V_0\|_\infty<m$,
ii) $V_0(x)=\gamma/|x|$, $x\in\mathbb{R}^n\setminus\{0\}$, with $\gamma\in\mathbb{R}$ such that $|\gamma|<(n-2)/2$,

and, in the particular case $n = 3$,

iii) $V_0^2 \in R$ with $\|V_0^2\|_R < 4\pi$.

Note that the admission of the relatively compact part V_1 of V, which is not subject to any relative norm bound, gives rise to complex eigenvalues. This was avoided in earlier papers by assuming that $V_1 = 0$ (see, e.g., [SSW40] for case i) and [Ves83] for case ii)).

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