Ergod. Th. & Dynam. Sys. (2006), **26**, 621–651
doi:10.1017/S0143385705000593

Lifts of Lipschitz maps and horizontal fractals in the Heisenberg group

ZOLTÁN M. BALOGH†, REGULA HOEFER-ISENEGGER† and
JEREMY T. TYSON‡

† *Department of Mathematics, University of Berne, Sidlerstrasse 5,*
3012 Berne, Switzerland
(e-mail: {zoltan.balogh,regula.hoefer-isenegger}@math.unibe.ch)
‡ *Department of Mathematics, University of Illinois, 1409 West Green Street,*
Urbana, IL 61801, USA
(e-mail: tyson@math.uiuc.edu)

(*Received* 19 *January* 2004 *and accepted in revised form* 2 *February* 2005)

Abstract. We consider horizontal iterated function systems in the Heisenberg group $\mathbb{H}^1$, i.e. collections of Lipschitz contractions of $\mathbb{H}^1$ with respect to the Heisenberg metric. The invariant sets for such systems are so-called *horizontal fractals*. We study questions related to connectivity of horizontal fractals and regularity of functions whose graph lies within a horizontal fractal. Our construction yields examples of horizontal BV (bounded variation) surfaces in $\mathbb{H}^1$ that are in contrast with the non-existence of horizontal Lipschitz surfaces which was recently proved by Ambrosio and Kirchheim (Rectifiable sets in metric and Banach spaces. *Math. Ann.* **318**(3) (2000), 527–555).

1. *Introduction*

In this paper we study connectivity of horizontal fractals in the Heisenberg group and regularity of selection maps into horizontal fractals.

Analysis of the Heisenberg group is motivated by its appearance in the analysis of several complex variables and in quantum mechanics. In addition, as the simplest non-Abelian example, the Heisenberg group serves as a testing ground for questions and conjectures on more general Carnot groups and sub-Riemannian spaces. Geometric measure theory and rectifiability play an important role in these settings in connection with sub-elliptic PDE's and control theory. For further information, we refer to [**25**, Ch. XIII and XIV] and [**14**]. For more recent results in the subject we refer to [**3, 5, 12, 13, 18**].

Let us recall that the (first) *Heisenberg group* $\mathbb{H} = \mathbb{H}^1$ is the unique non-Abelian Carnot group of rank two and dimension three. Explicitly, $\mathbb{H} = \mathbb{R}^3$ with the group law

$$(x, t) * (x', t') = (x + x', t + t' + 2\langle x, Jx'\rangle), \tag{1.1}$$

where $J : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the map

$$J(x_1, x_2) = (-x_2, x_1)$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^2$.

The sub-Riemannian nature of $\mathbb{H}$ is reflected in the so-called *horizontal distribution* $H\mathbb{H}$, which is the distinguished subbundle of the full tangent bundle $T\mathbb{H}$ defined by

$$H_p\mathbb{H} := \operatorname{span}\{X_p, Y_p\}.$$

Here X and Y denote the left-invariant vector fields in $\mathbb{H}$ whose values at a point $p = (x_1, x_2, t)$ are

$$X_p = \partial_{x_1} + 2x_2\partial_t, \quad Y_p = \partial_{x_2} - 2x_1\partial_t.$$

Equivalently, $H_p\mathbb{H}$ can be characterized as the kernel of the canonical contact form $d\tau = dt + 2x_1 dx_2 - 2x_2 dx_1$ on $\mathbb{H}$ at the point p.

We denote by $\tau_{p_0} : \mathbb{H} \to \mathbb{H}$, $p_0 \in \mathbb{H}$, the *left translation* $\tau_{p_0}(p) = p_0 * p$, and by $\delta_\epsilon : \mathbb{H} \to \mathbb{H}$, $\epsilon > 0$, the *dilation*

$$\delta_\epsilon(x, t) = (\epsilon x, \epsilon^2 t).$$

Two natural and equivalent metrics occupy a central role in the sub-Riemannian geometry of $\mathbb{H}$. The first is the so-called *control metric* d_{C} or *Carnot-Carathéodory metric* (*CC metric*). This metric is defined as a length metric using the horizontal vector fields X, Y. The second goes by several names in the literature (*gauge metric* or *Korányi metric*). We will refer to this latter metric as the *Heisenberg metric* and denote it by d_{H}. Explicitly,

$$d_{\mathrm{H}}(p, q) = |p^{-1} * q|_{\mathrm{H}}, \quad p, q \in \mathbb{H}, \tag{1.2}$$

where $*$ denotes the group law from (1.1) and $|\cdot|_{\mathrm{H}}$ denotes the Heisenberg norm given by

$$|(x, t)|_{\mathrm{H}} = (|x|^4 + t^2)^{1/4}. \tag{1.3}$$

In this paper we will work entirely with the metric d_{H}. The simple form of the expressions in (1.2), (1.3) makes this metric suitable for the computations which we will carry out.

Each of the metrics d_{C} and d_{H} is homogeneous, that is, $d(\delta_\epsilon p, \delta_\epsilon q) = \epsilon d(p, q)$ for $d = d_{\mathrm{C}}$ or $d = d_{\mathrm{H}}$. It follows that d_{C} and d_{H} are globally bi-Lipschitz equivalent. In fact,

$$\frac{1}{\sqrt{\pi}} d_{\mathrm{C}}(p, q) \le d_{\mathrm{H}}(p, q) \le d_{\mathrm{C}}(p, q) \tag{1.4}$$

for any two points $p, q \in \mathbb{H}$. The constant $1/\sqrt{\pi}$ in (1.4) may be explicitly calculated using the structure of the CC-geodesics in $\mathbb{H}$, see for example Bellaïche [**7**].

The principal objects of study in this paper are invariant sets for iterated function systems in $(\mathbb{H}, d_{\mathrm{H}})$. Recall that an *iterated function system* (for short, an *IFS*) on a complete metric space (X, d) is a finite collection

$$\mathcal{F} = \{f_1, \ldots, f_M\}$$

of contraction maps of (X, d) (i.e. Lipschitz maps with Lipschitz constant strictly less than one). The *invariant set* for $\mathcal{F}$ is the unique non-empty compact set in X which is invariant under the action of the elements of $\mathcal{F}$.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Iterated function systems were studied by Hutchinson [**16**] as a convenient language for describing the character of fractal objects and have since been used repeatedly in this context. The case of similarity maps on $X = \mathbb{R}^n$ gives rise to many of the standard examples of self-similar fractals such as the Cantor set, the von Koch snowflake curve and the Sierpinski gasket and carpet. See Falconer [**10, 11**] for additional information.

In order to ensure that our study of iterated function systems in the Heisenberg group has non-trivial content, it is necessary to begin with results guaranteeing the existence of suitable Lipschitz self-maps of $\mathbb{H}$. Our first theorem provides such a result. It asserts that every Lipschitz map of the plane, which preserves area up to a constant factor, may be lifted to a Lipschitz map of $\mathbb{H}$.

Definition 1.5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. A map $F : \mathbb{H} \to \mathbb{H}$ is called a *lift* of f if $\pi \circ F = f \circ \pi$.

Here $\pi : \mathbb{H} \to \mathbb{R}^2$ denotes the projection map

$$\pi(x, t) = x.$$

We now state our first theorem. Set $c = (2 + \sqrt{3})^{1/4} \approx 1.3899\ldots$.

THEOREM 1.6. (Existence and uniqueness of horizontal Lipschitz lifts) *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be* r*-Lipschitz with* $\det Df \equiv \lambda$ *almost everywhere. Then there exists a* cr*-Lipschitz lift* $F : (\mathbb{H}, d_{\mathrm{H}}) \to (\mathbb{H}, d_{\mathrm{H}})$.

If $\tilde{F}$ *is another Lipschitz lift of* f*, then* $\tilde{F}(x, t) = F(x, t + \tau)$ *for some* $\tau \in \mathbb{R}$.

Conversely, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is Lipschitz with Lipschitz lift, then there exists* $\lambda \in \mathbb{R}$ *so that* $\det Df \equiv \lambda$ *almost everywhere.*

As will be shown in the proof, an explicit formula for the Lipschitz lift F is

$$F(x, t) = (f(x), \lambda t + h_0(x)) \tag{1.7}$$

where

$$\nabla h_0 = 2(\lambda \cdot J - Df^* \cdot Jf) \tag{1.8}$$

almost everywhere.

The lifted map F from Theorem 1.6 is a *(generalized) contactomorphism*, i.e. F preserves the canonical Heisenberg contact form up to a multiplicative factor. Theorem 1.6 shows that the existence of such a lift is equivalent to the fact that the base map is (up to a multiplicative factor) a *symplectomorphism* of $\mathbb{R}^2$. The interplay between symplectic geometry in dimension $2n$ and contact geometry in dimension $2n + 1$ is a classical subject in (smooth) Riemannian geometry, dating back to the fundamental work of Boothby and Wang [**8**]. For appearances of this idea in connection with quasiconformal maps on the Heisenberg group, see Korányi and Reimann [**19, 20**].

For non-smooth maps, a theorem of Capogna and Tang [**9**] asserts that any L-Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $\det Df$ almost everywhere equal to a constant may be lifted to an L-Lipschitz map $F : (\mathbb{H}, d_{\mathrm{C}}) \to (\mathbb{H}, d_{\mathrm{C}})$. From the bi-Lipschitz equivalence (1.4) of the metrics d_{C} and d_{H} we deduce that $F : (\mathbb{H}, d_{\mathrm{H}}) \to (\mathbb{H}, d_{\mathrm{H}})$ is $\sqrt{\pi}\, L$-Lipschitz so the first statement of the above lifting theorem could be essentially deduced from the work of Capogna and Tang. However, our approach is quite different from the method in [**9**];

the advantage being that it yields also the second statement and the explicit formulas (1.7), (1.8) which we need subsequently. Let us mention also that our constant c is smaller than $\sqrt{\pi}$ but we suspect that the conclusion in Theorem 1.6 is still not sharp. Indeed, we do not know whether or not Theorem 1.6 holds with $c = 1$.

We notice here also that an alternative method for constructing Lipschitz self-maps of the Heisenberg group via flows generated by special vector fields, is provided by the work of Korányi and Reimann. It is rather amusing that affine self-maps of $\mathbb{R}^3$ that are Lipschitz in the Heisenberg metric arise always as lifts of affine self-maps of $\mathbb{R}^2$. See Proposition 2.3.

With the preparatory Theorem 1.6 in hand, we return to the subject of iterated function systems. As an immediate application we deduce the following existence theorem for horizontal iterated function systems in the Heisenberg group.

THEOREM 1.9. (Existence of horizontal IFSs) *Let $\mathcal{F} = \{f_1, \ldots, f_M\}$ be an iterated function system on $\mathbb{R}^2$, where each map f_i is r_i-Lipschitz for some $r_i < 1/c$ and satisfies* $\det Df_i \equiv \lambda_i$. *For each i, let F_i be a lift of f_i to $\mathbb{H}$.*

Then $\mathcal{F}_{\mathrm{H}} = \{F_1, \ldots, F_M\}$ is an iterated function system on $\mathbb{H}$. Denoting by K, respectively K_{H}, the invariant set for $\mathcal{F}$, respectively $\mathcal{F}_{\mathrm{H}}$, we have

$$\pi(K_{\mathrm{H}}) = K. \tag{1.10}$$

The invariant sets for IFSs on $\mathbb{H}$ of the type considered in Theorem 1.9 we call *horizontal fractals*. The bulk of this paper concerns the geometry of horizontal fractals. We study questions related to connectivity properties and regularity for functions whose graph lies within a horizontal fractal.

According to Theorem 1.6, the lift of a Lipschitz map of $\mathbb{R}^2$ is only defined up to the ambiguity of a vertical constant. The lifted iterated function system in Theorem 1.9 inherits the same ambiguity. More precisely, the family of all IFSs which are lifts of a given planar IFS $\mathcal{F}$ can be parameterized by a point in $\mathbb{R}^M$, where M denotes the cardinality of $\mathcal{F}$. Several of our results have a generic flavor; almost every lift (with respect to Lebesgue measure on $\mathbb{R}^M$) has a certain property. Our main result concerning the connectivity of K_{H} reads as follows.

THEOREM 1.11. *Let $\mathcal{F} = \{f_1, \ldots, f_M\}$ be an iterated function system in the plane such that* $\det Df_i \equiv \lambda_i$ *for $i = 1, \ldots, M$.*

(i) *There exists $\delta_1 = \delta_1(M)$ such that if*

$$\lambda_{\max} := \max\{|\lambda_1|, \ldots, |\lambda_M|\} < \delta_1$$

and $K = K(\mathcal{F})$ is connected, then $K_{\mathrm{H}} = K(\mathcal{F}_{\mathrm{H}})$ is connected for some horizontal lift $\mathcal{F}_{\mathrm{H}}$.

(ii) *There exists $\delta_2 = \delta_2(M)$ such that if $\lambda_{\max} < \delta_2$ and $\mathcal{F}$ is PCF (post-critically finite), then K_{H} is totally disconnected for almost every horizontal lift $\mathcal{F}_{\mathrm{H}}$.*

We conjecture that this statement holds even without the conditions $\lambda_{\max} < \delta_{1,2}$; see Remark 4.16.

The technical PCF condition will be treated in detail in §3.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Our final results concern the regularity properties of the function between K and K_{H}. To be more precise, consider the set-valued mapping α defined as

$$\alpha(x) := \pi^{-1}(x) \cap K_{\mathrm{H}}, \quad x \in K, \tag{1.12}$$

where K_{H} is a horizontal lift of an invariant set K for an IFS in $\mathbb{R}^2$. We show that each selection of this set-valued map is continuous at the 'irrational' points in K.

THEOREM 1.13. *Each selection β of the set-valued map α on $K = K(\mathcal{F})$ defined in (1.12) is continuous on $K \setminus V_*$.*

Here the members of the set $V_* \subset K$ are the so-called 'irrational' points of K i.e. points which have a unique symbolic representation (cf. §6). Recall that a function β is a *selection* of a set-valued map α if $\beta(x) \in \alpha(x)$ for all x.

Many horizontal fractals are obtained as lifts of classical self-similar examples (Cantor sets, snowflake curves, Sierpinski-type gaskets and carpets, and the like). A basic example which plays a crucial role in this paper is the *Heisenberg square* Q_{H}, which is the invariant set for the principal horizontal lift of the standard planar IFS

$$\mathcal{F} = \{f_0, f_1, f_2, f_3\},$$

where

$$f_0(x) = \tfrac{1}{2}x, \quad f_1(x) = \tfrac{1}{2}(x + e_1),$$
$$f_2(x) = \tfrac{1}{2}(x + e_2), \quad f_3(x) = \tfrac{1}{2}(x + e_1 + e_2).$$

Here $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis vectors in $\mathbb{R}^2$. The invariant set for $\mathcal{F}$ is the unit square $Q = [0, 1]^2$.

See §5 for pictures of several horizontal fractals, including Q_{H}.

The example of the Heisenberg square Q_{H} is not new. Strichartz [**26**] used Q_{H} (and versions of this object in much more general Carnot groups) to construct "dyadic-type" Carnot tilings. See also [**27**]. However, Strichartz obtained Q_{H} in a different way as the graph of an L^∞ function and not as a horizontal lift. Due to our different approach we obtain a more precise regularity result which we state as follows.

THEOREM 1.14. *Let Q_{H} be the principal horizontal lift of $Q = [0, 1]^2$ and let $\beta : Q \to \mathbb{H}$, $\beta(x) = (x, g(x))$, be any selection of the set-valued map $\alpha(x) = \pi^{-1}(x) \cap Q_{\mathrm{H}}$. Then $g : Q^o \to \mathbb{R}$ is a function of bounded variation.*

Here Q^o denotes the interior of Q.

In a subsequent paper [**6**] we consider the problem of calculating the Hausdorff dimension of horizontal fractals. As a consequence of our results in [**6**] we find that

$$\dim_{\mathrm{H}} Q_{\mathrm{H}} = \dim_{\mathrm{E}} Q_{\mathrm{H}} = \dim_{\mathrm{E}} Q = 2;$$

moreover, $0 < \mathcal{H}^2_{\mathrm{H}}(Q_{\mathrm{H}}) < \infty$. Here we denote by $\dim_{\mathrm{H}} A$ the Hausdorff dimension of a set A in $(\mathbb{H}, d_{\mathrm{H}})$, and by $\dim_{\mathrm{E}} A$ the Hausdorff dimension of a set A in $(\mathbb{R}^n, d_{\mathrm{E}})$, $n = 2, 3$. Furthermore, $\mathcal{H}^2_{\mathrm{H}}$ denotes the two-dimensional Hausdorff measure with respect to the metric d_{H}.

Combining this result and Theorem 1.14, we see that there exists a surface $S = g(Q^o)$ in $\mathbb{H}$ with

$$0 < \mathcal{H}^2_{\mathrm{H}}(S) < \infty \tag{1.15}$$

and g a function of bounded variation. This contrasts with a recent result of Ambrosio and Kirchheim [**2**, Theorem 7.2], which states the following: there are no surfaces $S = g(\Omega)$ in $\mathbb{H}$, where Ω is a domain in $\mathbb{R}^2$ and $\beta = (\mathrm{id}, g)$ is a Lipschitz map from Ω to $(\mathbb{H}, d_{\mathrm{H}})$, which satisfy (1.15). In Theorem 6.7 we strengthen this result by replacing d_{H} with d_{E}. In summary, while horizontal Lipschitz surfaces in $\mathbb{H}$ do not exist, there are horizontal BV (bounded variation) surfaces in $\mathbb{H}$. It would be of interest to know whether there exist horizontal surfaces of intermediate regularity.

1.16. *Overview.* The structure of this paper is as follows.

In §2 we prove that self-affine maps of $\mathbb{R}^3$ which are Lipschitz in the Heisenberg metric arise as horizontal lifts of self-affine maps of $\mathbb{R}^2$. The general nonlinear case (Theorem 1.6) is treated next. In contrast with the geometric construction of Capogna and Tang, we present an analytic argument which leads to the explicit formula (1.7) for the lift. The converse statement in Theorem 1.6 is obtained by using Pansu's differentiability theorem on Carnot groups.

In §3 we begin the main subject of the paper with the proof of Theorem 1.9 on horizontal iterated function systems. We collect in §3 a variety of basic results concerning the geometric and topological character of horizontal fractals.

In §4 we address the connectivity of horizontal fractals. In particular, Theorem 1.11 is proven here.

Section 5 is devoted to examples. We present here several horizontal lifts of classical examples and show some computer-generated pictures written in Maple.

In §6 we study the regularity of functions whose graph lies within a horizontal fractal. The main results in this section are Theorem 1.13 and Theorem 1.14.

2. *Lifts of Lipschitz maps from $\mathbb{R}^2$ to $\mathbb{H}$*

Recall that a function $f : X \to Y$ between metric spaces is called *r-Lipschitz*, $r > 0$, if

$$d(f(x), f(y)) \le r d(x, y) \tag{2.1}$$

for all $x, y \in X$. Moreover, f is *Lipschitz* if it is r-Lipschitz for some $r < \infty$. The infimum of those values r for which (2.1) holds for all $x, y \in X$ is called the *Lipschitz constant* of f; we denote this by $\mathrm{LIP}(f)$.

Assume now that $f : \mathbb{R}^n \to \mathbb{R}^n$ is an r-Lipschitz map. Hadamard's inequality implies that

$$|\det Df(x)| \le r^n \tag{2.2}$$

almost everywhere.

The first result of this section indicates that if an affine self-map of $\mathbb{R}^3$ is Lipschitz with respect to d_{H} then it must have a special form. In fact it necessarily appears as a lift of an affine self-map of $\mathbb{R}^2$.

PROPOSITION 2.3. *Let* $F : \mathbb{R}^3 \to \mathbb{R}^3$ *be an affine map of the form*

$$F(x, t) := (Ax + t \cdot a + b, d^{\mathrm{T}} x + ct + \tau),$$

where A *is a real* 2×2 *matrix,* $a, b, d \in \mathbb{R}^2$ *and* $c, \tau \in \mathbb{R}$*. Then* F *is Lipschitz with respect to the Heisenberg distance* d_{H} *if and only if the following relations hold:*

$$a = (0, 0), \quad d = -2A^{\mathrm{T}} Jb, \quad c = \det A. \tag{2.4}$$

The mapping F *is a similarity with respect to the Heisenberg metric if and only if the above relations hold and* A *is a similarity matrix of* $\mathbb{R}^2$*:*

$$A^{\mathrm{T}} A = c^2 \,\mathrm{Id} \quad \textit{or} \quad A^{\mathrm{T}} A = -c^2 \,\mathrm{Id},$$

where $c = \det A$.

Proof. Let $L := \mathrm{LIP}(F)$. Using the Lipschitz condition we have that

$$d_{\mathrm{H}}(F(0, 0), F(x, t)) \le L d_{\mathrm{H}}((0, 0), (x, t))$$

holds for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$. Writing this explicitly and using (1.2), (1.3) we obtain

$$|Ax + ta|^4 + (\langle d, x\rangle + tc + 2\langle Jb, Ax + ta\rangle)^2 \le L^4(|x|^4 + t^2). \tag{2.5}$$

Now set $x = 0$, which yields $|ta|^4 + |tc - 2\langle b, J(ta)\rangle|^2 \le L^4 \cdot t^2$ for all $t \in \mathbb{R}$. This shows that $a = 0$.

Setting $t = 0$ in the above inequality we find

$$|\langle d, x\rangle + 2\langle Jb, Ax\rangle| \le L^2 \cdot |x|^2,$$

for all $x \in \mathbb{R}^2$, from which we derive $d = -2A^{\mathrm{T}} Jb$.

Using the Lipschitz condition on F for points (x_1, t_1) and (x_2, t_2), the above relations, and the identities

$$\langle Ax_1, JA(x_2 - x_1)\rangle = \langle Ax_1, JAx_2\rangle = \det A\langle x_1, Jx_2\rangle,$$

we obtain

$$(c(t_2 - t_1) - 2\det A\langle x_1, Jx_2\rangle)^2 \le L^4(|x_1 - x_2|^4 + (t_2 - t_1 - 2\langle x_1, Jx_2\rangle)^2).$$

Choosing $t_2 - t_1 = 2\langle x_1, Jx_2\rangle$, where $x_1 \ne 0$ and $x_2 = x_1 + s \cdot Jx_1$, $s \in \mathbb{R}$, we find

$$4(c - \det A)^2 \le L^4 s^2 |x_1|^2.$$

Since s is arbitrary, we obtain the last relation $c = \det A$ as well.

The verification of the second statement in the proposition is left to the reader. □

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a general Lipschitz map. Without further assumptions, many different functions $F : \mathbb{H} \to \mathbb{H}$ serve as lifts of f. However, if we require in addition that F be Lipschitz with respect to d_{H}, then F is uniquely determined via the formulas in (1.7) and (1.8). Moreover, such a lift exists if and only if $\det Df$ is constant almost everywhere. This is the content of Theorem 1.6 which we now prove.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Proof of Theorem 1.6. The outline of the proof is as follows. First, we assume that F is a Lipschitz lift of f. Using Pansu's differentiability theorem from [**24**], we find an implicit formula for h_0 in (1.7) which we use to show that $\det Df \equiv \lambda$ almost everywhere. This leads to a more precise formula for h_0, which is very convenient for proving the existence of the required lift under the assumption $\det Df \equiv \lambda$ almost everywhere.

Assume then that f is a Lipschitz map of $\mathbb{R}^2$ with Lipschitz lift F to $\mathbb{H}$. By Rademacher's theorem, f is differentiable almost everywhere. Pansu's differentiability theorem [**24**] implies that F is P-differentiable at almost every point p_0 of $\mathbb{H}$. By definition, this means that there is a homogeneous homomorphism $D_{\mathrm{P}}F(p_0) : \mathbb{H} \to \mathbb{H}$ such that

$$\delta_{1/\epsilon} \circ \tau^{-1}_{F(p_0)} \circ F \circ \tau_{p_0} \circ \delta_\epsilon \tag{2.6}$$

converges locally uniformly to $D_{\mathrm{P}}F(p_0)$ as $\epsilon \to 0$. The *P-differential* of F at p_0 is the group homomorphism $D_{\mathrm{P}}F(p_0)$.

Pick a point $p_0 = (x_0, t_0) \in \mathbb{H}$ such that f is differentiable at x_0 and F is Pansu-differentiable at p_0 as well as at $(x_0, -t_0)$ (this last restriction is only a technical detail which we will use in (2.8)). Almost every point p_0 is of this type.

Our goal is to find the Pansu-differential $D_{\mathrm{P}}F$ at p_0. The projection π commutes throughout (2.6), whence

$$\pi \circ D_{\mathrm{P}}F(p_0) = Df(x_0) \circ \pi. \tag{2.7}$$

Since $D_{\mathrm{P}}F(p_0)$ is a group homomorphism, and in particular a Lipschitz self-mapping of $\mathbb{H}$ we have, by Proposition 2.3,

$$D_{\mathrm{P}}F(p_0)(e_j, 0) = (\partial_{x_j} f(x_0), 0) \quad j = 1, 2,$$

and $D_{\mathrm{P}}F(p_0)(0, 1) = (0, \mu)$ where $\mu = \det Df(x_0)$. Thus,

$$D_{\mathrm{P}}F(p_0) = \begin{pmatrix} Df(x_0) & 0 \\ 0 & \det Df(x_0) \end{pmatrix}.$$

Since F is a lift of f, we may write $F(x, t) = (f(x), h(x, t))$. To find a formula for $h : \mathbb{H} \to \mathbb{R}$, we use the t-coordinate of $D_{\mathrm{P}}F$ in (2.6) to obtain

$$\lim_{\epsilon \to 0} \epsilon^{-2} \begin{pmatrix} h(x_0 + \epsilon x, t_0 + 2\epsilon\langle x_0, Jx\rangle + \epsilon^2 t) \\ -h(x_0, t_0) - 2\langle f(x_0), Jf(x_0 + \epsilon x)\rangle \end{pmatrix} = \det Df(x_0) \cdot t \tag{2.8}$$

for all $(x, t) \in \mathbb{H}$. Setting $x = 0$, we find immediately that

$$\frac{\partial}{\partial t} h(x_0, t_0) \cdot t = \det Df(x_0) \cdot t$$

for all $t \in \mathbb{R}$, and hence

$$h(x_0, t_0) = \det Df(x_0) \cdot t_0 + h_0(x_0) \tag{2.9}$$

for some $h_0 : \mathbb{R}^2 \to \mathbb{R}$. Notice that (2.9) holds for all $t_0 \in \mathbb{R}$ and *a priori* only for almost every $x_0 \in \mathbb{R}^2$. The function h_0 is also *a priori* only almost everywhere defined. However, since h is clearly a continuous function we obtain from (2.9) that h_0 and thus $\lambda(x_0) := \det Df(x_0)$, are both continuous at almost every x_0. Moreover, we see that these functions have continuous extensions such that (2.9) holds everywhere. In what follows we work with these continuous extensions of h_0 and λ and under the assumption that (2.9) holds everywhere.

Now, set $t = 0$ in (2.8), drop one factor of ϵ in the denominator, and use (2.9) to obtain

$$\lim_{\epsilon\to 0}\left(\frac{h_0(x_0+\epsilon x) - h_0(x_0) + t_0(\lambda(x_0+\epsilon x) - \lambda(x_0))}{\epsilon} + 2\langle x_0, Jx\rangle\lambda(x_0+\epsilon x)\right)$$
$$= 2\langle f(x_0), J(Df(x_0)x)\rangle.$$

Using the above equality once with t_0 and once with $-t_0$ and summing, we deduce that

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{h_0(x_0+\epsilon x) - h_0(x_0)}{\epsilon} &= 2\langle f(x_0), J(Df(x_0)x)\rangle - 2\langle x_0, Jx\rangle\cdot\lambda(x_0)\\ &= 2(\lambda(x_0)\langle Jx_0, x\rangle - \langle Df(x_0)^*\cdot Jf(x_0), x\rangle).\end{aligned}$$

Thus, we have shown that $\nabla h_0(x_0)$ exists and satisfies

$$\nabla h_0(x_0) = 2(\lambda(x_0)J(x_0) - Df(x_0)^*\cdot Jf(x_0)) \tag{2.10}$$

for almost every x_0.

We will now show that $\det Df = \lambda$ is almost everywhere equal to a constant. For this we go back to the main equation (2.8) and use the explicit formula for h from (2.9) to obtain

$$\lim_{\epsilon\to 0}\epsilon^{-2}\begin{pmatrix}(\lambda(x_0+\epsilon x) - \lambda(x_0))(t_0 + 2\epsilon\langle x_0, Jx\rangle) + h_0(x_0+\epsilon x) - h_0(x_0)\\ +2\epsilon\lambda(x_0)\langle x_0, Jx\rangle - 2\langle f(x_0), Jf(x_0+\epsilon x)\rangle\end{pmatrix} = 0,$$

from whence we get, by dropping one factor of ϵ in the denominator,

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\lambda(x_0+\epsilon x) - \lambda(x_0)}{\epsilon}t_0 &= 2\langle f(x_0), J(Df(x_0)x)\rangle - 2\lambda(x_0)\langle x_0, Jx\rangle - \nabla h_0(x_0)x\\ &= 0\end{aligned}$$

for all $x \in \mathbb{R}^2$. Hence, by (2.10), $\lambda(x_0) = \det Df(x_0) \equiv \lambda$ for almost every $x_0 \in \mathbb{R}^2$. Using the continuity of λ we see that λ is constant.

The only thing left to prove is that a Lipschitz lift exists whenever $\det Df \equiv \lambda$. Assume that f is r-Lipschitz. Hadamard's determinant inequality (2.2) gives $|\lambda| \le r^2$. Set

$$F(x,t) = (f(x), \lambda t + h_0(x)),$$

where h_0 is given (up to an additive constant) by (2.10). We claim that F is cr-Lipschitz in the metric d_{H}, where $c = (2+\sqrt{3})^{1/4}$.

To see this we compute directly

$$d_{\mathrm{H}}(F(x,t), F(y,s))^4 = |f(y) - f(x)|^4 + |\lambda(s-t) + h_0(y) - h_0(x) - 2\langle f(x), Jf(y)\rangle|^2.$$

The difficult part of the proof will be to estimate the second term on the right side in the above equation. Set therefore

$$A := \lambda(s-t) + h_0(y) - h_0(x) - 2\langle f(x), Jf(y)\rangle.$$

In estimating the above expression we make use of (2.10) which holds only almost everywhere. To make our calculations formally correct we impose some initial restrictions

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

on the points $x, y \in \mathbb{R}^2$. Let us fix first $x \in \mathbb{R}^2$ and a value $\rho > 0$. Since (2.10) holds almost everywhere we conclude that for y in a full $\mathcal{L}^1$-measure set on the circle $\partial B(x, \rho)$ (2.10) holds at the points

$$x_0 = \xi(t) = \frac{y-x}{|y-x|}t + x$$

for $\mathcal{L}^1$ almost every $t \in [0, |y-x|]$. In what follows we work with such points $x, y \in \mathbb{R}^2$. Let us put $f = (u, v)$. Since h_0 is prescribed by its gradient, it is natural to write

$$\begin{aligned} h_0(y) - h_0(x) &= \int_0^{|y-x|} \langle \nabla h_0(\xi(t)), \xi'(t)\rangle\, dt \\ &= 2\int_0^{|y-x|} \langle v(\xi)\nabla u(\xi) - u(\xi)\nabla v(\xi), \xi'\rangle\, dt + 2\lambda \int_0^{|y-x|} \langle J(\xi), \xi'\rangle\, dt \\ &= 2\int_0^{|y-x|} \langle v(\xi)\nabla u(\xi) - u(\xi)\nabla v(\xi), \xi'\rangle\, dt - 2\lambda\langle x, Jy\rangle. \end{aligned}$$

Similarly, we write

$$\begin{aligned} \langle f(x), Jf(y)\rangle &= \langle f(x), J(f(y) - f(x))\rangle \\ &= \int_0^{|y-x|} \langle v(x)\nabla u(\xi) - u(x)\nabla v(\xi), \xi'\rangle\, dt. \end{aligned}$$

Combining the previous two calculations, we find

$$\begin{aligned} A^2 = \bigg| &\lambda(s - t - 2\langle x, Jy\rangle) \\ &+ 2\int_0^{|y-x|} \langle (v(\xi) - v(x))\nabla u(\xi) - (u(\xi) - u(x))\nabla v(\xi), \xi'\rangle\, dt\bigg|^2. \end{aligned} \tag{2.11}$$

Next, let p and q be a Hölder conjugate pair of exponents ($p^{-1} + q^{-1} = 1$) whose exact values will be chosen later. Using the estimates $(x+y)^2 \le px^2 + qy^2$, $x, y \ge 0$, and $|\lambda| \le r^2$ together with the Cauchy–Schwarz inequality, we find

$$\begin{aligned} A^2 &\le pr^4(s - t - 2\langle x, Jy\rangle)^2 \\ &\quad + 4q\bigg(\int_0^{|y-x|} |f(x) - f(\xi)|\sqrt{|\nabla u(\xi)|^2 + |\nabla v(\xi)|^2}\, dt\bigg)^2 \\ &\le pr^4(s - t - 2\langle x, Jy\rangle)^2 + 2qr^4|y-x|^4 \end{aligned} \tag{2.12}$$

since f is r-Lipschitz, and hence $|\nabla u(\xi)|, |\nabla u(\xi)| \le r$.

Putting everything together, we obtain

$$d_{\mathrm{H}}(F(x,t), F(y,s))^4 \le (1 + 2q)r^4|y-x|^4 + pr^4(s - t - 2\langle x, Jy\rangle)^2.$$

Choose the Hölder conjugate pair p and q so that $1 + 2q = p$. Then,

$$d_{\mathrm{H}}(F(x,t), F(y,s))^4 \le (2 + \sqrt{3})d_{\mathrm{H}}((x,t),(y,s))^4.$$

Since the above uniform estimate holds for a dense set of (x, t), (y, s), it holds everywhere by continuity. The proof is complete. □

Remark 2.13. From the proof of the theorem, it is clear that if

$$|h_0(y) - h_0(x) - 2\langle f(x), Jf(y)\rangle + 2\lambda\langle x, Jy\rangle| = 0 \tag{2.14}$$

for all $x, y \in \mathbb{R}^2$, then f and F have the same Lipschitz constant.

Example 2.15. The lifts of an affine map $f(x) = Ax + b$ are given explicitly as

$$F(x, t) = (f(x), \det A \cdot t - 2\langle Ax, Jb\rangle + \tau), \quad \tau \in \mathbb{R}. \tag{2.16}$$

Since $h_0(x) = -2\langle Ax, Jb\rangle$ satisfies (2.14), the Lipschitz constants of F and f agree. This, in conjunction with Proposition 2.3, shows that all affine maps that are Lipschitz in the Heisenberg metric arise as lifts of planar affine mappings.

PROPOSITION 2.17. *Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be Lipschitz maps with* $\det Df_i \equiv \lambda_i$ *almost everywhere, $i = 1, 2$. For each $i = 1, 2$ let F_i be a Lipschitz lift of f_i. Then $F_1 \circ F_2$ is a Lipschitz lift of $f_1 \circ f_2$.*

Proof. That $F_1 \circ F_2$ is a lift follows immediately from the definition. Moreover, a composition of Lipschitz functions is Lipschitz. □

3. *Iterated function systems and horizontal fractals in* $\mathbb{H}$

The lifting Theorem 1.6 is the key to the construction of horizontal iterated function systems on the Heisenberg group. The invariant sets for such systems are so-called *horizontal fractals* in $\mathbb{H}$. We first give the basic existence result for such systems and then proceed to describe various features of horizontal fractals in relation to the planar invariant set of the underlying IFS in $\mathbb{R}^2$.

We begin by reviewing the general theory of iterated function systems in metric spaces. A standard reference is [**17**, Chapter 1], whose notation and terminology we follow.

3.1. *Iterated function systems and invariant sets.* Let X be a complete metric space. A map $f : X \to X$ is a *contraction map* if $\mathrm{LIP}(f) < 1$.

Definition 3.2. An *iterated function system* (IFS) on X is a finite collection $\mathcal{F} = \{f_1, \ldots, f_M\}$ of contraction maps. A set $A \subset X$ is called an *invariant set* for $\mathcal{F}$ if

$$A = f_1(A) \cup \cdots \cup f_M(A). \tag{3.3}$$

The fundamental existence theorem for invariant sets of IFSs reads as follows.

THEOREM 3.4. *Let X be a complete metric space and let $\mathcal{F}$ be an iterated function system on X. Then there exists a unique non-empty compact invariant set $K = K(\mathcal{F})$ for $\mathcal{F}$.*

Henceforth, we use the phrase 'invariant set of $\mathcal{F}$' to refer to the specific set $K(\mathcal{F})$ whose existence is guaranteed by Theorem 3.4.

The proof of Theorem 3.4 uses the completeness of the space of all compact subsets of X with the Hausdorff metric. See [**17**, §1.1].

Theorems 1.6 and 3.4 immediately imply Theorem 1.9.

Proof of Theorem 1.9. That $\mathcal{F}_{\mathrm{H}}$ is an IFS on $\mathbb{H}$ is clear from the definitions. Since π is continuous, $\pi(K_{\mathrm{H}})$ is a non-empty compact set in $\mathbb{R}^2$. The identity

$$\pi(K_{\mathrm{H}}) = \bigcup_i \pi \circ F_i(K_{\mathrm{H}}) = \bigcup_i f_i(\pi(K_{\mathrm{H}}))$$

shows that $\pi(K_{\mathrm{H}})$ is an invariant set for $\mathcal{F}_{\mathrm{H}}$. Then (1.10) follows by the uniqueness assertion in Theorem 3.4. □

We call K_{H} a *horizontal lift* of K. Our main purpose is to study various topological and measure-theoretical properties of K_{H} in relation to corresponding properties of K.

Remarks 3.5. (1) In the case when the maps $f_i \in \mathcal{F}$ are affine contractions, the condition $r_i < 1/c$ may be weakened to $r_i < 1$. See Example 2.15.

(2) Since lifts of Lipschitz maps to $\mathbb{H}$ are not unique, it follows that horizontal lifts of invariant sets are not unique. Indeed, given an IFS $\mathcal{F} = \{f_1, \dots, f_M\}$ on $\mathbb{R}^2$ with invariant set K, the space of all lifted IFSs $\mathcal{F}_{\mathrm{H}} = \{F_1, \dots, F_M\}$ (and hence all horizontal lifts K_{H} of K) is naturally parameterized by an M-dimensional Euclidean space, namely, the t-coordinates of the fixed points of the lifted maps.

Definition 3.6. The *principal horizontal lift* of an IFS $\mathcal{F}$ on $\mathbb{R}^2$ is defined as the IFS $\mathcal{F}_{\mathrm{H}}$ on $\mathbb{H}$ for which all of the fixed points of the lifted maps have t-coordinate zero.

Remark 3.7. The necessity of the technical assumption $\mathrm{LIP}(f_i) < 1/c$ comes from the appearance of the factor c in Theorem 1.6. However, insofar as the existence of horizontal lifts is concerned, this assumption is not restrictive. Indeed, given any IFS $\mathcal{F} = \{f_1, \dots, f_M\}$ on $\mathbb{R}^2$ and any $m \geq 1$, the new IFS $\mathcal{F}^{(m)} := \{f_{i_1} \circ \cdots \circ f_{i_m} : 1 \leq i_1, \dots, i_m \leq M\}$, generates the same invariant set. Denoting by $r_{\max} < 1$ the maximum of the contraction ratios $r_1, \dots, r_M$, we have $\mathrm{LIP}(f) < r_{\max}^m$ for every $f \in \mathcal{F}^{(m)}$. Since $r_{\max}^m < 1/c$ for sufficiently large m, every invariant set K for an IFS in $\mathbb{R}^2$ admits a horizontal lift to $\mathbb{H}$, provided we are willing to view K as the invariant set for a finer collection of contractions as above.

Self-similar sets of Cantor type in the Heisenberg group have been considered earlier by Balogh [**3**] in connection with the distortion of Hausdorff dimension by quasiconformal maps.

3.8. *Symbolic dynamics.* The dynamical attributes of an iterated function system are encoded via its representation as a quotient of a standard sequence space. We study the connection between the symbolic representations of an IFS in $\mathbb{R}^2$ and its horizontal lifts to $\mathbb{H}$.

Let A be an alphabet consisting of the letters $1, \dots, M$. Let $W_m = A^m$, $m \geq 1$ (respectively $\Sigma = A^{\mathbb{N}}$) denote the space of words of length m (respectively words of infinite length) with letters drawn from A. We denote elements of these spaces by concatenation of letters, i.e. $w = w_1 w_2 \cdots w_m \in W_m$ or $w = w_1 w_2 \cdots \in \Sigma$, where $w_j \in A$ for each j. Let $W = \bigcup_{m \geq 1} W_m$ be the collection of all words of finite length. Denote the length of a word $w \in W$ by $|w|$. For fixed $i \in A$, let $\bar{i}$ be the infinite word $iii \cdots$.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Assume now that $\mathcal{F} = \{f_i\}_{i\in A}$ is an IFS in a complete metric space X with invariant set K. As before, denote by $r_i = \mathrm{LIP}(f_i) < 1$ the Lipschitz constant for f_i. For any finite word $w = w_1 \cdots w_m$ let

$$f_w = f_{w_1} \circ \cdots \circ f_{w_m}, \quad r_w = r_{w_1} \cdots r_{w_m}$$

and $K_w = f_w(K)$. Then $K = \bigcup_{w\in W_m} K_w$ for each m and

$$\max_{w\in W_m} \operatorname{diam} K_w \to 0, \text{ as } m \to \infty.$$

We also define K_w for infinite words $w = w_1 w_2 \cdots$ by setting $K_w = \cap_m K_{w_1\cdots w_m}$. In this case, K_w consists of a single point in K.

We denote by σ the shift map on Σ:

$$\sigma(w_1 w_2 w_3 \cdots) = w_2 w_3 \cdots .$$

We consider on Σ the product topology induced by the discrete topology on A and we define a map $p = p_{\mathcal{F}} : \Sigma \to K$ by setting $p(w)$ equal to the unique point in K_w. Then p is a continuous surjection from Σ to K given by

$$p(w) = \lim_{m\to\infty} f_{w_1\cdots w_m}(x_0), \quad w = w_1 w_2 \cdots \in \Sigma, \tag{3.9}$$

where x_0 is an arbitrarily chosen point in X. On the other hand, the mapping $p : \Sigma \to K$ in general is not injective. This is described in the following.

PROPOSITION 3.10. *$p(w) = p(w')$ for $w \neq w' \in \Sigma$ if and only if $p(\sigma^s w) = p(\sigma^s w') \in \bigcup_{i\neq j} K_i \cap K_j$, where $s = s(w, w') := \min\{m : w_m \neq w'_m\} - 1$.*

This is [**17**, Proposition 1.2.5]. Observe that $s = s(w, w')$ if and only if $w_i = w'_i$ for $1 \le i \le s$ and $w_{s+1} \neq w'_{s+1}$.

In the remainder of the paper, we typically work in the situation described in the following assumption.

ASSUMPTION 3.11. *$\mathcal{F} = \{f_i\}_{i\in A}$ is an iterated function system on $\mathbb{R}^2$ and $\mathcal{F}_{\mathrm{H}} = \{F_i\}_{i\in A}$ is a horizontal lift of $\mathcal{F}$ to $\mathbb{H}$.*

We denote by $p : \Sigma \to K$ and $p_{\mathrm{H}} : \Sigma \to K_{\mathrm{H}}$ the canonical surjections from sequence space onto the invariant sets for $\mathcal{F}$ and $\mathcal{F}_{\mathrm{H}}$ respectively.

LEMMA 3.12. *$p = \pi \circ p_{\mathrm{H}}$.*

Proof. This is an immediate consequence of the lifting identity $\pi \circ F = f \circ \pi$ and (3.9). □

3.13. *Post-critical finiteness and the open set condition.* Following Hutchinson [**16**] (but see also Moran [**22**]), we say that an IFS $\mathcal{F} = \{f_i\}_{i\in A}$ satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set O such that $f_j(O) \subset O$ for all j and $f_j(O) \cap f_k(O) = \emptyset$ for all $j \neq k$.

PROPOSITION 3.14. *Let $\mathcal{F}$ and $\mathcal{F}_{\mathrm{H}}$ be as in Assumption 3.11. If $\mathcal{F}$ satisfies the open set condition, then $\mathcal{F}_{\mathrm{H}}$ satisfies the open set condition.*

Proof. By (1.7), each lift $F_j \in \mathcal{F}_{\mathrm{H}}$ can be written in the form

$$F_j(x,t) = (f_j(x), \lambda_j t + h_j(x)),$$

where h_j satisfies the equation $\nabla h_j = 2(\lambda_j J - Df_j^* \cdot Jf_j)$.

Let O be an open set in $\mathbb{R}^2$ which verifies the OSC for $\mathcal{F}$. Then $K \subset \overline{O}$ [**17**, Exercise 1.2]. Choose $R > 0$ so large that $O \subset B(0,R)$, $\max\{|f_1(0)|,\ldots,|f_M(0)|\} \le R$ and $\max\{|h_1(0)|,\ldots,|h_M(0)|\} \le R^2$, and set

$$L := \frac{5R^2}{1 - r_{\max}^2}.$$

We claim that the open set $U := O \times (-L, L)$ verifies the OSC for $\mathcal{F}_{\mathrm{H}}$. Note that the sets $F_j(U)$ are pairwise disjoint since the corresponding sets $f_j(O)$ are. To complete the proof it suffices to show that $F_j(U) \subset U$ for each j. It is enough to show that

$$|\lambda_j t + h_j(x)| < L \tag{3.15}$$

for every $(x,t) \in U$. Applying Theorem 1.6, specifically (2.12), we deduce that

$$|h_j(x) - h_j(0) - 2\langle f_j(0), Jf_j(x)\rangle| \le \sqrt{2+\sqrt{3}}\, r_j^2 |x|^2$$

and so

$$|h_j(x)| < |h_j(0)| + 2|f_j(0)| \cdot |f_j(x)| + 2r_j^2|x|^2 \le 5R^2.$$

Since $|\lambda_j| \le r_j^2 \le r_{\max}^2$,

$$|\lambda_j t + h_j(x)| < r_{\max}^2 L + 5R^2 = L. \qquad \square$$

For a general IFS $\mathcal{F}$ on a metric space X, let $C(\mathcal{F}) := \bigcup_{i \ne j} K_i \cap K_j$ denote the *critical set* for the images $K_i = f_i(K)$. Then the *critical symbols*

$$\mathcal{C} := p^{-1}(C(\mathcal{F}))$$

and the *post-critical symbols*

$$\mathcal{P} := \bigcup_{m \ge 1} \sigma^m(\mathcal{C})$$

are defined as subsets of the sequence space Σ. An IFS is said to be *post-critically finite* (PCF) if it has finitely many post-critical symbols.

The *post-critical set* $V_0 := p(\mathcal{P})$ is defined as the image of the set of post-critical symbols†. Equivalently,

$$V_0 = \bigcup_{w \in W} f_w^{-1}(C(\mathcal{F})) \cap K. \tag{3.16}$$

Many classical examples (Cantor sets, the von Koch snowflake curve, the Sierpinski gasket) are invariant sets of PCF IFSs. The next lemma asserts the uniqueness of symbolic representatives of fixed points in post-critically finite systems. It appears as [**17**, Lemma 1.3.14].

† This terminology differs slightly from that of [**17**], where the terms 'critical' and 'post-critical set' refer to the subsets $\mathcal{C}$ and $\mathcal{P}$ of symbol space.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

LEMMA 3.17. *Let $\mathcal{F}$ be post-critically finite and let a_i be the fixed point for $f_i \in \mathcal{F}$. Then $p^{-1}(a_i) = \{\bar{i}\}$.*

Since points in the critical set have non-unique symbolic representatives, we have the following.

COROLLARY 3.18. *Let $\mathcal{F}$ be post-critically finite. Then the critical set $C(\mathcal{F})$ and the set of fixed points of the maps in $\mathcal{F}$ are disjoint.*

We now return to the setting of Assumption 3.11 and prove that post-critical finiteness of IFSs passes to horizontal lifts.

PROPOSITION 3.19. *Let $\mathcal{F}$ and $\mathcal{F}_{\mathrm{H}}$ be as in Assumption 3.11. If $\mathcal{F}$ is post-critically finite, then $\mathcal{F}_{\mathrm{H}}$ is post-critically finite.*

The converse of Proposition 3.19 is not true. See, for example the Cantor-type lift of the unit square Q in Example 5.1(i).

Proof. From the basic identity $\pi \circ F_j = f_j \circ \pi$ we deduce that the following diagram commutes:

$$\begin{tikzcd} & \Sigma \arrow[dl, "p_{\mathrm{H}}"'] \arrow[dr, "p"] & \\ K_{\mathrm{H}} \arrow[rr, "\pi"] & & K \end{tikzcd}$$

In particular we have that $\pi(C(\mathcal{F}_{\mathrm{H}})) = C(\mathcal{F})$. Since

$$p(\mathcal{C}_{\mathrm{H}}) = p \circ p_{\mathrm{H}}^{-1}(C(\mathcal{F}_{\mathrm{H}})) = \pi(C(\mathcal{F}_{\mathrm{H}})) = C(\mathcal{F}),$$

we have $\mathcal{C}_{\mathrm{H}} \subset \mathcal{C}_{\mathrm{E}}$ and so $\mathcal{P}_{\mathrm{H}} \subset \mathcal{P}_{\mathrm{E}}$. □

4. *Connectivity of horizontal lifts*

In this section we study the connectivity of invariant sets of horizontal lifts. We present here the proof of Theorem 1.11. Under the additional assumption that the planar IFS is affine we have more precise results. In Proposition 4.14 below we identify conditions on a planar affine IFS which imply that the principal horizontal lift is connected. Finally, in Proposition 4.18, we show that, under a set of hypotheses stronger than those of Theorem 1.11(ii), generic lifts of IFS are totally disconnected regardless of the size of the λ_i's.

For the proofs in this section, we use the following characterization of the connectivity of invariant sets, due to Hata [**15**] and presented in [**17**, Theorem 1.6.2]: K is connected if and only if for any $i, j \in A$ there is a chain of indices $i = i_0, i_1, \ldots, i_n = j$ in A so that $f_{i_{k-1}}(K) \cap f_{i_k}(K) \neq \emptyset$ for each $k = 1, \ldots, n$.

Proof of Theorem 1.11. To prove (i) observe that by assumption, Hata's condition holds for each pair $i, j \in A$ for the planar fractal K. Fix $i_0 \in A$ and set $\mathcal{I}_0 = \{i_0\}$. Consider the collection $\mathcal{I}_1 \subset A \setminus \mathcal{I}_0$ consisting of all i such that $f_i(K) \cap f_{i_0}(K) \neq \emptyset$.

We proceed recursively. Assuming that $\mathcal{I}_m$ has already been defined, let $\mathcal{I}_{m+1} \subset A \setminus (\mathcal{I}_0 \cup \cdots \cup \mathcal{I}_m)$ denote the collection of i for which $f_i(K) \cap f_j(K) \neq \emptyset$ for some $j \in \mathcal{I}_m$.

There may be more than one such index $j \in \mathcal{I}_m$ for which this condition is satisfied; choose one such index arbitrarily and call it the *parent* $\hat{i}$ of i.

After at most M steps all elements of A have been exhausted. Define a graph G whose vertices are the elements of A, where two vertices i and j are connected by an edge if $j \in \mathcal{I}_m$ and $i \in \mathcal{I}_{m+1}$ for some m, with $j = \hat{i}$. Note that G is a tree.

Our goal is to show that there is a choice of the parameter $\tau = (\tau_1, \ldots, \tau_M) \in \mathbb{R}^M$ so that Hata's condition is satisfied for the corresponding lift. To do that let us consider a typical junction point for the planar system. Choose a pair of elements $i, j \in A$ which are adjacent vertices in G. There exist points $x_i, x_j \in K$ such that $f_i(x_i) = f_j(x_j)$. We seek a choice of the parameter τ for which

$$F_i(x_i, t_i) = F_j(x_j, t_j) \tag{4.1}$$

for some $t_i, t_j \in \mathbb{R}$ with $(x_i, t_i), (x_j, t_j) \in K_{\mathrm{H}}$. If it is possible to find $\tau \in \mathbb{R}^M$ so that (4.1) holds simultaneously for all pairs (i, j) in question, then Hata's condition will also be satisfied for K_{H} and connectivity will follow.

Observe that (4.1) is equivalent to

$$h_i(x_i, t_i) = h_j(x_j, t_j),$$

where

$$h_i(x, t) = \lambda_i t + h_{0,i}(x) + \tau_i \tag{4.2}$$

is the t-coordinate of $F_i(x, t)$. This implies the equality

$$\lambda_i t_i + h_{0,i}(x_i) + \tau_i = \lambda_j t_j + h_{0,j}(x_j) + \tau_j. \tag{4.3}$$

It is important to notice that the values t_i, t_j depend on the choice of $\tau_1, \ldots, \tau_M$ as required by the condition $(x_i, t_i), (x_j, t_j) \in K_{\mathrm{H}}$.

To study this dependence we use the symbolic representations of $x_i, x_j \in K$. Let w_i, $w_j \in \Sigma$ be such that $pw_i = x_i$, $pw_j = x_j$. Writing

$$w_i = i_1 \cdots i_k \cdots, \quad w_j = j_1 \cdots j_k \cdots$$

gives

$$x_i = \lim_{k\to\infty} f_{i_1} \circ \cdots \circ f_{i_k}(x_0),$$

and

$$x_j = \lim_{k\to\infty} f_{j_1} \circ \cdots \circ f_{j_k}(x_0),$$

where $x_0 \in \mathbb{R}^2$ is fixed.

For the corresponding liftings we obtain

$$(x_i, t_i) = \lim_{k\to\infty} (F_{i_1} \circ \cdots \circ F_{i_k})(x_0, t_0)$$

and

$$(x_j, t_j) = \lim_{k\to\infty} (F_{j_1} \circ \cdots \circ F_{j_k})(x_0, t_0),$$

where $t_0 \in \mathbb{R}$ is again fixed. From the formula for $F_i(x, t)$ and Proposition 2.17 we deduce a formula for a finite composition of F_i's:

$$F_{i_1} \circ \cdots \circ F_{i_k}(x, t) = (f_{i_1} \circ \cdots f_{i_k}(x), \lambda_{i_1 \cdots i_k} \cdot t + h_{0,i_1 \cdots i_k}(x) + \tau_{i_1 \cdots i_k}).$$

We obtain from this the recurrence relations

$$\begin{aligned} \lambda_{i_1\cdots i_{k+1}} &= \lambda_{i_{k+1}} \cdot \lambda_{i_1\cdots i_k} \\ h_{0,i_1\cdots i_{k+1}} &= \lambda_{i_1\cdots i_k} \cdot h_{0,i_{k+1}} + h_{0,i_1\cdots i_k} \\ \tau_{i_1\cdots i_{k+1}} &= \lambda_{i_1\cdots i_k} \cdot \tau_{i_{k+1}} + \tau_{i_1\cdots i_k}. \end{aligned} \tag{4.4}$$

Explicit solutions to these recurrence relations are as follows:

$$\begin{aligned} \lambda_{i_1\cdots i_k} &= \prod_{j=1}^{k} \lambda_{i_j} \\ \tau_{i_1\cdots i_k} &= \sum_{r=1}^{k}\left(\prod_{l=1}^{r-1}\lambda_{i_l}\right)\cdot \tau_{i_r} \\ h_{0,i_1\cdots i_k} &= \sum_{r=1}^{k}\left(\prod_{l=1}^{r-1}\lambda_{i_l}\right)\cdot h_{0,i_r}. \end{aligned}$$

In the above relations the convention $\prod_{r=1}^{0} = 1$ has been used.

Taking limits as $k \to \infty$ we obtain

$$t_i = \sum_{r=1}^{\infty}\left(\prod_{l=1}^{r-1}\lambda_{i_l}\right)\cdot h_{0,i_r}(x_0) + \sum_{r=1}^{\infty}\left(\prod_{l=1}^{r-1}\lambda_{i_l}\right)\cdot \tau_{i_r}.$$

Using this explicit dependence of t_i on the parameters τ in (4.3) yields

$$\tau_i - \tau_j + \lambda_i \sum_{r=1}^{\infty}\left(\prod_{l=1}^{r-1}\lambda_{i_l}\right)\cdot \tau_{i_r} - \lambda_j \sum_{r=1}^{\infty}\left(\prod_{l=1}^{r-1}\lambda_{j_l}\right)\cdot \tau_{j_r} = u_{ij}, \tag{4.5}$$

where u_{ij} is independent of τ.

Let us express the infinite series in (4.5) as a linear function in the variables τ_i with coefficients depending on λ_i. The equation takes the form

$$\tau_i - \tau_j + \sum_{l=1}^{M} g_{ijl}(\lambda_1, \ldots, \lambda_M)\tau_l = u_{ij}, \tag{4.6}$$

where the function g_{ijl} verifies the estimate

$$|g_{ijl}(\lambda_1, \ldots, \lambda_M)| \leq \frac{2\lambda_{\max}}{1-\lambda_{\max}}.$$

Note that when $\lambda_1 = \cdots = \lambda_M = 0$ our equations read

$$\tau_i - \tau_j = u_{ij}.$$

Order the variables and the equations of the resulting system according to the hierarchy of indices from the beginning of our proof, so that indices in $\mathcal{I}_0$ are considered first, followed by indices in $\mathcal{I}_1, \mathcal{I}_2$, etc. The system in (4.6) has $M-1$ equations in M unknowns $\tau_1, \ldots, \tau_M$. Consider the coefficient matrix of this system. Observe that if we leave out the first column (corresponding to the base vertex $i_0 \in \mathcal{I}_0$) we obtain an $(M-1)\times(M-1)$ lower triangular matrix whose entries are all 1 on the diagonal. This implies that the system surely has solutions in the case $\lambda_1 = \cdots = \lambda_M = 0$. By the continuity of the coefficients with respect to the λ_i's this property persists for small λ_i's. This finishes the proof of the first statement.

To prove the second statement we show that the sets $F_i(K_{\mathrm{H}})$ are disjoint for $\mathcal{L}^M$ almost every $\tau = (\tau_1, \ldots, \tau_M) \in \mathbb{R}^M$. For an arbitrary pair of indices i, j (not necessarily related as in the first part of the proof) consider the set

$$Z_{ij} := \left\{ \tau \in \mathbb{R}^M : \begin{array}{c} F_i(K_{\mathrm{H}}) \cap F_j(K_{\mathrm{H}}) \neq \emptyset \\ \text{for the lift } \mathcal{F}_{\mathrm{H}} \text{ corresponding to } \tau \end{array} \right\}. \tag{4.7}$$

It suffices to show that $\mathcal{L}^M(Z_{ij}) = 0$ for all such i and j. In fact, we will show that Z_{ij} is contained in a finite union of affine subspaces of $\mathbb{R}^M$ of codimension one provided that the λ_i's are sufficiently small.

Our argument uses considerations from the first part of the proof. Notice first that the PCF condition implies that $f_i(K) \cap f_j(K)$ contains at most finitely many points. On the other hand, by looking at the vertical coordinates of elements of $F_i(K_{\mathrm{H}}) \cap F_j(K_{\mathrm{H}})$, we obtain condition (4.6), where the right side can take only finitely many values. When $\lambda_i = 0$ we obtain again the non-degenerate equation $\tau_i - \tau_j = u_{ij}$ which shows that Z_{ij} is contained in a finite union of hyperplanes. By the continuity of the coefficients of (4.6) with respect to the λ_i's, the non-degeneracy persists for small λ_i's. This completes the proof. □

In the following we give some more explicit sufficient conditions for the connectedness of K_{H}. We begin by introducing another class of IFS (which includes, for example, the von Koch curve) where a connected lift exists.

PROPOSITION 4.8. *Let $\mathcal{F} = \{f_1, \ldots, f_M\}$ be an IFS such that*

$$\lambda_1 + \cdots + \lambda_M \neq 1 \tag{4.9}$$

and

$$f_{i-1}(a_M) = f_i(a_1), \quad i = 2, \ldots, M, \tag{4.10}$$

where a_i denotes the fixed point of f_i. Then there exists a lift $\mathcal{F}_{\mathrm{H}}$ of $\mathcal{F}$ for which K_{H} is connected.

Proof. Write $F_i(x, t) = (f_i(x), \lambda_i t + h_{0,i}(x) + \tau_i)$. We may assume that $h_{0,i}(a_i) = 0$. The fixed point of F_i is

$$p_i = \left(a_i, \frac{\tau_i}{1 - \lambda_i}\right).$$

We now prove that we can find $\tau_1, \ldots, \tau_M$ such that K_{H} is connected. Using again Hata's condition of connectivity and (4.10), we obtain

$$F_{i-1}(p_M) = F_i(p_1) \quad i = 2, \ldots, M, \tag{4.11}$$

provided $\tau_1, \ldots, \tau_M$ are chosen to satisfy the linear system of equations with coefficient

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

matrix

$$\mathbf{M} := \begin{pmatrix} 1-\lambda_2/(1-\lambda_1) & -1 & & & & \lambda_1/(1-\lambda_M) \\ -\lambda_3/(1-\lambda_1) & 1 & -1 & & & \lambda_2/(1-\lambda_M) \\ \vdots & & \ddots & \ddots & & \vdots \\ -\lambda_{M-1}/(1-\lambda_1) & & & 1 & -1 & \lambda_{M-2}/(1-\lambda_M) \\ -\lambda_M/(1-\lambda_1) & & & & 1 & -1+\lambda_{M-1}/(1-\lambda_M) \end{pmatrix}. \quad (4.12)$$

The condition $\sum_{i=1}^{M}\lambda_i \neq 1$ implies that the rank of $\mathbf{M}$ is $M-1$, so the system is solvable. Hata's condition guarantees the existence of a connected lift. □

Remark 4.13. If $\sum_{i=1}^{M}\lambda_i = 1$ the rank of $\mathbf{M}$ is $M-2$. The conclusion of the proposition continues to hold if we assume in addition that $\sum_{i=1}^{M} h_{0,i}(a_1) - h_{0,i}(a_M) = 0$. We leave the details to the reader.

Next we consider affine IFSs.

PROPOSITION 4.14. *Let $\mathcal{F} = \{f_1, \dots, f_M\}$, $f_i(x) = A_i x + b_i$, be an affine iterated function system with invariant set K. For each i, let $a_i = (I-A_i)^{-1}(b_i)$ be the fixed point of f_i. Assume that:*

(i) $a_1 = 0$;

(ii) *for each $i = 2, \dots, M$, a_i is an eigenvector of $A_1 + A_i$ with eigenvalue* 1; *and*

(iii) *for each i, $\langle A_i a_i, J a_i\rangle = 0$.*

Let $\mathcal{F}_{\mathrm{H}} = \{F_i\}_{i\in A}$ be the principal horizontal lift. Then K_{H} and K are connected.

Proof. We will use again Hata's condition and verify that $F_i(0,0) = F_1(a_i,0)$ for each $i \geq 2$. From the condition $(A_1 + A_i)a_i = a_i$ we deduce that $f_1(a_i) = f_i(0)$. This implies the connectivity of K.

Using the fact that we are working with principal lifts in conjunction with (2.16) we compute that the horizontal lifts of f_i, $i = 2, \dots, M$, are

$$F_i(x,t) = (A_i x + b_i, \det A_i t - 2\langle A_i(x-a_i), J b_i\rangle).$$

Since the lift of f_1 is $F_1(x,t) = (A_1 x, \det A_1 t)$ it suffices to show that

$$2\langle A_i a_i, J b_i\rangle = 0.$$

But this is an immediate consequence of (iii) and the definition of a_i. □

COROLLARY 4.15. *Assume that $\mathcal{F}$ is a self-similar system so that $a_1 = 0$, $r_i = 1 - r_1$ for all $i = 2, \dots, M$, and $A_i = r_i \cdot I$ for all i. Then K_{H} and K are connected.*

The corollary implies, for example, that the principal horizontal lifts of the square Q and the Sierpinski gasket SG from Examples 5.1 have connected invariant sets.

Remark 4.16. The above results give an abundance of cases where a connected invariant set K_{H} exists whenever K is connected. We do not have any example of an IFS with connected K for which K_{H} is disconnected for all choices of τ. We conjecture that for an IFS $\mathcal{F}$ with connected invariant set K, there always exists a lift $\mathcal{F}_{\mathrm{H}}$ for which K_{H} is connected.

The final proposition of this section concerns the generic total disconnectivity of horizontal lifts. Here we impose the following assumption, which is stronger than post-critical finiteness.

Definition 4.17. An IFS $\mathcal{F}$ is *strongly post-critically finite* if every point in the post-critical set V_0 is a fixed point for an element of $\mathcal{F}$.

It is clear that strongly PCF systems are PCF. Hata's tree-like set [**17**, Example 1.2.9] is an example of a PCF system which is not strongly PCF.

PROPOSITION 4.18. *Let $\mathcal{F}$ be a strongly PCF iterated function system such that $\lambda_i+\lambda_j<1$ for all pairs $i, j, i \neq j$. Then the invariant set K_{H} is totally disconnected for almost every horizontal lift $\mathcal{F}_{\mathrm{H}}$.*

LEMMA 4.19. *Let $\mathcal{F}$ be a PCF system with horizontal lift $\mathcal{F}_{\mathrm{H}}$. Let a_k be the fixed point for some $f_k \in \mathcal{F}$ and let p_k be the fixed point for the corresponding map $F_k \in \mathcal{F}_{\mathrm{H}}$. Then $\pi^{-1}(a_k) = \{p_k\}$.*

Proof. Clearly $\pi(p_k) = a_k$. Suppose that $q \neq p_k$ satisfies $\pi(q) = a_k$. Then there exists a word $w \in \Sigma$, $w \neq \overline{k}$, so that $p_{\mathrm{H}}(w) = q$. Then $p(w) = a_k = p(\overline{k})$, contradicting Lemma 3.17. □

Proof of Proposition 4.18. As before, parameterize the set of lifts of $\mathcal{F}$ by $\tau \in \mathbb{R}^M$. The idea of the proof is the same as in the second part of Theorem 1.11. In fact, we obtain in this case an explicit form for equations in the system (4.6) which ensures that the sets Z_{ij} from (4.7) are contained in codimension-one affine subspaces of $\mathbb{R}^M$.

Let $\tau \in Z_{ij}$. Then $F_i(K_{\mathrm{H}}) \cap F_j(K_{\mathrm{H}})$ is non-empty; let (x_0, t_0) be an element of this set. Since $f_i(K) \cap f_j(K) \ni x_0$, the strong post-critical finiteness of $\mathcal{F}$ guarantees that $x_0 = f_i(a_k) = f_j(a_l)$ for the fixed points a_k, a_l of elements $f_k, f_l \in \mathcal{F}$. By Corollary 3.18, $i \neq k$ and $j \neq l$. By Lemma 4.19,

$$(x_0, t_0) = F_i(p_k) = F_j(p_l),$$

where $p_k = (a_k, \tau_k/(1-\lambda_k))$ and $p_l = (a_l, \tau_l/(1-\lambda_l))$ denote the fixed points of F_k and F_l, respectively. (See the proof of Proposition 4.8.) From (4.2) we deduce the affine relation

$$\tau_i - \tau_j + \frac{\lambda_i}{1-\lambda_k}\tau_k - \frac{\lambda_j}{1-\lambda_l}\tau_l = h_{0,j}(a_l) - h_{0,i}(a_k) \tag{4.20}$$

for each $\tau \in Z_{ij}$. To complete the proof, we show that (4.20) is never degenerate. From earlier remarks we see that (4.20) can be degenerate only if $i = l$ and $j = k$. But in this case (4.20) reads

$$\left(1 - \frac{\lambda_j}{1-\lambda_i}\right)\tau_i - \left(1 - \frac{\lambda_i}{1-\lambda_j}\right)\tau_j = h_{0,j}(a_i) - h_{0,i}(a_j).$$

The assumption $\lambda_i + \lambda_j < 1$ rules this out. □

Under the OSC assumption, $|\lambda_1| + \cdots + |\lambda_M| \leq 1$. We thus obtain the following consequence in the case $M \geq 3$.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

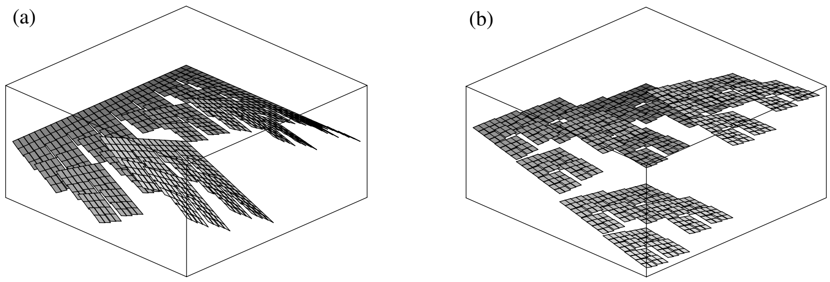

FIGURE 5.1. Horizontal lifts of $Q = [0, 1]^2$: (a) $\tau = (0, 0, 0, 0)$, (b) random choice of τ in $\mathbb{R}^4$.

COROLLARY 4.21. *Assume that $M \geq 3$. Let $\mathcal{F}$ be a strongly PCF iterated function system which satisfies the OSC. Assume that each map in $\mathcal{F}$ is orientation-preserving and non-degenerate (i.e. $\lambda_i > 0$). Then the invariant set K_{H} is totally disconnected for almost every horizontal lift $\mathcal{F}_{\mathrm{H}}$.*

5. *Examples*

We present here a few examples of horizontal lifts of iterated function systems. The first three examples are classical fractals generated by similarities, and example (4) is generated by affine maps.

Examples 5.1. Set $a_1 = 0, a_2 = e_1, a_3 = e_2$ and $a_4 = e_1 + e + 2$, and let $f_i, i = 1, 2, 3, 4$, be the rotation-free similarities of $\mathbb{R}^2$ with contraction ratio $r_i = \frac{1}{2}$ and fixed points a_i.

(1) The IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ has invariant set $K(\mathcal{F}) = Q = [0, 1]^2$. From (2.16) we derive the following expressions for the lifts F_i:

$$\begin{aligned} F_1(x_1, x_2, t) &= (\tfrac{1}{2}x_1, \tfrac{1}{2}x_x, \tfrac{1}{4}t + \tau_1), \\ F_2(x_1, x_2, t) &= (\tfrac{1}{2}x_1 + \tfrac{1}{2}, \tfrac{1}{2}x_2, \tfrac{1}{4}t - \tfrac{1}{2}x_2 + \tau_2), \\ F_3(x_1, x_2, t) &= (\tfrac{1}{2}x_1, \tfrac{1}{2}x_2 + \tfrac{1}{2}, \tfrac{1}{4}t + \tfrac{1}{2}x_1 + \tau_3), \\ F_4(x_1, x_2, t) &= (\tfrac{1}{2}x_1 + \tfrac{1}{2}, \tfrac{1}{2}x_2 + \tfrac{1}{2}, \tfrac{1}{4}t + \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 + \tau_4), \end{aligned}$$

where $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) \in \mathbb{R}^4$. Figure 5.1 shows an approximation (five iterations on the initial square $[0, 1]^2$) of the invariant set $Q_{\mathrm{H}} = K(\mathcal{F}_{\mathrm{H}})$ associated to two such lifted systems $\mathcal{F}_{\mathrm{H}}$. These examples satisfy the open set condition.

Figure 5.1(a) shows the principal horizontal lift (see Definition 3.6) associated with the choice $\tau = (0, 0, 0, 0)$. According to Corollary 4.15 this invariant set is connected. This particular example will play an important role in the final section of this paper.

Figure 5.1(b) shows the (totally disconnected) invariant set for the lift associated with a generic choice of τ in $\mathbb{R}^4$.

(2) The IFS $\mathcal{F} = \{f_1, f_2, f_3\}$ is the prototypical example of a PCF system. In this case the invariant set is the Sierpinski gasket SG. Figure 5.2 shows an approximation (five iterations) of the principal horizontal lift of SG to the Heisenberg group, which gives a connected lift (see Corollary 4.15).

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

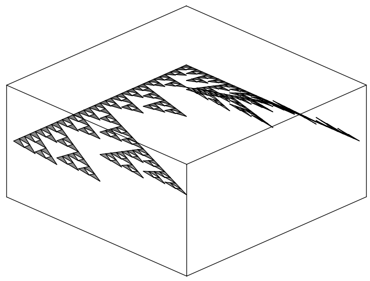

FIGURE 5.2. A lift of the Sierpinski gasket to $\mathbb{H}$.

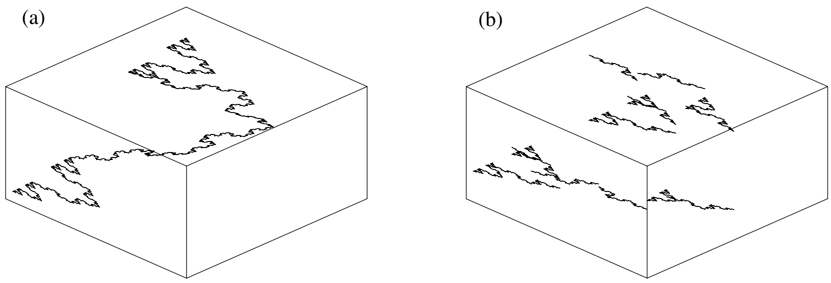

FIGURE 5.3. Lifts of the von Koch curve with (a) $\tau = (0, -\sqrt{3}/45, \sqrt{3}/15, -8\sqrt{3}/45)$ and (b) $\tau = (0, 0, 0, 0)$.

(3) The von Koch curve is a typical example for liftable IFSs treated by Proposition 4.8. The lifted functions are

$$F_1(x_1, x_2, t) = \left(\frac{x_1}{3}, \frac{x_2}{3}, \frac{t}{9} + \tau_1\right),$$

$$F_2(x_1, x_2, t) = \left(\frac{x_1}{6} - \frac{\sqrt{3}\,x_2}{6} + \frac{1}{3}, \frac{\sqrt{3}\,x_1}{6} + \frac{x_2}{6}, \frac{t}{9} - \frac{\sqrt{3}\,x_1}{9} - \frac{x_2}{9} + \tau_2\right),$$

$$F_3(x_1, x_2, t) = \left(\frac{x_1}{6} + \frac{\sqrt{3}\,x_2}{6} + \frac{1}{2}, \frac{-\sqrt{3}\,x_1}{6} + \frac{x_2}{6} + \frac{\sqrt{3}}{6}, \frac{t}{9} + \frac{2\sqrt{3}\,x_1}{9} + \tau_3\right),$$

$$F_4(x_1, x_2, t) = \left(\frac{x_1}{3} + \frac{2}{3}, \frac{x_2}{3}, \frac{t}{9} - \frac{4x_2}{9} + \tau_4\right),$$

with fixed points $p_1 = (0, 0, 9\tau_1/8)$ and $p_4 = (1, 0, 9\tau_4/8)$ of F_1 and F_4. Condition (4.11) from Proposition 4.8 is easily checked. Figure 5.3 shows the fifth-iterate approximation for (a) a connected horizontal lift and (b) for a random (generic) choice of τ which leads to an unconnected lift.

(4) The last example depicted in Figure 5.4 (approximation with ten iterations) is taken from Falconer [**11**, Example 11.4]. The two specific functions whose projections lead to a

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

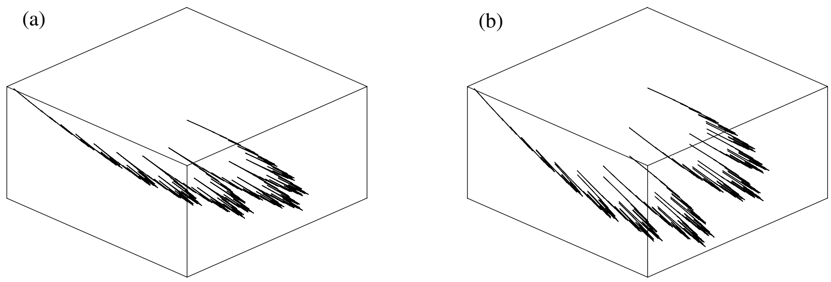

FIGURE 5.4. Lifts of some self-affine curve to $\mathbb{H}$: (a) connected with $\tau = (0, -3/20)$ and (b) disconnected with $\tau = (0, 0)$.

self-affine curve are

$$F_1(x_1, x_2, t) = \left(\frac{x_1}{2}, \frac{x_1}{4} + \frac{3x_2}{4}, \frac{3t}{8} + \tau_1\right),$$
$$F_2(x_1, x_2, t) = \left(\frac{x_1}{2} + \frac{1}{2}, -\frac{x_1}{4} + \frac{3x_2}{4} + \frac{1}{4}, \frac{3t}{8} + \frac{x_1}{2} - \frac{3x_2}{4} + \tau_2\right).$$

6. *Fiber structure of horizontal fractals and horizontal graphs in $\mathbb{H}$*

The first objective in this section is to prove Theorem 1.13 stated in the introduction. We start with some preparations.

6.1. *Symbolic uniqueness of irrational points.* For the duration of this subsection, we work in the context of a general complete metric space (X, d). Let K be the invariant set for an IFS $\mathcal{F}$ on X. Recall that $C(\mathcal{F}) = \bigcup_{i\neq j} K_i \cap K_j$ denotes the critical set for $\mathcal{F}$, while $\mathcal{C} = p^{-1}(C(\mathcal{F}))$ and $\mathcal{P} = \bigcup_{m\geq 1} \sigma^m(\mathcal{C})$ denote the critical and post-critical symbol sets in Σ.

Set $V_0 = p(\mathcal{P})$ to be the post-critical set. For $m \in \mathbb{N}$, let

$$V_m := \bigcup_{|w|\leq m} f_w(V_0) \quad \text{and} \quad V_* := \bigcup_{w\in W} f_w(V_0).$$

We view V_* as the set of 'rational' points in K. If V_0 is non-empty, then V_* is dense in K [**17**, Lemma 1.3.11]. If $\mathcal{F}$ is post-critically finite then V_* is countable.

LEMMA 6.2. *If $x \in K \setminus V_*$, then there exists a unique word in $p^{-1}(x)$.*

Proof. Suppose that $p(w) = x = p(w')$ for two distinct words $w, w' \in \Sigma$. By Proposition 3.10, $p(\sigma^s w) = p(\sigma^s w') \in C(\mathcal{F})$, where $s = s(w, w')$ is the maximal length of a common initial word of w and w' as defined in Proposition 3.10. Thus, $\sigma^s w \in \mathcal{C}$, which implies $p(\sigma^{s+1} w) \in V_0$ and

$$x = p(w) = f_{w_1\cdots w_{s+1}}(p(\sigma^{s+1} w)) \in V_{s+1}. \qquad \square$$

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

6.3. *Fiber structure of horizontal lifts.* We now specialize to the case when $\mathcal{F}$ and $\mathcal{F}_{\mathrm{H}}$ are IFSs satisfying Assumption 3.11.

LEMMA 6.4.
(i) *If $x \in K \setminus V_*$, then $\alpha(x)$ is a singleton.*
(ii) *If $x \in V_m \setminus V_{m-1}$ for some $m \geq 1$, then*

$$\operatorname{diam}_{\mathrm{H}} \alpha(x) \leq c r_{\max}^{m-1} \operatorname{diam}_{\mathrm{H}} K_{\mathrm{H}}.$$

Let β be a selection of α. Then
(iii) $\operatorname{diam}_{\mathrm{H}} \beta(F_v(K_{\mathrm{H}})) \leq c r_v \operatorname{diam}_{\mathrm{H}} K_{\mathrm{H}}$ *for each $v \in W$.*

Since $d_{\mathrm{H}} = \sqrt{d_{\mathrm{E}}}$ when restricted to a fiber $\pi^{-1}(x)$, $x \in \mathbb{R}^2$, we have the following corollary to part (ii) of Lemma 6.4.

COROLLARY 6.5. *If $x \in V_m \setminus V_{m-1}$ for some $m \geq 1$, then*

$$\operatorname{diam}_{\mathrm{E}} \alpha(x) \leq C r_{\max}^{2m},$$

where $C = c^2 \operatorname{diam}_{\mathrm{H}} K_{\mathrm{H}} / r^2$.

Corollary 6.5 will play an important role in the proof of Theorem 1.14.

Proof of Lemma 6.4. By applying a preliminary dilation, we may assume that the Heisenberg diameter of K_{H} is one.

Part (i) follows immediately from Lemma 6.2, since each element of $\pi^{-1}(x) \cap K_{\mathrm{H}}$ induces a distinct symbolic representative for x.

To prove (ii), assume that (x, t) and (x, t') are elements of K_{H} with $t \neq t'$. From (i) we see that $x \in V_{s+1}$; since $V_0 \subset V_1 \subset \cdots$ we must have $s + 1 \geq m$. Moreover, we find $w, w' \in \Sigma$, $w \neq w'$, such that $p(w) = x = p(w')$. Denote by $v \in W_s$ the maximal initial word common to w and w'. Then $(x, t), (x, t') \in F_v(K_{\mathrm{H}})$. Since F_v is a lift of f_v (see Proposition 2.17), it is $c r_v$-Lipschitz in the Heisenberg metric. Thus,

$$d_{\mathrm{H}}((x, t), (x, t')) \leq c r_v \leq c r_{\max}^{m-1},$$

as desired.

The proof of (iii) is similar. Let β be a selection of α. Given (x, t) and (x', t') in $F_v(K_{\mathrm{H}})$ we have

$$d_{\mathrm{H}}((x, t), (x', t')) \leq c r_v.$$

Thus $\operatorname{diam}_{\mathrm{H}} \alpha(F_v(K_{\mathrm{H}})) \leq c r_v$, where $\alpha(E) := \bigcup_{x \in E} \alpha(x)$, $E \subset \mathbb{R}^2$, and so *a fortiori* $\operatorname{diam}_{\mathrm{H}} \beta(F_v(K_{\mathrm{H}})) \leq c r_v$. □

Let us finally turn to the following proof.

Proof of Theorem 1.13. Let β be a selection of α and let $x \in K \setminus V_*$. Let $x^{(n)} \to x$ in K. By part (iii) of Lemma 6.4, it suffices to show that $s(w^{(n)}, w) \to \infty$ for any choice of words $w^{(n)}, w \in \Sigma$ with $p(w^{(n)}) = x^{(n)}$ and $p(w) = x$.

Suppose there exist words $w^{(n)}$ and w with $s(w^{(n)}, w) \leq C < \infty$ for all n. Passing to a subsequence, we may assume that $s(w^{(n)}, w) = k$ for all n and some $k \leq C$. Equivalently, $w_{k+1}^{(n)} \neq w_{k+1}$ for all n. Choose a limit $w^{(\infty)}$ for the sequence $(w^{(n)})$ in Σ. Then $p(w^{(\infty)}) = x$ by the continuity of p, but $w_{k+1}^{(\infty)} \neq w_{k+1}$. This contradicts the uniqueness of symbolic representations of $x \in K \setminus V_*$ asserted in Lemma 6.2. □

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

6.6. *Horizontal graphs in the Heisenberg group.* It is a question of some interest to determine the maximal degree of regularity of a horizontal graph in $\mathbb{H}$. Recall the notations $\mathcal{H}^\alpha_{\mathrm{H}}$ and $\mathcal{H}^\alpha_{\mathrm{E}}$ which stand for the α-dimensional Hausdorff measure with respect to the metric d_{H} (respectively d_{E}). Let Ω be a domain in $\mathbb{R}^2$ and let $\varphi : \Omega \to \mathbb{H}$. We say that $S := \varphi(\Omega)$ is a *horizontal surface* if $0 < \mathcal{H}^2_{\mathrm{H}}(S) < \infty$. If φ is a graph over $\mathbb{R}^2$, i.e. $\pi \circ \varphi = \mathrm{id}$, we say that S is a *horizontal graph*.

In an important recent work on rectifiability in metric spaces [**2**], Ambrosio and Kirchheim prove that there are no horizontal surfaces in the Heisenberg group $\mathbb{H}$ (with its Heisenberg metric d_{H}) which are Lipschitz images of planar domains. More precisely (see [**2**, Theorem 7.2]), $(\mathbb{H}, d_{\mathrm{H}})$ is purely 2-unrectifiable: $\mathcal{H}^2_{\mathrm{H}}(S) = 0$ for every $S = \varphi(\Omega)$ with $\varphi : \Omega \to (\mathbb{H}, d_{\mathrm{H}})$ Lipschitz, $\Omega \subset \mathbb{R}^2$.

In the case of graphs, the result of Ambrosio and Kirchheim can be strengthened as follows: there do not exist horizontal graphs in $\mathbb{R}^3$ (with the Euclidean metric d_{E}) which are Lipschitz images of planar domains. The next theorem gives a more precise version of this claim. It is a natural extension to the Lipschitz category of the well-known fact that there are no C^1 horizontal surfaces in the Heisenberg group (viewing the target as $\mathbb{R}^3$ with the Euclidean metric). The latter result follows, for example, from Pansu's isoperimetric inequality [**23**] which implies that the Heisenberg dimension of such a surface is three.

THEOREM 6.7. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain and let* $\varphi : \Omega \to (\mathbb{R}^3, d_{\mathrm{E}})$ *be a Lipschitz graph, i.e.* $\pi \circ \varphi = \mathrm{id}$. *Then* $\mathcal{H}^2_{\mathrm{H}}(S) = \infty$, *where* $S = \varphi(\Omega)$.

The theorem is false without the assumption that φ be a graph. For example, let Ω be a bounded domain and let $\varphi : \Omega \to \mathbb{R}^3$ be given by $\varphi(x) = (0, |x|)$. Then φ is Lipschitz and $0 < \mathcal{H}^2_{\mathrm{H}}(\varphi(\Omega)) < \infty$.

Proof. Since the conclusion is local in nature we may assume without loss of generality that Ω is bounded. In addition, we may assume that Ω contains the origin. Write $\varphi(x) = (x, g(x))$, $g : \Omega \to \mathbb{R}$, and define $G : \mathbb{R}^3 \to \mathbb{R}$ by $G(x, t) = g(x) - t$. Let $A := \{x \in \Omega : \nabla_{\mathrm{H}} G(x) \neq 0\}$; thus $\varphi(\Omega \setminus A)$ is the set of characteristic points of the surface $S = \varphi(\Omega)$.

Assume first that $\mathcal{H}^2_{\mathrm{E}}(A) = 0$. Fubini's theorem implies that

$$\mathcal{H}^1_{\mathrm{E}}(C_r \cap A) = 0$$

for almost every radius r such that $C_r \subset \Omega$, where C_r denotes the circle centered at the origin of radius r. Since

$$\nabla_{\mathrm{H}} G(x_1, x_2) = (\partial_1 \varphi - 2x_2, \partial_2 \varphi + 2x_1)$$

we find

$$\begin{aligned} 0 = \int_{C_r} \nabla_{\mathrm{H}} G \cdot ds &= \int_{C_r} \nabla \varphi \cdot ds - 2 \int_{C_r} Jx \cdot ds \\ &= 0 + 2 \int_0^{2\pi} (x_1 (x_2)' - (x_1)' x_2)\, d\theta \\ &= 4\pi r^2 > 0. \end{aligned}$$

From this contradiction we deduce that this case cannot occur.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Assume then that $\mathcal{H}^2_{\mathrm{E}}(A) > 0$. Without loss of generality we may assume that the tangent plane T_pS exists at each point $p \in \pi^{-1}(A) \cap S$. We claim that

$$\mathcal{H}^3_{\mathrm{H}}(S) > 0, \tag{6.8}$$

which clearly implies the desired conclusion.

To prove (6.8) we will use the mass distribution principle [**21**, Theorem 8.8] which states that the Hausdorff α-measure of a metric space X is positive provided there exists a positive Borel measure μ on X with *α-dimensional volume growth*: $\mu(B(x,r)) \le Cr^\alpha$ for every $x \in X$ and $r > 0$. In fact, we will show that the restriction of the Euclidean Hausdorff measure $\mathcal{H}^2_{\mathrm{E}}$ to $\pi^{-1}(A) \cap S$ has three-dimensional volume growth: there exists $C < \infty$ so that

$$\mathcal{H}^2_{\mathrm{E}}(B_{\mathrm{H}}(p,r) \cap S) \le Cr^3 \tag{6.9}$$

for every $p \in \pi^{-1}(A)$ and $r > 0$. (Note that the positivity of the measure is guaranteed since $\mathcal{H}^2_{\mathrm{E}}(\pi^{-1}(A) \cap S) \ge \mathcal{H}^2_{\mathrm{E}}(A) > 0$.)

The estimate in (6.9) follows from the Heisenberg box-ball theorem [**4**, (2.6)]. In our setting, this result states that

$$\mathrm{Box}(p, r/K) \subset B_{\mathrm{H}}(p,r) \subset \mathrm{Box}(p, Kr), \tag{6.10}$$

where $\mathrm{Box}(p,r)$ consists of those points $p + v \in \mathbb{R}^3$ for which $v = v^1_p + v^2_p$, $v^1_p \in H_p\mathbb{H}$, $|v^1_p| \le r$ and $v^2_p \in \mathbb{R}T$, $|v^2_p| \le r^2$. Since by assumption $p \in \pi^{-1}(A)$, p is non-characteristic and $H_p\mathbb{H} \ne T_pS$. From this the inequality

$$\mathcal{H}^2_{\mathrm{E}}(\mathrm{Box}(p,r) \cap S) \le Cr^3 \tag{6.11}$$

is easy to verify, and then (6.9) follows from (6.11) and (6.10). □

As the final result of this paper we prove Theorem 1.14, which asserts the existence of a horizontal BV surface in the Heisenberg group. The existence of such surfaces contrasts with the non-existence results from earlier in this section, specifically Theorem 6.7.

For simplicity, we have only stated Theorem 1.14 in the case of the Heisenberg square Q_{H}. The result holds also in other situations. For example, it holds for the principal horizontal lifts associated with certain other self-similar iterated function systems whose invariant sets have non-empty interior. The proof is similar and we leave the details to the reader.

Recall that a function $g : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, is a function of *bounded variation* if $g \in L^1(\Omega)$ and

$$|Dg|(\Omega) := \sup\left\{ \int_\Omega g \cdot \operatorname{div} F \, d\mathcal{L}^n : F \in C^\infty_0(\Omega, \mathbb{R}^n), \|F\| \le 1 \right\} \tag{6.12}$$

is finite. Equivalently, the distributional derivatives $\partial_i g$ exist as finite signed Radon measures. In this case we have the $\mathbb{R}^n$-valued signed measure $Dg = (\partial_1 g, \ldots, \partial_n g)$, whose total variation $|Dg|(\Omega)$ coincides with the value in (6.12). We denote by $\mathrm{BV}(\Omega)$ the space of functions of bounded variation on Ω and we denote by

$$\|g\|_{\mathrm{BV}} := \|g\|_{L^1(\Omega)} + |Dg|(\Omega)$$

the BV-norm of g. For the general theory of BV functions, see [**28**, Chapter 5] or [**1**].

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Proof of Theorem 1.14. Let $\beta : Q \to \mathbb{R}$, $\beta(x) = (x, g(x))$, be a selection of α. There exists a sequence of piecewise linear (discontinuous) functions $g_m : Q \to \mathbb{R}$ so that $g_m \to g$ in $L^1(Q)$. These approximations are given by

$$g_m = \sum_{w \in W_m} L_w \cdot \chi_{Q_w^o},$$

where the L_w's are affine maps and $Q_w^o = f_w(Q^o)$. The functions g_m are in $\mathrm{BV}(Q^o)$ and have

$$Dg_m = \sum_{w \in W_m} \nabla L_w \cdot \mathcal{L}^2 \llcorner Q_w^o + \sum_{w \in W_m} L_w \nu_w \mathcal{H}^1 \llcorner Q^o \cap \partial Q_w^o.$$

Here ν_w denotes the inward-pointing unit normal to the domain Q_w^o. See, for example, [**1**, Example 3.3].

It follows that

$$\|g_m\|_{L^1(Q)} = \sum_{w \in W_m} \int_{Q_w} |L_w| \, d\mathcal{L}^2 \tag{6.13}$$

and

$$|Dg_m|(Q^o) \le \sum_{w \in W_m} |\nabla L_w| \cdot |Q_w| + \sum_{\substack{w, \tilde{w} \in W_m \\ w \sim \tilde{w}}} \int_{\Gamma_{w\tilde{w}}} |L_w - L_{\tilde{w}}| \, d\mathcal{H}^1, \tag{6.14}$$

where $w \sim \tilde{w}$ if and only if the squares Q_w and $Q_{\tilde{w}}$ intersect along an edge and $\Gamma_{w\tilde{w}} = Q_w \cap Q_{\tilde{w}}$.

LEMMA 6.15. *There exist constants $A, B < \infty$ so that*

$$\sum_{w \in W_m} \int_{Q_w} |L_w| \, d\mathcal{L}^2 \le A + B$$

and

$$\sum_{w \in W_m} |\nabla L_w| \cdot |Q_w| \le B$$

for all m.

LEMMA 6.16. *Denoting by C the constant in Corollary 6.5, we have*

$$\sum_{\substack{w, \tilde{w} \in W_m \\ w \sim \tilde{w}}} \int_{\Gamma_{w\tilde{w}}} |L_w - L_{\tilde{w}}| \, d\mathcal{H}^1 \le 4C$$

for every m.

Using these two lemmas in conjunction with (6.13) and (6.14), we see that

$$\|g_m\|_{\mathrm{BV}} \le 2A + B + 4C < \infty$$

for each m. From the lower semicontinuity of the BV-norm [**1**, (3.11)], we deduce that $g \in \mathrm{BV}(Q^o)$ with $\|g\|_{\mathrm{BV}} \le 2A + B + 4C$. □

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Proof of Lemma 6.16. From Corollary 6.5, it follows that

$$|L_w(x) - L_{\tilde{w}}(x)| \leq C \cdot 2^{-2k} \tag{6.17}$$

for any $x \in \Gamma_{w\tilde{w}}$ provided $s(w, \tilde{w}) = k$. An elementary computation shows that the number of pairs of words $w, \tilde{w} \in W_m$ with $w \sim \tilde{w}$ and $s(w, \tilde{w}) = k$, is 2^{m+k+1}. Finally, $\mathcal{H}^1(\Gamma_{w\tilde{w}}) = 2^{-m}$ for any such pair $w, \tilde{w}$.

Thus,

$$\begin{aligned}\sum_{\substack{w,\tilde{w}\in W_m\\ w\sim\tilde{w}}} \int_{\Gamma_{w\tilde{w}}} |L_w - L_{\tilde{w}}|\, d\mathcal{H}^1 &\leq \sum_{k=0}^{m-1} \sum_{\substack{w,\tilde{w}\in W_m\\ w\sim\tilde{w}, s(w,\tilde{w})=k}} C \cdot 2^{-2k}\mathcal{H}^1(\Gamma_{w\tilde{w}})\\ &\leq C2^{-m}\sum_{k=0}^{m-1} 2^{-2k} \cdot 2^{m+k+1} \leq 4C. \qquad \square\end{aligned}$$

Proof of Lemma 6.15. Set $L_w(x) = \langle a_w, x\rangle + b_w$, where $a_w \in \mathbb{R}^2$ and $b_w \in \mathbb{R}$. Then $|L_w(x)| \leq |a_w| + |b_w|$ for $x \in Q$ and $|\nabla L_w| = |a_w|$.

Set

$$A_m = \max_{w\in W_m} |a_w|, \quad B_m = \max_{w\in W_m} |b_w|.$$

For $m = 0$ we have $L_\emptyset(x) = 0$ and so $A_0 = B_0 = 0$.

Next, we develop recursive inequalities for the expressions A_m and B_m. Recall from §5 the notation $\alpha_1 = 0, \alpha_1 = e_1, \alpha_2 = e_2$ and $\alpha_3 = e_1 + e_2$ for the fixed points of the maps in the planar IFS. For $w \in W$, $j = 1, 2, 3, 4$, and $x \in \mathbb{R}^2$, we have

$$L_{jw}(x) = \tfrac{1}{2}L_w(x) - \langle J\alpha_j, x\rangle - \tfrac{1}{4}L_w(\alpha_j).$$

Thus, $a_{jw} = \frac{1}{2}a_w - J\alpha_j$,

$$\begin{aligned}A_{m+1} &= \max_{w\in W_m} \{|a_{1w}|, |a_{2w}|, |a_{3w}|, |a_{4w}|\}\\ &\leq \max_{w\in W_m} (\tfrac{1}{2}|a_w| + \sqrt{2}) = \tfrac{1}{2}A_m + \sqrt{2},\end{aligned}$$

and

$$A_m \leq 2\sqrt{2}(1 - 2^{-m}) < 2\sqrt{2} =: A$$

for all m.

Similarly, $b_{jw} = \frac{1}{4}b_w - \frac{1}{4}\langle a_w, \alpha_j\rangle$,

$$B_{m+1} \leq \frac{1}{4}B_m + \frac{\sqrt{2}}{4}A_m \leq \frac{1}{4}B_m + 1,$$

and

$$B_m \leq \tfrac{4}{3}(1 - 4^{-m}) < \tfrac{4}{3} =: B$$

for all m.

The proof of the lemma is now completed by the estimates

$$\sum_{w\in W_m} \int_{Q_w} |L_w|\, d\mathcal{L}^2 \leq (A_m + B_m)|Q| \leq A + B$$

and

$$\sum_{w\in W_m} |\nabla L_w| \cdot |Q_w| \le A_m|Q| \le A. \qquad \square$$

Remark 6.18. Theorem 1.14 can be strengthened as follows: for each selection $\beta(x)$ of the set map $\alpha(x) = \pi^{-1}(x) \cap Q_{\mathrm{H}}$ corresponding to the principal horizontal lift Q_{H} of Q, the associated function $g : Q \to \mathbb{R}$ is in the class $\mathrm{SBV}(Q^o)$. Here $\mathrm{SBV}(\Omega)$, $\Omega \subset \mathbb{R}^n$, denotes the class of *special functions of bounded variation*, defined as those functions $g \in \mathrm{BV}(\Omega)$ whose derivative measure Dg has no Cantor part. See [**1**, §4.1].

To prove this claim, we use the following sufficient condition for membership in the class $\mathrm{SBV}(\Omega)$, which can be found as [**1**, Theorem 4.7].

THEOREM 6.19. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *and* $\theta : (0,\infty) \to (0,\infty)$ *be lower semicontinuous increasing functions satisfying*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty \quad \textit{and} \quad \lim_{t\to 0} \frac{\theta(t)}{t} = \infty. \tag{6.20}$$

Let $\Omega \subset \mathbb{R}^n$ *be open and bounded and let* $u_h \in \mathrm{SBV}(\Omega)$*,* $h \in \mathbb{N}$*, such that* $\|u_h\|_{\mathrm{BV}} \le M$*,* $u_h \to u$ *in* $L^1(\Omega)$*, and*

$$\int_\Omega \varphi(|\nabla u_h|)\, d\mathcal{L}^n + \int_{J(u_h)} \theta(\mathrm{jump}(u_h))\, d\mathcal{H}^{n-1} \le M \tag{6.21}$$

for some constant $M < \infty$*. Then,* $u \in \mathrm{SBV}(\Omega)$.

Here $J(u_h)$ denotes the *jump set* for u_h and $\mathrm{jump}(u_h)$ denotes the jump of u_h across the jump set; see [**1**, Definition 3.67].

In the current setting we consider the sequence g_m converging to g as in the proof of Theorem 1.14. Observe that the jump set for g_m is $Q^o \cap \bigcup_{w\in W_m} \partial Q_w$. It suffices to verify (6.21) for an appropriate choice of φ and θ. Let $\varphi(t) = t^2$ and $\theta(t) = t^{3/4}$. These functions satisfy the conditions in (6.20). By the method used in the proof of Theorem 1.14 we estimate

$$\int_Q \varphi(|\nabla g_m|)\, d\mathcal{L}^2 = \sum_{w\in W_m} \varphi(|\nabla L_w|) \cdot |Q_w| \le \varphi(A_m)|Q| \le A^2$$

and

$$\begin{aligned}
\int_{J(g_m)} \theta(\mathrm{jump}(g_m))\, d\mathcal{H}^1 &= \sum_{\substack{w,\tilde{w}\in W_m\\ w\sim\tilde{w}}} \int_{\Gamma_{w\tilde{w}}} \theta(|L_w - L_{\tilde{w}}|)\, d\mathcal{H}^1 \\
&\le \sum_{k=0}^{m-1} \sum_{\substack{w,\tilde{w}\in W_m\\ w\sim\tilde{w}, s(w,\tilde{w})=k}} \theta(C\cdot 2^{-2k})\mathcal{H}^1(\Gamma_{w\tilde{w}}) \\
&\le C^{3/4} 2^{-m} \sum_{k=0}^{m-1} 2^{-3k/2} \cdot 2^{m+k+1} \\
&\le 2C^{3/4} \sum_{k=0}^{\infty} 2^{-k/2} < \infty.
\end{aligned}$$

Thus, (6.21) holds with $M = A^2 + 2(2+\sqrt{2})C^{3/4}$.

https://doi.org/10.1017/S0143385705000593 Published online by Cambridge University Press

Acknowledgements. ZMB was supported by a grant from the Swiss NSF. RH-I was supported by the Marie Heim-Vögtlin Foundation (Swiss NSF). JTT was supported by NSF grant DMS 0228807. The research for this paper was done while JTT was a visitor at the University of Berne during 2003. He wishes to thank the department for its hospitality.

REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems (Oxford Mathematical Monographs)*. The Clarendon Press, Oxford University Press, New York, 2000.

[2] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.* **318**(3) (2000), 527–555.

[3] Z. M. Balogh. Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group. *J. Anal. Math.* **83** (2001), 289–312.

[4] Z. M. Balogh. Size of characteristic sets and functions with prescribed gradient. *J. reine angew. Math.* **564** (2003), 63–83.

[5] Z. M. Balogh, M. Rickly and F. Serra-Cassano. Comparison of Hausdorff measures with respect to the Euclidean and Heisenberg metric. *Publ. Mat.* **47** (2003), 237–259.

[6] Z. M. Balogh and T. J. Tyson. Hausdorff dimension of self-similar and self-affine fractals in the Heisenberg group. *Proc. London Math. Soc.* (3) **91** (2005), 153–183.

[7] A. Bellaïche. The tangent space in sub-Riemannian geometry. *Sub-Riemannian Geometry*. Eds. A. Bellaïche and J.-J. Risler. Birkhäuser, Basel, 1996, pp. 1–78.

[8] W. M. Boothby and H. C. Wang. On contact manifolds. *Ann. of Math.* (2) **68** (1958), 721–734.

[9] L. Capogna and P. Tang. Uniform domains and quasiconformal mappings on the Heisenberg group. *Manusacripta Math.* **86** (1995), 267–281.

[10] K. J. Falconer. *The geometry of fractal sets (Cambridge Tracts in Mathematics, 85)*. Cambridge University Press, Cambridge, 1986.

[11] K. J. Falconer. *Fractal geometry (Mathematical Foundations and Applications)*. John Wiley and Sons Ltd., Chichester, 1990.

[12] B. Franchi, R. Serapioni and F. Serra Cassano. Rectifiability and perimeter in the Heisenberg group. *Math. Ann.* **321** (2001), 479–531.

[13] N. Garofalo and D.-M Nhieu. Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.* **49** (1996), 1081–1144.

[14] M. Gromov. Carnot–Carathéodory spaces seen from within. *Sub-Riemannian Geometry (Progress in Mathematics, 144)*. Birkhäuser, Basel, 1996, pp. 79–323.

[15] M. Hata. On the structure of self-similar sets. *Japan J. Appl. Math.* **2**(2) (1985), 381–414.

[16] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.

[17] J. Kigami. *Analysis on fractals (Cambridge Tracts in Mathematics, 143)*. Cambridge University Press, Cambridge, 2001.

[18] B. Kirchheim and F. Serra Cassano. Rectifiability and parametrization of intrinsic regular surfaces in the Heisenberg group. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (5) **3** (2004), 871–896.

[19] A. Korányi and H. M. Reimann. Quasiconformal mappings on the Heisenberg group. *Invent. Math.* **80** (1985), 309–338.

[20] A. Korányi and H. M. Reimann. Foundations for the theory of quasiconformal mappings on the Heisenberg group. *Adv. Math.* **111** (1995), 1–87.

[21] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces (Cambridge Studies in Advanced Mathematics, 44)*. Cambridge University Press, Cambridge, 1995.

[22] P. A. P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.* **42** (1946), 15–23.

[23] P. Pansu. Une inégalité isopérimétrique sur le groupe de Heisenberg. *C. R. Acad. Sci. Paris Sér. I Math.* **295**(2) (1982), 127–130.

[24] P. Pansu. Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math.* (2) **129** (1989), 1–60.

[25] E. M. Stein with the assistance of Timothy S. Murphy. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals (Monographs in Harmonic Analysis, III)*. Princeton University Press, Princeton, NJ, 1993.

[26] R. S. Strichartz. Self-similarity on nilpotent Lie groups. *Geometric Analysis (Philadelphia, PA, 1991) (Contemporary Mathematics, 140)*. American Mathematical Society, Providence, RI, 1992, pp. 123–157.

[27] R. S. Strichartz. Self-similarity in harmonic analysis. *J. Fourier Anal. Appl.* **1**(1) (1994), 1–37.

[28] W. P. Ziemer. *Weakly Differentiable Functions (Graduate Texts in Mathematics, 120)*. Springer, New York, 1989.