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Universal equations for maximal isotropic Grassmannians

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ABSTRACT

The isotropic Grassmannian parametrizes isotropic subspaces of a vector space equipped with a quadratic form. In this paper, we show that any maximal isotropic Grassmannian in its Plücker embedding can be defined by pulling back the equations of $Gr_{\text{iso}}(3, 7)$ or $Gr_{\text{iso}}(4, 8)$.

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1. Introduction

The Grassmannian $Gr(k, V)$ is the space of k -dimensional subspaces of a vector space V . As an algebraic variety it is naturally embedded into the projective space $\mathbb{P}(\wedge^k V)$ via the Plücker embedding. The defining equations of this variety are quadrics known as the Plücker relations. The smallest nontrivial example is the Grassmannian $Gr(2, 4)$, which has a single defining equation known as the Klein quadric. For an arbitrary Grassmannian $Gr(k, V)$, the Plücker relations are still quadrics, but we need many quadrics of high rank to generate the defining ideal. However, in Kasman et al. (2008), Kasman et al. showed that if one only considers the Plücker relations that can be obtained from pulling back the Klein quadric, their vanishing locus is already the Grassmannian. In particular, since the Klein quadric has rank six, this implies that any Grassmannian can be set-theoretically defined by quadrics of rank six.

If a vector space is equipped with a nondegenerate quadratic form, the variety of isotropic linear subspaces is known as the isotropic Grassmannian. It is natural to ask if any isotropic Grassmannian can also be set-theoretically defined by pulling back the defining equations of a fixed small isotropic

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Grassmannian. In this paper we show that this is indeed the case when considering *maximal* isotropic subspaces in V .

Theorem 1.1. *Let \mathbb{K} be an algebraically closed field of characteristic not equal to 2, and V a \mathbb{K} -vector space equipped with a nondegenerate quadratic form. Then the isotropic Grassmannian $Gr_{iso}(\lfloor \frac{\dim V}{2} \rfloor, V)$ in its Plücker embedding can be defined set-theoretically by pulling back the defining equations of*

- $Gr_{iso}(3, 7)$ if V is odd-dimensional,
- $Gr_{iso}(4, 8)$ if V is even-dimensional.

Since the ideals of $Gr_{iso}(3, 7)$ and $Gr_{iso}(4, 8)$, and indeed of any isotropic Grassmannian, are generated by finitely many quadrics, this implies a universal bound on the ranks of the quadrics needed to set-theoretically define any maximal isotropic Grassmannian. Notably, this bound is precisely four.

In Draisma and Eggermont (2018), Draisma and Eggermont introduced the notion of a *Plücker variety*, of which the Grassmannian is a special case. They proved a universality result that generalizes the universality of the Klein quadric, which was recently further generalized by Nekrasov (2020). Our main theorem is a first step towards defining *isotropic Plücker varieties*, which should generalize the isotropic Grassmannian in the same way that Plücker varieties generalize Grassmannians.

Organization. The remainder of this article is organized as follows. Section 2 is about ordinary Grassmannians. In this section we recall the definition of the Grassmann cone and Grassmann cone preserving maps, and state the universality of the Klein quadric by Kasman et al. (Theorem 2.3). In Section 3 we provide the necessary background on quadratic forms before we introduce isotropic Grassmannians and isotropic Grassmann cone preserving maps (IGCP maps). In Section 4 we state and prove the Main Theorem 4.1, which states that if V has dimension at least 9, then starting with a $\lfloor \frac{\dim V}{2} \rfloor$ -form ω not in the isotropic Grassmann cone, there is an IGCP map mapping ω outside of the isotropic Grassmann cone. This immediately implies Theorem 1.1. A key ingredient in our proof is Proposition 4.7, which gives a characterization of forms in $\hat{Gr}_{iso}(\lfloor \frac{\dim V}{2} \rfloor, V)$. In Section 5 we show how the Main Theorem 4.1 fails for $\dim V$ at most 8, by giving counterexamples for $Gr_{iso}(3, 7)$ and $Gr_{iso}(4, 8)$. The first counterexample is related to the exceptional group G_2 and to the octonions. Finally, in Appendix A, we provide a computational proof that the ideals of $Gr_{iso}(3, 7)$ and $Gr_{iso}(4, 8)$ are generated by quadrics of rank at most four. Together with Theorem 1.1 this implies the same bound for any isotropic Grassmannian (Corollary 4.3). We also sketch an alternative proof of Corollary 4.3, which relies on a connection to the Cartan embedding instead of on Theorem 1.1.

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2. The ordinary Grassmannian

Let V be a finite-dimensional vector space over any field \mathbb{K} , and $k \leq \dim V$. The *Grassmann cone* is defined as

$$\hat{Gr}(k, V) := \{v_1 \wedge \dots \wedge v_k \mid v_1, \dots, v_k \in V\} \subset \bigwedge^k V,$$

where $\bigwedge^k V$ is the k 'th exterior power of V . For $\omega = v_1 \wedge \dots \wedge v_k \in \hat{Gr}(k, V)$, we will denote the corresponding subspace $\text{span}\{v_1, \dots, v_k\} \subseteq V$ as L_ω . If $\dim V = n$, we will sometimes write $\hat{Gr}(k, n)$ instead of $\hat{Gr}(k, V)$. Choosing a basis (e_1, \dots, e_n) of V induces coordinates $\{x_I \mid I \subset \{1, \dots, n\}, |I| = k\}$ on $\bigwedge^k V$, which are known as the *Plücker coordinates*. The Grassmann cone is a subvariety of $\bigwedge^k V$, and its defining equations are quadrics referred to as the *Plücker relations* (Shafarevich, 1994, (1.24)). In the case where $k = 2$ and $V = \mathbb{K}^4$, there exists only one Plücker relation, called the Klein quadric

$$P_{2,4} = x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}.$$

The *Grassmannian* is the projectivization of the Grassmann cone:

$$Gr(k, V) := \mathbb{P}(\hat{Gr}(k, V)) = (\hat{Gr}(k, V) \setminus \{0\}) / \mathbb{K}^\times \subseteq \mathbb{P}\left(\bigwedge^k V\right).$$

It is a projective variety whose defining equations are the Plücker relations.

Definition 2.1. A linear map $\varphi : \bigwedge^k V \rightarrow \bigwedge^q W$ is *Grassmann cone preserving* (GCP) if

$$\varphi(\hat{Gr}(k, V)) \subseteq \hat{Gr}(q, W).$$

Example 2.2. We give two examples of GCP maps.

- (1) If $f : V \rightarrow W$ is a linear map, the induced map $\bigwedge^k f : \bigwedge^k V \rightarrow \bigwedge^k W$ is Grassmann cone preserving.
- (2) For $\beta \in V^*$ the contraction map

$$\varphi_\beta : \bigwedge^k V \rightarrow \bigwedge^{k-1} \ker \beta \subseteq \bigwedge^{k-1} V$$

defined as

$$v_1 \wedge \dots \wedge v_k \mapsto \sum_{i=1}^k (-1)^{i-1} \beta(v_i) v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_k$$

is GCP. Below we give a coordinate-independent description of φ_β .

Proof. Let $v_1 \wedge \dots \wedge v_k \in \hat{Gr}(k, V) \setminus \{0\}$. Note that the vectors v_1, \dots, v_k are linearly independent. We distinguish two cases. First, assume that $\text{span}\{v_1, \dots, v_k\} \subseteq \ker \beta$. Then, $\varphi_\beta(v_1 \wedge \dots \wedge v_k) = 0 \in \hat{Gr}(k-1, \ker \beta)$. Now consider the case that $\ker \beta \cap \text{span}\{v_1, \dots, v_k\}$ has dimension $k-1$. After possibly replacing v_1, \dots, v_k with some v'_1, \dots, v'_k such that $v'_1 \wedge \dots \wedge v'_k = v_1 \wedge \dots \wedge v_k$, we can assume that $v_1 \notin \ker \beta$, but $v_2, \dots, v_k \in \ker \beta$. Then, $\varphi_\beta(v_1 \wedge \dots \wedge v_k) = \beta(v_1) v_2 \wedge \dots \wedge v_k$ is contained in $\hat{Gr}(k-1, \ker \beta)$. This proof also shows that φ_β takes values in $\bigwedge^{k-1} \ker \beta$. \square

The contraction map φ_β can also be described coordinate-independently. Recall that there is a natural isomorphism $\bigwedge^k V \cong \text{Alt}^k(V^*)$, where $\text{Alt}^k(V^*)$ is the space of alternating multilinear maps $V^* \times \dots \times V^* \rightarrow \mathbb{K}$. Under this identification, φ_β agrees with the map

$$\text{Alt}^k(V^*) \rightarrow \text{Alt}^{k-1}(V^*), \quad \omega \mapsto \omega(\beta, \cdot, \dots, \cdot).$$

Next, we recall the universality result by Kasman et al.

Theorem 2.3 (Kasman et al., 2008, Theorem 3.4). Let $\omega \in \bigwedge^k V$. Then $\omega \in \hat{Gr}(k, V)$ if and only if every GCP map to $\bigwedge^2 \mathbb{C}^4$ maps ω to $\hat{Gr}(2, 4)$.

In fact, Kasman et al. show that the GCP maps can be chosen from an explicit finite collection. We can rephrase Theorem 2.3 in terms of the Klein quadric, as follows.

Corollary 2.4. Any Grassmannian is set-theoretically defined by pullbacks of the Klein quadric $P_{2,4}$:

$$\hat{Gr}(k, V) = \left\{ \omega \in \bigwedge^k V \mid P_{2,4}(\varphi(\omega)) = 0 \quad \forall \varphi \in GCP \left(\bigwedge^k V, \bigwedge^2 \mathbb{C}^4 \right) \right\}.$$

3. Quadratic spaces and the isotropic Grassmannian

Throughout the remainder of the paper, we will work in a field \mathbb{K} of characteristic not 2. In this section, we will introduce quadratic spaces and isotropic Grassmannians, and establish several essential lemmas that we will later use to prove our main theorem.

3.1. Quadratic spaces

In this subsection we introduce quadratic spaces. The material is fairly standard, for a reference see Artin (1957, Chapter 3). For Lemma 3.6 we did not find a proof in the literature, so we opted to give a proof here.

A *quadratic space* refers to a vector space V equipped with a quadratic form—or equivalently, a symmetric bilinear form $\langle \cdot, \cdot \rangle$. We always assume that the bilinear form is nondegenerate. A vector $v \in V$ is considered *isotropic* if $\langle v, v \rangle = 0$. The set of all isotropic vectors in V is denoted by V_{iso} . The *orthogonal complement* of a subspace $L \subseteq V$ is defined as the space $L^\perp := \{v \in V \mid \langle v, u \rangle = 0, \forall u \in L\}$. We call a subspace $L \subseteq V$ *isotropic* if $L \subseteq L^\perp$, i.e. if $\langle u, v \rangle = 0$ for all $u, v \in L$. By polarization, using $\text{Char } \mathbb{K} \neq 2$, this is equivalent to $L \subseteq V_{\text{iso}}$. If L is isotropic but any proper superset $L' \supsetneq L$ is not isotropic, we refer to L as *maximal isotropic*.

Definition 3.1. We call a tuple $(e_1, e_{-1}, \dots, e_k, e_{-k})$ of vectors in V *hyperbolic* if $\langle e_i, e_{-i} \rangle = 1$ for $i = 1, \dots, k$, and $\langle e_i, e_j \rangle = 0$ if $i \neq -j$. Note that the e_i are necessarily linearly independent. If $2k = \dim V$, then we call $(e_1, e_{-1}, \dots, e_k, e_{-k})$ a *hyperbolic basis* of V .

Theorem 3.2. Let L be an isotropic subspace of V , and e_1, \dots, e_k a basis of L . Then we can find vectors $e_{-1}, \dots, e_{-k} \in V \setminus L$ such that $(e_1, e_{-1}, \dots, e_k, e_{-k})$ forms a hyperbolic tuple.

Proof. This is Artin (1957, Theorem 3.8) in the case where $U = L$ is isotropic. \square

Theorem 3.3 (See Artin (1957, Theorem 3.10)). All maximal isotropic subspaces of V have the same dimension, which is referred to as the *Witt index* of V .

Note that by Theorem 3.2, the Witt index can be at most $\lfloor \frac{\dim V}{2} \rfloor$. Moreover, this upper bound is attained when \mathbb{K} is algebraically closed, regardless of the chosen nondegenerate quadratic form.

Convention 3.4. From this point onward, we make the assumption that V has maximal Witt index $\lfloor \frac{\dim V}{2} \rfloor$. We will denote this Witt index by p .

Remark 3.5. If $\dim V = 2p$ is even, then by Theorem 3.2, V has a hyperbolic basis. Note that then a subspace L is maximal isotropic if and only if $L = L^\perp$.

If $\dim V = 2p + 1$ is odd, then V has a basis

$$B = (e_1, e_{-1}, \dots, e_p, e_{-p}, e_0) \tag{3.1}$$

such that $(e_1, e_{-1}, \dots, e_p, e_{-p})$ is hyperbolic and $\langle e_0, e_i \rangle = 0$ for all $i \neq 0$. We will call B hyperbolic as well. Note that $\langle e_0, e_0 \rangle \neq 0$ by nondegeneracy. If \mathbb{K} is algebraically closed we can rescale e_0 such that $\langle e_0, e_0 \rangle = 1$; in general we will write $c_0 := \frac{1}{2} \langle e_0, e_0 \rangle$. Note that we can also find a basis of V consisting of isotropic vectors, for instance by replacing e_0 by $e_0 + e_1 - c_0 e_{-1}$ in (3.1).

The following lemma will be used several times in the proof of our main theorem (to be precise: in Claim 4.8, Claim 4.10 and Claim 4.11).

Lemma 3.6. Let $W_1, W_2 \subseteq V$ be maximal isotropic subspaces. Then for any choice of decomposition

$$W_1 = (W_1 \cap W_2) \oplus U_1 \quad \text{and} \quad W_2 = (W_1 \cap W_2) \oplus U_2$$

the isomorphism $V \rightarrow V^*, v \mapsto \langle v, \cdot \rangle$ restricts to an isomorphism $U_1 \rightarrow U_2^*$. In particular, there exists a hyperbolic basis $e_1, e_{-1}, \dots, e_p, e_{-p}, (e_0)$ of V , such that

$$W_1 = \text{span}\{e_1, \dots, e_p\} \quad \text{and} \quad W_2 = \text{span}\{e_1, \dots, e_q, e_{-(q+1)}, \dots, e_{-p}\},$$

where $q = \dim(W_1 \cap W_2)$.

Proof. Note that U_1 and U_2 have the same dimension because the maximal isotropic subspaces W_1 and W_2 have the same dimension. Thus it suffices to show that the map $U_1 \rightarrow U_2^*$ is injective. Arguing by contradiction, assume there is some $u_1 \in U_1 \setminus \{0\}$ such that $\langle u_1, u_2 \rangle = 0$ for all $u_2 \in U_2$. Then it also holds that $\langle u_1, w_2 \rangle = 0$ for all $w_2 \in (W_1 \cap W_2) \oplus U_2 = W_2$ because $(W_1 \cap W_2) \oplus \text{span}\{u_1\} \subseteq W_1$ is isotropic. Since W_2 and $u_1 \in U_1 \subseteq W_1$ are isotropic, this implies that also $W_2 \oplus \text{span}\{u_1\}$ is isotropic. But W_2 is strictly contained in $W_2 \oplus \text{span}\{u_1\}$ because $u_1 \in U_1 \setminus \{0\}$, contradicting the fact that W_2 is maximal isotropic.

To see how the first statement implies the second one, choose a basis $\{e_1, \dots, e_p\}$ of W_1 such that $\{e_1, \dots, e_q\}$ forms a basis for $W_1 \cap W_2$. Let $U_1 = \text{span}\{e_{q+1}, \dots, e_p\}$ and choose some U_2 such that $W_2 = (W_1 \cap W_2) \oplus U_2$. It follows from the first part that there are unique $e_{-(q+1)}, \dots, e_{-p} \in U_2$ such that $\langle e_i, e_{-j} \rangle = \delta_{ij}$ for $i, j = q + 1, \dots, p$. Since W_1 and W_2 are isotropic, it holds that $W_1 \cap W_2 \subseteq U_1^\perp \cap U_2^\perp$, and $W_1 \cap W_2$ is a maximal isotropic subspace of $U_1^\perp \cap U_2^\perp$ by reasons of dimension. So there exist $e_{-1}, \dots, e_{-q}, (e_0) \in U_1^\perp \cap U_2^\perp$ such that $(e_1, e_{-1}, \dots, e_q, e_{-q}, (e_0))$ forms a hyperbolic basis of $U_1^\perp \cap U_2^\perp$. This completes the proof. \square

3.2. The isotropic Grassmann cone

We now introduce the isotropic Grassmann cone. We continue to work in a quadratic space V satisfying Convention 3.4.

Definition 3.7. For $k \leq p$, the isotropic Grassmann cone is defined as

$$\hat{Gr}_{\text{iso}}(k, V) := \{v_1 \wedge \dots \wedge v_k \mid \langle v_i, v_j \rangle = 0\} \subset \hat{Gr}(k, V) \subset \bigwedge^k V.$$

If $k = p$, then we call it the maximal isotropic Grassmann cone. Note that $\omega \in \hat{Gr}(k, V)$ lies in $\hat{Gr}_{\text{iso}}(k, V)$ if and only if $L_\omega \subset V$ is isotropic.

Definition 3.8. A linear map $\Phi : \bigwedge^k V \rightarrow \bigwedge^q W$ is isotropic Grassmann cone preserving (IGCP) if $\Phi(\hat{Gr}_{\text{iso}}(k, V)) \subseteq \hat{Gr}_{\text{iso}}(q, W)$.

We will only need one explicit family of IGCP maps; they are the analogue of the GCP maps from Example 2.2. Let $v \in V_{\text{iso}}$ be a non-zero isotropic vector. Define $V_v := v^\perp / \langle v \rangle$ (note that $\langle v \rangle \subseteq v^\perp$ because v is isotropic). It is easy to see that

$$\langle \bar{v}_1, \bar{v}_2 \rangle_{V_v} := \langle v_1, v_2 \rangle_V,$$

where $\bar{v}_i \in V_v$ denotes the equivalence class of $v_i \in v^\perp$ in V_v , is a well-defined nondegenerate bilinear form on V_v (i.e., the formula is independent of the choice of representatives $v_1, v_2 \in v^\perp$). Moreover, $(V_v, \langle \cdot, \cdot \rangle_{V_v})$ again has maximal Witt index. We denote by π_v the projection $v^\perp \rightarrow V_v$.

Definition 3.9. For $v \in V_{\text{iso}} \setminus \{0\}$ we define the linear map

$$\Phi_v : \bigwedge^k V \rightarrow \bigwedge^{k-1} V_v$$

as the following composition

$$\bigwedge^k V \xrightarrow{\varphi_v} \bigwedge^{k-1} v^\perp \xrightarrow{\bigwedge^{k-1} \pi_v} \bigwedge^{k-1} V_v,$$

where φ_v is the contraction map introduced in Example 2.2. Explicitly, this map is given by

$$\Phi_v(v_1 \wedge \dots \wedge v_k) = \sum_{j=1}^k (-1)^{j-1} \langle v, v_j \rangle \bar{v}_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge \bar{v}_k. \tag{3.2}$$

Since Φ_v is a composition of two GCP maps, it is itself GCP. By the same proof as Example 2.2, one readily sees that Φ_v is in fact IGCP. More explicitly, the following holds.

Lemma 3.10. For $\omega \in \hat{G}r_{iso}(k, V)$,

- (1) if $v \in L_\omega^\perp$, then $\Phi_v(\omega) = 0$,
- (2) if $v \notin L_\omega^\perp$, then $\Phi_v(\omega) \neq 0$, and

$$L_{\Phi_v(\omega)} = (L_\omega \cap v^\perp) / \langle v \rangle.$$

Proof. If $v \in L_\omega^\perp$, then $\langle v, w \rangle = 0$ for all $w \in L_\omega$. Therefore, $\varphi_v(\omega) = 0$ and hence also $\Phi_v(\omega) = 0$. This proves the first statement. For the second statement, suppose $v \notin L_\omega^\perp$ and choose a basis where $\omega = v_1 \wedge \dots \wedge v_k$, $L_\omega = \text{span}\{v_1, \dots, v_k\}$ and $L_\omega \cap v^\perp = \text{span}\{v_1, \dots, v_{k-1}\}$. By evaluating $\Phi_v(\omega)$ using (3.2), we obtain the result. \square

We now give a more coordinate-independent description of Φ_v . The bilinear form on V induces an isomorphism $V \xrightarrow{\cong} V^*$, $v \mapsto \langle v, \cdot \rangle$. Together with the natural isomorphism $\bigwedge^k V^* \cong \text{Alt}^k V$, this yields an isomorphism $b : \bigwedge^k V \xrightarrow{\cong} \text{Alt}^k V$. Then Φ_v is the composition

$$\bigwedge^k V \xrightarrow{b} \text{Alt}^k V \xrightarrow{\Phi_v^b} \text{Alt}^{k-1} V_v \xrightarrow{b^{-1}} \bigwedge^{k-1} V_v,$$

where the middle map Φ_v^b is given by the formula

$$\Phi_v^b(\omega^b)(\bar{v}_1, \dots, \bar{v}_{k-1}) = \omega^b(v, v_1, \dots, v_{k-1}). \tag{3.3}$$

Note that since ω^b is alternating, this does *not* depend on a choice of representatives $v_i \in V$ for $\bar{v}_i \in V_v$.

3.3. Two lemmas about IGCP maps

We finish this section by proving two lemmas that will play a central role throughout the proof of our main theorem. The first lemma states that no nonzero ω are annihilated by all IGCP maps:

Lemma 3.11. Let $\omega \in \bigwedge^k V$ with $0 < k < \dim V$. If $\Phi_v(\omega) = 0$ for all $v \in V_{iso} \setminus \{0\}$, then $\omega = 0$.

Proof. If $\Phi_v(\omega) = 0$ for all $v \in V_{iso} \setminus \{0\}$, then $\omega^b(v, v_2, \dots, v_k) = 0$ for all $v \in V_{iso} \setminus \{0\}$ and $v_2, \dots, v_k \in v^\perp$ due to (3.3). But then by Proposition 3.12 below, $\omega^b(w_1, \dots, w_k) = 0$ for all $w_1, \dots, w_k \in V$. So $\omega^b = 0$, and hence $\omega = 0$. This completes the proof. \square

Proposition 3.12. If $0 < k < \dim V$, then the set

$$\left\{ v \wedge v_2 \wedge \dots \wedge v_k \in \bigwedge^k V \mid v \in V_{iso} \setminus \{0\} \text{ and } v_2, \dots, v_k \in v^\perp \right\}$$

spans $\bigwedge^k V$.

Proof. Let \mathcal{S} be the span of the given set in $\bigwedge^k V$. We choose a hyperbolic basis $e_1, e_{-1}, \dots, e_p, e_{-p}, (e_0)$ for V . It suffices to show that each pure wedge $e_{i_1} \wedge \dots \wedge e_{i_k}$ is in \mathcal{S} . If there exists $j \neq 0$ such that $\#\{(j, -j) \cap \{i_1, \dots, i_k\}\} = 1$, then clearly $e_{i_1} \wedge \dots \wedge e_{i_k} \in \mathcal{S}$. So we only need to show that $e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_m} \wedge e_{-j_m} \in \mathcal{S}$ when $k = 2m$, or $e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_m} \wedge e_{-j_m} \wedge e_0 \in \mathcal{S}$ when $k = 2m + 1$, where $j_1, \dots, j_m \in \{1, \dots, p\}$.

If $m < p$, we choose $j_0 \in \{1, \dots, p\} \setminus \{j_1, \dots, j_m\}$. We define $\eta = e_{j_2} \wedge e_{-j_2} \wedge \dots \wedge e_{j_m} \wedge e_{-j_m}$ if k is even, and $\eta = e_{j_2} \wedge e_{-j_2} \wedge \dots \wedge e_{j_m} \wedge e_{-j_m} \wedge e_0$ if k is odd. Based on the definition of j_0 and \mathcal{S} , we have $(e_{j_0} + e_{j_1}) \wedge (e_{-j_0} - e_{-j_1}) \wedge \eta \in \mathcal{S}$. Expanding this expression, we obtain:

$$(e_{j_0} + e_{j_1}) \wedge (e_{-j_0} - e_{-j_1}) \wedge \eta = (e_{j_0} \wedge e_{-j_0} - e_{j_1} \wedge e_{-j_1}) \wedge \eta + (\text{terms in } \mathcal{S}).$$

Therefore, we conclude that

$$(e_{j_0} \wedge e_{-j_0} - e_{j_1} \wedge e_{-j_1}) \wedge \eta \in \mathcal{S}. \tag{3.4}$$

Similarly, by considering $(e_{j_0} + e_{-j_1}) \wedge (e_{-j_0} - e_{j_1}) \wedge \eta \in \mathcal{S}$, we can deduce

$$(e_{j_0} \wedge e_{-j_0} - e_{-j_1} \wedge e_{j_1}) \wedge \eta \in \mathcal{S}. \tag{3.5}$$

Subtracting (3.4) from (3.5) and using the anti-symmetry of \wedge , we obtain $2e_{j_1} \wedge e_{-j_1} \wedge \eta \in \mathcal{S}$. Since $\text{Char}(\mathbb{K}) \neq 2$ this shows that $e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_m} \wedge e_{-j_m} (\wedge e_0) \in \mathcal{S}$.

We still need to consider the case $m = p$; i.e. to show that $e_1 \wedge e_{-1} \wedge \dots \wedge e_p \wedge e_{-p} \in \mathcal{S}$ if $\dim V = 2p + 1$. For this we write $\eta = e_2 \wedge e_{-2} \wedge \dots \wedge e_p \wedge e_{-p}$ as before, and note that

$$2c_0 e_1 \wedge e_{-1} \wedge \eta = ((e_0 + e_1 - c_0 e_{-1}) \wedge (e_1 + c_0 e_{-1}) - e_0 \wedge e_1 - c_0 e_0 \wedge e_{-1}) \wedge \eta \in \mathcal{S},$$

where $c_0 = \frac{1}{2}(e_0, e_0)$. \square

The second lemma is a more technical variant of Lemma 3.11. We will use it to prove Claims 4.12 and 4.13 in the proof of the main theorem.

Lemma 3.13. *Assume $p \geq 2$ and $0 < k < \dim V$, and let $\omega \in \wedge^k V$ be nonzero. Let W and W' be maximal isotropic subspaces of V with $\dim(W \cap W') = p - 1$, and suppose that $\Phi_v(\omega) = 0$ for every isotropic $v \in W \cup W'$.*

- If $k > p$, then ω is of the form $\alpha \wedge \omega'$, where α lies in the one-dimensional space $\wedge^{p+1}(W + W')$.
- If $k \leq p$ then $\omega \in \wedge^k(W^\perp \cap W'^\perp)$.

Proof. We choose a hyperbolic basis of V such that $W = \text{span}\{e_1, e_2, \dots, e_p\}$ and $W' = \text{span}\{e_{-1}, e_2, \dots, e_p\}$. For $\{i_1, \dots, i_\ell\} \subset \{1, -1, \dots, p, -p, (0)\}$, we will write $V_{\hat{i}_1, \dots, \hat{i}_\ell}$ for $\text{span}\{e_i \mid i \notin \{i_1, \dots, i_\ell\}\} \subseteq V$.

We prove by induction on $i = 2, \dots, p + 1$ that

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_{i-1} \wedge \omega'_i + \omega''_i, \tag{3.6}$$

with

$$\omega'_i \in \wedge^{k-i} V_{\hat{1}, \hat{2}, \hat{3}, \dots, \widehat{i-1}, -\hat{1}} \quad \text{and} \quad \omega''_i \in \wedge^k V_{\hat{1}, -\hat{1}, -\hat{2}, \dots, -\widehat{i+1}},$$

and we put the first summand equal to zero if $i > k$.

First, let us show that (3.6) holds for $i = 2$. Indeed we can write

$$\omega = e_1 \wedge e_{-1} \wedge \omega'_2 + e_1 \wedge \alpha + e_{-1} \wedge \beta + \omega''_2$$

with

$$\omega'_2 \in \wedge^{k-2} V_{\hat{1}, -\hat{1}}, \quad \alpha, \beta \in \wedge^{k-1} V_{\hat{1}, -\hat{1}}, \quad \omega''_2 \in \wedge^k V_{\hat{1}, -\hat{1}}.$$

By assumption we have

$$\begin{aligned} 0 &= \Phi_{e_1}(\omega) = \Phi_{e_1}(e_{-1} \wedge \alpha), \\ 0 &= \Phi_{e_{-1}}(\omega) = \Phi_{e_{-1}}(e_1 \wedge \beta), \end{aligned}$$

hence $\alpha = \beta = 0$.

Next we assume (3.6) for some i , and want to show it for $i + 1$. We can write

$$\begin{aligned} \omega'_i &= e_i \wedge \omega'_{i+1} + e_{-i} \wedge \alpha' + \beta' \\ \omega''_i &= \omega''_{i+1} + e_i \wedge e_{-i} \wedge \alpha'' + e_{-i} \wedge \beta'', \end{aligned}$$

where

$$\begin{aligned} \omega'_{i+1} &\in \bigwedge^{k-i-1} V_{\hat{1}, \hat{2}, \dots, \hat{i}, -\hat{i}}, & \omega''_{i+1} &\in \bigwedge^k V_{\hat{1}, -\hat{1}, -\hat{2}, \dots, -\hat{i}}, \\ \alpha' &\in \bigwedge^{k-i-1} V_{\hat{1}, \hat{2}, \dots, \hat{i}, -\hat{i}, -\hat{i}}, & \alpha'' &\in \bigwedge^{k-2} V_{\hat{1}, \hat{i}, -\hat{1}, -\hat{2}, \dots, -\hat{i}}, \\ \beta' &\in \bigwedge^{k-i} V_{\hat{1}, \hat{2}, \dots, \hat{i}, -\hat{i}, -\hat{i}}, & \beta'' &\in \bigwedge^{k-1} V_{\hat{1}, \hat{i}, -\hat{1}, -\hat{2}, \dots, -\hat{i}}. \end{aligned}$$

We compute $0 = \Phi_{e_i}(\omega) = \bar{e}_1 \wedge \bar{e}_{-1} \wedge \bar{e}_2 \wedge \dots \wedge \bar{e}_{i-1} \wedge \overline{\alpha'} + \overline{\beta''}$, so we can conclude that $\alpha' = \beta'' = 0$. Next we compute

$$\begin{aligned} \Phi_{e_{i-e_1}}(\omega) &= \Phi_{e_{i-e_1}}(e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_{i-1} \wedge e_i \wedge \omega'_{i+1}) \\ &\quad + \Phi_{e_{i-e_1}}(e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_{i-1} \wedge \beta') \\ &\quad + \Phi_{e_{i-e_1}}(\omega''_{i+1}) \\ &\quad + \Phi_{e_{i-e_1}}(e_i \wedge e_{-i} \wedge \alpha''). \end{aligned}$$

The first and third summand are zero by Lemma 3.10. So we get

$$\Phi_{e_{i-e_1}}(\omega) = \bar{e}_1 \wedge \bar{e}_2 \wedge \dots \wedge \bar{e}_{i-1} \wedge \overline{\beta'} - \bar{e}_i \wedge \overline{\alpha''},$$

so $e_2 \wedge \dots \wedge e_{i-1} \wedge \beta' = \alpha''$. If we do the analogous computation for $\Phi_{e_{i-e_{-1}}}(\omega)$ we find that $e_2 \wedge \dots \wedge e_{i-1} \wedge \beta' = -\alpha''$. So $\beta' = \alpha'' = 0$, and we get

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_i \wedge \omega'_{i+1} + \omega''_{i+1},$$

which is exactly (3.6) for $i + 1$ instead of i .

Finally, note that the case $i = p + 1$ is exactly what we want. Indeed we have

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_p \wedge \omega' + \omega'',$$

with $\omega' \in \bigwedge^{k-p-1} V_{\hat{1}, \hat{2}, \dots, \hat{p}, -\hat{1}}$ and $\omega'' \in \bigwedge^k V_{\hat{1}, \hat{2}, \dots, \hat{p}, -\hat{1}} = \bigwedge^k (W^\perp \cap W'^\perp)$. But if $k \leq p$ the first summand is zero, and if $k > p$ the second summand is zero since $\dim V_{\hat{1}, \hat{2}, \dots, \hat{p}, -\hat{1}} = \dim V - p - 1 \leq p < k$. \square

4. Universality for maximal isotropic Grassmannians

4.1. Statement and consequences of the main result

For this entire section, let V be a quadratic space of maximal Witt index $p = \lfloor \frac{\dim V}{2} \rfloor$ over a field \mathbb{K} of characteristic not 2.

Main Theorem 4.1. *Assume $\dim V > 8$ and let $\omega \in \bigwedge^p V$. If for every isotropic vector $v \in V_{\text{iso}}$, the image of v under the isotropic Grassmann cone preserving map Φ_v lies in $\hat{G}r_{\text{iso}}(p - 1, V_v)$, then ω itself lies in $\hat{G}r_{\text{iso}}(p, V)$.*

Corollary 4.2. *For any $\omega \in \bigwedge^p V$, it holds that $\omega \in \hat{G}r_{\text{iso}}(p, V)$ if and only if*

- every IGCP map to $\bigwedge^3 \mathbb{K}^7$ maps ω to $\hat{G}r_{iso}(3, 7)$, if $\dim V = 2p + 1$,
- every IGCP map to $\bigwedge^4 \mathbb{K}^8$ maps ω to $\hat{G}r_{iso}(4, 8)$, if $\dim V = 2p$.

Proof of Corollary 4.2 assuming Main Theorem 4.1. One direction is straightforward from the definition of an IGCP map. To prove the other direction, we consider three cases depending on the dimension of V .

If $\dim V > 8$, we can repeatedly apply the Main Theorem 4.1 to obtain the desired result. If $\dim V = 8$ or $\dim V = 7$, we can just apply the assumption to the identity map (which is trivially IGCP). For $\dim V < 7$, we observe that the map $\varphi : \bigwedge^p V \rightarrow \bigwedge^{p+1}(V \oplus \text{span}\{e_{p+1}, e_{-p-1}\})$, which sends ω to $\omega \wedge e_{p+1}$, has the property that ω lies in the isotropic Grassmann cone if and only if $\varphi(\omega)$ lies in the isotropic Grassmann cone. By applying these maps iteratively until we reach $\bigwedge^3 \mathbb{K}^7$ or $\bigwedge^4 \mathbb{K}^8$, we complete the proof. \square

Similar to Kasman et al. (2008, Theorem 4.1), we obtain a statement about the ranks of quadrics defining the isotropic Grassmann cone, where we use the fact that $\hat{G}r_{iso}(p, 2p)$ has two irreducible components (Harris, 1992, Theorem 22.14).

Corollary 4.3. *The isotropic Grassmannian $\hat{G}r_{iso}(p, 2p + 1)$ in its Plücker embedding can be defined by quadrics of rank at most 4. Furthermore, both irreducible components of $\hat{G}r_{iso}(p, 2p)$ can be defined by linear equations and quadrics of rank at most 4.*

Proof. By Corollary 2.4 it suffices to show the statement is true for $\hat{G}r_{iso}(3, 7)$ and $\hat{G}r_{iso}(4, 8)$. This can be done by an explicit calculation, see Appendix A. \square

Remark 4.4. The statement in Corollary 4.3 can also be deduced using the Cartan embedding, see Appendix A.

Remark 4.5. A natural question arises: is there a similar result if we replace the symmetric form with a skew-symmetric form, focusing on Lagrangian Grassmannians? The answer, in the case of considering only the Lagrangian Grassmann cone preserving (LGCP) maps Φ_v for $v \in V$, defined as in Definition 3.9, is no.

To illustrate this, let us consider an 8-dimensional vector space V with basis (e_1, \dots, e_{-4}) and skew-symmetric form given by $\langle e_i, e_{-i} \rangle = 1$ for $i > 0$, $\langle e_i, e_{-i} \rangle = -1$ for $i < 0$, and all other pairings equal to 0.

Now, consider the vector $\alpha = e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + e_3 \wedge e_{-3} + e_4 \wedge e_{-4}$, and define

$$\omega = \alpha \wedge \alpha = 2 \sum_{1 \leq i < j \leq 4} e_i \wedge e_{-i} \wedge e_j \wedge e_{-j} \in \bigwedge^4 V.$$

It can be observed that ω does not lie in the Grassmann cone since $\omega \wedge \omega$ is a nonzero multiple of $e_1 \wedge e_{-1} \wedge e_2 \wedge e_{-2} \wedge e_3 \wedge e_{-3} \wedge e_4 \wedge e_{-4}$. However, upon explicit computation, it can be seen that every LGCP map Φ_v maps ω to zero, and thus it lies in the Lagrangian Grassmann cone.

This example can be generalized to any space of dimension $4m$ by considering $\omega = \alpha^{\wedge m} \in \bigwedge^{2m} V$. Hence, we have a counterexample to the analogue of the Main Theorem 4.1 (and even to the analogue of Lemma 3.11). However, it is not yet a counterexample to the analogue of Corollary 4.2, as there might be additional LGCP maps that could be considered.

4.2. Structure of the proof

The aim of this subsection is twofold. First, we aim to prove Proposition 4.7, which will serve as the key ingredient in proving the Main Theorem 4.1. Secondly, we will give an outline of the proof of the Main Theorem 4.1 to make it more accessible, as it involves some technical aspects.

We assume $p \geq 2$. Note that we can always decompose V as

$$V = V' \oplus \text{span}\{e_p, e_{-p}\}, \tag{4.1}$$

where (e_p, e_{-p}) is a hyperbolic tuple, and $\text{span}\{e_p, e_{-p}\} \subset V'^{\perp}$ where V' again has maximal Witt index. For the remaining part of this section we are fixing this decomposition. Any $\omega \in \bigwedge^p V$ can be uniquely written as

$$\omega = \omega_1 \wedge e_p \wedge e_{-p} + \omega_2 \wedge e_p + \omega_3 \wedge e_{-p} + \omega_4 \tag{4.2}$$

where $\omega_1 \in \bigwedge^{p-2} V'$, $\omega_2, \omega_3 \in \bigwedge^{p-1} V'$ and $\omega_4 \in \bigwedge^p V'$. The following observation shows that for $v \in V'$, a decomposition of ω maps to a decomposition of $\Phi_v(\omega)$.

Observation 4.6. *Let ω be as in (4.2). Then for any $v \in V'$ we have*

$$\Phi_v(\omega) =: \omega' = \omega'_1 \wedge \bar{e}_p \wedge \bar{e}_{-p} + \omega'_2 \wedge \bar{e}_p + \omega'_3 \wedge \bar{e}_{-p} + \omega'_4, \tag{4.3}$$

where $\omega'_i = \Phi_v(\omega_i)$.

Next, we give conditions for ω to be in the isotropic Grassmann cone.

Proposition 4.7. *Suppose we have written $\omega \in \bigwedge^p V$ in the form (4.2). Assume $\omega \in \hat{G}r_{\text{iso}}(p, V)$, then one of the following holds:*

- (1) $\omega_1 = \omega_3 = \omega_4 = 0$ and $\omega_2 \in \hat{G}r_{\text{iso}}(p - 1, V')$,
- (2) $\omega_1 = \omega_2 = \omega_4 = 0$ and $\omega_3 \in \hat{G}r_{\text{iso}}(p - 1, V')$,
- (3) $\omega_1 = 0$, and $\omega_2, \omega_3, \omega_4$ are nonzero. Then
 - $\omega_2, \omega_3 \in \hat{G}r_{\text{iso}}(p - 1, V')$, $\omega_4 \in \hat{G}r(p, V')$,
 - $L_{\omega_2} = L_{\omega_3} \subseteq L_{\omega_4}$.
 This case only occurs if $\dim V$ is odd.
- (4) $\omega_1, \omega_2, \omega_3, \omega_4$ are all nonzero. Then
 - $\omega_1 \in \hat{G}r_{\text{iso}}(p - 2, V')$, $\omega_2, \omega_3 \in \hat{G}r_{\text{iso}}(p - 1, V')$, $\omega_4 \in \hat{G}r(p, V')$,
 - $L_{\omega_2} \cap L_{\omega_3} = L_{\omega_1}$ and $L_{\omega_2} + L_{\omega_3} = L_{\omega_4}$.

Proof. We define $L' := L_{\omega} \cap V'$. Note that $p - 2 \leq \dim L' \leq p - 1$, where the second inequality holds since L' is an isotropic subspace of V' . We proceed by considering cases based on $\dim L'$. More precisely, we will show that (1), (2) or (3) hold if $\dim L' = p - 1$, and that (4) holds if $\dim L' = p - 2$.

Case 1. If $\dim L' = p - 1$, then L' is a maximal isotropic subspace of V' . Since L' has codimension one in L_{ω} , there exists a vector $v \in L_{\omega}$ such that $L_{\omega} = L' + \text{span}\{v\}$. Since L_{ω} is isotropic, we have $L_{\omega} \subseteq L_{\omega}^{\perp} \subseteq L'^{\perp}$. Therefore, $v \in L'^{\perp}$. We can write $v = w + v'$, where $w \in \text{span}\{e_p, e_{-p}\}$ and $v' \in V'$. Moreover, note that $w \in L'^{\perp}$, and therefore, $v' = v - w \in L'^{\perp}$. If $\dim V$ is even, then $L' = L'^{\perp} \cap V'$, hence $v' \in L'$. Consequently, we have

$$L_{\omega} = L' + \text{span}\{w + v'\} = L' + \text{span}\{w\}.$$

Since ω is isotropic, the vector w is also isotropic. Thus, we can conclude that either

$$w \in \text{span}\{e_p\} \quad \text{or} \quad w \in \text{span}\{e_{-p}\}.$$

So we conclude

$$L_{\omega} = L' + \text{span}\{e_p\} \quad \text{or} \quad L_{\omega} = L' + \text{span}\{e_{-p}\},$$

and therefore

$$\omega = \omega_2 \wedge e_p \quad \text{or} \quad \omega = \omega_3 \wedge e_{-p},$$

where $\omega_2, \omega_3 \in \hat{G}r_{iso}(p-1, V')$.

If $\dim V$ is odd, there is a possibility that $v' \notin L'$. Nevertheless, we still have

$$L_\omega = L' + \text{span}\{w + v'\}.$$

Writing $w = \lambda e_p + \mu e_{-p}$, we obtain that $2\lambda\mu + \langle v', v' \rangle = 0$. Since L' is maximal isotropic in V' , the vector v' cannot be isotropic. Hence, we have $\lambda \neq 0 \neq \mu$. Consequently, we can write

$$\omega = \omega' \wedge (\lambda e_p + \mu e_{-p} + v'),$$

where $L_\omega = L'$. By doing so, we have expressed ω in the form (4.2), with $\omega_1 = 0, \omega_2 = \lambda\omega', \omega_3 = \mu\omega'$, and $\omega_4 = \omega' \wedge v'$. One can verify that this proves all the claims in (3).

Case 2. If $\dim L' = p-2$, we can write $L_\omega = \text{span}\{e_p + u, e_{-p} + v\} \oplus L'$ for some $u, v \in V'$. We choose (v_1, \dots, v_{p-2}) as a basis of L' and express ω as

$$\begin{aligned} \omega &= v_1 \wedge \dots \wedge v_{p-2} \wedge (e_p + u) \wedge (e_{-p} + v) \\ &=: \omega_1 \wedge (e_p + u) \wedge (e_{-p} + v) \\ &= \omega_1 \wedge e_p \wedge e_{-p} - \omega_1 \wedge v \wedge e_p + \omega_1 \wedge u \wedge e_{-p} + \omega_1 \wedge u \wedge v \\ &=: \omega_1 \wedge e_p \wedge e_{-p} + \omega_2 \wedge e_p + \omega_3 \wedge e_{-p} + \omega_4. \end{aligned}$$

To show that all ω_i are nonzero, we need to show that $\{v_1, v_2, \dots, v_{p-2}, u, v\}$ are linearly independent. We already know that $\{v_1, \dots, v_{p-2}\}$ are linearly independent since they form a basis of L' . Furthermore, v is also linearly independent from $\{v_1, \dots, v_{p-2}\}$; otherwise $e_{-p} \in L_\omega$, but this would imply $\langle e_p + u, e_{-p} \rangle = 0$. Hence, we need to show that u is linearly independent from $\{v_1, \dots, v_{p-2}, v\}$. Assuming $u = \lambda v + v'$, where $v' \in L'$, we obtain

$$\omega = v_1 \wedge \dots \wedge v_{p-2} \wedge (e_p + \lambda v) \wedge (e_{-p} + v),$$

where the vectors $\{v_1, \dots, v_{p-2}, e_p + \lambda v, e_{-p} + v\}$ are all isotropic. In particular, the pairing $\langle v, v \rangle = 0$. However, this implies $\langle e_p + \lambda v, e_{-p} + v \rangle = 1$, contradicting the isotropy of L_ω . Hence, the vectors $\{v_1, v_2, \dots, v_{p-2}, u, v\}$ are linearly independent, implying that all ω_i are nonzero. Note that by definition all ω_i belong to the corresponding Grassmann cone. Furthermore, since u and v are isotropic and $\langle v_i, v_j \rangle = 0, \langle u, v_i \rangle = 0, \langle v, v_i \rangle = 0$ for all i, j , we can conclude that ω_1, ω_2 and ω_3 are isotropic. This proves the first statement in (4). The second statement follows from the definition and linear independence of $\{v_1, v_2, \dots, v_{p-2}, u, v\}$. \square

For the proof of the Main Theorem 4.1, we will fix $\omega \in \wedge^p V$ satisfying the assumption. We decompose ω as in (4.2). Then ω has one of the following zero patterns:

	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4	
(0)	0	0	0	0	*	0	0	0	(8)
(1)	0	0	0	*	*	0	0	*	(9)
(2)	0	0	*	0	*	0	*	0	(10)
(3)	0	0	*	*	*	0	*	*	(11)
(4)	0	*	0	0	*	*	0	0	(12)
(5)	0	*	0	*	*	*	0	*	(13)
(6)	0	*	*	0	*	*	*	0	(14)
(7)	0	*	*	*	*	*	*	*	(15)

The proof splits into the Claims 4.8–4.14 which are based on the different zero patterns. First, we will show that zero patterns (0), (1), (3), (5), (6), (8)–(14) and (7) (if V is even-dimensional) are not possible:

- Claim 4.8 shows that the only possible zero patterns are (2), (4), (7) and (15), with (7) only occurring when V has odd dimension.

Note that the highlighted zero patterns align with the cases in Proposition 4.7. We proceed by proving that the Main Theorem 4.1 is true if $\omega \in \bigwedge^p V$ has one of the highlighted zero patterns as follows:

- Claim 4.9 proves that the Main Theorem 4.1 is true if ω has zero pattern (2) or (4).
- Claim 4.10 and Claim 4.11 show that if ω has zero pattern (7) or (15), there are three possibilities for the dimension of the intersection $L_{\omega_2} \cap L_{\omega_3}$:
 - (a) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$ (when $\dim V$ is even);
 - (b) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$ (when $\dim V$ is odd);
 - (c) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 1$ (when $\dim V$ is odd).
- Claim 4.12 shows that the Main Theorem 4.1 holds for case (a).
- Claim 4.13 shows that the Main Theorem 4.1 holds for case (b).
- Claim 4.14 shows that the Main Theorem 4.1 holds for case (c).

4.3. Proof of the main theorem

We will now prove the Main Theorem 4.1 following the strategy we just explained.

Proof of the Main Theorem 4.1.

Let $\omega \neq 0$ satisfy the assumption of Main Theorem 4.1. Trivially, ω cannot have zero pattern (0).

Claim 4.8. ω cannot have zero pattern (1), (3), (5), (6), or (8)-(14). If $\dim V$ is even it also cannot have zero pattern (7).

Proof.

Step 1. If $\omega_1 \neq 0$, then ω_2, ω_3 and ω_4 are also nonzero. In other words, ω cannot have zero patterns (8)-(14).

Proof. If $\omega_1 \neq 0$, according to Lemma 3.11, there exists $v \in V'_{iso}$ such that $\Phi_v(\omega_1) \neq 0$. Therefore, applying case (4) of Proposition 4.7 to

$$\Phi_v(\omega) = \Phi_v(\omega_1) \wedge e_p \wedge e_{-p} + \Phi_v(\omega_2) \wedge e_p + \Phi_v(\omega_3) \wedge e_{-p} + \Phi_v(\omega_4)$$

we can conclude that $\Phi_v(\omega_2), \Phi_v(\omega_3)$ and $\Phi_v(\omega_4)$ are nonzero. This implies that ω_2, ω_3 and ω_4 are nonzero as well. \square

Step 2. If $\omega_4 \neq 0$, then either ω has zero pattern (15), or $\dim V$ is odd and ω has zero pattern (7). In other words, ω cannot have zero patterns (1), (3), (5), and also not (7) if $\dim V$ is even.

Proof. As before, by Lemma 3.11 there exists $v \in V'_{iso}$ such that $\Phi_v(\omega_4) \neq 0$. The result follows by applying Proposition 4.7 to $\Phi_v(\omega)$ as before. \square

Step 3. ω cannot have zero pattern (6).

Proof. Assume ω has zero pattern (6). Our goal is to find a vector $v \in V'_{iso}$ such that $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$. Then $\omega' := \Phi_v(\omega)$ also has the property that $\omega'_1 = 0 = \omega'_4$ but $\omega'_2 \neq 0 \neq \omega'_3$. So by Proposition 4.7 ω' is not in $\hat{G}_{r_{iso}}(p - 1, V_v)$, which is a contradiction with the assumption of the Main Theorem 4.1. We consider two cases:

Case 1. Assume $L_{\omega_2} + L_{\omega_3} \subsetneq V'$. This case holds if $\dim V$ is odd, and also if $\dim V$ is even except when $L_{\omega_2} \cap L_{\omega_3} = 0$. Since V is spanned by isotropic vectors, we can find an isotropic vector v that does not lie in the linear subspace $L_{\omega_2} + L_{\omega_3}$. Then we have the desired property that $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$ by Lemma 3.10.

Case 2. Assume $L_{\omega_2} + L_{\omega_3} = V'$. In this case, $\dim V$ is even and $L_{\omega_2} \cap L_{\omega_3} = 0$. By Lemma 3.6 we can choose a hyperbolic basis such that

$$\omega_2 = \alpha e_1 \wedge \cdots \wedge e_{p-1} \quad \text{and} \quad \omega_3 = \beta e_{-1} \wedge \cdots \wedge e_{-p+1}.$$

Taking $v := e_1 + e_{-2}$, we have $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$, satisfying the desired property. \square

We now have considered all cases, and the proof of Claim 4.8 is complete. \square

We now know that ω has one of the highlighted zero patterns. Next, we prove that the Main Theorem 4.1 holds if ω has zero pattern (2) or (4).

Claim 4.9. *The Main Theorem 4.1 is true if ω has zero pattern (2) or (4).*

Proof. Let ω have zero pattern (2). Then $\omega = \omega_3 \wedge e_{-p}$. For $v = e_p$, by (3.2), we have $\Phi_{e_p}(\omega) = \pm\omega_3$, which by assumption lies in $\hat{G}r_{\text{iso}}(p-1, V_v) = \hat{G}r_{\text{iso}}(p-1, V')$ and therefore also $\omega \in \hat{G}r_{\text{iso}}(p, V)$. If ω has zero pattern (4) we proceed analogously, using $v = e_{-p}$. \square

For the rest of the proof, we assume that ω has zero pattern (15) or (7); where (7) can only occur if $\dim V$ is odd.

Claim 4.10. *The intersection $L_{\omega_2} \cap L_{\omega_3}$ is nonzero.*

Proof. Assume by contradiction that $L_{\omega_2} \cap L_{\omega_3} = 0$. Then by Lemma 3.6 we can find a hyperbolic basis of V' such that

$$\omega_2 = \alpha e_1 \wedge \cdots \wedge e_{p-1} \quad \text{and} \quad \omega_3 = \beta e_{-1} \wedge \cdots \wedge e_{-p+1}.$$

If $\dim V$ is even we take $v = e_1 + e_{-2}$. By Observation 4.6 we get

$$\Phi_v(\omega) =: \omega' = \omega'_1 \wedge \bar{e}_p \wedge \bar{e}_{-p} + \omega'_2 \wedge \bar{e}_p + \omega'_3 \wedge \bar{e}_{-p} + \omega'_4.$$

Note that in the quotient space $\bigwedge^{p-1}(e_1 + e_{-2})^\perp / \langle e_1 + e_{-2} \rangle$, ω'_2 and ω'_3 have only the basis vector $\bar{e}_1 = \bar{e}_{-2}$ in common, thus

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) = 1.$$

Since $\omega' \in \hat{G}r_{\text{iso}}(p-1, V_v)$, we can conclude by Proposition 4.7 that the intersection $L_{\omega'_2} \cap L_{\omega'_3} = L_{\omega'_1}$, in particular

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) = p - 3.$$

This contradicts our assumption $\dim V > 8$, which for $\dim V$ even implies $p > 4$.

If $\dim V$ is odd we take $v = e_0 + e_{-1} - c_0 e_1$, where $c_0 = \frac{1}{2} \langle e_0, e_0 \rangle$. Then we find

$$\omega'_2 = \alpha \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-1} \quad \text{and} \quad \omega'_3 = -c_0 \beta \bar{e}_{-2} \wedge \cdots \wedge \bar{e}_{-p+1}.$$

Note that also in the quotient space $\bigwedge^{p-1}(e_0 + e_{-1} - c_0 e_1)^\perp / \langle e_0 + e_{-1} - c_0 e_1 \rangle$, we have that

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) = 0.$$

But by inspecting cases (3) and (4) of Proposition 4.7, we see that

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) \in \{p - 2, p - 3\},$$

which again is a contradiction since if $\dim V$ is odd, our assumption $\dim V > 8$ implies $p > 3$. \square

We will use this result and distinguish if the dimension of V is odd or even.

Claim 4.11. *One of the following holds.*

- (a) $\dim V$ is even, and $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$,
- (b) $\dim V$ is odd, and $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$,
- (c) $\dim V$ is odd, and $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 1$ and thus $L_{\omega_2} = L_{\omega_3}$.

Proof. We write $q := \dim(L_{\omega_2} \cap L_{\omega_3}) > 0$. By Lemma 3.6 we can find a hyperbolic basis of V' such that

$$\omega_2 = e_1 \wedge \cdots \wedge e_q \wedge \tilde{\omega}_2 \quad \text{and} \quad \omega_3 = e_1 \wedge \cdots \wedge e_q \wedge \tilde{\omega}_3,$$

where $\tilde{\omega}_2, \tilde{\omega}_3 \in \hat{G}r_{\text{iso}}(p - q - 1, \tilde{V})$ and $\tilde{V} = \text{span}\{e_{q+1}, e_{-q-1}, \dots, e_{p-1}, e_{-p+1}\}$. Now if we choose v as any negative indexed basis vector e_{-i} and write $w' = \Phi_v(\omega)$, by Observation 4.6 we get

$$\omega'_2 = \pm \bar{e}_1 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \hat{e}_i \wedge \bar{e}_{i+1} \wedge \cdots \wedge \bar{e}_q \wedge \tilde{\omega}_2,$$

$$\omega'_3 = \pm \bar{e}_1 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \hat{e}_i \wedge \bar{e}_{i+1} \wedge \cdots \wedge \bar{e}_q \wedge \tilde{\omega}_3.$$

Therefore we have $L_{\omega'_2} \cap L_{\omega'_3} = \text{span}\{\bar{e}_1, \dots, \bar{e}_{i-1}, \hat{e}_i, \bar{e}_{i+1}, \dots, \bar{e}_q\}$ and it holds that

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) = q - 1.$$

If $\dim V$ is even, then

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) = p - 3,$$

hence we conclude $q = p - 2$, as desired. Similarly, if $\dim V$ is odd, then

$$\dim(L_{\omega'_2} \cap L_{\omega'_3}) \in \{p - 2, p - 3\}.$$

We conclude $q \in \{p - 2, p - 1\}$. In other words, either $q = p - 2$, or $L_{\omega_2} = L_{\omega_3}$. \square

We will finish the proof by a case analysis of the cases in Claim 4.11. For the first two cases we need Lemma 3.13.

Claim 4.12. *The Main Theorem 4.1 holds in case (a).*

Proof. Observe that for every $v \in L_{\omega_2} \cup L_{\omega_3}$, either $\Phi_v(\omega_2)$ or $\Phi_v(\omega_3)$ is zero. By Proposition 4.7 and hence the possible zero patterns, this implies that also $\Phi_v(\omega_1)$ and $\Phi_v(\omega_4)$ are zero. Thus, applying Lemma 3.13 to them yields that $\omega_1 \in \bigwedge^{p-1}(L_{\omega_2}^\perp \cap L_{\omega_3}^\perp) = \bigwedge^{p-1}(L_{\omega_2} \cap L_{\omega_3})$ and $\omega_4 \in \bigwedge^{p+1}(L_{\omega_2} + L_{\omega_3})$. After choosing a hyperbolic basis for V' with $L_{\omega_2} = \text{span}\{e_1, e_2, \dots, e_{p-1}\}$ and $L_{\omega_3} = \text{span}\{e_{-1}, e_2, \dots, e_{p-1}\}$ and defining $W := \text{span}\{e_1, e_{-1}, e_p, e_{-p}\}$, we can write

$$\omega = e_2 \wedge \cdots \wedge e_{p-1} \wedge \eta \tag{4.4}$$

for some $\eta \in \bigwedge^2 W$. Note that $L_{\omega_2} \cap L_{\omega_3} = \text{span}\{e_2, \dots, e_{p-1}\}$ is isotropic and orthogonal to W . Now, choose $v = e_{-p+1}$. Then we have $V_v \cong V_v'' \oplus W$, where $V_v'' = \text{span}\{e_2, e_{-2}, \dots, e_{p-1}, e_{-p+1}\}$. Using (4.4) we can write

$$\Phi_v(\omega) = \pm \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-2} \wedge \eta.$$

By assumption we have $\Phi_v(\omega) \in \hat{G}r_{\text{iso}}(p - 1, V_v)$. So we find that $\eta \in \hat{G}r_{\text{iso}}(2, W)$, which in turn implies $\omega \in \hat{G}r_{\text{iso}}(p, V)$. \square

Claim 4.13. *The Main Theorem 4.1 holds in case (b).*

Proof. We choose a hyperbolic basis of V' with $L_{\omega_2} = \text{span}\{e_1, e_2, \dots, e_{p-1}\}$ and $L_{\omega_3} = \text{span}\{e_{-1}, e_2, \dots, e_{p-1}\}$. Applying Lemma 3.13 to ω_1 and ω_4 we get $\omega_1 \in \bigwedge^{p-2} (L_{\omega_2}^\perp \cap L_{\omega_3}^\perp)$ and $\omega_4 = \nu e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_{p-1}$ for some $\nu \in \mathbb{K}^\times$. So we can write

$$\begin{aligned} \omega = & \left(\mu_0 e_2 \wedge \dots \wedge e_{p-1} + e_0 \wedge \sum_{i=2}^{p-1} \mu_i e_2 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_{p-1} \right) \wedge e_p \wedge e_{-p} \\ & + \alpha e_1 \wedge e_2 \wedge \dots \wedge e_{p-1} \wedge e_p + \beta e_{-1} \wedge e_2 \wedge \dots \wedge e_{p-1} \wedge e_{-p} \\ & + \nu e_1 \wedge e_{-1} \wedge e_2 \wedge \dots \wedge e_{p-1}. \end{aligned}$$

Picking $\nu = e_{-2}$ yields

$$\begin{aligned} \Phi_\nu(\omega) = & \left(\mu_0 e_3 \wedge \dots \wedge e_{p-1} - e_0 \wedge \sum_{i=3}^{p-1} \mu_i e_3 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_{p-1} \right) \wedge e_p \wedge e_{-p} \\ & - \alpha e_1 \wedge e_3 \wedge \dots \wedge e_{p-1} \wedge e_p - \beta e_{-1} \wedge e_3 \wedge \dots \wedge e_{p-1} \wedge e_{-p} \\ & + \nu e_1 \wedge e_{-1} \wedge e_3 \wedge \dots \wedge e_{p-1}. \end{aligned}$$

By assumption $\omega' = \Phi_\nu(\omega) \in \hat{G}r_{\text{iso}}(p-1, V_\nu)$. Thus, by Proposition 4.7 one of the cases (1) – (4) holds. Clearly, cases (1) and (2), and since $L_{\omega'_2} \neq L_{\omega'_3}$, also case (3), are not possible. Thus, case (4) holds, which implies that $\omega'_1 \in \hat{G}r_{\text{iso}}(p-3, V'_\nu)$ and $L_{\omega'_1} = L_{\omega'_2} \cap L_{\omega'_3}$. In coordinates, this means that $\mu_i = 0$ for $i = 3, \dots, p-1$, and that $\mu_0 \neq 0$. The same argument¹ with $\nu = e_{-3}$ shows that also $\mu_2 = 0$. We now have written ω as in (4.4), and can proceed exactly as in Claim 4.12. \square

Claim 4.14. *The Main Theorem 4.1 holds in case (c).*

Proof. The proof is divided into several steps:

- Step 1 shows that $\omega_1 = 0$.
- Steps 2-4 show that $\omega_4 = e_1 \wedge \dots \wedge e_{p-1} \wedge u$ for some $u \in V'$.
 - Step 2 shows that we can write:

$$\omega_4 = e_1 \wedge \dots \wedge e_{p-1} \wedge u + \sum_{j_1, \dots, j_\ell} \mu_j e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0),$$

where we write $p = 2\ell$ or $p = 2\ell + 1$, and the factor $\wedge e_0$ only appears in the latter case.

- Step 3 shows that all μ_j are equal and thus:

$$\omega_4 = e_1 \wedge \dots \wedge e_{p-1} \wedge u + \mu \sum_{j_1, \dots, j_\ell} e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0).$$

- Step 4 shows that $\mu = 0$.
- Step 5 then concludes that $\omega \in \hat{G}r_{\text{iso}}(p, V)$.

Step 1. $\omega_1 = 0$

Proof. Note that for every $\nu \in V'_{\text{iso}}$, if we consider $\omega' := \Phi_\nu(\omega) \in \hat{G}r_{\text{iso}}(p-1, V_\nu)$, we have $L_{\omega'_2} = L_{\omega'_3}$. This implies that either $\omega' = 0$, or ω' is in case (3) of Proposition 4.7. In both cases we conclude $\omega'_1 = 0$. So we proved $\Phi_\nu(\omega_1) = 0$ for each $\nu \in V'_{\text{iso}}$, which by Lemma 3.11 implies that $\omega_1 = 0$. \square

¹ here we use $p \geq 4$.

Now we choose a hyperbolic basis of V' such that $L_{\omega_2} = L_{\omega_3} = \text{span}\{e_1, \dots, e_p\}$. For any $v \in V_{\text{iso}}$, we can apply Proposition 4.7 to $\Phi_v(\omega)$ and find:

- (1) If $v \in L_{\omega_2}$, we have $\Phi_v(\omega_4) = 0$.
- (2) If $v \in V_{\text{iso}} \setminus L_{\omega_2}$, we have $\Phi_v(\omega_4) \in \hat{G}r(p-1, V_v)$ with $L_{\omega_2} \subset L_{\Phi_v(\omega_4)}$.

Step 2. ω_4 is of the form

$$e_1 \wedge \dots \wedge e_{p-1} \wedge u + \sum_{j_1, \dots, j_\ell} \mu_J e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0),$$

where we write $p = 2\ell$ or $p = 2\ell + 1$, and the factor $\wedge e_0$ only appears in the latter case.

Proof. We will write

$$\omega_4 = \sum_{i_1, \dots, i_p} \lambda_{i_1, \dots, i_p} e_{i_1} \wedge \dots \wedge e_{i_p},$$

where we always order the indices as follows: $1, -1, 2, -2, \dots, p-1, -p+1, 0$. We will abbreviate $\lambda_{i_1, \dots, i_p}$ to λ_I , where $I = \{i_1, \dots, i_p\} \subset \{1, -1, 2, -2, \dots, p-1, -p+1, 0\}$. If we choose $v = e_i$, then (1) tells us that

$$0 = \Phi_v(\omega_4) = \sum_{-i \in I, i \notin I} \pm \lambda_I \bar{e}_{I \setminus \{-i\}} \in \bigwedge^{p-1} e_i^\perp / \langle e_i \rangle,$$

where the occurring vectors $\bar{e}_{I \setminus \{-i\}}$ are linearly independent. So if $-i \in I$ but $i \notin I$ then $\lambda_I = 0$. On the other hand, if we choose $v = e_{-i}$, then (2) tells us that

$$\Phi_v(\omega_4) = \sum_{-i \notin I, i \in I} \pm \lambda_I \bar{e}_{I \setminus \{i\}} \in \bigwedge^{p-1} e_{-i}^\perp / \langle e_{-i} \rangle$$

is of the form

$$\bar{e}_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \bar{e}_{p-1} \wedge u$$

for some $u \in V'$. So if $i \in I$ but $-i \notin I$, then $\lambda_I = 0$, unless $\{1, 2, \dots, p-1\} \subset I$. Together with the above, this implies the claim. \square

Step 3. All μ_I are equal, so we can write

$$\omega_4 = e_1 \wedge \dots \wedge e_{p-1} \wedge u + \mu \sum_{j_1, \dots, j_\ell} e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0).$$

Proof. Take $v = e_i - e_j$ with i, j positive. Then $\Phi_v(\omega_1) = 0$ and $\Phi_v(\omega_2) = \Phi_v(\omega_3) = 0$, hence by Proposition 4.7 we get $\Phi_v(\omega_4) = 0$. But

$$\begin{aligned} \varphi_v(\omega_4) &= - \sum_{i \in J} \mu_J e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_i \wedge \hat{e}_{-i} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0) \\ &\quad + \sum_{j \in J} \mu_J e_{j_1} \wedge e_{-j_1} \wedge \dots \wedge e_j \wedge \hat{e}_{-j} \wedge \dots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0). \end{aligned}$$

After projecting to $\bigwedge^{p-1} (e_i - e_j)^\perp / \langle e_i - e_j \rangle$ we get

$$\begin{aligned} 0 &= \Phi_v(\omega_4) = - \sum_{i \in J, i \notin J'} \mu_J \bar{e}_{j_1} \wedge \bar{e}_{-j_1} \wedge \cdots \wedge \bar{e}_i \wedge \hat{e}_{-i} \wedge \cdots \wedge \bar{e}_{j_\ell} \wedge \bar{e}_{-j_\ell} (\wedge \bar{e}_0) \\ &\quad + \sum_{j \in J, i \notin J'} \mu_J \bar{e}_{j_1} \wedge \bar{e}_{-j_1} \wedge \cdots \wedge \bar{e}_j \wedge \hat{e}_{-j} \wedge \cdots \wedge \bar{e}_{j_\ell} \wedge \bar{e}_{-j_\ell} (\wedge \bar{e}_0) \\ &= \bar{e}_i \wedge \sum_{i, j \notin J'} (-\mu_{J' \cup \{i\}} + \mu_{J' \cup \{j\}}) \bar{e}_{j'_1} \wedge \bar{e}_{-j'_1} \wedge \cdots \wedge \bar{e}_{j'_{\ell-1}} \wedge \bar{e}_{-j'_{\ell-1}} (\wedge \bar{e}_0). \end{aligned}$$

So we find that $\mu_{J' \cup \{i\}} = \mu_{J' \cup \{j\}}$ for every $J' \subset \{1, \dots, \hat{i}, \dots, \hat{j}, \dots, p-1\}$. Letting i and j vary yields the result. \square

Step 4. $\mu_4 = 0$

Proof. Finally, take $v = e_0 + e_{-1} - c_0 e_1$, where as before $c_0 = \frac{1}{2} \langle e_0, e_0 \rangle$. If $p = 2\ell$ we can write

$$\Phi_v(\omega_4) = \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-1} \wedge \tilde{u} + \mu \sum_{j \geq 1} (\bar{e}_{-1} + c_0 \bar{e}_1) \wedge \bar{e}_{j_2} \wedge \bar{e}_{-j_2} \wedge \cdots \wedge \bar{e}_{j_\ell} \wedge \bar{e}_{-j_\ell}$$

and if $p = 2\ell + 1$ we have

$$\begin{aligned} \Phi_v(\omega_4) &= \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-1} \wedge \tilde{u} + \mu \sum_{j \geq 1} (\bar{e}_{-1} + c_0 \bar{e}_1) \wedge \bar{e}_{j_2} \wedge \bar{e}_{-j_2} \wedge \cdots \wedge \bar{e}_{j_\ell} \wedge \bar{e}_{-j_\ell} \wedge \bar{e}_0 \\ &\quad + 2c_0 \mu \sum_J \bar{e}_{j_1} \wedge \bar{e}_{-j_1} \wedge \cdots \wedge \bar{e}_{j_\ell} \wedge \bar{e}_{-j_\ell}. \end{aligned}$$

In both cases we have $L_{\Phi_v(\omega_4)} \supset \text{span}\{\bar{e}_2, \dots, \bar{e}_{p-1}\}$ by (2) from which we conclude $\mu = 0$. \square

Step 5. $\omega \in \hat{G}r_{\text{iso}}(p, V)$

Proof. Since $\omega_1 = 0$ and $\omega_4 = e_1 \wedge \cdots \wedge e_{p-1} \wedge u$ for some $u \in V'$, we can write

$$\omega = e_1 \wedge \cdots \wedge e_{p-1} \wedge u'$$

for some $u' \in \text{span}\{e_0, e_p, e_{-1}, \dots, e_{-p+1}, e_{-p}\}$. Choose $v = e_{-1}$, then we have

$$\Phi_v(\omega) = \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-1} \wedge \overline{u'} \in \hat{G}r_{\text{iso}}(p-1, V_v)$$

hence $\langle u', u' \rangle = 0$ and $\langle e_j, u' \rangle = 0$ for all $j = 2, \dots, p-1$. Replacing $v = e_{-1}$ with $v = e_{-2}$ yields that also $\langle e_1, u' \rangle = 0$, hence $\omega \in \hat{G}r_{\text{iso}}(p, V)$. \square

This proves Claim 4.14. \square

5. Counterexamples in small dimensions

In the Main Theorem 4.1, we assumed that $\dim V > 8$. In this section, we will show that this assumption is actually necessary. In both $\bigwedge^3 \mathbb{K}^7$ and $\bigwedge^4 \mathbb{K}^8$, we will give a p -form ω that does not lie in the isotropic Grassmannian, but which maps to the isotropic Grassmannian upon applying any IGCP map Φ_v . For case of simplicity, we assume the underlying field \mathbb{K} is either \mathbb{C} or \mathbb{R} .

5.1. Counterexample in dimension 7

Let V be a 7-dimensional vector space over \mathbb{K} with a fixed basis $\{e_0, e_1, e_2, e_3, e_{-1}, e_{-2}, e_{-3}\}$, and a quadratic form given by the matrix

$$J = \left(\begin{array}{c|cc} -\frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & I_3 \\ \hline 0 & I_3 & 0 \end{array} \right).$$

Choose

$$\omega_7 := e_1 \wedge e_2 \wedge e_3 + e_{-1} \wedge e_{-2} \wedge e_{-3} + e_0 \wedge (e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + e_3 \wedge e_{-3}).$$

One verifies that $\omega_7 \notin \hat{Gr}(3, V)$, so in particular $\omega_7 \notin \hat{Gr}_{iso}(3, V)$. In Claim 5.2 below we will show that every Φ_v maps ω_7 to the isotropic Grassmann cone $\hat{Gr}_{iso}(2, 5)$. One could verify this by a direct computation for an arbitrary isotropic vector v . However, we will exploit the fact that ω_7 is sufficiently symmetric (Claim 5.1), so it suffices to do the computation for one fixed $v \in V_{iso}$.

Consider the algebraic group

$$\begin{aligned} SO(V) &= \{\phi \in SL(V) \mid \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \forall x, y \in V\} \\ &= \{A \in SL(7, \mathbb{K}) \mid A^T J A = J\} \end{aligned}$$

and its subgroup

$$G = \text{stab}(\omega_7) = \{\phi \in SO(V) \mid \phi \cdot \omega_7 = \omega_7\}.$$

Claim 5.1. *The action of G on V_{iso} is transitive.*

Proof. Take any $v_0 \in V_{iso}$. We want to show that its orbit $G \cdot v_0$ has dimension six. Then $G \cdot v_0$ is a full-dimensional subvariety of the irreducible 6-dimensional variety V_{iso} , and hence is equal to V_{iso} . For this we use the formula

$$\dim(G \cdot v_0) = \dim G - \dim(\text{stab}_G(v_0)),$$

where $\text{stab}_G(v_0) = \{\phi \in G \mid \phi \cdot v_0 = v_0\}$ is the stabilizer. We will compute both terms $\dim G$ and $\dim(\text{stab}_G(v_0))$ by switching to Lie algebras.

The Lie algebra of $SO(V)$ is given by

$$\begin{aligned} \mathfrak{so}(V) &= \{X \in \mathfrak{sl}(7, \mathbb{K}) \mid X^T J + J X = 0\} \\ &= \left\{ \left(\begin{array}{c|cc} 0 & -2y^T & -2x^T \\ \hline x & a & b \\ \hline y & c & -a^T \end{array} \right) \mid x, y \in \mathbb{K}^3, a, b, c \in \mathbb{K}^{3 \times 3}, b + b^T = c + c^T = 0 \right\}. \end{aligned}$$

We introduce the following notation.

$$\text{For } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ we write } l_x := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

We can compute the Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)$ of G as follows:

$$\begin{aligned} \mathfrak{g} &= \{X \in \mathfrak{so}(V) \mid X \cdot \omega_7 = 0\} \\ &= \left\{ \left(\begin{array}{c|cc} 0 & -2y^T & -2x^T \\ \hline x & a & l_y \\ \hline y & l_x & -a^T \end{array} \right) \mid x, y \in \mathbb{K}^3, a \in \mathfrak{sl}(3, \mathbb{K}) \right\}. \end{aligned}$$

Observe that $\dim \mathfrak{g} = 3 + 3 + 8 = 14$ and since $\dim G = \dim \mathfrak{g}$, G has also dimension 14. For the stabilizer, if we take $v_0 = e_{-3} \in V_{\text{iso}}$, we see that

$$\text{stab}_{\mathfrak{g}}(v_0) = \{X \in \mathfrak{so}(V) \mid X \cdot e_{-3} = 0\}$$

is the set of matrices in \mathfrak{g} whose final column is zero, which has dimension 8. So $\dim(G \cdot v_0) = 14 - 8 = 6 = \dim V_{\text{iso}}$, as desired. \square

Claim 5.2. For every $v \in V_{\text{iso}}$, it holds that $\Phi_v(\omega_7) \in \hat{G}r_{\text{iso}}(2, V_v)$.

Proof. By the previous claim, it suffices to prove the claim for one fixed $v_0 \in V_{\text{iso}}$. Indeed, then any $v \in V_{\text{iso}}$ is of the form $f \cdot v_0$ for some $f \in \text{stab}(\omega_7)$, and we get

$$\Phi_v(\omega_7) = \Phi_{f \cdot v_0}(f \cdot \omega_7) = f \cdot \Phi_{v_0}(\omega_7) \in Gr_{\text{iso}}(2, V_v),$$

since $f \cdot \omega := (\wedge^p f)(\omega)$ and $\Phi_{f(v_0)} \circ (\wedge^3 f) = (\wedge^2 f) \circ \Phi_{v_0}$ for every $f \in SO(V)$. So we take $v_0 = e_{-3}$, and readily compute

$$\Phi_{e_{-3}}(\omega_7) = \bar{e}_1 \wedge \bar{e}_2 \in \hat{G}r_{\text{iso}}(2, e_{-3}^\perp / (e_{-3})). \quad \square$$

In summary, this shows how the Main Theorem 4.1 fails for $\hat{G}r_{\text{iso}}(3, 7)$: by Claim 5.2 ω_7 satisfies the assumption but is itself not in $\hat{G}r_{\text{iso}}(3, 7)$. In particular, this means that $\hat{G}r_{\text{iso}}(3, 7)$ cannot be defined by pulling back the equations of $\hat{G}r_{\text{iso}}(2, 5)$ along IGCP maps of the form Φ_v . We originally constructed our counterexample by analyzing where our proof fails if $\dim V = 7$. However, it turned out, that ω_7 is interesting also from different points of view, which we will discuss in the following remarks.

Remark 5.3. In 1900, Engel (1900) showed that if ω is a generic 3-form on \mathbb{C}^7 , its symmetry group is isomorphic to the exceptional group G_2 , and that such a 3-form gives rise to a bilinear form β_ω . If we choose coordinates such that ω agrees with our form ω_7 , then this group G_2 is precisely the stabilizer G we computed in Claim 5.1, and β_ω is up to scaling equal to our bilinear form given by J . For more about G_2 , we refer the reader to Fontanals (2018).

Remark 5.4. Alternatively we can construct ω_7 as the triple product on the split octonions. Here we will follow the notation from Baez and Huerta (2014). Recall that the space \mathbb{H} of quaternions is the 4-dimensional real vector space with basis $\{1, i, j, k\}$, equipped with a bilinear associative product specified by Hamilton’s formula

$$i^2 = j^2 = k^2 = ijk = -1.$$

The conjugate of a quaternion $x = a + bi + cj + dk$ is given by $\bar{x} = a - bi - cj - dk$. We also have a quadratic form given by $Q_{\mathbb{H}}(x) := x\bar{x} = \bar{x}x = a^2 + b^2 + c^2 + d^2$. The space of split octonions is the vector space $\mathbb{O}_s := \mathbb{H} \oplus \mathbb{H}$ with a bilinear (but nonassociative) product given by

$$(a, b)(c, d) := (ac + d\bar{b}, \bar{a}d + cb).$$

The conjugate of an octonion (a, b) is given by $\overline{(a, b)} = (\bar{a}, -b)$, and we define a quadratic form $Q_{\mathbb{O}_s}$, of signature (4, 4), by $Q_{\mathbb{O}_s}(x) = x\bar{x} = \bar{x}x$; or equivalently $Q_{\mathbb{O}_s}((a, b)) = Q_{\mathbb{H}}(a) - Q_{\mathbb{H}}(b)$. We will write

$$\begin{aligned} e_0 &:= (1, 0) & e_1 &:= (i, 0) & e_2 &:= (j, 0) & e_3 &:= (k, 0) \\ e_4 &:= (0, 1) & e_5 &:= (0, i) & e_6 &:= (0, j) & e_7 &:= (0, k). \end{aligned}$$

Let $\mathbb{O}_{\text{im}} = \{x \in \mathbb{O}_s \mid \bar{x} = -x\} = \text{span}\{e_1, \dots, e_7\}$ denote the imaginary split octonions. On \mathbb{O}_{im} we can define a cross product given by the commutator:

$$x \times y := \frac{1}{2}(xy - yx),$$

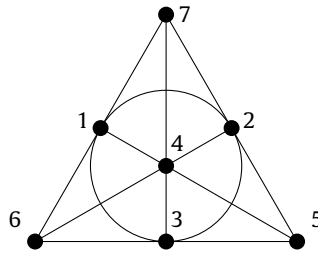
and a triple product $T : \mathbb{O}_{\text{Im}} \times \mathbb{O}_{\text{Im}} \times \mathbb{O}_{\text{Im}} \rightarrow \mathbb{R}$, given by

$$T(x, y, z) := \langle x, y \times z \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form coming from $Q_{\mathbb{O}_3}$. This triple product is an alternating trilinear form, and hence can be identified with an element $\omega \in \wedge^3 V^*$, where $V = \mathbb{O}_{\text{Im}}$. Explicitly, writing $e_i^* \in V^*$ for the dual vector to e_i , we have

$$\begin{aligned} \omega = & e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_4^* \wedge e_5^* + e_1^* \wedge e_6^* \wedge e_7^* + e_2^* \wedge e_4^* \wedge e_6^* \\ & - e_2^* \wedge e_5^* \wedge e_7^* + e_3^* \wedge e_4^* \wedge e_7^* + e_3^* \wedge e_5^* \wedge e_6^*. \end{aligned}$$

Note that the terms in ω correspond to the lines in the Fano plane:



This ω agrees with ω_7 up to a change of basis. Explicitly, if we substitute

$$\begin{aligned} e_0 &\mapsto \frac{e_4^*}{\sqrt{2}}, & e_1 &\mapsto \frac{e_1^* + e_5^*}{\sqrt{2}}, & e_2 &\mapsto \frac{e_2^* + e_6^*}{\sqrt{2}}, & e_3 &\mapsto \frac{e_3^* + e_7^*}{\sqrt{2}}, \\ e_{-1} &\mapsto \frac{e_1^* - e_5^*}{\sqrt{2}}, & e_{-2} &\mapsto \frac{e_2^* - e_6^*}{\sqrt{2}}, & e_{-3} &\mapsto \frac{e_3^* - e_7^*}{\sqrt{2}} \end{aligned}$$

into ω_7 , we recover ω (up to scaling).

5.2. Counterexample in dimension 8

Let V be an 8-dimensional vector space with basis $\{e_1, e_2, e_3, e_4, e_{-1}, e_{-2}, e_{-3}, e_{-4}\}$, and quadratic form given by the matrix

$$J = \left(\begin{array}{c|c} 0 & I_4 \\ \hline I_4 & 0 \end{array} \right).$$

Choose

$$\begin{aligned} \omega_8 := & 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 + 2e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4} \\ & + e_1 \wedge e_2 \wedge e_{-1} \wedge e_{-2} + e_1 \wedge e_3 \wedge e_{-1} \wedge e_{-3} + e_1 \wedge e_4 \wedge e_{-1} \wedge e_{-4} \\ & + e_2 \wedge e_3 \wedge e_{-2} \wedge e_{-3} + e_2 \wedge e_4 \wedge e_{-2} \wedge e_{-4} + e_3 \wedge e_4 \wedge e_{-3} \wedge e_{-4}. \end{aligned} \tag{5.1}$$

One can verify that $\omega_8 \notin \hat{Gr}(4, V)$, so in particular $\omega_8 \notin \hat{Gr}_{\text{iso}}(4, V)$. As before, we consider the algebraic group

$$\begin{aligned} SO(V) = & \{ \phi \in SL(V) \mid \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \forall x, y \in V \} \\ = & \{ A \in SL(8, \mathbb{K}) \mid A^T J A = J \} \end{aligned}$$

and its subgroup

$$G := \text{stab}(\omega_8) = \{\phi \in SO(V) \mid \phi \cdot \omega_8 = \omega_8\}.$$

Claim 5.5. *The action of G on V_{iso} is transitive.*

Proof. Take any $v_0 \in V_{\text{iso}}$; we want to show that its orbit $G \cdot v_0$ has dimension equal to $\dim V_{\text{iso}} = 7$. The Lie algebra of $SO(V)$ is given by

$$\begin{aligned} \mathfrak{so}(V) &= \{X \in \mathfrak{sl}(8, \mathbb{K}) \mid X^T J + JX = 0\} \\ &= \left\{ \left(\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array} \right) \mid a, b, c \in \mathbb{K}^{4 \times 4}, b + b^T = c + c^T = 0 \right\}. \end{aligned}$$

We introduce the following notation

$$\text{for } b = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{pmatrix} \text{ write } \tilde{b} := \begin{pmatrix} 0 & -b_{34} & b_{24} & -b_{23} \\ b_{34} & 0 & -b_{14} & b_{13} \\ -b_{24} & b_{14} & 0 & -b_{12} \\ b_{23} & -b_{13} & b_{12} & 0 \end{pmatrix}.$$

We can compute the Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)$ of G as follows:

$$\begin{aligned} \mathfrak{g} &= \{X \in \mathfrak{so}(V) \mid X \cdot \omega_8 = 0\} \\ &= \left\{ \left(\begin{array}{c|c} a & b \\ \hline b & -a^T \end{array} \right) \mid a \in \mathfrak{sl}(4, \mathbb{K}), b + b^T = 0 \right\}. \end{aligned}$$

As before $\dim G = \dim \mathfrak{g} = 21$. For the stabilizer, if we take $v_0 = e_{-4} \in V_{\text{iso}}$, we see that

$$\text{stab}_{\mathfrak{g}}(v_0) = \{X \in \mathfrak{g} \mid X \cdot e_{-4} = 0\}$$

is the set of matrices in \mathfrak{g} whose final column is zero, which has dimension 14. So $\dim(G \cdot v_0) = 21 - 14 = 7 = \dim V_{\text{iso}}$, as desired. \square

As before, we conclude the following claim.

Claim 5.6. *For every $v \in V_{\text{iso}}$, it holds that $\Phi_v(\omega_8) \in \hat{G}r_{\text{iso}}(3, V_v)$.*

Proof. As in Claim 5.2, it suffices to prove the claim for one fixed $v \in V_{\text{iso}}$. Taking $v = e_{-1}$, we compute that

$$\Phi_v(\omega_8) = 2\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_4 \in \hat{G}r_{\text{iso}}(3, e_{-1}^\perp / \langle e_{-1} \rangle). \quad \square$$

In summary, this shows how the Main Theorem 4.1 fails for $\hat{G}r_{\text{iso}}(4, 8)$. As before, this means that $\hat{G}r_{\text{iso}}(4, 8)$ cannot be defined by pulling back the equations of $\hat{G}r_{\text{iso}}(3, 6)$ along IGCP maps of the form Φ_v .

Remark 5.7. The Lie algebra \mathfrak{g} defined above is in fact isomorphic to $\mathfrak{so}(7)$. An explicit isomorphism $\mathfrak{so}(7) \rightarrow \mathfrak{g}$ can be given by

$$\left(\begin{array}{c|ccc|ccc} 0 & -2y_1 & -2y_2 & -2y_3 & -2x_1 & -2x_2 & -2x_3 \\ \hline x_1 & a_{11} & a_{12} & a_{13} & 0 & b_{12} & b_{13} \\ x_2 & a_{21} & a_{22} & a_{23} & -b_{12} & 0 & b_{23} \\ x_3 & a_{31} & a_{32} & a_{33} & -b_{13} & -b_{23} & 0 \\ \hline y_1 & 0 & c_{12} & c_{13} & -a_{11} & -a_{21} & -a_{31} \\ y_2 & -c_{12} & 0 & c_{23} & -a_{12} & -a_{22} & -a_{32} \\ y_3 & -c_{13} & -c_{23} & 0 & -a_{13} & -a_{23} & -a_{33} \end{array} \right) \mapsto \left(\begin{array}{ccc|ccc|ccc} d_{11} & a_{12} & a_{13} & -c_{23} & 0 & -y_3 & y_2 & x_1 \\ a_{21} & d_{22} & a_{23} & c_{13} & y_3 & 0 & -y_1 & x_2 \\ a_{31} & a_{32} & d_{33} & -c_{12} & -y_2 & y_1 & 0 & x_3 \\ b_{23} & -b_{13} & b_{12} & d_{44} & -x_1 & -x_2 & -x_3 & 0 \\ \hline 0 & -x_3 & x_2 & y_1 & -d_{11} & -a_{21} & -a_{31} & -b_{23} \\ x_3 & 0 & -x_1 & y_2 & -a_{12} & -d_{22} & -a_{32} & b_{13} \\ -x_2 & x_1 & 0 & y_3 & -a_{13} & -a_{23} & -d_{33} & -b_{12} \\ -y_1 & -y_2 & -y_3 & 0 & c_{23} & -c_{13} & c_{12} & -d_{44} \end{array} \right),$$

where for the left hand side we used the notation from Section 5.1, and in the right hand side we have

$$d_{11} := \frac{a_{11} - a_{22} - a_{33}}{2}, d_{22} := \frac{-a_{11} + a_{22} - a_{33}}{2}, d_{33} := \frac{-a_{11} - a_{22} + a_{33}}{2}, d_{44} := \frac{a_{11} + a_{22} + a_{33}}{2}.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Ranks of defining quadrics

In Section A.1, we will finish the proof of Corollary 4.3, by verifying the following fact:

Claim A.8. $\hat{G}r_{iso}(3, 7)$, as well as both irreducible components of $\hat{G}r_{iso}(4, 8)$, can be set-theoretically defined by quadrics of rank at most 4.

In Section A.2, we explain how Corollary 4.3 follows from the literature on isotropic Grassmannians, in particular the Cartan embedding.

A.1. Computational approach

Our verification is based on an algorithm, which we implemented in Macaulay2 Grayson and Stillman. We sketch the steps of the algorithm below. Let X be either $\hat{G}r_{iso}(3, 7)$, or one of the components of $\hat{G}r_{iso}(4, 8)$.

- (1) Compute the ideal I defining X by parametrizing an open subset and performing a Gröbner basis computation. The ideal I is generated by linear equations and quadrics.
- (2) Get rid of the linear equations by substituting variables.
- (3) View the space I_2 of quadrics in I as a representation of $SO(V)$, and decompose it into weight spaces.
- (4) Find a highest weight vector $p \in I_2$ of minimal rank.
- (5) Compute the subrepresentation generated by p , using the lowering operators in $\mathfrak{so}(V)$.
- (6) If we generated all of I_2 , we are done.
- (7) Otherwise, find the highest weight space we did not yet generate, let p be a quadric of minimal rank in it, and return to step (5).

By construction, the $SO(V)$ -orbits of the quadrics p we found give sufficiently many equations to define X . Since acting with $SO(V)$ does not change the rank of a quadric, it follows that if each of our quadrics has rank at most 4, then X can be defined by quadrics of rank at most 4. For $\hat{G}r_{iso}(3, 7)$, our algorithm returned the following quadrics:

$$\begin{aligned} &x_{0,1,2}^2 + 2x_{1,2,3}x_{1,2,-3}, \\ &x_{0,1,2}(x_{1,2,-2} + x_{1,3,-3}) + 2x_{1,2,3}x_{0,1,-3}, \\ &x_{0,1,3}x_{0,1,-3} + x_{0,1,2}x_{0,1,-2}, \\ &x_{1,2,3}(x_{0,1,-1} + x_{0,2,-2} - x_{0,3,-3}) + x_{0,1,2}(x_{2,3,-2} + x_{1,3,-1}), \\ &x_{0,1,-1}^2 - (x_{0,2,-2} + x_{0,3,-3})^2 + 2(x_{1,3,-3} + x_{1,2,-2})(x_{3,-1,-3} + x_{2,-1,-2}). \end{aligned}$$

For one of the components of $\hat{Gr}_{iso}(4, 8)$, we found the following quadrics:

$$\begin{aligned} &x_{1,2,3,-3}^2 - x_{1,2,3,4}x_{1,2,-3,-4}, \\ &2x_{1,2,3,-3}x_{1,3,4,-1} - x_{1,2,3,4}(x_{1,2,-1,-2} - x_{1,3,-1,-3} - x_{1,4,-1,-4}), \\ &(x_{1,4,-1,-4} + x_{2,4,-2,-4} - x_{3,4,-3,-4})^2 - 4x_{3,4,-1,-2}x_{1,2,-3,-4}. \end{aligned}$$

Since all quadrics listed above have rank at most 4, and since both components of $\hat{Gr}_{iso}(4, 8)$ are isomorphic, our verification is now complete.

A.2. Rank 4 quadrics via the Cartan embedding

In this section we will sketch an alternative proof that $Gr_{iso}(p, 2p + 1)$ and the connected components of $Gr_{iso}(p, 2p)$, in their Plücker embedding, are defined by linear equations and quadrics of rank at most 4, using the Cartan embedding (sometimes called spinor embedding), cf. Cartan (1981) or Harnad and Balogh (2021, Appendix E). The proof follows by combining the following facts:

- The image of the Cartan embedding is defined by quadrics (Cartan, 1981).
- The Plücker embedding factors as the Cartan embedding followed by a degree two Veronese embedding (Balogh et al. (2021, Theorem 2.1) and Cardinali and Pasini (2013, Theorem 1)).
- The image of a degree two Veronese embedding is defined by quadrics of rank 3 and 4.

The idea is that the Veronese embedding turns the quadratic equations of the Cartan embedding into linear equations, so the only quadrics we need are the ones coming from the Veronese embedding.

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