# Spectral Theory of the Klein-Gordon Equation in Pontryagin Spaces 

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#### Abstract

In this paper we investigate an abstract Klein-Gordon equation by means of indefinite inner product methods. We show that, under certain assumptions on the potential which are more general than in previous works, the corresponding linear operator $A$ is self-adjoint in the Pontryagin space $\mathcal{K}$ induced by the so-called energy inner product. The operator $A$ possesses a spectral function with critical points, the essential spectrum of $A$ is real with a gap around 0 , and the non-real spectrum consists of at most finitely many pairs of complex conjugate eigenvalues of finite algebraic multiplicity; the number of these pairs is related to the 'size' of the potential. Moreover, A generates a group of bounded unitary operators in the Pontryagin space $\mathcal{K}$. Finally, the conditions on the potential required in the paper are illustrated for the Klein-Gordon equation in $\mathbb{R}^{n}$; they include potentials consisting of a Coulomb part and an $L_{p}$-part with $n \leq p<\infty$.


## 1. Introduction

The motion of a relativistic spinless particle of mass $m$ and charge $e$ in an electrostatic field with potential $q$ is described by the Klein-Gordon equation

$$
\begin{equation*}
\left(\left(\frac{\partial}{\partial t}-\mathrm{i} e q\right)^{2}-\Delta+m^{2}\right) \psi=0 \tag{1.1}
\end{equation*}
$$

where the velocity of light has been normalized to 1 ; here $\psi$ is a complex-valued function of $t \in \mathbb{R}$ and of $x \in \mathbb{R}^{n}$. An abstract model for this equation is obtained if we replace the strictly positive self-adjoint operator generated by the differential expression $-\Delta+m^{2}$ in the function space $L_{2}\left(\mathbb{R}^{n}\right)$ by a strictly positive self-adjoint operator $H_{0}$ in a Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \cdot)$ and the operator of multiplication by the function eq in $L_{2}\left(\mathbb{R}^{n}\right)$ by a symmetric operator $V$ in $\mathcal{H}$ :

$$
\begin{equation*}
\left(\left(\frac{\mathrm{d}}{\mathrm{~d} t}-\mathrm{i} V\right)^{2}+H_{0}\right) u=0 \tag{1.2}
\end{equation*}
$$

[^0]here $u$ is a function of $t$ with values in $\mathcal{H}$. The abstract Klein-Gordon equation (1.2) can be transformed into a first order differential equation for a vector function $\mathbf{x}$ with two components in an appropriate product Hilbert space $\mathcal{G}$ and a linear operator $A$ in $\mathcal{G}$ :
\[

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}=\mathrm{i} A \mathbf{x} \tag{1.3}
\end{equation*}
$$

\]

This can be achieved by different substitutions leading to different operators $A$; however, in general this is not possible with a self-adjoint operator $A$ in a Hilbert space $\mathcal{G}$.

The operator considered in the present paper arises from the abstract Klein-Gordon equation (1.2) by means of the substitution

$$
\begin{equation*}
x=u, \quad y=-\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} t} u \tag{1.4}
\end{equation*}
$$

which leads to a first order differential equation for $\mathbf{x}=\left(\begin{array}{ll}x y\end{array}\right)^{\mathrm{t}}$ of the form

$$
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}=\mathrm{i} \widehat{A} \mathbf{x}, \quad \widehat{A}=\left(\begin{array}{cc}
0 & I  \tag{1.5}\\
H_{0}-V^{2} & 2 V
\end{array}\right) .
$$

Since both operators $H_{0}$ and $V$ are in general unbounded, the block operator matrix $\widehat{A}$ in (1.5) may not even be densely defined nor closed. To this end, suitable assumptions have to be imposed on the potential $V$ so that we can associate a closed operator $A$ with the block operator matrix $\widehat{A}$. If the potential $V$ is not small, $\widehat{A}$ does not exhibit symmetry in any Hilbert space. However, formally, if we introduce the so-called energy inner product $\langle\cdot, \cdot\rangle$ which, for suitable elements $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}}, \mathbf{x}^{\prime}=\left(x^{\prime} y^{\prime}\right)^{\mathrm{t}}$ of $\mathcal{H} \oplus \mathcal{H}$, is given by

$$
\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left(\left(\begin{array}{cc}
H_{0}-V^{2} & 0  \tag{1.6}\\
0 & I
\end{array}\right) \mathbf{x}, \mathbf{x}^{\prime}\right)=\left(\left(H_{0}-V^{2}\right) x, x^{\prime}\right)+\left(y, y^{\prime}\right)
$$

then it is not difficult to see that $\widehat{A}$ is symmetric with respect to $\langle\cdot, \cdot\rangle$ :

$$
\left.\left\langle\widehat{A} \mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left(\begin{array}{cc}
0 & H_{0}-V^{2} \\
H_{0}-V^{2} & 2 V
\end{array}\right) \mathbf{x}, \mathbf{x}^{\prime}\right) .
$$

The inner product $\langle\cdot, \cdot\rangle$ is in general indefinite; under our assumptions on the potential $V$, it is negative definite on a subspace of finite dimension so that the space $\mathcal{G}$ equipped with $\langle\cdot, \cdot\rangle$ becomes a so-called Pontryagin space.

For the Klein-Gordon equation in $\mathbb{R}^{n}$, the operator $\widehat{A}$ in the energy inner product $\langle\cdot, \cdot\rangle$ has been studied in a number of papers, see, e.g., [SSW40, Lun73a, Lun73b, Eck76, Eck80, Sch76, Kak76, Wed77, Wed78, Jon79, Naj80a, Naj80b, Naj83, Bac04], and the unpublished manuscript [LN96] ${ }^{1}$; some of these works also consider the corresponding abstract operator $\widehat{A}$ in a Pontryagin space, but under more restrictive assumptions on the potential $V$.

The operator $A$ and the energy inner product $\langle\cdot, \cdot\rangle$ studied in this paper are related to other operators associated with the abstract Klein-Gordon equation (see [LNT06]). They arise from the second order differential equation (1.2) by means of the substitution

$$
\begin{equation*}
x=u, \quad y=\left(-\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} t}-V\right) u \tag{1.7}
\end{equation*}
$$

[^1]which leads to a first order differential equation (1.3) for $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}}$ of the form

$$
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}=\mathrm{i}\left(\begin{array}{cc}
V & I  \tag{1.8}\\
H_{0} & V
\end{array}\right) \mathbf{x}
$$

The operator $A_{1}$, for example, is obtained from (1.8) as the closure of the block operator matrix therein in the Hilbert space $\mathcal{G}_{1}=\mathcal{H} \oplus \mathcal{H}$; it turns out to be symmetric with respect to the so-called charge inner product $[\cdot, \cdot]$, which is defined on elements $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}}, \mathbf{x}^{\prime}=\left(\begin{array}{ll}x^{\prime} y^{\prime}\end{array}\right)^{\mathrm{t}}$ of $\mathcal{G}_{1}=\mathcal{H} \oplus \mathcal{H}$ by a relation of the form

$$
\left[\mathbf{x}, \mathbf{x}^{\prime}\right]=\left(\left(\begin{array}{ll}
0 & I  \tag{1.9}\\
I & 0
\end{array}\right) \mathbf{x}, \quad \mathbf{x}^{\prime}\right)=\left(x, y^{\prime}\right)+\left(y, x^{\prime}\right)
$$

Independently of the potential $V$, the charge inner product is in general negative on an infinite dimensional subspace and hence leads to a so-called Krein space. The energy inner product is related to the charge inner product as follows:

$$
\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left[A_{1} \widetilde{W} \mathbf{x}, \widetilde{W} \mathbf{x}^{\prime}\right], \quad \widetilde{W}=\left(\begin{array}{cc}
I & 0  \tag{1.10}\\
-V & I
\end{array}\right)
$$

for suitable elements $\mathbf{x}, \mathbf{x}^{\prime}$ of the Pontryagin space $(\mathcal{G},\langle\cdot, \cdot\rangle)$; under our assumptions on $V$, the operator $\widetilde{W}: \mathcal{G} \rightarrow \mathcal{G}_{1}$ is bounded. The spectral properties of the operator $A_{1}$ and of another operator $A_{2}$ associated with (1.8) in the charge inner product and their relations to the operator $A$ are investigated in a separate paper (see [LNT06]).

The present paper is organized as follows: In the next Sect. 2 we briefly review results from the theory of self-adjoint operators in Pontryagin spaces. In Sect. 3 we associate the operator $A$ with (1.5); it acts in the space $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$, where $\mathcal{H}_{1 / 2}$ is the Hilbert space given by $\mathcal{D}\left(H_{0}^{1 / 2}\right)$ with norm $\left\|H_{0}^{1 / 2} \cdot\right\|$. We show that if
(i) $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ (i.e., $S=V H_{0}^{-1 / 2}$ is bounded) and
(ii) $I-S^{*} S$ is boundedly invertible,
then the operator

$$
A=\left(\begin{array}{cc}
0 & I \\
H & 2 V
\end{array}\right), \quad \mathcal{D}(A)=\mathcal{D}(H) \oplus \mathcal{H}_{1 / 2}
$$

is closed and boundedly invertible in $\mathcal{G}$; here $H$ is the self-adjoint operator in $\mathcal{H}$ given by $H=H_{0}^{1 / 2}\left(I-S^{*} S\right) H_{0}^{1 / 2}$. In Sect. 4 we introduce the indefinite inner product $\langle\cdot, \cdot\rangle$ on $\mathcal{G}$ and we prove that under the above assumptions (i) and (ii) the space $\mathcal{G}$ equipped with this inner product is a Krein space $\mathcal{K}$ and $A$ is a self-adjoint operator in $\mathcal{K}$ with non-empty resolvent set. In addition, we study the relation (1.10) of the energy inner product $\langle\cdot, \cdot \cdot\rangle$ with the operator $A_{1}$ and the corresponding charge inner product $[\cdot, \cdot]$. Section 5 contains the main result about the spectral properties of $A$. Under the additional assumption
(iii) $S=V H_{0}^{-1 / 2}=S_{0}+S_{1}$ with $\left\|S_{0}\right\|<1$ and a compact operator $S_{1}$,
we show that $\mathcal{K}$ is a Pontryagin space of index $\kappa$, where $\kappa$ is the number of negative eigenvalues of $I-S^{*} S$, the operator $A$ possesses a spectral function with at most finitely many critical points, the non-real spectrum of $A$ consists of at most $\kappa$ pairs of complex conjugate eigenvalues, and the essential spectrum of $A$ is real and has a gap of size at
least $2\left(1-\left\|S_{0}\right\|\right) m$ around 0 . Moreover, the operator $A$ generates a strongly continuous group $(\exp (\mathrm{i} A t))_{t \in \mathbb{R}}$ of unitary operators in the Pontryagin space $\mathcal{K}$ and hence the Cauchy problem

$$
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}=\mathrm{i} A \mathbf{x}, \quad \mathbf{x}(0)=\mathbf{x}_{0}
$$

has a unique solution for all initial values $\mathbf{x}_{0} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}$. Since $\infty$ is not a critical point for a self-adjoint operator in a Pontryagin space, the group $(\exp (\mathrm{i} A t))_{t \in \mathbb{R}}$ is uniformly bounded in $\mathcal{K}$; therefore the time-asymptotic behaviour of the solution $\mathbf{x}$ and hence of the solution of the abstract Klein-Gordon equation (1.2) is the same as in a Hilbert space. This is not the case for the self-adjoint operator $A_{1}$ in the Krein space $\mathcal{G}_{1}$ since there $\infty$ is a critical point (see [LNT06]).

Finally, in Sect. 6, we consider the Klein-Gordon equation in $\mathbb{R}^{n}$ and present sufficient conditions for the above assumptions. In particular, we show that our results apply to potentials $V$ of the form $V=V_{0}+V_{1}$ with a Coulomb part $V_{0}(x)=\gamma /|x|, x \in \mathbb{R}^{n} \backslash\{0\}$, with $\gamma<(n-2) / 2$ and $V_{1} \in L_{p}\left(\mathbb{R}^{n}\right)$ with $n \leq p<\infty$.

## 2. Preliminaries

1. Notations and definitions from spectral theory. For a closed linear operator $A$ in a Hilbert space $\mathcal{G}$ with domain $\mathcal{D}(A)$ we denote by $\rho(A), \sigma(A)$, and $\sigma_{\mathrm{p}}(A)$ its resolvent set, spectrum, and point spectrum (or set of eigenvalues), respectively. For $\lambda \in \sigma_{\mathrm{p}}(A)$ the algebraic eigenspace of $A$ at $\lambda$ is denoted by $\mathcal{L}_{\lambda}(A)$. The operator $A$ is called Fredholm if its kernel is finite dimensional and its range is finite codimensional (and hence closed), see, e.g., [GGK90, Chapter IV, §5.1]. The essential spectrum of $A$ is defined by

$$
\sigma_{\text {ess }}(A):=\{\lambda \in \mathbb{C}: A-\lambda \text { is not Fredholm }\}
$$

An eigenvalue $\lambda_{0} \in \sigma_{\mathrm{p}}(A)$ is called of finite type if $\lambda_{0}$ is isolated (i.e., a punctured neighbourhood of $\lambda_{0}$ belongs to $\rho(A)$ ) and $A-\lambda_{0}$ is Fredholm or, equivalently, the corresponding Riesz projection is finite dimensional.
2. Linear spaces with inner products. A Krein space $(\mathcal{K},[\cdot, \cdot])$ is a linear space $\mathcal{K}$ which is equipped with an (indefinite) inner product (i.e., a hermitian sesquilinear form) $[\cdot, \cdot]$ such that $\mathcal{K}$ can be written as

$$
\begin{equation*}
\mathcal{K}=\mathcal{G}_{+}[\dot{+}] \mathcal{G}_{-}, \tag{2.1}
\end{equation*}
$$

where $\left(\mathcal{G}_{ \pm}, \pm[\cdot, \cdot]\right)$ are Hilbert spaces and $[\dot{+}]$ means that the sum of $\mathcal{G}_{+}$and $\mathcal{G}_{-}$is direct and $\left[\mathcal{G}_{+}, \mathcal{G}_{-}\right]=\{0\}$. The norm topology on a Krein space $\mathcal{K}$ is the norm topology of the orthogonal sum of the Hilbert spaces $\mathcal{G}_{ \pm}$in (2.1). It can be shown that this norm topology is independent of the particular decomposition (2.1); all topological notions in $\mathcal{K}$ refer to this norm topology and $\|\cdot\|$ denotes any of the equivalent norms.

Krein spaces often arise as follows: In a given Hilbert space $(\mathcal{G},(\cdot, \cdot))$, every bounded self-adjoint operator $G$ in $\mathcal{G}$ with $0 \in \rho(G)$ induces an inner product

$$
\begin{equation*}
[x, y]:=(G x, y), \quad x, y \in \mathcal{G}, \tag{2.2}
\end{equation*}
$$

such that $(\mathcal{G},[\cdot, \cdot])$ becomes a Krein space; here, in the decomposition (2.1), we can choose $\mathcal{G}_{+}$as the spectral subspace of $G$ corresponding to the positive spectrum of
$G$ and $\mathcal{G}_{-}$as the spectral subspace of $G$ corresponding to the negative spectrum of $G$. A subspace $\mathcal{L}$ of a linear space $\mathcal{K}$ with inner product $[\cdot, \cdot]$ is called non-degenerated if there exists no $x \in \mathcal{L}, x \neq 0$, such that $[x, \mathcal{L}]=0$, otherwise $\mathcal{L}$ is called degenerated; note that a Krein space $\mathcal{K}$ is always non-degenerated, but it may have degenerated subspaces. An element $x \in \mathcal{K}$ is called positive (non-negative, negative, non-positive, neutral, respectively) if $[x, x]>0(\geq 0,<0, \leq 0,=0$, respectively); a subspace of $\mathcal{K}$ is called positive (non-negative, etc., respectively), if all its nonzero elements are positive (non-negative, etc., respectively). For the definition and simple properties of Krein spaces and linear operators therein we refer to [Bog74, Lan82, AI89].
3. Self-adjoint operators in Krein spaces. For a closed linear operator $A$ in a Krein space $\mathcal{K}$ with dense domain $\mathcal{D}(A)$, the (Krein space) adjoint $A^{+}$of $A$ is the densely defined operator in $\mathcal{K}$ given by

$$
\mathcal{D}\left(A^{+}\right):=\{y \in \mathcal{K}:[A \cdot, y] \text { is a continuous linear functional on } \mathcal{D}(A)\}
$$

and the relation

$$
[A x, y]=\left[x, A^{+} y\right], \quad x \in \mathcal{D}(A), \quad y \in \mathcal{D}\left(A^{+}\right)
$$

The operator $A$ is called symmetric if $A \subset A^{+}$and self-adjoint if $A=A^{+}$. The spectrum of a self-adjoint operator $A$ in a Krein space $\mathcal{K}$ is always symmetric to the real axis; note that both the spectrum $\sigma(A)$ or the resolvent set $\rho(A)$ may be empty. An orthogonal projection $P$ in a Krein space $\mathcal{K}$ is a self-adjoint projection in $\mathcal{K}$; note that orthogonal projections in a Krein space may have norm $>1$.

If for a self-adjoint operator $A$ in a Krein space $\mathcal{K}$ with $\lambda_{0} \in \sigma_{\mathrm{p}}(A)$ all the eigenvectors at $\lambda_{0}$ are positive (negative, respectively), then $\lambda_{0}$ is called an eigenvalue of positive (negative, respectively) type. A positive or negative eigenvector $x_{0}$ of $A$ at $\lambda_{0}$ does not have any associated vectors. Consequently, if for an eigenvector $x_{0}$ at $\lambda_{0}$ there exists an element $x_{1}$ such that $\left(A-\lambda_{0}\right) x_{1}=x_{0}$, then $x_{0}$ is neutral.
4. Self-adjoint operators in Pontryagin spaces. If in some decomposition (2.1) one of the components $\mathcal{G}_{ \pm}$is of finite dimension, it is of the same dimension in all such decompositions, and the Krein space ( $\mathcal{K},[\cdot, \cdot]$ ) is called a Pontryagin space. For the Pontryagin spaces $\mathcal{K}$ occurring in this paper, the negative component $\mathcal{G}_{-}$is of finite dimension, say $\kappa$; in this case, $\mathcal{K}$ is called a Pontryagin space with negative index say $\kappa$. If $\mathcal{K}$ arises from a Hilbert space $\mathcal{G}$ by means of a self-adjoint operator $G$ with inner product (2.2), then $\mathcal{K}$ is a Pontryagin space with negative index $\kappa$ if and only if the negative spectrum of the invertible operator $G$ consists of exactly $\kappa$ eigenvalues, counted according to their multiplicities. In a Pontryagin space $\mathcal{K}$ with negative index $\kappa$ each non-positive subspace is of dimension $\leq \kappa$, and a non-positive subspace is maximal non-positive (that is, it is not properly contained in another non-positive subspace) if and only if it is of dimension say $\kappa$.

If $\mathcal{L}$ is a non-degenerated linear space with inner product $[\cdot, \cdot]$ such that for a $\kappa$ dimensional subspace $\mathcal{L}_{-}$we have

$$
[x, x]<0, \quad x \in \mathcal{L}_{-}, \quad x \neq 0
$$

but there is no $(\kappa+1)$-dimensional subspace with this property, then there exists a Pontryagin space $\mathcal{K}$ with negative index $\kappa$ such that $\mathcal{L}$ is a dense subset of $\mathcal{K}$. This means
that $\mathcal{L}$ can be completed to a Pontryagin space in a similar way as a pre-Hilbert space can be completed to a Hilbert space.

The spectrum of a self-adjoint operator in a Pontryagin space is real with the possible exception of at most $\kappa$ non-real pairs of eigenvalues $\lambda, \bar{\lambda}$ of finite type; this estimate can be improved by taking multiplicities into account (see (2.3) below). According to a theorem of Pontryagin, a self-adjoint operator $A$ in a Pontryagin space with negative index $\kappa$ has a $\kappa$-dimensional invariant non-positive subspace $\mathcal{L}_{-}^{\max }$ :

$$
\mathcal{L}_{-}^{\max } \subset \mathcal{D}(A), \quad A \mathcal{L}_{-}^{\max } \subset \mathcal{L}_{-}^{\max }
$$

the subspace $\mathcal{L}_{-}^{\max }$ can be chosen such that $\operatorname{Im}\left(\sigma\left(A \mid \mathcal{L}_{-}^{\max }\right)\right) \geq 0$. Then the points of $\sigma\left(A \mid \mathcal{L}_{-}^{\max }\right)$ are the eigenvalues of $A$ in the closed upper half plane with a non-positive eigenvector. We denote the set of all eigenvalues of $A$ with a non-positive eigenvector by $\sigma_{0}(A)$; for a point $\lambda \in \sigma_{0}(A)$, the maximal dimension of a non-positive subspace of $\mathcal{L}_{\lambda}(A)$ is denoted by $\kappa_{\lambda}^{-}(A)$. Concerning the non-real spectrum of $A$, the closed linear span of all the algebraic eigenspaces $\mathcal{L}_{\lambda}(A)$ corresponding to the eigenvalues $\lambda$ of $A$ in the open upper (or lower) half plane is a neutral subspace of $\mathcal{K}$; for all such points $\lambda$ the algebraic eigenspaces $\mathcal{L}_{\lambda}(A), \mathcal{L}_{\bar{\lambda}}(A)$ are skewly linked, that is, to each nonzero $x \in \mathcal{L}_{\lambda}(A)$ there exists a $y \in \mathcal{L}_{\bar{\lambda}}(A)$ such that $[x, y] \neq 0$ and to each nonzero $y \in \mathcal{L}_{\bar{\lambda}}(A)$ there exists an $x \in \mathcal{L}_{\lambda}(A)$ such that $[x, y] \neq 0$. In particular, $\operatorname{dim} \mathcal{L}_{\bar{\lambda}}(A)=\operatorname{dim} \mathcal{L}_{\lambda}(A)$ and the Jordan structure of $A$ in $\mathcal{L}_{\lambda}(A)$ and in $\mathcal{L}_{\bar{\lambda}}(A)$ is the same. Further, the relation

$$
\begin{equation*}
\kappa=\sum_{\lambda \in \sigma_{0}(A) \cap \mathbb{R}} \kappa_{\lambda}^{-}(A)+\sum_{\lambda \in \sigma(A) \cap \mathbb{C}^{+}} \operatorname{dim} \mathcal{L}_{\lambda}(A) \tag{2.3}
\end{equation*}
$$

holds, which yields estimates for the number of points of $\sigma_{0}(A)$. All (real) points $\lambda \in$ $\sigma(A) \backslash \sigma_{0}(A)$ are spectral points of positive type, by which we mean that they are either eigenvalues of positive type or, if they belong to the continuous spectrum, that for each sequence $\left(x_{n}\right) \subset \mathcal{D}(A)$,

$$
\left\|x_{n}\right\|=1, \quad(A-\lambda) x_{n} \rightarrow 0 \Longrightarrow \quad \liminf _{n \rightarrow \infty}\left[x_{n}, x_{n}\right]>0
$$

5. Spectral functions of self-adjoint operators in Pontryagin spaces. If $q$ denotes the minimal polynomial or the characteristic polynomial of the restriction $A \mid \mathcal{L}_{-}^{\max }$, and $q^{*}$ is the polynomial given by $q^{*}(z)=\overline{q(\bar{z})}, z \in \mathbb{C}$, then the polynomial $q^{*} q$ is independent of the particular choice of the invariant subspace $\mathcal{L}_{-}^{\max }$, and it is not hard to show that $\left[q^{*}(A) q(A) x, x\right] \geq 0, x \in \mathcal{D}\left(A^{2 \kappa}\right)$. As a consequence, a self-adjoint operator $A$ in a Pontryagin space possesses a spectral function with possible critical points (see [KL63] and also [Lan82]). In order to introduce it, we call a bounded or unbounded real interval $\Gamma \subset \mathbb{R}$ admissible for the operator $A$ if the end points of $\Gamma$ do not belong to $\sigma_{0}(A)$. Then, for every admissible interval $\Gamma$, there exists an orthogonal projection $E(\Gamma)$ in $\mathcal{K}$ such that the range $E(\Gamma) \mathcal{K}$ is invariant under $A$ and

$$
\sigma(A \mid E(\Gamma) \mathcal{K}) \subset \bar{\Gamma}, \quad \sigma(A \mid(I-E(\Gamma)) \mathcal{K}) \cap \mathbb{R} \subset \overline{\mathbb{R} \backslash \Gamma}
$$

Moreover, the mapping $\Gamma \mapsto E(\Gamma)$ from the semiring $\mathcal{R}_{A}$ of all admissible intervals into the space of all bounded linear operators in $\mathcal{K}$ is a homomorphism, that is, for $\Gamma_{1}, \Gamma_{2} \in \mathcal{R}_{A}$,

$$
E\left(\Gamma_{1} \cap \Gamma_{2}\right)=E\left(\Gamma_{1}\right) E\left(\Gamma_{2}\right), \quad E\left(\Gamma_{1} \cup \Gamma_{2}\right)=E\left(\Gamma_{1}\right)+E\left(\Gamma_{2}\right)-E\left(\Gamma_{1} \cap \Gamma_{2}\right),
$$

and

$$
E(\emptyset)=0, \quad E(\mathbb{R}) \mathcal{K}=\left(\sum_{\lambda \in \sigma(A) \backslash \mathbb{R}} \mathcal{L}_{\lambda}(A)\right)^{[\perp]} ;
$$

here ${ }^{[\perp]}$ denotes the orthogonal complement with respect to the indefinite inner product. The critical points of the spectral function $E$ are those points $\lambda \in \mathbb{R}$ for which the inner product $[\cdot, \cdot]$ is indefinite on $E(\Gamma) \mathcal{K}$ for each $\Gamma \in \mathcal{R}_{A}$ containing $\lambda$; all critical points of $E$ belong to $\sigma_{0}(A)$.

If an interval $\Gamma \in \mathcal{R}_{A}$ does not contain points of $\sigma_{0}(A)$, then the range $E(\Gamma) \mathcal{K}$ is a positive subspace of $\mathcal{K}$ and hence a Hilbert space. Therefore, with the exception of the points of $\sigma_{0}(A) \cap \mathbb{R}$, the spectral behaviour of $A$ is that of a self-adjoint operator in a Hilbert space. In particular, for an admissible interval $\Gamma$ with $\Gamma \cap \sigma_{0}(A)=\emptyset$,

$$
A E(\Gamma)=\int_{\Gamma} \lambda E(d \lambda)
$$

here, if $A$ is an unbounded operator and $\Gamma$ is an unbounded interval, the expressions on either side coincide as unbounded operators.

Given a point $\lambda_{0} \in \sigma_{0}(A) \cap \mathbb{R}$, we choose an admissible interval $\Gamma=[\alpha, \beta]$ such that $[\alpha, \beta] \cap \sigma_{0}(A)=\left\{\lambda_{0}\right\}$. If $\mathcal{L}_{\lambda_{0}}(A)$ is non-degenerated (e.g., if $\lambda_{0}$ is an eigenvalue of negative type), then the strong limits

$$
\lim _{\mu \nearrow \lambda_{0}} E([\alpha, \mu]), \quad \lim _{\mu \searrow \lambda_{0}} E([\mu, \beta])
$$

exist. They can be considered as spectral projections $E\left(\left[\alpha, \lambda_{0}\right)\right)$ and $E\left(\left(\lambda_{0}, \beta\right]\right)$ of $A$ corresponding to the intervals $\left[\alpha, \lambda_{0}\right)$ and ( $\left.\lambda_{0}, \beta\right]$, respectively, and the decomposition

$$
\begin{equation*}
E(\Gamma) \mathcal{K}=E\left(\left[\alpha, \lambda_{0}\right)\right) \mathcal{K}[\dot{+}] \mathcal{L}_{\lambda_{0}}(A)[\dot{+}] E\left(\left(\lambda_{0}, \beta\right]\right) \mathcal{K} \tag{2.4}
\end{equation*}
$$

holds. If, however, $\mathcal{L}_{\lambda_{0}}$ is degenerated, then at least one of the quantities

$$
\limsup _{\mu \nearrow \lambda_{0}}\|E([\alpha, \mu])\| \quad \text { or } \quad \limsup _{\mu \searrow \lambda_{0}}\|E([\mu, \beta])\|
$$

is infinite and the subspace $\mathcal{L}_{\lambda_{0}}(A)$ cannot be split off as in (2.4).
If $A$ is an unbounded self-adjoint operator in a Pontryagin space $\mathcal{K}$, we choose a bounded admissible interval $\Gamma$ which contains all the real points of $\sigma_{0}(A)$ and we consider the space

$$
\mathcal{L}^{1}:=E(\Gamma) \mathcal{K}[\dot{+}] \sum_{\lambda \in \sigma(A) \backslash \mathbb{R}} \mathcal{L}_{\lambda}(A) .
$$

It is a Pontryagin space with negative index $\kappa$ that reduces $A$ and the restriction $A_{1}:=$ $A \mid \mathcal{L}^{1}$ is a bounded operator. The orthogonal complement $\mathcal{L}^{0}$ of $\mathcal{L}^{1}$ in $\mathcal{K}$ is a Hilbert space with respect to the inner product $[\cdot, \cdot]$ and the decomposition

$$
\mathcal{K}=\mathcal{L}^{1}[\dot{+}] \mathcal{L}^{0}
$$

yields a corresponding orthogonal decomposition of the operator $A$ :

$$
\begin{equation*}
A=A_{1}[\dot{+}] A_{0} . \tag{2.5}
\end{equation*}
$$

Here $A_{1}$ is a bounded self-adjoint operator in the Pontryagin space $\mathcal{L}^{1}$ with negative index $\kappa$ and $A_{0}$ is a self-adjoint operator in the Hilbert space $\mathcal{L}^{0}$. Thus, the study of an unbounded self-adjoint operator in a Pontryagin space can always be reduced to the study of a bounded self-adjoint operator in a Pontryagin space and of an unbounded self-adjoint operator in a Hilbert space.

A bounded operator in a Pontryagin space $\mathcal{K}$ is called unitary if it maps $\mathcal{K}$ onto itself and

$$
[U x, U y]=[x, y], \quad x, y \in \mathcal{K} .
$$

Using the decomposition (2.5), it readily follows that a self-adjoint operator $A$ in a Pontryagin space generates a group $(\exp (\mathrm{i} t A))_{t \in \mathbb{R}}$ of unitary operators in $\mathcal{K}$ and that this group is exponentially bounded, that is,

$$
\|\exp (\mathrm{i} t A)\| \leq C \mathrm{e}^{\gamma|t|}, \quad t \in \mathbb{R}
$$

with positive constants $C$ and $\gamma$. This was first proved by M.A. Naĭmark in [Naĭ66].

## 3. An Operator Associated with the Abstract Klein-Gordon Equation

Let $(\mathcal{H},(\cdot, \cdot))$ be a Hilbert space with corresponding norm $\|\cdot\|, H_{0}$ a strictly positive self-adjoint operator in $\mathcal{H}, H_{0} \geq m^{2}>0$, and $V$ a symmetric operator in $\mathcal{H}$. By means of the operator $H_{0}$ we introduce the Hilbert space $\left(\mathcal{H}_{1 / 2},(\cdot, \cdot)_{1 / 2}\right)$ as

$$
\begin{equation*}
\mathcal{H}_{1 / 2}:=\mathcal{D}\left(H_{0}^{1 / 2}\right), \quad(x, y)_{1 / 2}:=\left(H_{0}^{1 / 2} x, H_{0}^{1 / 2} y\right), \quad x, y \in \mathcal{H}_{1 / 2} \tag{3.1}
\end{equation*}
$$

In the orthogonal sum $\mathcal{G}:=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$ with norm

$$
\|\mathbf{x}\|_{\mathcal{G}}=\left(\left\|H_{0}^{1 / 2} x\right\|^{2}+\|y\|^{2}\right)^{1 / 2}, \quad \mathbf{x}=(x y)^{\mathrm{t}} \in \mathcal{G}
$$

we consider the block operator matrix $\widehat{A}$, formally given by

$$
\widehat{A}:=\left(\begin{array}{cc}
0 & I  \tag{3.2}\\
H_{0}-V^{2} & 2 V
\end{array}\right),
$$

which arises from the differential equation (1.2) by means of the substitution (1.4) (see (1.5)).

In order to associate a well-defined operator with the entry $H_{0}-V^{2}$ in $\widehat{A}$, we make the following assumption:
Ass. (i) $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$;
this condition implies that the operator

$$
\begin{equation*}
S:=V H_{0}^{-1 / 2} \tag{3.3}
\end{equation*}
$$

is everywhere defined and bounded on $\mathcal{H}$.
In the next section we need that the operator associated with the formal expression $H_{0}-V^{2}$ is boundedly invertible. In order to assure this we also assume
Ass. (ii) $1 \in \rho\left(S^{*} S\right)$, that is, the operator $I-S^{*} S$ is boundedly invertible.

In this case the operator $H_{0}^{-1 / 2}\left(I-S^{*} S\right)^{-1} H_{0}^{-1 / 2}$ is everywhere defined, injective, bounded and self-adjoint in $\mathcal{H}$; therefore the operator

$$
\begin{equation*}
H:=H_{0}^{1 / 2}\left(I-S^{*} S\right) H_{0}^{1 / 2}, \quad \mathcal{D}(H)=\left\{x \in \mathcal{H}_{1 / 2}:\left(I-S^{*} S\right) H_{0}^{1 / 2} x \in \mathcal{H}_{1 / 2}\right\} \tag{3.4}
\end{equation*}
$$

is self-adjoint and boundedly invertible in $\mathcal{H}$. The operator $H$ can also be considered as a densely defined closed operator from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$, for which we use the same symbol $H$ : it is densely defined because $\mathcal{D}(H)$ is dense in $\mathcal{H}$ and the inclusion $\mathcal{H}_{1 / 2} \hookrightarrow \mathcal{H}$ is continuous; it is closed since the middle factor is closed in $\mathcal{H}$, the left factor is boundedly invertible in $\mathcal{H}$ and the right factor is boundedly invertible as an operator from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$ (see [Kat66, Sect. III.5.2]).

Remark 3.1. The operator $H$ in $\mathcal{H}$ (from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$, respectively) can also be defined by means of quadratic forms if we replace the conditions (i) and (ii) by Ass. ( $\mathbf{i}^{\prime}$ ) $V$ is $H_{0}^{1 / 2}$-bounded with relative bound less than 1 .

In fact, Assumption (i) is equivalent to the fact that $V$ is $H_{0}^{1 / 2}$-bounded, that is, $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ and there exist constants $a, b \geq 0$, such that

$$
\begin{equation*}
\|V x\| \leq a\|x\|+b\left\|H_{0}^{1 / 2} x\right\|, \quad x \in \mathcal{D}\left(H_{0}^{1 / 2}\right) \tag{3.5}
\end{equation*}
$$

In Assumption ( $\mathrm{i}^{\prime}$ ) it is required, in addition, that (3.5) holds with $b<1$, or, equivalently (see [Kat82, Sect. V.4.1]), there exist constants $a^{\prime}, b^{\prime} \geq 0, b^{\prime}<1$, such that

$$
\begin{equation*}
\|V x\|^{2} \leq a^{\prime 2}\|x\|^{2}+b^{\prime 2}\left\|H_{0}^{1 / 2} x\right\|^{2}, \quad x \in \mathcal{D}\left(H_{0}^{1 / 2}\right) \tag{3.6}
\end{equation*}
$$

If we introduce the forms

$$
\begin{aligned}
\mathbf{h}[x, y] & :=\left(H_{0}^{1 / 2} x, H_{0}^{1 / 2} y\right), \quad x, y \in \mathcal{D}\left(H_{0}^{1 / 2}\right), \\
\mathbf{v}_{2}[x, y] & :=(V x, V y), \quad x, y \in \mathcal{D}(V),
\end{aligned}
$$

then (3.6) (and hence ( $\mathrm{i}^{\prime}$ )) implies that the form $\mathbf{v}_{2}$ is $\mathbf{h}$-bounded with relative formbound less than 1. Then, according to [Kat82, Theorem VI.3.9], the form $\operatorname{sum} \mathbf{h}+\mathbf{v}_{2}$ is closed and symmetric, and the entry $H_{0}-V^{2}$ in (3.7) can be defined by means of the self-adjoint operator in $\mathcal{H}$ induced by the form sum $\mathbf{h}+\mathbf{v}_{2}$. Our choice of the conditions (i) and (ii) rather than ( $\mathrm{i}^{\prime}$ ) is due to the fact that Assumption (ii) is needed in the next section for other reasons.

With the formal matrix $\widehat{A}$ in (3.2) we now associate the block operator matrix $A$ in $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$ defined by

$$
A=\left(\begin{array}{cc}
0 & I  \tag{3.7}\\
H & 2 V
\end{array}\right), \quad \mathcal{D}(A)=\mathcal{D}(H) \oplus \mathcal{D}\left(H_{0}^{1 / 2}\right)
$$

Lemma 3.2. If $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ and $1 \in \rho\left(S^{*} S\right)$, then the operator $A$ from (3.7) is boundedly invertible, and hence closed in $\mathcal{G}$, with

$$
A^{-1}=\left(\begin{array}{cc}
-2 H^{-1} V & H^{-1}  \tag{3.8}\\
I & 0
\end{array}\right) .
$$

Proof. By the assumptions, $H$ is a boundedly invertible operator from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$. Hence formally the inverse of $A$ is given by (3.8). It remains to be shown that $A^{-1}$ is a bounded operator in $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$.

This follows from the facts that $H^{-1}$ is a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{1 / 2}$, the identity $I$ is bounded as an operator from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$ since the inclusion $\mathcal{H}_{1 / 2} \hookrightarrow \mathcal{H}$ is continuous, and $H^{-1} V=H_{0}^{-1 / 2}\left(I-S^{*} S\right)^{-1} H_{0}^{-1 / 2} V$ is a bounded operator in $\mathcal{H}_{1 / 2}$. For the latter, we observe that $V$ is bounded from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$ by the first assumption, $\left(I-S^{*} S\right)^{-1} H_{0}^{-1 / 2}$ is bounded in $\mathcal{H}$ by the second assumption, and $H_{0}^{-1 / 2}$ is bounded from $\mathcal{H}$ to $\mathcal{H}_{1 / 2}$.

The operator $A$ is related to another operator associated with the Klein-Gordon equation (1.2) which is formally given by (1.8) and arises from the substitution (1.7): In the orthogonal sum $\mathcal{G}_{1}:=\mathcal{H} \oplus \mathcal{H}$ we consider the operator

$$
\widehat{A_{1}}:=\left(\begin{array}{cc}
V & I  \tag{3.9}\\
H_{0} & V
\end{array}\right)
$$

with domain

$$
\mathcal{D}\left(\widehat{A}_{1}\right):=\left\{\binom{x}{y} \in \mathcal{H} \oplus \mathcal{H}: x \in \mathcal{D}(V) \cap \mathcal{D}\left(H_{0}\right), y \in \mathcal{D}(V)\right\}
$$

It has been shown in [LNT06, Thm. 3.1] that Assumption (i) implies that $\widehat{A}_{1}$ is closable with closure $A_{1}$ given by

$$
\begin{align*}
\mathcal{D}\left(A_{1}\right) & =\left\{\binom{x}{y} \in \mathcal{H} \oplus \mathcal{H}: x \in \mathcal{D}\left(H_{0}^{1 / 2}\right), H_{0}^{1 / 2} x+S^{*} y \in \mathcal{D}\left(H_{0}^{1 / 2}\right)\right\}  \tag{3.10}\\
A_{1}\binom{x}{y} & =\binom{V x+y}{H_{0}^{1 / 2}\left(H_{0}^{1 / 2} x+S^{*} y\right)} \tag{3.11}
\end{align*}
$$

In order to establish the relation between $A$ and $A_{1}$, we introduce the unbounded operator $W$ from $\mathcal{G}_{1}=\mathcal{H} \oplus \mathcal{H}$ to $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$ as

$$
W:=\left(\begin{array}{cc}
I & 0 \\
V & I
\end{array}\right), \quad \mathcal{D}(W):=\mathcal{H}_{1 / 2} \oplus \mathcal{H}
$$

Its inverse

$$
W^{-1}=\left(\begin{array}{cc}
I & 0 \\
-V & I
\end{array}\right)
$$

(denoted by $\widetilde{W}$ in (1.10)) is a bounded operator from $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$ to $\mathcal{G}_{1}=\mathcal{H} \oplus \mathcal{H}$ since $V$ is a bounded operator from $\mathcal{H}_{1 / 2}$ to $\mathcal{H}$ by Assumption (i).

Lemma 3.3. If Assumptions (i) and (ii) are satisfied, then

$$
\begin{equation*}
A=W A_{1} W^{-1} \tag{3.12}
\end{equation*}
$$

Proof. Using the description of the domain of $A_{1}$ from (3.10) and the fact that for $y \in \mathcal{H}_{1 / 2} \subset \mathcal{D}(V)$ we have $S^{*} y=\overline{H_{0}^{-1 / 2} V} y=H_{0}^{-1 / 2} V y \in \mathcal{H}_{1 / 2}$, we find

$$
\begin{aligned}
\mathcal{D} & \left(W A_{1} W^{-1}\right) \\
& =\left\{\binom{x}{y} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}:\binom{x}{-V x+y} \in \mathcal{D}\left(A_{1}\right), A_{1}\binom{x}{-V x+y} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}\right\} \\
& =\left\{\binom{x}{y} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}: H_{0}^{1 / 2} x+S^{*}(-V x+y) \in \mathcal{D}\left(H_{0}^{1 / 2}\right), V x-V x+y \in \mathcal{H}_{1 / 2}\right\} \\
& =\left\{\binom{x}{y} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}: H_{0}^{1 / 2} x-S^{*} V H_{0}^{-1 / 2} H_{0}^{1 / 2} x \in \mathcal{D}\left(H_{0}^{1 / 2}\right), y \in \mathcal{H}_{1 / 2}\right\} \\
& =\left\{\binom{x}{y} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}:\left(I-S^{*} S\right) H_{0}^{1 / 2} x \in \mathcal{D}\left(H_{0}^{1 / 2}\right), y \in \mathcal{H}_{1 / 2}\right\} \\
& =\mathcal{D}(H) \oplus \mathcal{H}_{1 / 2}=\mathcal{D}(A) .
\end{aligned}
$$

That the operators $A$ and $W A_{1} W^{-1}$ coincide is seen as follows: for $x \in \mathcal{D}(H)$ and $y \in \mathcal{H}_{1 / 2}=\mathcal{D}\left(H_{0}^{1 / 2}\right)$ we have, observing (3.11) and (3.4),

$$
\begin{aligned}
\left(\begin{array}{cc}
I & 0 \\
V & I
\end{array}\right) A_{1}\binom{x}{-V x+y} & =\left(\begin{array}{cc}
I & 0 \\
V & I
\end{array}\right)\binom{V x-V x+y}{H_{0}^{1 / 2}\left(H_{0}^{1 / 2} x+S^{*}(-V x+y)\right)} \\
& =\binom{y}{H_{0}^{1 / 2}\left(H_{0}^{1 / 2} x+S^{*}(-V x+y)\right)+V y} \\
& =\binom{y}{H x+H_{0}^{1 / 2} S^{*} y+V y}=\binom{y}{H x+2 V y} \\
& =A\binom{x}{y}
\end{aligned}
$$

where we have used that $y \in \mathcal{H}_{1 / 2} \subset \mathcal{D}(V)$ and $H_{0}^{1 / 2} S^{*}=\left(S H_{0}^{1 / 2}\right)^{*}=V^{*} \supset V$.

## 4. Indefinite Inner Products

In this section we always suppose that Assumptions (i) and (ii) are satisfied and we consider the operator $A$ from (3.7). Obviously, $A$ is not symmetric with respect to the Hilbert space inner product of $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$. However, it exhibits symmetry with respect to another inner product which is, in general, indefinite. This so-called energy inner product on $\mathcal{G}$ is defined as

$$
\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle:=\left(H_{0}^{1 / 2} x, H_{0}^{1 / 2} x^{\prime}\right)-\left(V x, V x^{\prime}\right)+\left(y, y^{\prime}\right)
$$

for $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}}, \mathbf{x}^{\prime}=\left(x^{\prime} y^{\prime}\right)^{\mathrm{t}} \in \mathcal{G}$, which, using $S=V H_{0}^{-1 / 2}$, can also be written as

$$
\begin{equation*}
\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left(\left(I-S^{*} S\right) H_{0}^{1 / 2} x, H_{0}^{1 / 2} x^{\prime}\right)+\left(y, y^{\prime}\right) \tag{4.1}
\end{equation*}
$$

Lemma 4.1. Under Assumptions (i) and (ii), the space $\mathcal{K}:=(\mathcal{G},\langle\cdot, \cdot\rangle)$ is a Krein space. If, additionally, the number of negative eigenvalues of the operator $I-S^{*} S$ in $\mathcal{H}$ is finite, say $\kappa$, then $\mathcal{K}$ is a Pontryagin space with negative index $\kappa$.

Proof. Due to Assumptions (i) and (ii), the operator $I-S^{*} S$ is bounded and self-adjoint in $\mathcal{H}$ with $0 \in \rho\left(I-S^{*} S\right)$. Hence $\mathcal{H}$ equipped with the inner product $\left(\left(I-S^{*} S\right) \cdot, \cdot\right)$ is a Krein space (see Sect. 2.2); if, in addition, the number of negative eigenvalues of $I-S^{*} S$ in $\mathcal{H}$ is finite, say $\kappa$, it is a Pontryagin space with negative index $\kappa$. Now the claim follows since $H_{0}^{1 / 2}: \mathcal{H}_{1 / 2} \rightarrow \mathcal{H}$ is an isomorphism.

Remark 4.2. If $\|S\|<1$, that is, $\kappa=0$, then $\mathcal{K}$ is a Hilbert space.
Theorem 4.3. Suppose that $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ and that $1 \in \rho\left(S^{*} S\right)$. Then $A$ is a selfadjoint operator in the Krein space $\mathcal{K}$ with $\rho(A) \neq \emptyset$.

Proof. For $\mathbf{x}=\left(\begin{array}{ll}x & y)^{\mathrm{t}} \in \mathcal{D}(A)=\mathcal{D}(H) \oplus \mathcal{D}\left(H_{0}^{1 / 2}\right) \text {, we obtain, using (4.1), }\end{array}\right.$

$$
\begin{aligned}
\langle A \mathbf{x}, \mathbf{x}\rangle & =\left\langle\binom{ y}{H x+2 V y},\binom{x}{y}\right\rangle \\
& =\left(\left(I-S^{*} S\right) H_{0}^{1 / 2} y, H_{0}^{1 / 2} x\right)+(H x+2 V y, y) \\
& =(y, H x)+(H x, y)+2(V y, y),
\end{aligned}
$$

which is real. Thus the operator $A$ is symmetric in $\mathcal{K}$. In order to prove that $A$ is selfadjoint in $\mathcal{K}$, it remains to be shown that $\rho(A)$ contains a real point $\mu$ (then $(A-\mu)^{-1}$ is bounded and symmetric in $\mathcal{K}$ and hence self-adjoint in $\mathcal{K}$ ). Hence, to complete the proof, it suffices to show that $0 \in \rho(A)$. For $\mathbf{f}=(f g)^{\mathrm{t}} \in \mathcal{H}_{1 / 2} \oplus \mathcal{H}$, the equation $A \mathbf{x}=\mathbf{f}$ with $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}} \in \mathcal{D}(A)=\mathcal{D}(H) \oplus \mathcal{H}_{1 / 2}$ is equivalent to

$$
\begin{array}{r}
y=f \\
H x+2 V y=g .
\end{array}
$$

Since $1 \in \rho\left(S^{*} S\right), H$ is boundedly invertible (see (3.4)) and so the second equation with $y=f$ has a unique solution $x \in \mathcal{D}(H)$, whence $A \mathbf{x}=\mathbf{f}$ has the unique solution

$$
\mathbf{x}=\binom{H^{-1}(-2 V f+g)}{f} \in \mathcal{D}(H) \oplus \mathcal{H}_{1 / 2}=\mathcal{D}(A)
$$

which proves that $0 \in \rho(A)$.
The energy inner product $\langle\cdot, \cdot\rangle$ on $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{G}$ is related to the so-called charge inner product $[\cdot, \cdot]$ on $\mathcal{G}_{1}=\mathcal{H} \oplus \mathcal{H}$ which is given by

$$
\begin{equation*}
\left[\mathbf{x}, \mathbf{x}^{\prime}\right]:=\left(x, y^{\prime}\right)+\left(y, x^{\prime}\right)=\left(G \mathbf{x}, \mathbf{x}^{\prime}\right) \tag{4.2}
\end{equation*}
$$

with

$$
G:=\left(\begin{array}{ll}
0 & I \\
I & 0
\end{array}\right) .
$$

Obviously, the space $\mathcal{K}_{1}:=\left(\mathcal{G}_{1},[\cdot, \cdot]\right)$ is a Krein space for which the positive and negative components in the decomposition (2.1) have the same dimension; in particular, if $\mathcal{H}$ is infinite dimensional (as it is the case for the Klein-Gordon equation), then both components are infinite dimensional.

The operator $\widehat{A}_{1}$ given by (3.9) is symmetric with respect to the charge inner product in $\mathcal{K}_{1}$ since, for $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}} \in \mathcal{D}\left(\hat{A}_{1}\right)=\mathcal{D}\left(H_{0}\right) \oplus \mathcal{D}(V)$,

$$
\left[\widehat{A}_{1} \mathbf{x}, \mathbf{x}\right]=\left(G \widehat{A}_{1} \mathbf{x}, \mathbf{x}\right)=\left(H_{0} x, x\right)+(V y, x)+(x, V y)+(y, y),
$$

which is real. Moreover, it has been shown in [LNT06] that, under Assumption (i), the closure $A_{1}$ of $\widehat{A}_{1}$ given by (3.10), (3.11) is self-adjoint in $\mathcal{K}_{1}$ with $\rho\left(A_{1}\right) \neq \emptyset$.

Proposition 4.4. Between the indefinite inner products $\langle\cdot, \cdot\rangle$ of $\mathcal{G}$ and $[\cdot, \cdot]$ of $\mathcal{G}_{1}$ the following relations hold:
i) $\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left[A_{1} W^{-1} \mathbf{x}, W^{-1} \mathbf{x}^{\prime}\right], \quad \mathbf{x} \in W \mathcal{D}\left(A_{1}\right), \mathbf{x}^{\prime} \in \mathcal{G}$,
ii) $\left\langle A \mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left[A_{1}^{2} W^{-1} \mathbf{x}, W^{-1} \mathbf{x}^{\prime}\right], \quad \mathbf{x} \in \mathcal{D}(A), A \mathbf{x} \in W \mathcal{D}\left(A_{1}\right), \mathbf{x}^{\prime} \in \mathcal{G}$.

Proof. i) Let $\mathbf{x}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}} \in W \mathcal{D}\left(A_{1}\right), \mathbf{x}^{\prime}=\left(x^{\prime} y^{\prime}\right)^{\mathrm{t}} \in \mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$, and set

$$
\mathbf{u}=\binom{u}{v}:=W^{-1} \mathbf{x}=\binom{x}{-V x+y} \in \mathcal{D}\left(A_{1}\right) .
$$

Then we have

$$
\mathbf{x}=W \mathbf{u}=\binom{x}{y}=\binom{u}{V u+v},
$$

and the left-hand side of i) becomes

$$
\begin{aligned}
\left\langle\mathbf{x}, \mathbf{x}^{\prime}\right\rangle & =\left(H_{0}^{1 / 2} x, H_{0}^{1 / 2} x^{\prime}\right)-\left(V x, V x^{\prime}\right)+\left(y, y^{\prime}\right) \\
& =\left(H_{0}^{1 / 2} u, H_{0}^{1 / 2} x^{\prime}\right)-\left(V u, V x^{\prime}\right)+\left(V u+v, y^{\prime}\right)
\end{aligned}
$$

Using (3.11) and the fact that $x^{\prime} \in \mathcal{H}_{1 / 2}$, we can rewrite the right hand side of i) as

$$
\begin{aligned}
{\left[A_{1} W^{-1} \mathbf{x}, W^{-1} \mathbf{x}^{\prime}\right] } & =\left[A_{1}\binom{u}{v},\binom{x^{\prime}}{-V x^{\prime}+y^{\prime}}\right)=\left[\binom{V u+v}{H_{0}\left(u+T^{*} v\right)},\binom{x^{\prime}}{-V x^{\prime}+y^{\prime}}\right] \\
& =\left(V u+v,-V x^{\prime}+y^{\prime}\right)+\left(H_{0}^{1 / 2}\left(u+T^{*} v\right), H_{0}^{1 / 2} x^{\prime}\right) \\
& =\left(V u+v,-V x^{\prime}+y^{\prime}\right)+\left(H_{0}^{1 / 2} u, H_{0}^{1 / 2} x^{\prime}\right)+\left(H_{0}^{1 / 2} T^{*} v, H_{0}^{1 / 2} x^{\prime}\right) .
\end{aligned}
$$

Since $T=V H_{0}^{-1}$, the last summand equals $\left(v, T H_{0} x^{\prime}\right)=\left(v, V x^{\prime}\right)$ and i) follows.
ii) Let $\mathbf{x} \in \mathcal{D}(A)$ be such that $A \mathbf{x} \in W \mathcal{D}\left(A_{1}\right)$ and hence $W^{-1} A \mathbf{x} \in \mathcal{D}\left(A_{1}\right)$. Since, by the operator equality (3.12), we have $W^{-1} A x=A_{1} W^{-1} x$, it follows that $A_{1} W^{-1} \mathbf{x} \in$ $\mathcal{D}\left(A_{1}\right)$ and further, by i),

$$
\left\langle A \mathbf{x}, \mathbf{x}^{\prime}\right\rangle=\left[A_{1} W^{-1} A \mathbf{x}, W^{-1} \mathbf{x}^{\prime}\right]=\left[A_{1}^{2} W^{-1} \mathbf{x}, W^{-1} \mathbf{x}^{\prime}\right]
$$

for arbitrary $\mathbf{x}^{\prime} \in \mathcal{G}$.
Lemma 4.5. Let $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$. Then the set $W \mathcal{D}\left(A_{1}\right)$ is dense in $\mathcal{G}$.

Proof. By (3.10), we have

$$
W \mathcal{D}\left(A_{1}\right)=\left\{\binom{x}{V x+y}: x \in \mathcal{D}(\bar{V}), x+T^{*} y \in \mathcal{D}\left(H_{0}\right)\right\} .
$$

Hence if $\left(x_{0} y_{0}\right)^{\mathrm{t}} \in \mathcal{G}$ is orthogonal to $W \mathcal{D}\left(A_{1}\right)$ with respect to the Hilbert space inner product in $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$, then

$$
\begin{equation*}
\left(\binom{x}{V x+y}, \quad\binom{x_{0}}{y_{0}}\right)_{\mathcal{G}}=\left(H_{0}^{1 / 2} x, H_{0}^{1 / 2} x_{0}\right)+\left(V x+y, y_{0}\right)=0 \tag{4.3}
\end{equation*}
$$

for all $(x y)^{\mathrm{t}} \in W \mathcal{D}\left(A_{1}\right)$. If we choose $x=0$ and $y \in \mathcal{D}(V)$, then $T^{*} y=\overline{H_{0}^{-1} V} y=$ $H_{0}^{-1} V y \in \mathcal{D}\left(H_{0}\right)$ and hence $(0 y)^{\mathrm{t}} \in W \mathcal{D}\left(A_{1}\right)$. Now (4.3) shows that $y_{0}$ is orthogonal in $\mathcal{H}$ to the dense subset $\mathcal{D}(V)$ and thus $y_{0}=0$. If we choose $x \in \mathcal{D}\left(H_{0}\right)$ and $y=0$, then $(x 0)^{\mathrm{t}} \in W \mathcal{D}\left(A_{1}\right)$ because $\mathcal{D}\left(H_{0}\right) \subset \mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ by assumption. Since $H_{0}$ is bijective, (4.3) implies that $x_{0}$ is orthogonal to $\mathcal{H}$ and hence $x_{0}=0$.

Remark 4.6. If the operator $I-S^{*} S$ has only finitely many, say $\kappa$, negative eigenvalues, the Pontryagin space $\mathcal{K}$ and the operator $A$ can also be introduced by means of the operator $A_{1}$ in the space $\mathcal{G}_{1}$ as follows. By Proposition 4.4 i), the indefinite inner product $\langle\cdot, \cdot\rangle$ is defined on the dense subset $W \mathcal{D}\left(A_{1}\right)$ of $\mathcal{G}$. Since $I-S^{*} S$ has only $\kappa$ negative eigenvalues, the form $\left[A_{1} \cdot, \cdot\right]$ on $\mathcal{D}\left(A_{1}\right) \subset \mathcal{G}$, and hence $\langle\cdot, \cdot\rangle$ on $W \mathcal{D}\left(A_{1}\right)$, has $\kappa$ negative squares (see [LNT06]). Therefore $\mathcal{K}$ is the Pontryagin space completion of $W \mathcal{D}\left(A_{1}\right) \subset \mathcal{G}$ with respect to the inner product $\langle\cdot, \cdot\rangle ;$ the operator $A$ can now be defined by the relation in Proposition 4.4 ii).

## 5. Spectral Properties of the Operator $A$

In this section we exploit the self-adjointness of the operator $A$ with respect to the indefinite inner product $\langle\cdot, \cdot\rangle$. We show that $A$ possesses a spectral function with at most finitely many critical points, we investigate the structure of the spectrum of $A$ and consider the solvability of an abstract Cauchy problem for $A$ (and hence for the Klein-Gordon equation).

In order to guarantee that $\mathcal{K}$ is a Pontryagin space, in addition to the Assumptions (i) and (ii), we suppose that

Ass. (iii) $S=V H_{0}^{-1 / 2}=S_{0}+S_{1}$ with $\left\|S_{0}\right\|<1$ and a compact operator $S_{1}$.
To study the spectrum and essential spectrum of $A$ under Assumption (iii), the following lemma for the particular case $\|S\|<1$ is useful.

Lemma 5.1. Suppose that $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ and $1 \in \rho\left(S^{*} S\right)$. Define the quadratic pencil $L$ of bounded operators in $\mathcal{H}$ by

$$
\begin{equation*}
L(\lambda):=I-S^{*} S+\lambda\left(S^{*} H_{0}^{-1 / 2}+H_{0}^{-1 / 2} S\right)-\lambda^{2} H_{0}^{-1}, \quad \lambda \in \mathbb{C} . \tag{5.1}
\end{equation*}
$$

If $\|S\|<1$, then $\rho(A)=\rho(L)$ and, with $\alpha:=(1-\|S\|) m$,

$$
(-\alpha, \alpha) \subset \rho(A)
$$

Proof. By Lemma 3.2, the operator $A$ has a bounded inverse $A^{-1}$. By the spectral mapping theorem (see [EE87, Thm. IX.2.3]), we have

$$
\lambda \in \rho(A) \Longleftrightarrow \mu:=\lambda^{-1} \in \rho\left(A^{-1}\right)
$$

If $\|S\|<1$, then the operator $\Gamma:=I-S^{*} S$ is uniformly positive. Hence, by (3.4), we can write $H=H_{0}^{1 / 2} \Gamma H_{0}^{1 / 2}, H^{-1}=H_{0}^{-1 / 2} \Gamma^{-1 / 2} \Gamma^{-1 / 2} H_{0}^{-1 / 2}$, and thus we can factorize the inverse $A^{-1}$ given by (3.8) as

$$
\begin{aligned}
A^{-1} & =\left(\begin{array}{cc}
-2 H^{-1} V & H^{-1} \\
I & 0
\end{array}\right) \\
& =\left(\begin{array}{ccc}
H_{0}^{-1 / 2} \Gamma^{-1 / 2} & 0 \\
0 & I
\end{array}\right)\left(\begin{array}{cc}
-2 \Gamma^{-1 / 2} H_{0}^{-1 / 2} V & \Gamma^{-1 / 2} H_{0}^{-1 / 2} \\
I & 0
\end{array}\right)
\end{aligned}
$$

here the right factor is an operator from $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$ and the left factor is an operator from $\mathcal{H} \oplus \mathcal{H}$ back to $\mathcal{G}=\mathcal{H}_{1 / 2} \oplus \mathcal{H}$. If we exchange the order of the factors and define the auxiliary operator $B$ in $\mathcal{H} \oplus \mathcal{H}$ by

$$
\begin{aligned}
B: & =\left(\begin{array}{cc}
-2 \Gamma^{-1 / 2} H_{0}^{-1 / 2} V & \Gamma^{-1 / 2} H_{0}^{-1 / 2} \\
I & 0
\end{array}\right)\left(\begin{array}{ccc}
H_{0}^{-1 / 2} \Gamma^{-1 / 2} & 0 \\
0 & I
\end{array}\right) \\
& =\left(\begin{array}{cc}
-\Gamma^{-1 / 2} S^{*} H_{0}^{-1 / 2} \Gamma^{-1 / 2}-\Gamma^{-1 / 2} H_{0}^{-1 / 2} S \Gamma^{-1 / 2} & \Gamma^{-1 / 2} H_{0}^{-1 / 2} \\
H_{0}^{-1 / 2} \Gamma^{-1 / 2} & 0
\end{array}\right),
\end{aligned}
$$

then $\rho\left(A^{-1}\right) \backslash\{0\}=\rho(B) \backslash\{0\}$; here we have used that $H_{0}^{-1 / 2} V H_{0}^{-1 / 2}=S^{*} H_{0}^{-1 / 2}=$ $H_{0}^{-1 / 2} S$. For $\mu \in \mathbb{C}, \mu \neq 0$, we have $\mu \in \rho(B)$ if and only if for every $(f g)^{\mathrm{t}} \in \mathcal{H} \oplus \mathcal{H}$ there exists a unique $\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{t}} \in \mathcal{H} \oplus \mathcal{H}$ such that

$$
\begin{aligned}
\left(-\Gamma^{-1 / 2} S^{*} H_{0}^{-1 / 2} \Gamma^{-1 / 2}-\Gamma^{-1 / 2} H_{0}^{-1 / 2} S \Gamma^{-1 / 2}-\mu\right) x+\Gamma^{-1 / 2} H_{0}^{-1 / 2} y & =f \\
H_{0}^{-1 / 2} \Gamma^{-1 / 2} x-\mu y & =g
\end{aligned}
$$

If we divide both equations by $\mu(\neq 0)$ and insert the second into the first, we see that this is equivalent to

$$
\begin{aligned}
-\Gamma^{-1 / 2}\left(\frac{1}{\mu}\left(S^{*} H_{0}^{-1 / 2}+H_{0}^{-1 / 2} S\right)+\Gamma-\frac{1}{\mu^{2}} H_{0}^{-1}\right) \Gamma^{-1 / 2} x & =\frac{1}{\mu} f+\frac{1}{\mu^{2}} \Gamma^{-1 / 2} H_{0}^{-1 / 2} g \\
y & =\frac{1}{\mu}\left(H_{0}^{-1 / 2} \Gamma^{-1 / 2} x-g\right)
\end{aligned}
$$

Since $\Gamma^{-1 / 2}$ is bounded and boundedly invertible, the latter is equivalent to the fact that $\mu$ belongs to the resolvent set of the operator pencil given by

$$
\frac{1}{\mu}\left(S^{*} H_{0}^{-1 / 2}+H_{0}^{-1 / 2} S\right)+\Gamma-\frac{1}{\mu^{2}} H_{0}^{-1}
$$

or, equivalently, $\lambda=\mu^{-1} \in \rho(L)$ with $L$ given by (5.1). This completes the proof of $\rho(A)=\rho(L)$. Finally, for $\lambda \in \mathbb{R},|\lambda|<(1-\|S\|) m$, the estimate

$$
\begin{aligned}
& \left\|S^{*} S-\lambda\left(S^{*} H_{0}^{-1 / 2}+H_{0}^{-1 / 2} S\right)+\lambda^{2} H_{0}^{-1}\right\| \\
& \quad<\|S\|^{2}+2(1-\|S\|) m \frac{1}{m}\|S\|+(1-\|S\|)^{2} m^{2} \frac{1}{m^{2}}=1
\end{aligned}
$$

and (5.1) show that $\lambda \in \rho(L)=\rho(A)$.

Theorem 5.2. Suppose that $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$, that $S=V H_{0}^{-1 / 2}=S_{0}+S_{1}$ with $\left\|S_{0}\right\|<1$ and a compact operator $S_{1}$, and that $1 \in \rho\left(S^{*} S\right)$. Then we have:
i) $\mathcal{K}$ is a Pontryagin space with finite negative index $\kappa$, where $\kappa$ is the number of negative eigenvalues of the operator $I-S^{*} S$.
ii) The self-adjoint operator $A$ in $\mathcal{K}$ has a spectral function with at most finitely many critical points.
iii) The non-real spectrum of $A$ is symmetric with respect to the real axis and consists of at most $\kappa$ pairs of eigenvalues $\lambda, \bar{\lambda}$ of finite type; the algebraic eigenspaces corresponding to $\lambda$ and $\bar{\lambda}$ are isomorphic.
iv) The linear span of all the algebraic eigenspaces corresponding to the eigenvalues of $A$ in the upper (or lower) half plane is a neutral subspace of $\mathcal{K}$ and

$$
\kappa=\sum_{\lambda \in \sigma_{0}(A) \cap \mathbb{R}} \kappa_{\lambda}^{-}(A)+\sum_{\lambda \in \sigma(A) \cap \mathbb{C}^{+}} \operatorname{dim} \mathcal{L}_{\lambda}(A) ;
$$

here $\sigma_{0}(A)$ denotes the set of all eigenvalues of $A$ with non-positive eigenvector.
v) The points of $\sigma(A) \backslash \sigma_{0}(A)$ (which are all real) are spectral points of positive type.
vi) The essential spectrum $\sigma_{\text {ess }}(A)$ is real and

$$
\sigma_{\mathrm{ess}}(A) \cap(-\alpha, \alpha)=\emptyset,
$$

where $\alpha:=\left(1-\left\|S_{0}\right\|\right) m$.
vii) The operator A generates a strongly continuous group $(\exp (\mathrm{i} A t))_{t \in \mathbb{R}}$ of unitary operators in $\mathcal{K}$ and hence the Cauchy problem

$$
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}=\mathrm{i} A \mathbf{x}, \quad \mathbf{x}(0)=\mathbf{x}_{0}
$$

has the unique solution $\mathbf{x}(t)=\exp (\mathrm{i} A t) \mathbf{x}_{0}, t \in \mathbb{R}$, for all initial values $\mathbf{x}_{0} \in \mathcal{K}$.
Proof. i) Assumption (iii) on $S$ implies that

$$
I-S^{*} S=I-S_{0}^{*} S_{0}+K
$$

where $I-S_{0}^{*} S_{0}$ is uniformly positive and $K$ is a compact operator in $\mathcal{H}$. Thus $I-S^{*} S$ has only a finite number $\kappa$ of negative eigenvalues and hence $\mathcal{K}$ is a Pontryagin space of finite negative index $\kappa$ by Lemma 4.1.
ii), iii), iv), v), and vii) are immediate consequences of Theorem 4.3 and of i) by [Lan82] (see also Sects. 2.4 and 2.5).
vi) We define an operator $A_{0}$ in $\mathcal{G}$ by

$$
A_{0}:=\left(\begin{array}{cc}
0 & I  \tag{5.2}\\
H_{0}^{1 / 2}\left(I-S_{0}^{*} S_{0}\right) H_{0}^{1 / 2} & 2 V
\end{array}\right)
$$

By the spectral mapping theorem (see [EE87, Thm. IX.2.3]), we have

$$
\begin{align*}
& \lambda \in \sigma_{\mathrm{ess}}(A) \Longleftrightarrow \mu:=\lambda^{-1} \in \sigma_{\mathrm{ess}}\left(A^{-1}\right), \\
& \lambda \in \sigma_{\mathrm{ess}}\left(A_{0}\right) \Longleftrightarrow \mu:=\lambda^{-1} \in \sigma_{\mathrm{ess}}\left(A_{0}^{-1}\right) . \tag{5.3}
\end{align*}
$$

The difference $A^{-1}-A_{0}^{-1}$ is compact since $S_{1}$ is compact by assumption; in fact,

$$
A^{-1}-A_{0}^{-1}=A^{-1}\left(A_{0}-A\right) A_{0}^{-1}=A^{-1}\left(\begin{array}{cc}
0 & 0 \\
H_{0}^{1 / 2}\left(S_{1}^{*} S_{0}+S_{0}^{*} S_{1}\right) H_{0}^{1 / 2} & 0
\end{array}\right) A_{0}^{-1}
$$

which is compact since $S_{1}^{*} S_{0}+S_{0}^{*} S_{1}$ is compact and

$$
\begin{aligned}
A^{-1}\left(\begin{array}{cc}
0 & 0 \\
0 & H_{0}^{1 / 2}
\end{array}\right) & =\left(\begin{array}{cc}
0 & H^{-1} H_{0}^{1 / 2} \\
0 & 0
\end{array}\right)=\left(\begin{array}{ccc}
0 & H_{0}^{-1 / 2}\left(I-S^{*} S\right) \\
0 & 0
\end{array}\right) \\
\left(\begin{array}{cc}
H_{0}^{1 / 2} & 0 \\
0 & 0
\end{array}\right) A_{0}^{-1} & =\left(\begin{array}{ccc}
-2\left(I-S_{0}^{*} S_{0}\right)^{-1} H_{0}^{-1 / 2} V & \left(I-S_{0}^{*} S_{0}\right)^{-1} H_{0}^{-1 / 2} \\
0 & 0
\end{array}\right)
\end{aligned}
$$

are bounded. By iii), $\sigma(A)$ has empty interior as a subset of $\mathbb{C}$. In addition, part iii) applied to $A$ and $A_{0}$ shows that each of the at most two components of $\mathbb{C} \backslash \sigma(A)$ contains a point in $\rho\left(A_{0}\right)$. Hence, by [RS78, Lemma XIII.4], $\sigma_{\text {ess }}\left(A^{-1}\right)=\sigma_{\text {ess }}\left(A_{0}^{-1}\right)$ and thus, by (5.3),

$$
\begin{equation*}
\sigma_{\mathrm{ess}}(A)=\sigma_{\mathrm{ess}}\left(A_{0}\right) \tag{5.4}
\end{equation*}
$$

Now Lemma 5.1 applied to $A_{0}$ shows that $(-\alpha, \alpha) \subset \rho\left(A_{0}\right)$ and, consequently, $\sigma_{\text {ess }}(A) \cap$ $(-\alpha, \alpha)=\sigma_{\text {ess }}\left(A_{0}\right) \cap(-\alpha, \alpha)=\emptyset$.

The special cases that $S$ is compact or that $\|S\|<1$ in Theorem 5.2 have been considered before (see, e.g., [LN96] and [Naj79], respectively):

Remark 5.3. Suppose that $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ and $1 \in \rho\left(S^{*} S\right)$.
i) If $V$ is $H_{0}^{1 / 2}$-compact, then

$$
\sigma_{\mathrm{ess}}(A)=\left\{\lambda \in \mathbb{C}: \lambda^{2} \in \sigma_{\mathrm{ess}}\left(H_{0}\right)\right\}
$$

ii) If $\left\|V H_{0}^{-1 / 2}\right\|<1$, then $\kappa=0, \mathcal{K}$ is a Hilbert space, $A$ is self-adjoint in this Hilbert space, and

$$
\sigma(A) \cap(-\alpha, \alpha)=\emptyset
$$

with $\alpha=\left(1-\left\|V H_{0}^{-1 / 2}\right\|\right) m$.
Proof. i) If $V$ is $H_{0}^{1 / 2}$-compact, we can choose $S_{0}=0$ and $S_{1}=V H_{0}^{-1 / 2}$ in Assumption (iii). Then (5.4), (5.3), and (5.2) show that

$$
\lambda \in \sigma_{\mathrm{ess}}(A) \Longleftrightarrow \lambda^{-1} \in \sigma_{\mathrm{ess}}\left(A_{0}^{-1}\right)=\sigma_{\mathrm{ess}}\left(\left(\begin{array}{cc}
-2 H_{0}^{-1} V & H_{0}^{-1} \\
I & 0
\end{array}\right)\right)
$$

If we define

$$
D:=\left(\begin{array}{cc}
0 & H_{0}^{-1} \\
I & 0
\end{array}\right)
$$

then

$$
A_{0}^{-1}-D=\left(\begin{array}{cc}
-2 H_{0}^{-1} V & H_{0}^{-1} \\
I & 0
\end{array}\right)-\left(\begin{array}{cc}
0 & H_{0}^{-1} \\
I & 0
\end{array}\right)=\left(\begin{array}{cc}
-2 H_{0}^{-1} V & 0 \\
0 & 0
\end{array}\right)
$$

is compact by assumption. Moreover, by Theorem 5.2 iii) and v), $\sigma\left(A_{0}^{-1}\right)$ has empty interior as a subset of $\mathbb{C}$ and $\mathbb{C} \backslash \sigma\left(A_{0}^{-1}\right)$ consists of only one component containing points in $\rho(D)$ (e.g., all non-real points of $\mathbb{C} \backslash \sigma\left(A_{0}^{-1}\right)$ ). Hence [RS78, Lemma XIII.4] shows that $\sigma_{\text {ess }}\left(A_{0}^{-1}\right)=\sigma_{\text {ess }}(D)$. Now the fact that

$$
\lambda^{-1} \in \sigma_{\mathrm{ess}}(D) \Longleftrightarrow \lambda^{-2} \in \sigma_{\mathrm{ess}}\left(H_{0}^{-1}\right)
$$

which is not difficult to check (see, e.g., [HM01]), completes the proof.
ii) is immediate from Theorem 5.2 v ) if we choose $S_{0}=V H_{0}^{-1 / 2}$ and $S_{1}=0$ in Assumption (iii).

Remark 5.4. If, under Assumption (iii), Assumption (ii) is not satisfied, that is, $1 \in$ $\sigma_{\mathrm{p}}\left(S^{*} S\right)$, then 0 is an isolated eigenvalue of finite multiplicity of the self-adjoint operator $H$ and, with $\mathcal{N}_{0}:=\operatorname{ker} H=\operatorname{ker}\left(I-S^{*} S\right) H_{0}^{1 / 2}$, the subspace $\mathcal{N}_{0} \oplus\{0\}$ is the isotropic subspace of the inner product space $(\mathcal{K},\langle\cdot, \cdot\rangle)$. Then the factor space $\widetilde{\mathcal{K}}=\mathcal{K} / \mathcal{N}_{0}$ is a Pontryagin space with negative index again given by the number of negative eigenvalues of $I-S^{*} S$. Since ker $A \equiv \mathcal{N}_{0} \oplus\{0\}$, the operator $A$ induces a self-adjoint operator $\widetilde{A}$ in this Pontryagin space $\widetilde{\mathcal{K}}$. Then all claims of Theorem 5.2 remain true for $\widetilde{A}$.

## 6. Assumptions for the Klein-Gordon Equation in $\mathbb{R}^{\boldsymbol{n}}$

In this section we consider the example of the Klein-Gordon equation in $\mathbb{R}^{n}$ for which $\mathcal{H}=L_{2}\left(\mathbb{R}^{n}\right)$ with norm $\|\cdot\|_{2}$ and scalar product $(\cdot, \cdot)_{2}, H_{0}=-\Delta+m^{2}$, and $V$ stands for the operator of multiplication by a function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}$. In this case, sufficient conditions on the potential $V$ will be established that guarantee the assumptions
Ass. (i) $\mathcal{D}\left(H_{0}^{1 / 2}\right) \subset \mathcal{D}(V)$ or, equivalently, $V: \mathcal{H}_{1 / 2} \rightarrow \mathcal{H}$ is bounded,
Ass. (iii) $S=V H_{0}^{-1 / 2}=S_{0}+S_{1}$ with $\left\|S_{0}\right\|<1$ and a compact operator $S_{1}$, which were used in the previous sections. Obviously, (iii) is stronger than (i). Note that, according to Remark 5.4, Assumption (ii), that is, $1 \in \rho\left(S^{*} S\right)$, is not an essential restriction and thus will not be considered here.

It is well-known (see [Tri92, Sects. 1.3.1, 1.3.2]) that for $H_{0}=-\Delta+m^{2}$ the space $\mathcal{H}_{1 / 2}=\mathcal{D}\left(H_{0}^{1 / 2}\right)$ is the Sobolev space of order 1 associated with $L_{2}\left(\mathbb{R}^{n}\right)$ :

$$
\mathcal{H}_{1 / 2}=W_{2}^{1}\left(\mathbb{R}^{n}\right)
$$

Hence Assumption (i) holds if and only if $W_{2}^{1}\left(\mathbb{R}^{n}\right) \subset \mathcal{D}(V)$ or, equivalently, if there exist constants $a, b \geq 0$ such that

$$
\begin{equation*}
\|V u\|_{2} \leq a\|u\|_{2}+b\left\|\left(-\Delta+m^{2}\right)^{1 / 2} u\right\|_{2}, \quad u \in W_{2}^{1}\left(\mathbb{R}^{n}\right) \tag{6.1}
\end{equation*}
$$

this is equivalent to the $\left(-\Delta+m^{2}\right)$-form boundedness of $V^{2}$, that is,

$$
\left(V^{2} u, u\right)_{2} \leq a(u, u)_{2}+b\left(\left(-\Delta+m^{2}\right) u, u\right)_{2}, \quad u \in W_{2}^{1}\left(\mathbb{R}^{n}\right)
$$

Assumption (iii) holds if $V=V_{0}+V_{1}$, where $W_{2}^{1}\left(\mathbb{R}^{n}\right) \subset \mathcal{D}\left(V_{i}\right)$ for $i=0,1, V_{0}$ satisfies (6.1) with $a, b \geq 0$ such that

$$
\begin{equation*}
\frac{a}{m}+b<1 \tag{6.2}
\end{equation*}
$$

that is, $S_{0}=V_{0}\left(-\Delta+m^{2}\right)^{-1 / 2}$ is a strict contraction, and $S_{1}=V_{1}\left(-\Delta+m^{2}\right)^{-1 / 2}$ is compact.

Many different sufficient conditions for the relative boundedness as well as for the relative compactness of a multiplication operator with respect to $\left(-\Delta+m^{2}\right)^{1 / 2}$ have been established (see, e.g., [Kat82, RS75, Sim71] and the more specialized references therein). In the following we formulate two well-known sufficient conditions in terms of $L_{p}$-spaces and Rollnik classes. We start with a well-known relative compactness result, which, for $p=3$, goes back to Brezis and Kato (see [BK79]).

Theorem 6.1. If $n \geq 3$ and $V \in L_{p}\left(\mathbb{R}^{n}\right)$ with $n \leq p<\infty$, then $V$ is $\left(-\Delta+m^{2}\right)^{1 / 2}$ compact. ${ }^{2}$

Proof. The operator of multiplication with $V$ in $L_{2}\left(\mathbb{R}^{n}\right)$ is $\left(-\Delta+m^{2}\right)^{1 / 2}$-compact if and only if $V\left(-\Delta+m^{2}\right)^{-1 / 2}$ is compact, that is, if $\left(u_{m}\right)$ is a sequence in the form domain $W_{2}^{1}\left(\mathbb{R}^{n}\right)$ of $-\Delta+m^{2}$ that converges weakly to 0 , then $\left(V u_{m}\right)$ converges strongly to 0 in $L_{2}\left(\mathbb{R}^{n}\right)$. Now let $n \leq p<\infty$ and $W \in L_{p}\left(\mathbb{R}^{n}\right)$, and set $q:=p /(p-2)$. By Hölder's inequality and the boundedness of the embedding of the Sobolev space $W_{2}^{1}\left(\mathbb{R}^{n}\right)$ into $L_{2 q}\left(\mathbb{R}^{n}\right)$ (which holds since $p \geq n$, see [EE87, Theorem V.3.7]), we have

$$
\begin{equation*}
\|W u\|_{2}^{2} \leq\|W\|_{p}^{2}\|u\|_{2 q}^{2} \leq c^{2}\|W\|_{p}^{2}\|u\|_{2,1}^{2}, \quad u \in W_{2}^{1}\left(\mathbb{R}^{n}\right) \tag{6.3}
\end{equation*}
$$

here $c$ is the norm of the embedding of $W_{2}^{1}\left(\mathbb{R}^{n}\right)$ into $L_{2 q}\left(\mathbb{R}^{n}\right)$. Assume now that $\left(u_{m}\right) \subset$ $W_{2}^{1}\left(\mathbb{R}^{n}\right)$ converges weakly to 0 and let $\varepsilon>0$. Since $C_{0}^{\infty}\left(\mathbb{R}^{n}\right) \subset L_{p}\left(\mathbb{R}^{n}\right)$ is dense, there exists a function $V_{\varepsilon} \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ such that $\left\|V-V_{\varepsilon}\right\|_{p}<\varepsilon$. Let $\Omega_{\varepsilon}:=\operatorname{supp} V_{\varepsilon}$ and choose $C_{\varepsilon} \geq 0$ such that $\left|V_{\varepsilon}\right| \leq C_{\varepsilon}$. Then we obtain, using (6.3) with $W=V-V_{\varepsilon}$,

$$
\begin{equation*}
\left\|V u_{m}\right\|_{2} \leq\left\|\left(V-V_{\varepsilon}\right) u_{m}\right\|_{2}+\left\|V_{\varepsilon} u_{m}\right\|_{2} \leq c \varepsilon\left\|u_{m}\right\|_{2,1}+C_{\varepsilon}\left\|u_{m} \mid \Omega_{\varepsilon}\right\|_{2} . \tag{6.4}
\end{equation*}
$$

Since $\left(u_{m}\right)$ converges weakly in $W_{2}^{1}\left(\mathbb{R}^{n}\right)$, it is a bounded sequence in $W_{2}^{1}\left(\mathbb{R}^{n}\right)$. Hence, choosing $\varepsilon$ sufficiently small, the first term can be made arbitrarily small. The second term becomes arbitrarily small for sufficiently large $m$ : in fact, $W_{2}^{1}\left(\Omega_{\varepsilon}\right)$ is compactly embedded in $L_{2}\left(\Omega_{\varepsilon}\right)$ since $\Omega_{\varepsilon}$ is bounded (see [EE87, Theorem V.3.7]) and thus ( $u_{m} \mid \Omega_{\varepsilon}$ ) converges to 0 strongly in $L_{2}\left(\Omega_{\varepsilon}\right)$.

For $n=3$, a criterion for the relative form-boundedness of $V^{2}$ with respect to $-\Delta+m^{2}$ can be formulated in terms of Rollnik potentials, see [RS75], [Sim71]: A measurable function $W: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is said to belong to the class $R$ of Rollnik potentials if

$$
\|W\|_{R}^{2}:=\int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{|W(x) \| W(y)|}{|x-y|^{2}} \mathrm{~d} x \mathrm{~d} y<\infty
$$

Theorem 6.2. If $n=3$ and $V: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is a measurable function such that $V^{2} \in$ $R+L_{\infty}\left(\mathbb{R}^{3}\right)$, then $V$ is $\left(-\Delta+m^{2}\right)^{1 / 2}$-bounded (with relative bound 0 ). In particular, if $V^{2} \in R$, we have

$$
\left\|V\left(-\Delta+m^{2}\right)^{-1 / 2}\right\| \leq \sqrt{\frac{\left\|V^{2}\right\|_{R}}{4 \pi}}
$$

[^2]Proof. The first statement may be found in [RS75, Theorem X.19], for the proof see [Sim71, Theorem I.21]. For the second claim, we note that, by [Sim71, (I.13)],

$$
\begin{aligned}
\left(|V|\left(-\Delta+m^{2}\right)^{-1}|V| u, u\right)_{2} & \leq \frac{1}{4 \pi}\left(\int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{\left|V^{2}(x)\right| \mathrm{e}^{-m|x-y|}\left|V^{2}(y)\right|}{|x-y|^{2}} \mathrm{~d} x \mathrm{~d} y\right)^{1 / 2}\|u\|_{2}^{2} \\
& \leq \frac{1}{4 \pi}\left\|V^{2}\right\|_{R}\|u\|_{2}^{2}, \quad u \in \mathcal{D}(|V|)=\mathcal{D}(V),
\end{aligned}
$$

which implies that $\left\|\left(-\Delta+m^{2}\right)^{-1 / 2}|V|\right\| \leq \sqrt{\left\|V^{2}\right\|_{R} /(4 \pi)}$. Hence the densely defined operator $\left(-\Delta+m^{2}\right)^{-1 / 2}|V|$ is bounded and

$$
\begin{aligned}
\left\|V\left(-\Delta+m^{2}\right)^{-1 / 2}\right\| & =\left\||V|\left(-\Delta+m^{2}\right)^{-1 / 2}\right\|=\left\|\left(|V|\left(-\Delta+m^{2}\right)^{-1 / 2}\right)^{*}\right\| \\
& =\left\|\overline{\left(-\Delta+m^{2}\right)^{-1 / 2}|V|}\right\| \leq \sqrt{\frac{\left\|V^{2}\right\|_{R}}{4 \pi}}
\end{aligned}
$$

follows.
Remark 6.3. For $n=3$, Theorem 6.1 shows that every $V \in L_{p}\left(\mathbb{R}^{3}\right)+L_{\infty}\left(\mathbb{R}^{3}\right)$ with $3 \leq$ $p<\infty$ is $\left(-\Delta+m^{2}\right)^{1 / 2}$-bounded (with relative bound 0 ). This condition is more restrictive than the condition $V^{2} \in R+L_{\infty}\left(\mathbb{R}^{3}\right)$ in Theorem 6.2. Indeed, $V \in L_{p}\left(\mathbb{R}^{3}\right)+L_{\infty}\left(\mathbb{R}^{3}\right)$ with $p \geq 3$ implies that $V^{2} \in L_{p / 2}\left(\mathbb{R}^{3}\right)+L_{p}\left(\mathbb{R}^{3}\right)+L_{\infty}\left(\mathbb{R}^{3}\right) \subset R+L_{\infty}\left(\mathbb{R}^{3}\right)$ since $L_{q}\left(\mathbb{R}^{3}\right)+L_{\infty}\left(\mathbb{R}^{3}\right) \subset R+L_{\infty}\left(\mathbb{R}^{3}\right)$ for $q \geq 3 / 2$ (see [Sim71, Corollary I.2]).

The Coulomb potential $V(x)=\gamma /|x|, x \in \mathbb{R}^{n} \backslash\{0\}$, does not have relative bound 0 with respect to $\left(-\Delta+m^{2}\right)^{1 / 2}$; therefore neither Theorem 6.1 nor Theorem 6.2 apply to it. In this case, however, Assumption (i) is an immediate consequence of the Hardy inequality.

Proposition 6.4. The Coulomb potential $V(x)=\gamma /|x|, x \in \mathbb{R}^{n} \backslash\{0\}$, with $\gamma \in \mathbb{R}$ satisfies Assumption (i) for $n \geq 3$; in fact,

$$
\left\|V\left(-\Delta+m^{2}\right)^{-1 / 2}\right\| \leq \frac{2|\gamma|}{n-2}
$$

Proof. The classical Hardy inequality (see [HLP88, Theorem 330]) shows that, for $u \in W_{2}^{1}\left(\mathbb{R}^{n}\right)$,

$$
\|V u\|_{2}^{2} \leq \frac{4 \gamma^{2}}{(n-2)^{2}}\|\nabla u\|_{2}^{2} \leq \frac{4 \gamma^{2}}{(n-2)^{2}}\left\|\left(-\Delta+m^{2}\right)^{1 / 2} u\right\|_{2}^{2},
$$

which yields the desired estimate.
As a consequence of Theorems 6.1, 6.2, and Proposition 6.4, we obtain:
Example 6.5. Let $n \geq 3$. Assumption (iii) (and hence (i)) is satisfied if

$$
V=V_{0}+V_{1},
$$

where $V_{1} \in L_{p}\left(\mathbb{R}^{n}\right)$ with $n \leq p<\infty$, and for $V_{0}$ one of the following holds:
i) $V_{0} \in L_{\infty}\left(\mathbb{R}^{n}\right)$ with $\left\|V_{0}\right\|_{\infty}<m$,
ii) $V_{0}(x)=\gamma /|x|, x \in \mathbb{R}^{n} \backslash\{0\}$, with $\gamma \in \mathbb{R}$ such that $|\gamma|<(n-2) / 2$,
and, in the particular case $n=3$,
iii) $V_{0}^{2} \in R$ with $\left\|V_{0}^{2}\right\|_{R}<4 \pi$.

Note that the admission of the relatively compact part $V_{1}$ of $V$, which is not subject to any relative norm bound, gives rise to complex eigenvalues. This was avoided in earlier papers by assuming that $V_{1}=0$ (see, e.g., [SSW40] for case i) and [Ves83] for case ii)).

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[^1]:    ${ }^{1}$ This manuscript was the starting point for the present paper; unfortunately, Professor Branko Najman died in August 1996.

[^2]:    2 We thank W.D. Evans for communicating this result to us.

