The very clean theoretical predictions for the rare decays \( K \rightarrow \pi \nu \bar{\nu} \) and \( K_L \rightarrow \pi^0 \ell^+ \ell^- \) are reviewed, and their various theoretical inputs summarized. The less favorable situation for \( K_L \rightarrow \mu^+ \mu^- \) is also commented.

### I. INTRODUCTION

The rare decays \( K \rightarrow \pi \nu \bar{\nu} \) and \( K_L \rightarrow \pi^0 \ell^+ \ell^- \), driven by semi-leptonic flavor-changing neutral currents (FCNC), are exceptionally clean probes of the flavor structure of the Standard Model, or of the still elusive New Physics. Concerted theoretical efforts have brought the SM predictions to an impressive level of accuracy (for the current experimental situation, see [1]). In this section, the main theoretical ingredients are briefly reviewed, while the situation for each mode is summarized in the following sections.

#### A. FCNC electroweak structure

FCNC arise at one loop in the electroweak theory. The processes driving the rare semi-leptonic K decays are the W box, Z and \( \gamma \) penguins, see Fig.1, and lead to the amplitudes

\[
A_K \left( K_L \rightarrow \pi^0 X \right) = \sum_{q=u,c,t} \left( \text{Im} \lambda_q + \varepsilon \text{Re} \lambda_q \right) y_q^X (m_q),
\]

\[
A_K \left( K^+ \rightarrow \ell^+ \ell^- \right) = \sum_{q=u,c,t} \left( \text{Re} \lambda_q + i \text{Im} \lambda_q \right) y_q^X (m_q),
\]

with \( X = \nu \bar{\nu}, \ell^+ \ell^- \) and \( \lambda_q = V_{q \mu}^* V_{q \nu} \). In standard terminology, the \( \varepsilon \) part is the indirect CP-violating piece (ICPV), while the \( \text{Im} \lambda_q \) part is called direct CP-violating (DCPV). The amplitude for \( K \) is obtained from the \( K_L \) one by interchanging \( \text{Im} \lambda_q \leftrightarrow \text{Re} \lambda_q \).

Without the dependence of the loop functions \( y_q^X \) on the quark masses, CKM unitarity would imply vanishing FCNC (GIM mechanism). Now, looking at these dependences, combined with the scaling of the CKM elements, one can readily get a handle on the importance of each quark contribution, and thereby on the cleansness achievable for the decay once QCD effects are included.

For \( X = \nu \bar{\nu} \), only the Z penguin and W box enter, \( y_{12}^{\nu \bar{\nu}} \sim m_2^2 \), and light-quark contributions are suppressed. Since, in addition, \( \varepsilon \sim 10^{-3} \) and \( \text{Re} \lambda_t \sim \text{Im} \lambda_t \), ICPV is very small. For \( K^+ \), the c-quark contribution is suppressed from the loop, but enhanced by \( \text{Re} \lambda_c \gg \lambda_t \), and ends up being comparable to the \( t \)-quark contribution.

#### B. QCD corrections

Having identified the relevant electroweak structures, QCD effects have now to be included. This is done in three main steps:

- **Step 1**: Integration of heavy degrees of freedom (top, W, Z), including perturbative QCD effects above \( M_W \). This generates local FCNC operators (Fig.1 with \( t \)-quark), and Fermi-type four-fermion local operators.

- **Step 2**: Resummation of QCD corrections (running down). At the \( \epsilon \) threshold (similar for \( b, \tau \)), four-fermion operators are combined to form closed c-loops, which are then replaced by a tower of effective interactions in increasing powers of \( \left( \text{external momentum} / \text{charm mass} \right) \), Fig.2a. The lowest order consists again of the dimension-six FCNC operators, while dimension-eight operators are corrections scaling naively like \( m_{W}^2 / m_c^2 \sim 15\% \).

  These first two steps (the OPE) can, in principle, be achieved to any desired level of precision within perturbative QCD, though the computation of the required multiloop diagrams represents a formidable task at higher orders. Still, this is unavoidable in order to reduce theoretical errors, in particular scale dependencies. At this stage, one has obtained the complete Hamiltonian, i.e. all the effective operators, with the short-distance physics encoded in their Wilson coefficients.

- **Step 3**: To get the amplitudes, the matrix elements of these operators between meson states remain to be calculated.
estimated. To this end, one makes use of the symmetries of QCD, as embodied in Chiral Perturbation Theory (ChPT), to relate the desired matrix elements to experimentally known quantities.

For the most interesting dimension-six semi-leptonic operators, the matrix elements can be related to those of $K_{23}, K_{L3}$ decays (taking into account isospin-breaking corrections). Contributions from four light-quark operators ($Q_1, \ldots, Q_5$) are represented directly in terms of meson fields in ChPT, such that non-local $u$-quark loops are represented as meson loops (Fig.2b,c). The price to pay is the introduction of some unknown low-energy constants ($G_{8,27,\ldots}$), to be extracted from experiment. In particular, $G_8$ is fixed from $B(K \to \pi \pi)$, accounting for the large non-perturbative $\Delta I = 1/2$ effects. For dimension-eight operators, an approximate matching is done with the ChPT representation of the $u$-quark contributions.

II. THE $K^+ \to \pi^+ \nu \bar{\nu}$ AND $K_L \to \pi^0 \nu \bar{\nu}$ DECAYS

Thanks to the suppression of light-quark effects, these modes are the cleanest and their rates are precisely predicted within the SM. Their branching ratios read:

$$B^\nu_{+} = \kappa_+ \left( \frac{\text{Im} \lambda_1}{\lambda^5} X_t \right)^2 + \left| \frac{\text{Re} \lambda_1}{\lambda^5} X_t + \frac{\text{Re} \lambda_3}{\lambda} P_{u,c} \right|^2,$$

$$B^\nu_{L} = \kappa_L \left| \frac{\text{Im} \lambda_1}{\lambda^5} X_t \right|^2,$$

with $P_{u,c} = P_{+} + \delta P_{u,c}$. The Wilson coefficient of the dimension-six FCNC operator $Q^\nu = (\bar{s}d)_{V-A} (\bar{\nu} \bar{\nu})_{V-A}$ arising from the top-quark loop is known at NLO, $X_t = 1.646 \pm 0.041$. For the charm-quark, the contribution to this operator has recently been obtained at NNLO, $P_c = 0.37 \pm 0.04$. Residual $c$-quark effects from dimension-eight operators, along with long-distance $u$-quark effects, amount to a small correction $\delta P_{u,c} = 0.04 \pm 0.02$. The matrix elements of $Q^\nu$ are known from $K_{L3}$, including the leading isospin-breaking corrections, and are encoded into $\kappa_L = 2.29 \pm 0.03 \cdot 10^{-10}$ and $\kappa_+ = 5.26 \pm 0.06 \cdot 10^{-11}$ for $\lambda = 0.225$. Finally, for $K_L \to \pi^0 \nu \bar{\nu}$, ICPV is of about 1% while the CP-conserving contribution arising from box diagrams is less than 0.01%.

The SM predictions are then

\[
B(K_L \to \pi^0 \nu \bar{\nu}) = (2.7 \pm 0.4) \cdot 10^{-11}, \\
B(K^+ \to \pi^+ \nu \bar{\nu}) = (8.4 \pm 1.0) \cdot 10^{-11}.
\]

The error on $K_L \to \pi^0 \nu \bar{\nu}$ is dominated by $\text{Im} \lambda_1$, while for $K^+ \to \pi^+ \nu \bar{\nu}$, it breaks down to scales (13%), $m_c(22\%)$, CKM, $\rho_S$, $m_t (37\%)$ and matrix-elements from $K_{L3}$ and light-quark contributions (28%). Further improvements are thus possible through a better knowledge of $m_c$, of the isospin-breaking in the $K \to \pi$ form-factors, or by a lattice study of higher-dimensional operators.

As the determination of $\lambda_1$ from general UT fits to $B$ physics data is already very precise, and expected to be further improved in the near future, the main interest of the $K \to \pi \nu \bar{\nu}$ decays is to test the CKM paradigm for CP-violation in the SM. Indeed, these modes do offer a
produced at LO through the finite two-loop process
space suppressed $2^\ell$. In ChPT, loops are small and one just needs
$K$ (Fig.2c). Higher order corrections are estimated using
$\ell$ contribution from $Q$ sign from NA48 measurements as
state, the latter in a $1^-$ state. Only the $0^+$ state.

Indirect CP-violation is related to $K_S \to \pi^0\ell^+\ell^-$,
for which the long-distance photon penguin dominates
(Fig.2b). In ChPT, loops are small and one just needs to
fix a counterterm, $a_S$ [13]. This can be done up to a
sign from NA48 measurements as $|a_S| = 1.2 \pm 0.2$ [13].
Producing $\ell^+\ell^-$ in a $1^-$ state, it interferes with the
contribution from $Q_{7V}$, arguably constructively [14 17].

The CP-conserving (CPC) contribution from $Q_{1,...,6}$
proceeds through two-photons, i.e. produces the lepton
pair in either a helicity-suppressed $0^+\ell^+\ell^-$ or phase-
space suppressed $2^\ell$ state. Only the $0^+$ state is
produced at LO through the finite two-loop process
$K_L \to \pi^0\ell^+\ell^-$, $\ell^+\ell^-$, $P = \pi, K$ (Fig.2c).
Higher order corrections are estimated using
$K_L \to \pi^0\gamma\gamma$ experimental data for both the $0^+\ell^+$
and $2^\ell$ contributions [14].

Altogether, the branching ratios are

$$B^{\ell^+\ell^-} = \left( C_{dir}^\ell + C_{int}^\ell |a_S|^2 + C_{mix}^\ell |a_S|^2 + C_{\gamma\gamma}^\ell \right) \times 10^{-12},$$

with the coefficients given in Table I. Interestingly, these
coefficients obey $C_{\gamma}^\ell / C_{\pi}^\ell \approx 0.23$ due to the
phase-space suppression, but for the helicity-suppressed contributions arising from the $Q_{7A}$ operator (DCPV) and from $\gamma\gamma$
(CPC). This maintains the sensitivity of $B^{\ell^+\ell^-}$ on the
interesting short-distance physics at the same level as
$B^{\ell^+\ell^-}$. Further, it allows in principle to disentangle the
$Q_{7V}$ and $Q_{7A}$ contributions from the measurements of
both modes. This is illustrated in Fig.3a, where the hy-
perbola corresponds to a common rescaling of both $y_{7V}$
and $y_{7A}$ [16]. As discussed in [16 17], this plane is particu-
larly interesting to look for signals of New Physics, and
identify its precise nature [14].

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$C_{dir}^\ell$</th>
<th>$C_{int}^\ell$</th>
<th>$C_{mix}^\ell$</th>
<th>$C_{\gamma\gamma}^\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell = e$</td>
<td>$(4.62 \pm 0.24) (y_{7V}^2 + y_{7A}^2)$</td>
<td>$(11.3 \pm 0.3) y_{7V}$</td>
<td>$14.5 \pm 0.5$, $\approx 0$</td>
<td></td>
</tr>
<tr>
<td>$\ell = \mu$</td>
<td>$(1.09 \pm 0.05) (y_{7V}^2 + 2.32 y_{7A}^2)$</td>
<td>$(2.63 \pm 0.06) y_{7V}$</td>
<td>$3.36 \pm 0.20$, $5.2 \pm 1.6$</td>
<td></td>
</tr>
</tbody>
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TABLE I: Coefficients encoding the various contributions to $B^{\ell^+\ell^-}$

for constructive (destructive) interference (Fig.3a).
Overall, the error on $a_S$ is currently the most limit-

ing and better measurements of $K_S \to \pi^0\ell^+\ell^-$
would be welcomed (Fig.3b). Also, better measurements of
$K_L \to \pi^0\gamma\gamma$ would help reduce the error on the $0^+\ell^$
and $2^\ell$ contributions. Alternatively, they can be
partially cut away through energy cuts or Dalitz plot
analyses [14 16 17].

The integrated forward-backward (or lepton-energy)
asymmetry [17 18]

$$A_{FB}^\mu = \frac{N(E_{E^\ell} > E_{E^\ell}) - N(E_{E^\ell} < E_{E^\ell})}{N(E_{E^\ell} > E_{E^\ell}) + N(E_{E^\ell} < E_{E^\ell})},$$

is generated by the interference between CP-conserving
and CP-violating amplitudes. It cannot be reliably esti-

mated at present for $\ell = e$ because of the poor theoretical
control on the $2^\ell$ contribution. The situation is better
for $\ell = \mu$, for which this part is negligible,

$$A_{FB}^\mu = (1.3 (1) y_{7V} \pm 1.72 |a_S|) \times 10^{-12} / B^{\ell^+\ell^-},$$

i.e., $A_{FB,SM}^\mu \approx 20\%(-12\%)$ for constructive (destructive)
interference [17]. Interestingly, though the error is large,
$A_{FB}^\mu$ can be used to fix the sign of $a_S$.

IV. THE $K_L \to \mu^+\mu^-$ DECAY

For this mode, the short-distance (SD) piece from $t$
and c-quarks is known to NLO and NNLO [10], respecti-

vely. Indirect CP-violation is negligible. The long-distance
(LD) contribution from $Q_{1,...,6}$ matrix elements proceeds
again through two-photons. Still, there are three differ-

ences with respect to the $K_L \to \pi^0\ell^+\ell^-$ decays.

First, the contribution from the imaginary part of the
$\gamma\gamma$ loop, estimated from $K_L \to \gamma\gamma$, is much larger
than SD, and already accounts for the bulk of the experimental
$K_L \to \mu^+\mu^-$ rate. Second, while the charged meson loop
in $K_L \to \pi^0\ell^+\ell^-$ acts like a cut-off, and a finite result is
found, now the two photons arise from the axial anomaly,
and $K_L \to \pi^0, \eta, \eta' \to \gamma\gamma \to \mu^+\mu^-$ is divergent, requir-
ing unknown counterterms. To estimate them, though
still with a large error, one can use the experimental in-
formation on $K_L \to \gamma\gamma\gamma$ together with the perturbative
behavior of the $s\bar{d} \to u\bar{u} \to \gamma\gamma$ loop [20]. Finally, SD and
LD produce the same $0^-$ state and thus interfere. This
interference, which depends on the sign of $A(K_L \to \gamma \gamma)$, is presumably constructive\(^{[21]}\). Better measurements of $K_S \to \pi^0 \gamma \gamma$ or $K^+ \to \pi^+ \gamma \gamma$ could settle this sign.

$K_L \to \mu^+ \mu^-$ is thus obviously not as clean as $K \to \pi \nu \bar{\nu}$ or $K_L \to \pi^0 \ell^+ \ell^-$. Nevertheless, being measured precisely, it can still lead to interesting constraints in some specific scenarios like SUSY at large $\tan \beta$\(^{[11]}\).

**V. CONCLUSION**

Thanks to the numerous theoretical efforts, the four rare decays, $K \to \pi \nu \bar{\nu}$ and $K_L \to \pi^0 \ell^+ \ell^-$, now provide for one of the cleanest and most sensitive tests of the Standard Model. These modes are promising not only to get clear signals of New Physics – or to severely constrain it –, but also to uncover the nature of the possible New Physics at play through the specific pattern of deviations they would exhibit with respect to the SM predictions.

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[1] See the status reports from CERN and KEK Collaborations, these proceedings.