

Curves intersecting in a circuit pattern

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ABSTRACT

We show that the cycle relation between Dehn twists about curves in a circuit detects whether the circuit bounds an embedded disc. This is done by determining the isomorphism type of the group generated by said Dehn twists for various surfaces.

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1. Introduction

A very well-known fact in the theory of mapping class groups is that relations between Dehn twists about pairs of curves detect whether the curves intersect zero, one, or at least two times [1, Section 3.5]. More precisely, let α_1 and α_2 be curves in a surface S , and let T_i be the Dehn twist about α_i . Then α_1 and α_2 are disjoint (up to homotopy) if and only if the associated Dehn twists satisfy the *commutation relation* $T_1T_2 = T_2T_1$. Similarly, α_1 and α_2 intersect precisely once (again, up to homotopy) if and only if the associated Dehn twists satisfy the *braid relation* $T_1T_2T_1 = T_2T_1T_2$. In all other cases, T_1 and T_2 generate a free subgroup of the mapping class group of S . This article will be on another instance of the observation that the presence or absence of certain relations between Dehn twists has consequences about the constellation of the involved curves.

Recent work on so-called bouquets of curves (families $\alpha_1, \dots, \alpha_n$ of curves intersecting, up to isotopy, in one common point) shows that pairwise braid relations and the so-called *cycle relation* $T_iT_jT_kT_i = T_jT_kT_iT_j$ or $T_iT_kT_jT_i = T_kT_jT_iT_k$ between all triples $\alpha_i, \alpha_j, \alpha_k$ of distinct curves is equivalent to the family $\alpha_1, \dots, \alpha_n$

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of pairwise non-isotopic curves forming a bouquet [2, Theorem 1]. Here, we seek to characterize circuits of curves in a very similar way, featuring longer cycle relations.

A family $\alpha_1, \dots, \alpha_n$ of n curves in a surface S is said to form a *circuit* if each curve α_i intersects α_{i-1} and α_{i+1} precisely once, and is disjoint from all other curves α_j , where indices are taken modulo n . A circuit $\alpha_1, \dots, \alpha_n$ is said to *bound a disc* if the complement $S \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ has a connected component homeomorphic to a disc. We say that a circuit $\alpha_1, \dots, \alpha_n$ *bounds an embedded closed disc* in S if there is an embedding of a closed disc such that its boundary gets mapped to points in the union $\alpha_1 \cup \dots \cup \alpha_n$.

We write $\mathcal{M}(S)$ for the mapping class group (consisting of isotopy classes of orientation-preserving homeomorphisms fixing the boundary) of an orientable surface S . Circuits of curves have previously been studied by Labruère, who showed that if $\alpha_1, \dots, \alpha_n$ is a circuit bounding an embedded closed disc Δ such that when travelling in the counter-clockwise manner around Δ the curves appear in the order $\alpha_1, \dots, \alpha_n$, then the cycle relation $T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$ holds [3, Proposition 2]. In fact, her result is considerably stronger, see Proposition 4.6 below. It is worth emphasizing the fact that Δ being embedded is important: if the boundary of Δ has self-intersections, then the cycle relation may not hold.

Another result that will feature in this article is by Mortada, asserting that if S is a certain neighbourhood (denoted below by M^n , see Fig. 5) of a circuit $\alpha_1, \dots, \alpha_n$, then the homomorphism $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{M}(S)$, mapping the standard generator s_i to T_i for all i , is injective [4, Theorem 5.5.4]. Here, $\mathcal{A}(\tilde{A}_{n-1})$ is an Artin group, see Section 2. We will prove this result and generalize it (see Theorem 1.2 and Proposition 4.2 below) in order to prove our main result.

Theorem 1.1. *Let $\alpha_1, \dots, \alpha_n$ be a circuit of $n \geq 3$ curves in a surface S . Then the circuit $\alpha_1, \dots, \alpha_n$ bounds an embedded closed disc Δ if and only if one of the two cycle relations*

$$T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$$

or

$$T_1 \cdots T_n T_1 \cdots T_{n-2} = T_2 \cdots T_n T_1 \cdots T_{n-1}$$

holds. The first relation corresponds to the curves appearing in the cyclic order $\alpha_1, \dots, \alpha_n$ when travelling in the counter-clockwise manner around Δ , and the second corresponds to the other cyclic order.

The left-to-right direction is being taken care of by Labruère's result, whereas Mortada's shows part of the right-to-left direction. In fact, the main topological insight allowing us to prove Theorem 1.1 is the following positive answer to Conjecture 5.5.5 in his thesis, which asserts that the homomorphism $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{M}(S)$ mapping s_i to T_i is injective whenever S is a regular neighbourhood of $\alpha_1 \cup \dots \cup \alpha_n$. We prove the result by constructing suitable branched coverings of annuli by regular neighbourhoods of circuits in order to apply the Birman-Hilden theorem.

Theorem 1.2. *Let S be a regular neighbourhood of a circuit $\alpha_1, \dots, \alpha_n$ of $n \geq 3$ curves. Then the subgroup of $\mathcal{M}(S)$ generated by the Dehn twists T_i about α_i is geometrically isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$.*

A *geometric embedding* of an Artin group into the mapping class group of a surface is an injective homomorphism mapping the standard generators to Dehn twists, and a *geometric isomorphism* is a bijective geometric embedding or its inverse. It is an open question what Artin groups geometrically embed into the mapping class group of a surface, although partial answers are plentiful. For instance, the group generated by Dehn twists about two curves intersecting two or more times is isomorphic to a free group on two generators. Hence, the free group of two generators (which is the Artin group $\mathcal{A}(\tilde{A}_1)$ by convention)

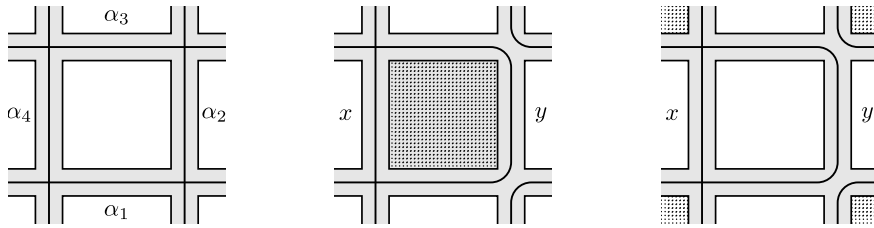
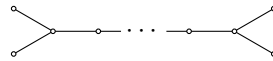


Fig. 1. A neighbourhood of the curves $\alpha_1, \dots, \alpha_n$ and the resulting curves $x = \alpha$ and $y = T_3T_2(\alpha_1)$ that must cobound one of the dotted bigons.

geometrically embeds [1, Theorem 3.14]. More generally, free groups geometrically embed [5, Theorem 1.1]. Perron-Vannier [6, Théorème 1] showed that both $\mathcal{A}(A_n)$ and $\mathcal{A}(D_n)$ geometrically embed.

On the other hand, because two Dehn twists can only generate \mathbb{Z}, \mathbb{Z}^2 , a quotient of $\mathcal{A}(A_2)$, or a free group [1, Section 3.5.2], the Artin group $\mathcal{A}(\Gamma)$ does not geometrically embed if Γ contains a weight different from 2, 3, ∞ . Labruère showed that $\mathcal{A}(\tilde{D}_{n-1})$ does not geometrically embed [3, Theorem], where \tilde{D}_{n-1} is the graph



with n vertices for $n \geq 5$, and shortly after, Wajnryb showed that neither do the exceptional groups $\mathcal{A}(E_6), \mathcal{A}(E_7), \mathcal{A}(E_8)$ [7, Theorem 3].

In the same article, Wajnryb appears to claim that Labruère also showed that $\mathcal{A}(\tilde{A}_{n-1})$ does not geometrically embed [7, Theorem 2]. But this is not true (it contradicts Theorem 1.2). Possibly, the confusion comes from Labruère considering not a regular neighbourhood S of a family of curves intersecting in a circuit pattern, but rather a surface $S \cup \Delta$ containing an additional embedded closed disc (in our notation of Section 4.1 below, Labruère considers the surface $N^n \cup \Delta^1$). By Theorem 1.1, considering $S \cup \Delta$ instead of S causes the introduction of a cycle relation.

Presumably, Theorem 1.1 could be proven without any Artin group theory by considering the action of Dehn twists on curves in the surfaces, as the standard cycle relation $T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$ is equivalent to the commutation relation

$$T_n \cdots T_2 T_3^{-1} \cdots T_n^{-1} \cdot T_1 = T_1 \cdot T_n \cdots T_2 T_3^{-1} \cdots T_n^{-1}$$

(see [8, Section 1] for a justification of this). It thus suffices to prove that if for the homeomorphism $h = T_n \cdots T_3$ the curves $h(\alpha_2)$ and α_1 are disjoint (up to homotopy), then the circuit $\alpha_1, \dots, \alpha_n$ bounds an embedded disc. This idea is illustrated in Fig. 1, where we consider a neighbourhood (later called N^4) of the circuit in the case that the cycle relation $T_4T_3T_2T_1T_4T_3 = T_3T_2T_1T_4T_3T_2$ holds. The strategy of proving the result pictorially has a few drawbacks, however. Most notably, is not very convincing to rely solely on such pictures, as the reasoning for the presence of discs seems very prone to error, in particular for some of the surfaces that are less easily drawn in a flat way. In the author’s opinion, the approach taken in this text is more insightful and reliable.

A brief summary of sections is as follows. Section 2 is a collection of prerequisites from Artin group theory and its relation to the theory of mapping class groups. In Section 3 we discuss possible regular neighbourhoods of circuits and prove Theorem 1.2. In our path toward a proof of Theorem 1.1 we get sidetracked in Section 4 and investigate the effect of all (possibly non-embedded) discs on the relational theory of the group $\mathcal{G}(S)$ generated by the Dehn twists T_1, \dots, T_n (note that $\mathcal{G}(S)$ depends not only on S but also on the circuit $\alpha_1, \dots, \alpha_n$). This allows us to determine many isomorphism types of $\mathcal{G}(S)$ for the surfaces S arising from regular neighbourhoods by gluing in discs, see Table 1, which is interesting in its

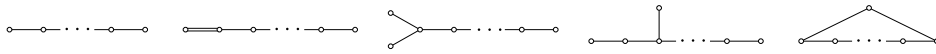


Fig. 2. The graphs $A_n, B_n, D_n, E_n, \tilde{A}_{n-1}$. All five graphs have n vertices.

own right. The reader mainly interested in Theorem 1.1 may wish to focus on Subsections 4.1, 4.2, 4.3, and skip the rest of Section 4. Finally, in Section 5, we prove Theorem 1.1.

2. Artin groups and mapping class groups

2.1. Artin groups

To an undirected multigraph Γ (i.e., a graph with any number, infinity included, of edges between two vertices, but no edge between a vertex and itself), we associate the corresponding *Artin group* $\mathcal{A}(\Gamma)$ by describing a presentation. The generators of $\mathcal{A}(\Gamma)$ are the vertices of Γ , and the relations are $sts \dots = tst \dots$, where the words on both sides have length $n_{st} + 2$, where n_{st} is the number of edges between s and t . The numbers $n_{st} + 2$ are sometimes referred to as *weights*. Explicitly, if there is no edge between s and t then they satisfy the *commutation relation* $st = ts$ (weight 2), if s and t are joined by a single edge then s and t satisfy the *braid relation* $sts = tst$ (weight 3), and so on. It is customary that s and t satisfying no such relation is allowed. This corresponds to infinitely many edges joining s and t (weight ∞). We call an Artin group $\mathcal{A}(\Gamma)$ *irreducible* if Γ is a connected graph. A few example graphs Γ are listed in Fig. 2.

2.2. The Birman exact sequence

For a surface S , we write $C(S, n)$ for the configuration space of n points in S [1, Section 9.1.2]. In order to relate Artin groups to mapping class groups of surfaces, a very useful tool is the Birman exact sequence.

Lemma 2.1 ([1, Theorem 9.1]). *Let S be a surface without marked points such that the identity component of the group of orientation-preserving homeomorphisms $S \rightarrow S$ keeping the boundary ∂S fixed, is simply connected. Let S_n be the surface obtained from S by marking n points in the interior of S . Then there is an exact sequence*

$$1 \longrightarrow \pi_1(C(S, n)) \longrightarrow \mathcal{M}(S_n) \longrightarrow \mathcal{M}(S) \longrightarrow 1,$$

where the homomorphism $\mathcal{M}(S_n) \rightarrow \mathcal{M}(S)$ is obtained from forgetting that the marked points are marked.

The Birman exact sequence is commonly used to prove that mapping class groups of certain surfaces are generated by finitely many Dehn twists [1, Theorem 4.1]. Another application is the following classical Lemma.

Lemma 2.2 ([1, Section 9.1]). *Let $n \geq 2$, and let Δ be a closed disc and Δ_n be Δ with n marked points. Then the groups $\mathcal{A}(A_{n-1})$, $\pi_1(C(\Delta, n))$, and $\mathcal{M}(\Delta_n)$, are pairwise isomorphic.*

Beware that the isomorphism between $\mathcal{A}(A_{n-1})$ and $\mathcal{M}(\Delta_n)$ is not geometric, because the generators are mapped to half-twists (which are not Dehn twists). It turns out that the Birman exact sequence can also be applied to the annulus Z rather than the disc Δ . This yields the following crucial result for our work here.

Lemma 2.3. *Let $n \geq 2$, and let Z be an annulus and let Z_n be Z with n marked points. Then the groups $\mathcal{A}(B_n)$, $\pi_1(C(Z, n))$, and the kernel of the homomorphism $\mathcal{M}(Z_n) \rightarrow \mathcal{M}(Z)$ forgetting that the marked points are marked, are pairwise isomorphic.*

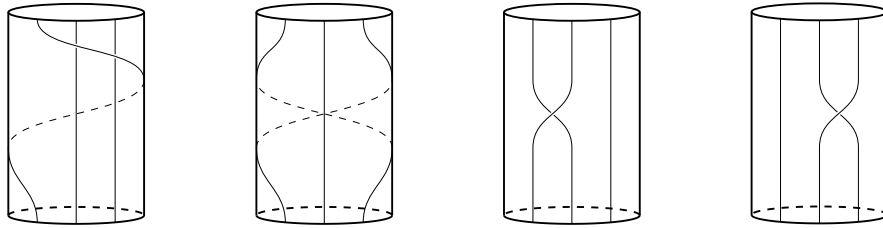


Fig. 3. The images of the elements t, s_0, s_1, s_2 of $\mathcal{A}(B_3)$ in $\pi_1(C(Z, 3))$.

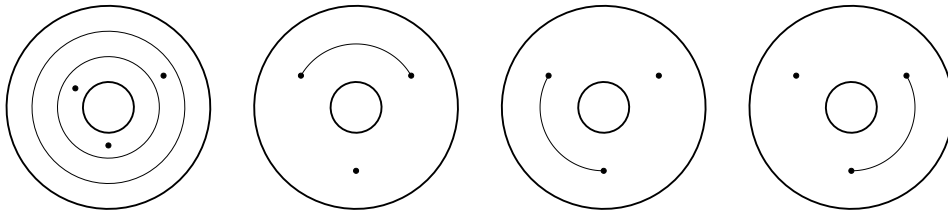


Fig. 4. The images of the elements t, s_0, s_1, s_2 of $\mathcal{A}(B_3)$ in $\mathcal{M}(Z_n)$.

Proof. In [9, Theorem One], it is shown that $\mathcal{A}(B_n)$ is isomorphic to the fundamental group $\pi_1(C(Z, n))$ of the configuration space of n points in the annulus Z . Because the identity component of the space of orientation-preserving homeomorphisms of Z keeping the boundary fixed is contractible [10, Lemma 0.10], we can apply Lemma 2.1 to show that $\pi_1(C(Z, n))$ embeds into the mapping class group $\mathcal{M}(Z_n)$ of the annulus with n marked points. More specifically, by exactness of the Birman exact sequence, the image of $\pi_1(C(Z, n))$ is the kernel of the homomorphism $\mathcal{M}(Z_n) \rightarrow \mathcal{M}(Z)$ forgetting that the marked points are marked. \square

Remark 2.4. The group $\pi_1(C(Z, n))$ can be thought of as n -stranded braids in $Z \times [0, 1]$. Projecting to the central curve of Z at each height in $[0, 1]$, but remembering which strand goes over and which goes under, yields diagrams of elements of $\pi_1(C(Z, n))$, similarly as for the ordinary braid group $\pi_1(C(\Delta, n))$, where one usually projects to a diameter of Δ at each height. Let us refer to the generators of $\mathcal{A}(B_n)$ by the symbols t, s_1, \dots, s_{n-1} . We also write $s_0 = \delta s_{n-1} \delta^{-1}$, where $\delta = t s_1 \dots s_{n-1}$. The isomorphism $\mathcal{A}(B_n) \rightarrow \pi_1(C(Z, n))$ maps the elements $t, s_0, s_1, \dots, s_{n-1}$ to the annular braids depicted in Fig. 3 for the case $n = 3$, and stacks the braids from bottom to top when reading the word in $\mathcal{A}(B_n)$ from left to right.

Remark 2.5. We can also explicitly describe the images of the elements $t, s_0, s_1, \dots, s_{n-1}$ in $\mathcal{M}(\Delta_n)$. The element t maps to a product $T_\alpha T_\beta^{-1}$ of two Dehn twists about two curves α (inner) and β (outer) depicted in Fig. 4. More importantly for us, the elements s_i map to so-called *half-twists* [1, Section 9.1.3] about the arcs depicted in Fig. 4.

We are now ready to describe the group $\mathcal{A}(\tilde{A}_{n-1})$ appearing in Theorem 1.2 quite explicitly.

Lemma 2.6 ([11, Section 1]). *Let t, s_1, \dots, s_{n-1} be the standard generators of $\mathcal{A}(B_n)$ corresponding to the vertices in Fig. 2 from left to right, and let $s_n = \delta s_{n-1} \delta^{-1}$, where $\delta = t s_1 \dots s_{n-1}$. Then the subgroup G of $\mathcal{A}(B_n)$ generated by the s_i is isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$ under an isomorphism mapping the s_i to the standard generators. Moreover, G is the kernel of the homomorphism $\mathcal{A}(B_n) \rightarrow \mathbb{Z}$ mapping t to one and all other generators to zero.*

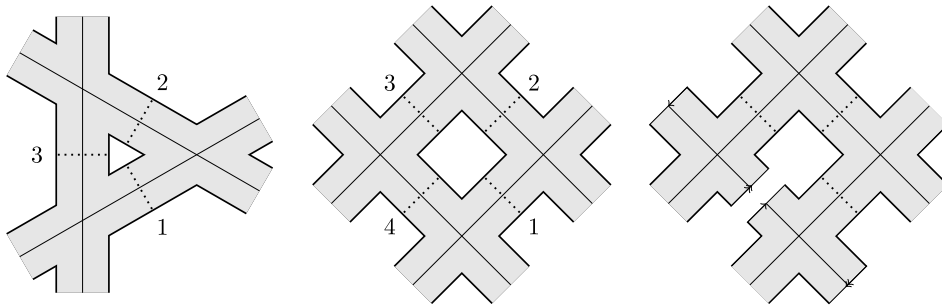


Fig. 5. The neighbourhoods N^3 , N^4 , M^4 . Opposite ends of the strips are identified, unless indicated otherwise.

2.3. Birman-Hilden theory

In the 1970s, Birman and Hilden proved highly influential results about fibre-preserving isotopies. Their work spawned an entire research area referred to as Birman-Hilden theory, recently surveyed by Margalit and Winarski [12]. Only a very small part of this theory will find its way into this text.

For an orientation-preserving involution ι on a surface S , let us write S/ι for the quotient of S by ι with the images of the fixed points of ι marked. With this notation, we have the following reformulation of the classical Birman-Hilden theorem.

Lemma 2.7. *Let the surface S have at least one boundary component, and let ι be a continuous involution on S with finitely many fixed points. Suppose that ι leaves the curves α_i invariant as sets and restricts to a reflection of each α_i . Then there is a well-defined homomorphism $\mathcal{G}(S) \rightarrow \mathcal{M}(S/\iota)$ mapping the Dehn twists T_i about the α_i to half-twists.*

Proof. Let f be a symmetric homeomorphism of S , i.e., a homeomorphism that commutes with ι . Then f induces a homeomorphism \bar{f} on the quotient S/ι . The Birman-Hilden theorem [13, Theorem 1] asserts that the mapping class of \bar{f} does not depend on the symmetric representative of the mapping class of f . The Dehn twist T_i about a simple closed curve α_i is symmetric up to isotopy, so we obtain a homomorphism $\mathcal{G}(S) \rightarrow \mathcal{M}(S/\iota)$. Moreover, ι restricts to an involution of an annular neighbourhood of α_i exchanging the two boundary components. Hence, \bar{T}_i is a half-twist. \square

It is worth pointing out the nontrivial part of the proof of the result referred to as the Birman-Hilden theorem in Farb-Margalit’s book [1, Section 9.4.1] is Lemma 2.7, formulated in a slightly different way [1, Proposition 9.4]. The involutions ι they use yield well-defined maps from the braid group $\mathcal{A}(A_n)$ on $n + 1$ strands to the group generated by Dehn twists about a chain of n curves (each curve intersecting the previous and the next).

3. Neighbourhoods of circuits

This section is concerned with the proof of Theorem 1.2. Let $\alpha_1, \dots, \alpha_n$ be a circuit. Up to orientation-preserving homeomorphism, there are two possible regular neighbourhoods of the union $\alpha_1 \cup \dots \cup \alpha_n$, see Fig. 5. One way to see this is as follows. There is only one possible regular neighbourhood of the smaller set $\alpha_1 \cup \dots \cup \alpha_{n-1}$. Now the curve α_n might sit in the regular neighbourhood in two different ways.

If n is odd, those two ways lead to regular neighbourhoods $N^{\mathcal{G}}_n$ and $N^{\mathcal{G}}_n$ that are related by an orientation-reversing homeomorphism. For brevity, we will abbreviate $N^{\mathcal{G}}_n$ by the symbol N^n and usually not talk about $N^{\mathcal{G}}_n$ explicitly, as all the results about N^n carry over to $N^{\mathcal{G}}_n$ by enumerating the α_i in the opposite order. The left-hand side of Fig. 11 below displays a drawing of the surface N^n embedded into \mathbb{R}^3 .

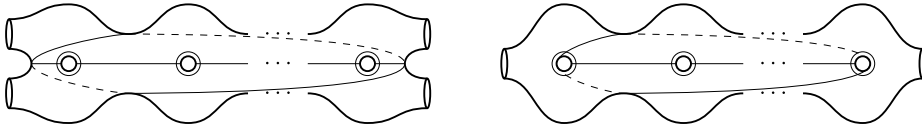


Fig. 6. Another view of the surfaces N^n and M^n for even n .

If n is even, the two possible neighbourhoods N^n and M^n are related to themselves via an orientation-reversing homeomorphism, so orientation is less of a concern in this case. The neighbourhoods N^n and M^n differ, for example, in their number of boundary components: N^n has four and M^n just two. A drawing of the surfaces N^n and M^n embedded into \mathbb{R}^3 can be found in Fig. 6.

Now let S be any of the above regular neighbourhoods of $\alpha_1 \cup \dots \cup \alpha_n$. Notice that in each case, S can be thought of as a union of n cross-shaped pieces, see Fig. 5. Turning all those pieces by an angle of π yields a well-defined involution ι of S . We will call ι the *cross-involution*. Note that in the drawings from Figs. 6 and 11, the cross-involution is a rotation about the x -axis by an angle of π .

Proof of Theorem 1.2. With the notation from Section 2.3, the surface S/ι is an annulus with n marked points in each case. This may be verified by counting the number of boundary components of S/ι and computing its Euler characteristic. By Lemma 2.7, there exists a homomorphism $\varphi: \mathcal{G}(S) \rightarrow \mathcal{M}(Z_n)$ mapping the Dehn twists T_i to half-twists. By Lemma 2.6, the image of φ is isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$. But the inverse homomorphism $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{G}(S)$ mapping the generator s_i to T_i is well-defined. Hence, φ is an isomorphism. \square

4. Gluing in discs

In this section, we glue in discs to regular neighbourhoods of circuits in order to obtain more geometric embeddings of Artin groups, which we will need in order to prove the right-to-left direction of Theorem 1.1. In fact, the results in this section are much stronger than what is needed. Each possible combination of discs that can be glued in to neighbourhoods is investigated separately, and a few interesting relations between Dehn twists about circuits are discussed.

4.1. Building circuit surfaces

We now adopt the perspective that the circuit $\alpha_1, \dots, \alpha_n$ stays fixed while the surface S containing it varies. We will write $\mathcal{G}(S)$ for the subgroup of $\mathcal{M}(S)$ generated by the Dehn twists T_i about α_i . The inclusion homomorphism theorem asserts the following.

Lemma 4.1 ([1, Theorem 3.18]). *Suppose S and S' are closed and connected subsurfaces of a surface $S \cup S'$ with disjoint interiors. Let K be the kernel of the inclusion-induced homomorphism $\mathcal{M}(S) \rightarrow \mathcal{M}(S \cup S')$. Then:*

- (i) *If $S' = \Delta_1$ is a once-marked disc with $\partial\Delta_1 \subset \partial S$, then K is cyclically generated by the Dehn twist T_α about the boundary curve of Δ_1 .*
- (ii) *If $S' = Z$ is an annulus with $\partial Z \subset \partial S$, then K is cyclically generated by $T_\alpha T_\beta^{-1}$, where T_α and T_β are Dehn twists about the boundary curves α and β of Z , respectively.*
- (iii) *If S' is neither a disc, a once-marked disc, nor an annulus, then K is trivial.*

In particular, K is a subgroup of the centre of $\mathcal{M}(S)$.

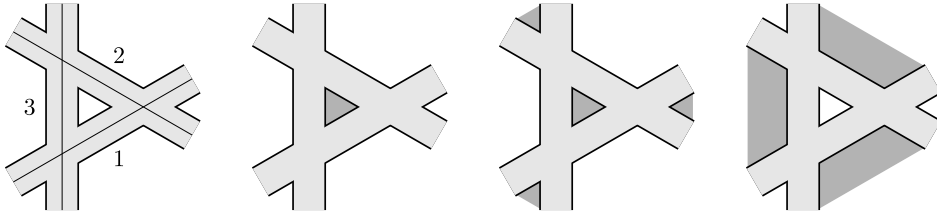


Fig. 7. The surfaces N^3 , $N^3 \cup \Delta^1$, $N^3 \cup 2\Delta^1$, $N^3 \cup \Delta^2$.

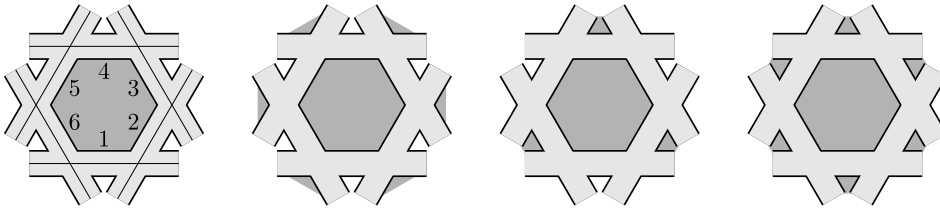


Fig. 8. The surfaces $N^6 \cup \Delta_{\circlearrowleft}^1$, $N^6 \cup 2\Delta_{\circlearrowleft}^1$, $N^6 \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowleft}^1$, $N^6 \cup \Delta_{\circlearrowleft}^1 \cup 2\Delta_{\circlearrowleft}^1$.

Note that the inclusion homomorphism theorem makes no assertion in the case that S' is a disc Δ . This means that to compare $\mathcal{G}(S \cup \Delta)$ to $\mathcal{G}(S)$ is expected to require more creativity than to compare $\mathcal{G}(S \cup S')$ to $\mathcal{G}(S)$ for other surfaces S' .

Not all boundary components of neighbourhoods of $\alpha_1 \cup \dots \cup \alpha_n$ are equivalent. Considering intersection points between the α_i as vertices of polygons allows us to make the following observation. If n is odd, then two boundary components of N^n are exchanged by the cross-involution. They both have the property that if they are capped by a disc, then $\alpha_1, \dots, \alpha_n$ bound an n -gon. We will denote such a disc by Δ^1 , and the union of two such discs by $2\Delta^1$. Both of these will make the circuit bound embedded closed discs. The third boundary component can be capped off by a disc Δ^2 . The curves $\alpha_1, \dots, \alpha_n$ bound a $2n$ -gon in $S \cup \Delta^2$. Note that the circuit does not bound a closed embedded disc in $N^n \cup \Delta^2$. For brevity, we will sometimes say that the boundary of Δ^1 is an n -gon and that the boundary of Δ^2 is a $2n$ -gon. See Fig. 7 for a visual description of the various discs.

If n is even, the situation is qualitatively different. All four boundary components of N^n are n -gons, and both boundary components of M^n are $2n$ -gons. But they differ in the following way. Travelling around the boundary component in the counter-clockwise direction, as seen from the centre of the disc, the curves might appear in the order $\alpha_1, \dots, \alpha_n$ or the other way around. In the first case, we will write $\Delta_{\circlearrowleft}$, and $\Delta_{\circlearrowright}$ otherwise, with the appropriate superscript numbers. See Fig. 8 for the different discs in N^n . We will usually abbreviate $\Delta_{\circlearrowleft}$ by Δ with the appropriate superscript number, and we will abbreviate $\Delta_{\circlearrowleft}^2 \cup \Delta_{\circlearrowleft}^2$ by $2\Delta^2$. A list of all surfaces that are obtained by gluing in discs to a regular neighbourhood of $\alpha_1 \cup \dots \cup \alpha_n$ can be found in Table 1. Note that gluing in any disc into N^n makes the circuit bound an embedded closed disc, and no disc in M^n has this effect.

4.2. Extending the cross-involution

It turns out that for certain surfaces S containing a neighbourhood of $\alpha_1, \dots, \alpha_n$, the cross-involution of the neighbourhood extends to S . In these cases, by very similar reasoning as in the proof of Theorem 1.2, we are able to determine the group $\mathcal{G}(S)$.

Proposition 4.2. *Let S be a regular neighbourhood of a circuit of $n \geq 3$ curves $\alpha_1, \dots, \alpha_n$, and let S' be a $2n$ -gon Δ^2 (such a disc does not exist if n is even and $S = N^n$), or the union $2\Delta^2$ of two $2n$ -gons (such a union only exists for even n and $S = M^n$). Then $\mathcal{G}(S \cup S')$ is geometrically isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$.*

Table 1

Various surfaces S obtained from regular neighbourhoods of a circuit $\alpha_1, \dots, \alpha_n$ by gluing in discs. In this table, $n \geq 3$ and $k \geq 1$. Moreover, “not Artin” is short for “not geometrically isomorphic to an Artin group”.

S	$\mathcal{G}(S)$	Reference
N^n	$\mathcal{A}(\tilde{A}_{n-1})$	Theorem 1.2
$N^n \cup \Delta^1$	$\mathcal{A}(D_n)$	Proposition 4.6
$N^n \cup 2\Delta^1$	$\mathcal{A}(A_{n-1})$	Proposition 4.3
$N^{2k+1} \cup \Delta^2$	$\mathcal{A}(\tilde{A}_{2k})$	Proposition 4.2
$N^{2k+1} \cup \Delta^1 \cup \Delta^2$	not Artin	Proposition 4.17
$N^{2k+1} \cup 2\Delta^1 \cup \Delta^2$	not Artin	Proposition 4.10
$N^{2k+4} \cup \Delta^1_{\circ} \cup \Delta^1_{\circ}$	not Artin	Proposition 4.11
$N^{2k+4} \cup 2\Delta^1_{\circ} \cup \Delta^1_{\circ}$	not Artin	Proposition 4.11
$N^{2k+4} \cup 2\Delta^1_{\circ} \cup 2\Delta^1_{\circ}$	not Artin	Proposition 4.11
$N^4 \cup \Delta^1_{\circ} \cup \Delta^1_{\circ}$	$\mathcal{A}(A_3)$	Proposition 4.12
$N^4 \cup 2\Delta^1_{\circ} \cup \Delta^1_{\circ}$	$\mathcal{A}(A_2)$	Proposition 4.12
$N^4 \cup 2\Delta^1_{\circ} \cup 2\Delta^1_{\circ}$	$\text{SL}(2, \mathbb{Z})$	Proposition 4.12
M^{2k+2}	$\mathcal{A}(\tilde{A}_{2k+1})$	Theorem 1.2
$M^{2k+2} \cup \Delta^2$	$\mathcal{A}(\tilde{A}_{2k+1})$	Proposition 4.2
$M^{2k+2} \cup 2\Delta^2$	$\mathcal{A}(\tilde{A}_{2k+1})$	Proposition 4.2

Proof. In each case, the cross-involution ι extends to $S \cup S'$ in a straightforward fashion. In the case that $S' = \Delta^2$ is a $2n$ -gon, ι gets one additional fixed point, so $(S \cup S')/\iota$ is a disc Δ_{n+1} with $n + 1$ marked points. Again, the homomorphism $\varphi: \mathcal{G}(S) \rightarrow \mathcal{M}(\Delta_{n+1})$ is well-defined by Lemma 2.7. The images of the T_i under φ fix the last puncture. Because Δ minus one point is homotopy equivalent to the annulus Z , we have that the subgroup of $\pi_1(C(\Delta, n + 1))$ fixing one strand is isomorphic to $\pi_1(C(Z, n))$. By Lemmas 2.3 and 2.6, the image of φ is isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$. Since the inverse homomorphism is well-defined, the result follows for this case.

Similarly, if $S' = 2\Delta^2$, we get a well-defined homomorphism $\varphi: \mathcal{G}(S) \rightarrow \mathcal{M}(\Sigma_{n+2})$, where Σ_{n+2} is a sphere with $n + 2$ marked points. The images of the T_i fix two of the punctures. Because the group of orientation-preserving homeomorphisms of Σ is not simply connected, there is no Birman exact sequence. That is, we cannot apply Lemma 2.1 directly to get a description of $\mathcal{M}(\Sigma_{n+2})$. However, it is straightforward to show that the kernel of the map $\pi_1(C(\Sigma, n + 2)) \rightarrow \mathcal{M}(\Sigma_{n+2})$ is generated by the map rotating the $n + 2$ marked points by a full twist [1, Section 9.1]. Because such a full twist cannot be expressed by just n generators, we have that the image of φ is isomorphic to the subgroup of $\pi_1(C(\Sigma, n + 2))$ fixing two strands. But because Σ minus two points is homotopy equivalent to Z , we get that the image of φ is isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$, as desired. \square

Recall that for odd n we abbreviate N°_n by N^n , and for all n , we abbreviate Δ^1_{\circ} by Δ^1 . Using these conventions allows us to concisely state the following result which is essentially equivalent to a version of the Birman-Hilden theorem stated in the book by Farb and Margalit [1, Theorem 9.2].

Proposition 4.3. *For $n \geq 3$, the group $\mathcal{G}(N^n \cup 2\Delta^1)$ is geometrically isomorphic to the braid group $\mathcal{A}(A_{n-1})$ on n strands.*

Proof. Let us write $S = N^n \cup 2\Delta^1$. The cross-involution ι extends to $N^n \cup 2\Delta^1$ with no additional fixed points. This yields a well-defined homomorphism $\varphi: \mathcal{G}(S) \rightarrow \mathcal{M}(\Delta_n)$ mapping the T_i to half-twists by Lemma 2.7. Since these half-twists generate $\mathcal{M}(\Delta_n)$, Lemma 2.2 yields that the image of φ is isomorphic to $\mathcal{A}(A_{n-1})$. \square

Remark 4.4. It is not difficult to show that the kernel of the inclusion-induced homomorphism $\mathcal{G}(N^n) \rightarrow \mathcal{G}(N^n \cup 2\Delta^1)$ is normally generated by the relation $T_n \cdots T_2 = T_{n-1} \cdots T_1$. One possible strategy is to explicitly compute the kernel of the inclusion-induced homomorphism $\pi_1(C(Z, n)) \rightarrow \pi_1(C(\Delta, n))$.

4.3. The cycle relation

The surface the current subsection is about is $N^n \cup \Delta^1$. Luckily for us, this case has almost entirely been solved by Labruère, and the rest can be extracted from work by Baader and Lönne.

If the circuit $\alpha_1, \dots, \alpha_n$ bounds an n -gon $\Delta_{\circlearrowleft}^1$ or $\Delta_{\circlearrowright}^1$, we will say that $\alpha_1, \dots, \alpha_n$ form a *cycle*. The *standard cycle relation* between the Dehn twists T_1, \dots, T_n is $T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$. One can show that this relation is equivalent to the commutation relation $T_1 f = f T_1$ where $f = (T_n \cdots T_3) T_2 (T_n \cdots T_3)^{-1}$ [8, Section 1]. Using this representation of the standard cycle relation it becomes a routine task to verify that the standard cycle relation holds in the surface $N^n \cup \Delta^1$. Similarly, the *reverse cycle relation* $T_1 \cdots T_n T_1 \cdots T_{n-2} = T_2 \cdots T_n T_1 \cdots T_{n-1}$, holds in the surface $N_{\circlearrowright}^n \cup \Delta_{\circlearrowright}^1$. But Labruère made an even stronger observation.

Lemma 4.5 ([3, Proposition 2]). *The kernel of the homomorphism $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{G}(N^n \cup \Delta^1)$ mapping the standard generators s_i to T_i is normally generated by the cycle relation.*

Proposition 4.6. *For $n \geq 3$, the group $\mathcal{G}(N^n \cup \Delta^1)$ is geometrically isomorphic to $\mathcal{A}(D_n)$.*

Proof. Let s_1, \dots, s_n be the standard generators of $\mathcal{A}(D_n)$ read from left to right in Fig. 2. Using Lemma 4.5, one can verify that an explicit isomorphism $\mathcal{A}(D_n) \rightarrow \mathcal{G}(N^n \cup \Delta^1)$ is given by $s_1 \mapsto (T_n \cdots T_3)^{-1} T_1 (T_n \cdots T_3)$ and $s_i \mapsto T_i$ for $i \geq 2$. \square

Remark 4.7. Baader and Lönne prove the considerably more general but also less easily digested result that the secondary braid group is invariant via a geometric isomorphism under elementary conjugation [8, Section 4]. Indeed, by Lemma 4.5, the group $\mathcal{G}(N^n \cup \Delta^1)$ is geometrically isomorphic to the secondary braid group [8, Definition 1] associated to the positive braid word $\sigma_1 \sigma_2 \sigma_1^{n-2} \sigma_2 \sigma_1$, whereas $\mathcal{A}(D_n)$ is geometrically isomorphic to the group associated to $\sigma_1^2 \sigma_2 \sigma_1^{n-2} \sigma_2$.

4.4. Inhomogeneous relations

Sadly, we do not manage to compute the remaining groups $\mathcal{G}(S)$ up to isomorphism. We will, however, get to know the groups well enough to exclude the possibility of them being geometrically isomorphic to an Artin group.

A relation $t = t'$ is called *homogeneous* if the exponent sums of t and t' agree. Otherwise, the relation $t = t'$ is called *inhomogeneous*. The strategy in the current subsection will be to find inhomogeneous relations in $\mathcal{G}(S)$. The following elementary result allows us to conclude that $\mathcal{G}(S)$ is not geometrically isomorphic to an Artin group.

Lemma 4.8. *If $\mathcal{G}(S)$ has an inhomogeneous relation, then $\mathcal{G}(S)$ is not geometrically isomorphic to an Artin group.*

Proof. We argue contrapositively: The map sending each generator of an Artin group $\mathcal{A}(\Gamma)$ to one extends to a homomorphism $\mathcal{A}(\Gamma) \rightarrow \mathbb{Z}$ because all relations in $\mathcal{A}(\Gamma)$ are homogeneous, and hence also hold in \mathbb{Z} . \square

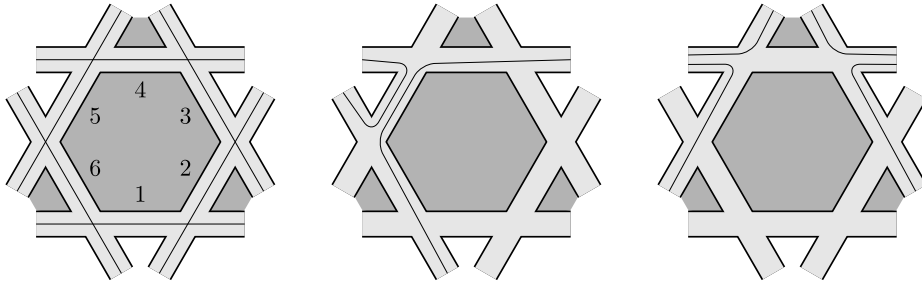


Fig. 9. The curves $\alpha_1, \dots, \alpha_6$, the curve $T_3T_4T_5T_6(\alpha_1)$, and the curve $\beta = T_6T_5T_4T_3^2T_4T_5T_6(\alpha_1)$, all in the surface $N^6 \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$.

An effective way to produce inhomogeneous relations is to apply the classical result called the chain relation. Recall that a *chain* of curves is a family $\alpha_1, \dots, \alpha_n$ of n curves such that α_i intersects α_j exactly once if $j = i \pm 1$ and zero times otherwise.

Lemma 4.9 ([1, Proposition 4.12]). *Let $\alpha_1, \dots, \alpha_n$ be a chain of n curves, and let T_i be the Dehn twist about α_i . If n is even, let β be the boundary curve of a regular neighbourhood of $\alpha_1 \cup \dots \cup \alpha_n$. Similarly, if n is odd, let β_1, β_2 be the two boundary curves. Then:*

- (i) *If n is even, then $(T_n \cdots T_1)^{2n+2} = T_\beta$.*
- (ii) *If n is odd, then $(T_n \cdots T_1)^{n+1} = T_{\beta_1}T_{\beta_2}$.*

Proposition 4.10. *For odd $n \geq 3$, the relation $(T_{n-1} \cdots T_1)^{2n} = 1$ holds in the group $\mathcal{G}(N^n \cup 2\Delta^1 \cup \Delta^2)$. In particular, it is not geometrically isomorphic to an Artin group.*

Proof. Note that the boundary of a regular neighbourhood of the chain $\alpha_1, \dots, \alpha_{n-1}$ is null-homotopic. The proposition now follows from Lemma 4.9 and Lemma 4.8. \square

Proposition 4.11. *The group $\mathcal{G}(N^n \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1)$ is not geometrically isomorphic to an Artin group if $n \geq 6$, and neither is $\mathcal{G}(S)$ for any supersurface S of $N^n \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$.*

Proof. Suppose the curves are arranged as in Fig. 9. Let $\beta = T_n \cdots T_3T_3 \cdots T_n(\alpha_1)$. Then $\beta \cup \alpha_1$ is the boundary of a regular neighbourhood of $\alpha_3 \cup \dots \cup \alpha_{n-1}$, see Fig. 9. By the Lemma 4.9, the relation $(T_{n-1} \cdots T_3)^{n-2} = T_1T_\beta$ follows. Because T_β is conjugate to T_1 by the formula $T_{f(\alpha_1)} = fT_1f^{-1}$ [1, Fact 3.7] for $f = T_n \cdots T_3T_3 \cdots T_n$, the relation in question is inhomogeneous for $n \geq 6$. Lemma 4.8 leads us to the desired conclusion.

If the indices of the curves are shifted by one from the ones in Fig. 9, we instead end up with the relation $(T_{n-2} \cdots T_2)^{n-2} = T_nT_{\beta'}$ where $\beta' = T_{n-1} \cdots T_2T_2 \cdots T_{n-1}(\alpha_n)$, which is also inhomogeneous for $n \geq 6$.

The statement about supersurfaces follows from the fact that the inclusion-induced homomorphisms preserve inhomogeneous relations. \square

4.5. Pathological cases

The case $n = 4$ becomes strange when too many discs are glued in, because some of the curves become isotopic. The relations from the proof of Proposition 4.11 do not reflect this, so we cover this case separately.

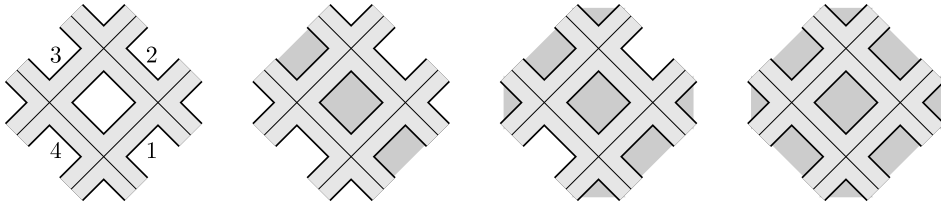


Fig. 10. The surfaces N^4 , $N^4 \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$, $N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$, $N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup 2\Delta_{\circlearrowright}^1$.

Proposition 4.12. *The following statements hold.*

- (i) *The group $\mathcal{G}(N^4 \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1)$ is geometrically isomorphic to $\mathcal{A}(A_3)$.*
- (ii) *The group $\mathcal{G}(N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1)$ is geometrically isomorphic to $\mathcal{A}(A_2)$,*
- (iii) *The group $\mathcal{G}(N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup 2\Delta_{\circlearrowright}^1)$ is isomorphic to $SL(2, \mathbb{Z})$, and not geometrically isomorphic to an Artin group.*

Proof. Suppose the curves and discs are arranged as in Fig. 10. We consider each surface S separately.

- (i) Let $S = N^4 \cup \Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$. Because α_2 and α_4 are isotopic in S , we have that $\mathcal{G}(S)$ is generated by T_1, T_2, T_3 . Moreover, S is a regular neighbourhood of $\alpha_1, \alpha_2, \alpha_3$, so $\mathcal{G}(S)$ is isomorphic to $\mathcal{A}(A_3)$ [1, Section 9.4.1].
- (ii) Let $S = N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup \Delta_{\circlearrowright}^1$. In addition to α_2 being isotopic to α_4 from the previous case, α_1 is also isotopic to α_3 . So $\mathcal{G}(S)$ is generated by T_1, T_2 . Moreover, S is a regular neighbourhood of $\alpha_1 \cup \alpha_2$. Hence, $\mathcal{G}(S)$ is isomorphic to $\mathcal{A}(A_2)$ [1, Theorem 9.2].
- (iii) Let $S = N^4 \cup 2\Delta_{\circlearrowleft}^1 \cup 2\Delta_{\circlearrowright}^1$. Then S is just a torus with meridian α_1 and α_2 . It is well-known that $\mathcal{M}(S)$ is generated by T_1, T_2 , and that it is isomorphic to $SL(2, \mathbb{Z})$ [1, Theorem 2.5]. Moreover, the inhomogeneous relation $(T_1 T_2)^6 = 1$ (see [1, Section 3.5]) shows that $\mathcal{G}(S)$ is not geometrically isomorphic to an Artin group.

If the indices of the curves are instead shifted by one, the same arguments hold. \square

4.6. One last surface

Up to orientation-reversing homeomorphism, we have now glued in every possible combination of discs, except one. For this final surface $S = N^n \cup \Delta^1 \cup \Delta^2$, the strategy of finding inhomogeneous relations failed, so the proof that $\mathcal{G}(S)$ is not geometrically isomorphic to an Artin group turns out to be the most involved argument in this text. Toward a contradiction, we will assume that $\mathcal{G}(S)$ is geometrically isomorphic to an Artin group $\mathcal{A}(\Gamma)$. We then exclude all possibilities for the graph Γ . Lemma 4.13 below is a statement about Coxeter groups that helps achieve this for most graphs.

The Coxeter group $\mathcal{C}(\Gamma)$ is obtained from $\mathcal{A}(\Gamma)$ by adding the relations $s^2 = 1$ for all generators s . If $\mathcal{C}(\Gamma)$ is finite, we will say that $\mathcal{A}(\Gamma)$ is of *finite type*. Otherwise, $\mathcal{A}(\Gamma)$ is of *infinite type*. The finite Coxeter groups were classified by Coxeter himself [14, Theorem†]. They are groups of the form $\mathcal{C}(\Gamma)$, where $\Gamma = A_n, B_n, D_n$ for arbitrary n , $\Gamma = E_n$ for $n = 6, 7, 8$, or a few more graphs that do not appear in this text. See Fig. 2 for a list of the mentioned graphs.

Lemma 4.13 ([15, Theorem 0.4 and Table 3]). *Let $n \geq 3$ with $n \neq 4$. If there exists a surjective homomorphism $\mathcal{C}(D_n) \rightarrow \mathcal{C}(\Gamma)$, then Γ is either the one-vertex graph A_1 , the graph A_{n-1} , or the graph D_n .*

Next, we need two results about the group-theoretic properties of the Artin groups $\mathcal{A}(A_{n-1})$ and $\mathcal{A}(D_n)$. The first result asserts that $\mathcal{A}(A_{n-1})$ is “geometrically co-Hopfian”.

Lemma 4.14. *Let $n \geq 2$. Every injective homomorphism $\mathcal{A}(A_{n-1}) \rightarrow \mathcal{A}(A_{n-1})$ such that the image of a standard generator is conjugate to a standard generator is an isomorphism.*

Proof. Think of $\mathcal{A}(A_{n-1})$ as the group $\mathcal{G}(S)$ generated by $\alpha_1, \dots, \alpha_{n-1}$, where S is the surface $N^n \cup 2\Delta^1$, see Proposition 4.3. Then a homomorphism as in the assumption corresponds to an injective homomorphism $\varphi: \mathcal{G}(S) \rightarrow \mathcal{G}(S)$ mapping each T_i to a Dehn twist T'_i about a curve α'_i . Because φ is injective, the α'_i are pairwise non-isotopic. Moreover, the curves $\alpha'_1, \dots, \alpha'_{n-1}$ form a chain because consecutive curves satisfy the braid relation [1, Section 3.5.2]. Hence, by the change of coordinates principle [1, Section 1.3.3], there exists a homeomorphism f of S such that $\alpha'_i = f(\alpha_i)$. Thus, φ is given by conjugation by f , and hence is an isomorphism. \square

Remark 4.15. Bell and Margalit in fact describe all the injective homomorphisms from the n -strand braid group $\mathcal{A}(A_{n-1})$ to itself, even the non-geometric ones, for $n \geq 4$ [16, Main Theorem 1]. Their uniform description of these homomorphisms does not hold for $n = 3$ because $\mathcal{A}(A_2)$ modulo its centre is not co-Hopfian (it is isomorphic to the free product $\mathbb{Z}/2 * \mathbb{Z}/3$).

Our final lemma in this Section asserts that finite type Artin groups are “Hopfian”.

Lemma 4.16. *Every surjective homomorphism from a finite type Artin group onto itself is an isomorphism.*

Proof. Because finite type Artin groups are residually finite [17, Corollary 1.2] they are also Hopfian [18, Theorem IV.4.10]. \square

Proposition 4.17. *For odd $n \geq 3$, the group $\mathcal{G}(N^n \cup \Delta^1 \cup \Delta^2)$ is not geometrically isomorphic to an Artin group.*

Proof. Write $S = N^n \cup \Delta^1 \cup \Delta^2$. Suppose toward a contradiction that $\mathcal{G}(S)$ is geometrically isomorphic to $\mathcal{A}(\Gamma)$ for a graph Γ . Recall that by Proposition 4.6, the group $\mathcal{G}(N^n \cup \Delta^1)$ is geometrically isomorphic to $\mathcal{A}(D_n)$. Thus, the inclusion-induced homomorphism $\mathcal{G}(N^n \cup \Delta^1) \rightarrow \mathcal{G}(S)$ gives rise to a surjective homomorphism $\mathcal{C}(D_n) \rightarrow \mathcal{C}(\Gamma)$ (note that we use here that the isomorphism $\mathcal{A}(D_n) \rightarrow \mathcal{G}(N^n \cup \Delta^1)$ is geometric). From Lemma 4.13 it follows that Γ is either A_1 , A_{n-1} , or D_n . We will now rule out each of those graphs.

We first argue that $\mathcal{G}(S)$ contains a strict subgroup isomorphic to $\mathcal{A}(A_{n-1})$. Consider the plastic view of N^n as on the left of Fig. 11. Capping of the top and right boundary components with discs yields the surface S on the right. Now rotating about the x -axis by an angle of π yields an involution ι of S . Suppose the curves $\alpha_1, \dots, \alpha_n$ are numbered such that α_n is the right-most curve. Then ι preserves $\alpha_1, \dots, \alpha_{n-1}$, but not α_n . Thus, the strict subgroup of $\mathcal{G}(S)$ generated by T_1, \dots, T_{n-1} is isomorphic to $\mathcal{A}(A_{n-1})$. This excludes the case $\Gamma = A_1$ immediately, and an application of Lemma 4.14 excludes the case $\Gamma = A_{n-1}$.

Next, we show that the inclusion-induced homomorphism $\mathcal{G}(N^n \cup \Delta) \rightarrow \mathcal{G}(S)$ is not injective. To this end, consider the boundary curve β of the chain $\alpha_1, \dots, \alpha_{n-1}$ in $N^n \cup \Delta$. Then β intersects α_n twice. But the image of β under the inclusion map $N^n \cup \Delta \rightarrow S$ does not intersect the image of α_n . Hence, the commutator $T_\beta T_n T_\beta^{-1} T_n^{-1}$ is a non-trivial element of the kernel. By Lemma 4.16, Γ cannot be D_n , excluding all possibilities for Γ . \square

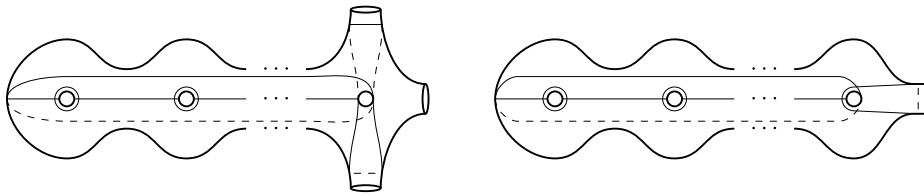


Fig. 11. Another view of the surfaces N^n and $N^n \cup \Delta^1 \cup \Delta^2$ for odd n .

5. Proof of the main theorem

This short final section is about gluing in punctured discs and annuli to the surfaces from Table 1 and collecting the relevant results in this text to prove Theorem 1.1.

Proposition 5.1. *Let S be a surface containing a circuit $\alpha_1, \dots, \alpha_n$. Suppose that $\mathcal{G}(S)$ is geometrically isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$. Let Δ_1 be a once-marked disc whose interior is disjoint from the interior of S , with $\partial\Delta_1 \subset \partial S$. Then the inclusion-induced homomorphism $\mathcal{G}(S) \rightarrow \mathcal{G}(S \cup \Delta_1)$ is an isomorphism. Similarly, if Z is an annulus whose interior is disjoint from the interior of S , with $\partial Z \subset \partial S$, then the inclusion-induced homomorphism $\mathcal{G}(S) \rightarrow \mathcal{G}(S \cup Z)$ is an isomorphism.*

Proof. Charney and Peifer show that for $n \geq 3$, the centre of $\mathcal{A}(\tilde{A}_{n-1})$ is trivial [11, Proposition 1.3]. It now follows from Lemma 4.1 that the inclusion-induced homomorphisms $\mathcal{G}(S) \rightarrow \mathcal{G}(S \cup \Delta_1)$ and $\mathcal{G}(S) \rightarrow \mathcal{G}(S \cup Z)$ are injective and hence isomorphisms. \square

Proof of Theorem 1.1. We prove the right-to-left implication, as the left-to-right implication follows from Labruère's result [3, Proposition 2]. Contrapositively, suppose that the circuit $\alpha_1, \dots, \alpha_n$ does not bound an embedded closed disc. In other words, the complement of a regular neighbourhood of $\alpha_1, \dots, \alpha_n$ in S is a union of surfaces that are not embedded discs. Let S' be the union of such a neighbourhood with all the non-embedded discs in its complement. Theorem 1.2 and Proposition 4.2 imply that $\mathcal{G}(S')$ is geometrically isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$. The complement of S' in S is a union of surfaces that are not discs, so by Proposition 5.1 and Lemma 4.1, it follows that also $\mathcal{G}(S)$ is geometrically isomorphic to $\mathcal{A}(\tilde{A}_{n-1})$. But the cycle relation does not hold in this group. Indeed, as remarked above, the centre of $\mathcal{A}(\tilde{A}_{n-1})$ is trivial, whereas the quotient of $\mathcal{A}(\tilde{A}_{n-1})$ by the normal subgroup generated by the cycle relation is isomorphic to $\mathcal{A}(D_n)$ (see Lemma 4.5 and Proposition 4.6), which has infinite cyclic centre [19, Satz 7.2]. \square

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