On Feferman's operational set theory OST

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Abstract

We study OST and some of its most important extensions primarily from a proof-theoretic perspective, determine their consistency strengths by exhibiting equivalent systems in the realm of traditional set theory and introduce a new and interesting extension of OST which is conservative over ZFC.

Keywords: Operational set theory, explicit mathematics, proof theory, classical and constructive set theories

1 Introduction

In the seventies Feferman introduced explicit mathematics as an appropriate logical framework for formalizing Bishop-style constructive mathematics. However, soon it turned out that it also played an important role in reductive proof theory and as an axiomatic approach to abstract computability. The seminal paper Feferman [9] presents the general program of explicit mathematic and the famous theory T_0; Feferman [10] deals with and lays the foundations for later work about the connections between explicit mathematics and generalized recursion theory. In Feferman and Jäger [13] and Jäger and Strahm [19] the proof theory of the non-constructive μ-operator and the Suslin operator in an explicit context are studied; Jäger and Strahm [18, 20] deal with various forms of explicit reflections, in particular with Mahloness and analogues of Π_3 reflection.

It is evident from these publications that explicit mathematics has a strong set-theoretic flavor. Nevertheless, as far as precise formal systems are concerned, only a little has been done in this direction for quite some time. Beeson [7] presents an interesting computation system based on set theory, formulated as a theory of sets and rules. Feferman [11], the starting point of the following considerations, introduces the system OST of operational set theory, motivated by the aim to develop a common language for small large cardinal notions as in classical set theory, admissible set and recursion theory, constructive set theory, explicit mathematics, constructive type theory and recursive ordinal notation systems. This is achieved by "expanding the language of set theory to allow us to talk about general set-theoretical operations and formulating the large cardinal notions in question in terms of operational closure conditions; this is a partial adaptation of explicit mathematics notions to the set-theoretical framework" (quotation from [11]).

Cantini and Crosilla [8] is about a constructive set theory with operations COST, which may be considered as a constructive version of OST, and may be regarded as providing a bridge between Aczel's constructive set theory CZF, see, e.g., Aczel [1, 2, 3], and explicit mathematics. Finally, there is Feferman [12] in which variants of the systems of Feferman [11] are described closer in syntax to original explicit mathematics.

In the present article we study OST and some of its most important extensions primarily from a proof-theoretic perspective, determine their consistency strengths by exhibiting equivalent systems in the realm of traditional set theory and introduce a new and interesting extension of OST which is conservative over ZFC.

2 Feferman's theory OST

The theory OST is formulated in the language $\mathcal{L}^\circ$ which extends the usual language $\mathcal{L}$ of set theory by the possibility to treat all objects as operations and to apply them freely to each other. Actually, we will present a minor syntactic variant of Feferman's original formulation.

The language $\mathcal{L}$ is a typical language of admissible or classical set theory with symbols for the element and identity relations as its only relation symbols. In addition, we have set variables $a, b, c, f, g, u, v, w, x, y, z, \ldots$ (possibly with subscripts) and the constant ω for the first infinite ordinal. The only terms of $\mathcal{L}$ are the variables and the constant ω; the formulas of $\mathcal{L}$ are defined as usual.

$\mathcal{L}^\circ$, the language of OST, augments $\mathcal{L}$ by the unary relation symbol $\downarrow$ (defined), the binary function symbol $\circ$ for partial term application and the following constants: (i) the combinators k and s; (ii) $\top$, $\bot$, **el**, **non**, **dis** and **e** for logical operations; (iii) $\mathbb{S}$, $\mathbb{R}$ and $\mathbb{C}$ for set-theoretic operations. The meaning of these constants follows from the axioms below.

The *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. The variables and constants of $\mathcal{L}^\circ$ are terms of $\mathcal{L}^\circ$.

2. If s and t are terms of $\mathcal{L}^\circ$, then so is $\circ(s,t)$.

In the following we often abbreviate $\circ(s,t)$ as $(s \circ t)$, as (st) or – if no confusion arises – simply as st. We also adopt the convention of association to the left so that $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 s_2)\dots s_n)$. In addition, we often write $s(t_1,\dots,t_n)$ for $st_1 \dots t_n$ if this seems more intuitive. Moreover, we frequently make use of the vector notation $\vec{s}$ as shorthand for a finite string $s_1,\dots,s_n$ of $\mathcal{L}^\circ$ terms whose length is either not important or evident from the context.

As you can see, self-application is possible and meaningful, but it is not necessarily total, and there may be terms which do not denote an object. We make use of the definedness predicate $\downarrow$ to single out those which do, and $(t\downarrow)$ is read "t is defined" or "t has a value".

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \dots)$ of $\mathcal{L}^\circ$ are generated as follows:

1. All expressions of the form $(s \in t)$, $(s = t)$ and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$; the so-called *atomic* formulas.

2. If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$, $(A \lor B)$ and $(A \land B)$.

3. If A is a formula and t a term of $\mathcal{L}^\circ$ which does not contain x, then $(\exists x \in t)A$, $(\forall x \in t)A$, $\exists x A$ and $\forall x A$ are formulas of $\mathcal{L}^\circ$.

Since we will be working within classical logic, the remaining logical connectives can be defined as follows:

$$(A \to B) \ := \ (\neg A \lor B) \quad \text{and} \quad (A \leftrightarrow B) \ := \ ((A \to B) \land (B \to A)).$$

We will often omit parentheses and brackets whenever there is no danger of confusion. The free variables of t and A are defined in the conventional way; the closed $\mathcal{L}^\circ$ terms and closed $\mathcal{L}^\circ$ formulas, also called $\mathcal{L}^\circ$ sentences, are those which do not contain free variables.

Given an $\mathcal{L}^\circ$ formula A and a variable u not occurring in A, we write A^u for the result of replacing each unbounded set quantifier $\exists x(\dots)$ and $\forall x(\dots)$ in A by $(\exists x \in u)(\dots)$ and $(\forall x \in u)(\dots)$, respectively.

Suppose now that $\vec{u} = u_1,\dots,u_n$ and $\vec{s} = s_1,\dots,s_n$. Then $A[\vec{s}/\vec{u}]$ is the $\mathcal{L}^\circ$ formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the $\mathcal{L}^\circ$ terms $\vec{s}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. If the $\mathcal{L}^\circ$ formula A is written as $B[\vec{u}]$, then we often simply write $B[\vec{s}]$ instead of $B[\vec{s}/\vec{u}]$. Further variants of this notation will be obvious.

The logic of OST is the (classical) *logic of partial terms* due to Beeson [5, 6]; see also Troelstra and van Dalen [25], where $E(t)$ is written instead of $(t\downarrow)$. By the strictness axioms of this logic the formula $(s = t)$ implies that both, s and t, are defined. Partial equality of terms is introduced by

$$(s \simeq t) \;:=\; (s\downarrow \vee\, t\downarrow \;\rightarrow\; s = t)$$

and says that if either s or t denotes anything, then they both denote the same object.

The non-logical axioms of OST comprise axioms about the applicative structure of the universe, some basic set-theoretic properties, the representation of elementary logical connectives as operations and operational set existence axioms. They divide into four groups.

I. Applicative axioms.

(1) $\mathsf{k} \neq \mathsf{s}$,

(2) $\mathsf{k}xy = x$,

(3) $\mathsf{s}xy\downarrow \;\wedge\; \mathsf{s}xyz \simeq (xz)(yz)$.

Thus the universe is a partial combinatory algebra. We have λ-abstraction and thus can introduce for each $\mathcal{L}^\circ$ term t a term $\lambda x.t$ whose variables are those of t other than x such that

$$\lambda x.t\downarrow \;\wedge\; (\lambda x.t)y \simeq t[y/x].$$

Furthermore, there exists a closed $\mathcal{L}^\circ$ term fix, a so-called fixed point operator, with

$$\mathsf{fix}(f)\downarrow \;\wedge\; (\mathsf{fix}(f) = g \;\rightarrow\; gx \simeq f(g, x)).$$

II. Basic set-theoretic axioms. They state that: (i) there is the empty set; (ii) there are unordered pairs and unions; (iii) ω is the first infinite ordinal; (iv) all objects are extensional,

$$a = b \;\leftrightarrow\; \forall x(x \in a \;\leftrightarrow\; x \in b),$$

and (iv) $\in$-induction is available for arbitrary formulas $A[x]$ of $\mathcal{L}^\circ$,

$$(\mathcal{L}^\circ\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \;\rightarrow\; \forall x A[x].$$

To increase readability, we will freely use standard set-theoretic terminology from now on; for example

$$a \subset b \;:=\; (\forall x \in a)(x \in b) \quad \text{and} \quad \mathit{Tran}(a) \;:=\; (\forall x \in a)(x \subset a).$$

If $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ stands for the collection of all sets satisfying A; an expression of the form $\{x \in s : A[x]\}$ is used as shorthand for $\{x : x \in s \wedge A[x]\}$. A collection $\{x : A[x]\}$ may be (extensionally equal to) a set, but this is not necessarily the case. Hence some care is required when working with such expressions, and we define:

$$
\begin{aligned}
t \in \{x : A[x]\} &\;:=\; A(t), \\
t = \{x : A[x]\} &\;:=\; t\downarrow \;\wedge\; \forall x(x \in t \leftrightarrow A[x]).
\end{aligned}
$$

In particular, we set

$$\mathbb{B} \;:=\; \{x : x = \top \vee x = \bot\} \quad \text{and} \quad \mathbb{V} \;:=\; \{x : x\downarrow\}$$

so that $\mathbb{B}$ stands for the unordered pair consisting of the truth values $\top$ and $\bot$, which is a set by the previous axioms. $\mathbb{V}$ denotes the collection of all sets, but is not a set itself. The shorthand notations, for n an arbitrary natural number,

$$
\begin{aligned}
(f : a \to b) &\;:=\; (\forall x \in a)(fx \in b), \\
(f : a^{n+1} \to b) &\;:=\; (\forall x_1, \ldots, x_{n+1} \in a)(f(x_1, \ldots, x_{n+1}) \in b)
\end{aligned}
$$

express that f, in the operational sense, is a unary and $(n{+}1)$-ary mapping from a to b, respectively. They do not say, however, that f is a unary or $(n{+}1)$-ary function in the set-theoretic sense (see below).

In the previous definition the set variables a and/or b may be replaced by $\mathbb{V}$ and/or $\mathbb{B}$. So, for example, $(f : a \to \mathbb{V})$ means that f is total on a, and $(f : \mathbb{V} \to b)$ means that f maps all sets into b. If we have $(f : a \to \mathbb{B})$, we may regard f as a *definite predicate* on a. The n-ary *Boolean operations* are those f for which $(f : \mathbb{B}^n \to \mathbb{B})$.

III. Logical operations axioms.

(1) $\top \neq \bot$,

(2) $(\mathbf{el} : \mathbb{V}^2 \to \mathbb{B}) \;\wedge\; \forall x \forall y(\mathbf{el}(x, y) = \top \;\leftrightarrow\; x \in y)$,

(3) $(\mathbf{non} : \mathbb{B} \to \mathbb{B}) \;\wedge\; (\forall x \in \mathbb{B})(\mathbf{non}(x) = \top \;\leftrightarrow\; x = \bot)$,

(4) $(\mathbf{dis} : \mathbb{B}^2 \to \mathbb{B}) \;\wedge\; (\forall x, y \in \mathbb{B})(\mathbf{dis}(x, y) = \top \;\leftrightarrow\; (x = \top \vee y = \top))$,

(5) $(f : a \to \mathbb{B}) \;\to\; (\mathbf{e}(f, a) \in \mathbb{B} \;\wedge\; (\mathbf{e}(f, a) = \top \;\leftrightarrow\; (\exists x \in a)(fx = \top)))$.

The Δ_0 *formulas* of $\mathcal{L}^\circ$ are those $\mathcal{L}^\circ$ formulas which do not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers. Hence they are the usual Δ_0 formulas of set theory, possibly containing additional constants. The logical operations make it possible to represent all Δ_0 formulas by constant $\mathcal{L}^\circ$ terms.

Lemma 1 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every Δ_0 formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that the axioms introduced so far yield*

$$t_A\downarrow \;\wedge\; (t_A : \mathbb{V}^n \to \mathbb{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t_A(\vec{x}) = \top).$$

This result is also mentioned in Feferman [11]; its proof is straightforward and left to the reader. After having introduced the final group of axioms of OST, we will also formulate a representation property concerning a wider class of $\mathcal{L}^\circ$ formulas; see Lemma 3 below.

IV. Operational set-theoretic axioms.

(1) Separation for definite operations:

$$(f : a \to \mathbb{B}) \;\to\; (\mathbb{S}(f,a)\downarrow \;\wedge\; \forall x(x \in \mathbb{S}(f,a) \;\leftrightarrow\; (x \in a \;\wedge\; fx = \top))).$$

(2) Replacement:

$$(f : a \to \mathbb{V}) \;\to\; (\mathbb{R}(f,a)\downarrow \;\wedge\; \forall x(x \in \mathbb{R}(f,a) \;\leftrightarrow\; (\exists y \in a)(x = fy))).$$

(3) Choice:

$$\exists x(fx = \top) \;\to\; (\mathbb{C}f\downarrow \;\wedge\; f(\mathbb{C}f) = \top).$$

This finishes the description of the non-logical axioms of OST. A significant strengthening of OST is obtained by adding the operational form of the power set axiom. To do so, we extend $\mathcal{L}^\circ$ to the language $\mathcal{L}^\circ(\mathbb{P})$ by introducing the fresh constant $\mathbb{P}$ and add the axiom

$$(\mathbb{P}) \qquad (\mathbb{P} : \mathbb{V} \to \mathbb{V}) \;\wedge\; \forall x \forall y(x \in \mathbb{P}y \;\leftrightarrow\; x \subset y).$$

Accordingly, $\mathsf{OST}(\mathbb{P})$ is the operational set theory which comprises the axioms of OST plus operational power set $(\mathbb{P})$, everything formulated for the language $\mathcal{L}^\circ(\mathbb{P})$.

Definition 2 *The* $e\Sigma$ formulas *of* $\mathcal{L}^\circ(\mathbb{P})$ *are inductively defined as follows:*

1. *If s and t are $\mathcal{L}^\circ(\mathbb{P})$ terms, then $(s \in t)$, $(s = t)$ and $(t\downarrow)$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.*

2. *If s and t are variables or constants, then $(s \notin t)$ and $(s \neq t)$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.*

3. *If A and B are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$, then so are $(A \vee B)$ and $(A \wedge B)$.*

4. *If A is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$ and t a term of $\mathcal{L}^\circ(\mathbb{P})$ which does not contain x, then $(\exists x \in t)A$ and $\exists x A$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.*

5. *If A is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$ and t a constant or a variable other than x, then $(\forall x \in t)A$ is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$.*

The $e\Sigma$ formulas of $\mathcal{L}^\circ$ are exactly the $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$ in which the constant $\mathbb{P}$ does not occur.

Hence the $e\Sigma$ formulas, i.e. the extended Σ formulas, of $\mathcal{L}^\circ$ and $\mathcal{L}^\circ(\mathbb{P})$ are as the Σ formulas of set theory with positive occurrences of arbitrary $\mathcal{L}^\circ$ terms respectively $\mathcal{L}^\circ(\mathbb{P})$ terms permitted as well. They can be represented in OST and $\mathsf{OST}(\mathbb{P})$, but only in a form weaker than the Δ_0 formulas.

Lemma 3 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every $e\Sigma$ formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that OST proves*

$$t_A\downarrow \;\wedge\; \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

Clearly, we also have the analogous result for the $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$ and the theory $\mathsf{OST}(\mathbb{P})$.

The proof of this lemma can be easily reconstructed from Feferman [11]. Together with the set-theoretic axiom of OST it immediately implies the following corollary, also taken from [11].

Corollary 4 *There exist closed $\mathcal{L}^\circ$ terms $\emptyset$ for the empty set,* **uopa** *for forming unordered pairs,* **un** *for forming unions,* **p** *for forming ordered pairs and* **prod** *for forming Cartesian products. In addition, there are closed $\mathcal{L}^\circ$ terms $\mathbf{p}_L$ and $\mathbf{p}_R$ which act as projections with respect to* **p***, i.e.*

$$\mathbf{p}_L(\mathbf{p}(a,b)) = a \quad \textit{and} \quad \mathbf{p}_R(\mathbf{p}(a,b)) = b.$$

To comply with the set-theoretic conventions, we generally write $\{a, b\}$ instead of $\mathbf{uopa}(a, b)$, $\cup a$ instead of $\mathbf{un}(a)$, $\langle a, b\rangle$ instead of $\mathbf{p}(a, b)$ and $a \times b$ instead of $\mathbf{prod}(a, b)$. Remember that ω is a constant for the first infinite ordinal and belongs to the base language $\mathcal{L}$.

We end this section with a few remarks concerning the relationship between functions in the set-theoretic sense and operations in the sense of our form of term application. Similar questions for similar operational set theories are also discussed in Beeson [7] and in Cantini and Crosilla [8].

It is well-known (see, for example, Barwise [4]) that there are Δ_0 formulas $Rel(a)$ and $Fun(a)$ of our basic language $\mathcal{L}$, stating that the set a is a binary relation and function, respectively, in the typical set-theoretic sense. It can also be expressed in Δ_0 form that a is a relation with domain b, abbreviated as $Dom(a) = b$, and that a is a relation with range b, abbreviated as $Ran(a) = b$. If $Fun(a)$ holds and u belongs to the domain of a we write $a\,'u$ for the unique v such that $\langle u, v\rangle \in a$.

Lemma 5 *There exist closed $\mathcal{L}^\circ$ terms* **dom**, **ran**, **op** *and* **fun** *so that* OST *proves the following assertions:*

1. $\mathbf{dom}(f){\downarrow} \;\wedge\; \mathbf{ran}(f){\downarrow} \;\wedge\; \mathbf{op}(f){\downarrow}$.
2. $Rel(a) \;\rightarrow\; (Dom(a) = \mathbf{dom}(a) \;\wedge\; Ran(a) = \mathbf{ran}(a))$.
3. $(Fun(f) \;\wedge\; a \in \mathbf{dom}(f)) \;\rightarrow\; f\,'a = \mathbf{op}(f, a)$.
4. $(f : a \to \mathbb{V}) \;\rightarrow\; (Fun(\mathbf{fun}(f, a)) \;\wedge\; \mathbf{dom}(\mathbf{fun}(f, a)) = a)$.
5. $(f : a \to \mathbb{V}) \;\rightarrow\; (\forall x \in a)(\mathbf{fun}(f, a)\,'x = fx)$.

This lemma, whose proof can also be found in Feferman [11], implies that: (i) each set-theoretic function can be translated into an operation acting on the same domain and yielding the same values; (ii) to each operation total on a set a corresponds a set-theoretic function with domain a so that the values of this operation and of this function on a agree.

3 The consistency strength of OST

We plan to determine the consistency strengths of the operational set theory OST by relating it to well-known systems of admissible set theory. We start off from Kripke-Platek set theory plus infinity, hereinafter called $\mathsf{KP}\omega$, and then add the axiom of constructibility. For further reading about $\mathsf{KP}\omega$, its proof-theoretic analysis and some interesting subsystems and extensions consult, for example, Jäger [15, 16] and Rathjen [22].

$\mathsf{KP}\omega$ is formulated in our basic language $\mathcal{L}$, its underlying logic is classical first order logic with equality, and its non-logical axioms are: extensionality, pair, union, infinity (i.e. the assertion that ω is the least infinite ordinal), $\in$-induction for arbitrary formulas $A[x]$ of $\mathcal{L}$,

$$(\mathcal{L}\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \;\rightarrow\; \forall x A[x],$$

as well as Δ_0 separation and Δ_0 collection, i.e.

$$(\Delta_0\text{-}\mathsf{Sep}) \qquad \exists x(x = \{y \in a : B[y]\}),$$

$$(\Delta_0\text{-}\mathsf{Col}) \qquad (\forall x \in a)\exists y C[x, y] \rightarrow \exists z(\forall x \in a)(\exists y \in z)C[x, y]$$

for arbitrary Δ_0 formulas $B[u]$ and $C[u, v]$ of $\mathcal{L}$. The theory $\mathsf{KP}\omega + (\mathsf{AC})$ is the extension of $\mathsf{KP}\omega$ obtained by adding, for each parameter a, the axiom of choice

$$(\mathsf{AC}) \quad (\forall x \in a)(x \neq \emptyset) \rightarrow \exists f(\mathit{Fun}(f) \wedge \mathit{Dom}(f) = a \wedge (\forall x \in a)(f\,'x \in x)).$$

The language of $\mathsf{KP}\omega + (\mathsf{AC})$ is a sublanguage of the language of OST, and it is easy to see that OST proves all axioms of $\mathsf{KP}\omega + (\mathsf{AC})$. Hence $\mathsf{KP}\omega + (\mathsf{AC})$ is a subsystem of OST, as has already been remarked in Feferman [11].

Theorem 6 *The theory* $\mathsf{KP}\omega + (\mathsf{AC})$ *is contained in* OST.

Proof. Clearly, the axioms of $\mathsf{KP}\omega$ about extensionality, the existence of pairs and unions, infinity and $\in$-induction are provable in OST. Each instance of Δ_0 separation is a direct consequence of Lemma 1 and operational separation.

To deal with Δ_0 collection, let $A[\vec{u}, v, w]$ be a Δ_0 formula of $\mathcal{L}$ with at most the variables $\vec{u}, v, w$ free and suppose that $\vec{u}$ is a sequence of length n. We work informally in OST and assume that

$$(1) \qquad (\forall x \in a)\exists y A[\vec{u}, x, y].$$

In view of Lemma 1 we know that there is a closed $\mathcal{L}^\circ$ term t_A so that

$$(2) \qquad t_A\!\downarrow \;\wedge\; (t_A : \mathbb{V}^{n+2} \rightarrow \mathbb{B}) \;\wedge\; \forall\vec{u}\forall v\forall w(A[\vec{u}, v, w] \leftrightarrow t_A(\vec{u}, v, w) = \top).$$

Thus from (1) and (2) we immediately obtain

$$(\forall x \in a)\exists y(t_A(\vec{u}, x, y) = \top),$$

therefore our operational set-theoretic axiom about choice implies

$$(3) \qquad (\forall x \in a)(\mathbb{C}(\lambda y.t_A(\vec{u}, x, y))\!\downarrow \;\wedge\; t_A(\vec{u}, x, \mathbb{C}(\lambda y.t_A(\vec{u}, x, y))) = \top).$$

Assertions (2) and (3) thus yield

$$(4) \qquad (\forall x \in a)(\mathbb{C}(\lambda y.t_A(\vec{u}, x, y))\!\downarrow \;\wedge\; A[\vec{u}, x, \mathbb{C}(\lambda y.t_A(\vec{u}, x, y))])$$

from which we also deduce that

$$(\lambda x.\mathbb{C}(\lambda y.t_A(\vec{u}, x, y)) : a \to \mathbb{V}).$$

Finally, we apply operational replacement and therefore know

$$\mathbb{R}(\lambda x.\mathbb{C}(\lambda y.t_A(\vec{u}, x, y)), a) \downarrow, \tag{5}$$

$$(\forall x \in a)(\mathbb{C}(\lambda y.t_A(\vec{u}, x, y)) \in \mathbb{R}(\lambda x.\mathbb{C}(\lambda y.t_A(\vec{u}, x, y)), a). \tag{6}$$

Choosing $\mathbb{R}(\lambda x.\mathbb{C}(\lambda y.t_A(\vec{u}, x, y)), a)$, which has a value according to (5), as a witness for z, we have thus shown, in view of (4) and (6), that

$$\exists z(\forall x \in a)(\exists y \in z)A[\vec{u}, x, y],$$

and Δ_0 collection is validated. Now we consider (AC) and pick a set a whose elements are non-empty,

$$(\forall x \in a)(x \neq \emptyset).$$

Trivially, this line can be rewritten as

$$(\forall x \in a)\exists y((\lambda z.\mathbf{el}(z, x))y = \top).$$

Hence operational choice yields

$$(\forall x \in a)(\mathbb{C}(\lambda z.\mathbf{el}(z, x))\downarrow \ \wedge \ (\lambda z.\mathbf{el}(z, x))(\mathbb{C}(\lambda z.\mathbf{el}(z, x))) = \top)$$

and thus, after some obvious modifications,

$$(\forall x \in a)(\mathbb{C}(\lambda z.\mathbf{el}(z, x))\downarrow \ \wedge \ \mathbb{C}(\lambda z.\mathbf{el}(z, x)) \in x).$$

This means that $\lambda x.\mathbb{C}(\lambda z.\mathbf{el}(z, x))$ is an operation total on a, mapping each element x of a to an element of x. By Lemma 5 we therefore can be sure that there exists a set-theoretic function f so that $Dom(f) = a$ and $f\,'x \in x$ for all elements x of a. This establishes (AC) and finishes the proof of our theorem. $\square$

Now we turn to the upper bound of the proof-theoretic strength of OST. The collections of Σ and Π formulas of $\mathcal{L}$ are defined canonically, and an $\mathcal{L}$ formula A is Δ over $\mathsf{KP}\omega$ provided that for some Σ formula B of $\mathcal{L}$ and some Π formula C of $\mathcal{L}$, both with exactly the same free variables as A, $\mathsf{KP}\omega$ proves the equivalence of A, B and C.

We will embed OST into the theory $\mathsf{KP}\omega + (V{=}L)$, which is the extension of $\mathsf{KP}\omega$ by the famous *axiom of constructibility*; this axiom will allow us to cope with operational choice. The crucial part of this embedding is the interpretation of the application relation ($ab \simeq c$) by means of a suitable Σ predicate which will be gained via definition by Σ recursion.

Feferman suggests in [11] to interpret the applicative structure of OST in the codes for Σ_1 definable functions, obtained by uniformizing the Σ_1 predicates. Here we choose a different route and provide a direct inductive definition of the application operation. Apart from being more direct, this way of reducing OST to $\mathsf{KP}\omega + (V{=}L)$ has the advantage that it can be directly adapted, see Section 6.1, to dealing with a strong extension of OST.

Ordinals are defined in $\mathsf{KP}\omega$ by a Δ_0 formula $Ord(a)$ of $\mathcal{L}$. We use lower case Greek letters $\alpha, \beta, \gamma, \delta \ldots$ (possibly with subscripts) for ordinals and write $(\alpha < \beta)$ for $(\alpha \in \beta)$. Furthermore, $(a \in L_\alpha)$ states that the set a is an element of the αth level L_α of the constructible hierarchy, and $(a <_L b)$ means that a is smaller than b according to the well-ordering $<_L$ on L. It is well-known that the assertions $(a \in L_\alpha)$ and $(a <_L b)$ are Δ over $\mathsf{KP}\omega$; see, e.g., Barwise [4] or Kunen [21].

The following approach is motivated by the one in Feferman and Jäger [13] and Jäger and Strahm [18] and begins with some notational preparations. For any natural number n greater than 0 we select (i) a Δ_0 formula $Tup_n(a)$ formalizing that a is an ordered n-tuple and (ii) a Δ_0 formula $(a)_i = b$ formalizing that b the projection of a on its ith component, $i \leq n$, so that

$$Tup_n(a) \,\land\, (a)_1 = b_1 \,\land\, \ldots \,\land\, (a)_n = b_n \;\rightarrow\; a = \langle b_1, \ldots, b_n \rangle.$$

Then we fix pairwise different sets $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\top}$, $\widehat{\bot}$, $\widehat{\mathbf{el}}$, $\widehat{\mathbf{non}}$, $\widehat{\mathbf{dis}}$, $\widehat{\mathbf{e}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, $\widehat{\mathbb{C}}$ and for later use (see Section 4) a further set $\widehat{\mathbb{P}}$ which all do not belong to the collection of ordered pairs and triples; they will later act as the codes of the corresponding constants of $\mathcal{L}^\circ$ and $\mathcal{L}^\circ(\mathbb{P})$, respectively. We are going to code the $\mathcal{L}^\circ$ terms $\mathsf{k}x$, $\mathsf{s}x$, $\mathsf{s}xy$, $\ldots$ by the ordered tuples $\langle \widehat{\mathsf{k}}, x \rangle$, $\langle \widehat{\mathsf{s}}, x \rangle$, $\langle \widehat{\mathsf{s}}, x, y \rangle$, $\ldots$ of the corresponding form. For example, to satisfy $\mathsf{k}xy = x$ we interpret $\mathsf{k}x$ as $\langle \widehat{\mathsf{k}}, x \rangle$, and "$\langle \widehat{\mathsf{k}}, x \rangle$ applied to y" is taken to be x.

Next let R be a fresh 4-place relation symbol and extend $\mathcal{L}$ to the language $\mathcal{L}(R)$ with expressions $R(\alpha, a, b, c)$ as additional atomic formulas. We also abbreviate

$$R^{<\alpha}(a, b, c) \;:=\; (\exists \beta < \alpha) R(\beta, a, b, c).$$

For finding the required interpretation of the application operation of OST within $\mathsf{KP}\omega + (V{=}L)$ we work with a specific $\mathcal{L}(R)$ formula, introduced in the following definition. Afterwards, this formula together with Σ recursion will help to provide what we need.

Definition 7 *We choose* $\mathfrak{A}[R,\alpha,a,b,c]$ *to be the* $\mathcal{L}(R)$ *formula defined as*

$$\mathfrak{A}[R,\alpha,a,b,c] \quad := \quad c \in L_\alpha \ \wedge \ \mathfrak{B}[R,\alpha,a,b,c],$$

where $\mathfrak{B}[R,\alpha,a,b,c]$ *is an auxiliary* $\mathcal{L}(R)$ *formula given as the disjunction of the following clauses:*

(1) $a = \widehat{\mathsf{k}} \ \wedge \ c = \langle \widehat{\mathsf{k}}, b \rangle$,

(2) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathsf{k}} \ \wedge \ (a)_2 = c$,

(3) $a = \widehat{\mathsf{s}} \ \wedge \ c = \langle \widehat{\mathsf{s}}, b \rangle$,

(4) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathsf{s}} \ \wedge \ c = \langle \widehat{\mathsf{s}}, (a)_2, b \rangle$,

(5) $Tup_3(a) \ \wedge \ (a)_1 = \widehat{\mathsf{s}} \ \wedge$
$$(\exists x, y \in L_\alpha)(R^{<\alpha}((a)_2, b, x) \ \wedge \ R^{<\alpha}((a)_3, b, y) \ \wedge \ R^{<\alpha}(x, y, c)),$$

(6) $a = \widehat{\mathbf{el}} \ \wedge \ c = \langle \widehat{\mathbf{el}}, b \rangle$,

(7) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{el}} \ \wedge \ (a)_2 \in b \ \wedge \ c = \widehat{\top}$,

(8) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{el}} \ \wedge \ (a)_2 \notin b \ \wedge \ c = \widehat{\bot}$,

(9) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\bot}$,

(10) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\top}$,

(11) $a = \widehat{\mathbf{dis}} \ \wedge \ c = \langle \widehat{\mathbf{dis}}, b \rangle$,

(12) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(13) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\bot} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(14) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\bot} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\bot}$,

(15) $a = \widehat{\mathbf{e}} \ \wedge \ c = \langle \widehat{\mathbf{e}}, b \rangle$,

(16) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{e}} \ \wedge \ (\exists x \in b) R^{<\alpha}((a)_2, x, \widehat{\top}) \ \wedge \ c = \widehat{\top}$,

(17) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{e}} \ \wedge \ (\forall x \in b) R^{<\alpha}((a)_2, x, \widehat{\bot}) \ \wedge \ c = \widehat{\bot}$,

(18) $a = \widehat{\mathbb{S}} \ \wedge \ c = \langle \widehat{\mathbb{S}}, b \rangle$,

(19) $Tup_2(a) \land (a)_1 = \widehat{\mathbb{S}} \land (\forall x \in b)(R^{<\alpha}((a)_2, x, \widehat{\top}) \lor R^{<\alpha}((a)_2, x, \widehat{\bot})) \land$

$(\forall x \in c)(x \in b \land R^{<\alpha}((a)_2, x, \widehat{\top})) \land$

$(\forall x \in b)(R^{<\alpha}((a)_2, x, \widehat{\top}) \to x \in c)$,

(20) $a = \widehat{\mathbb{R}} \land c = \langle \widehat{\mathbb{R}}, b \rangle$,

(21) $Tup_2(a) \land (a)_1 = \widehat{\mathbb{R}} \land (\forall x \in b)(\exists y \in c) R^{<\alpha}((a)_2, x, y) \land$

$(\forall y \in c)(\exists x \in b) R^{<\alpha}((a)_2, x, y)$,

(22) $a = \widehat{\mathbb{C}} \land R^{<\alpha}(b, c, \widehat{\top}) \land (\forall x \in L_\alpha)(x <_L c \to \neg R^{<\alpha}(b, x, \widehat{\top})) \land$

$(\forall \beta < \alpha)(\forall x \in L_\beta) \neg R^{<\beta}(b, x, \widehat{\top})$.

We immediately see that $\mathfrak{A}[R, \alpha, a, b, c]$ is Δ over $\mathsf{KP}\omega$ with respect to the language $\mathcal{L}(R)$. It is also easy ro verify that $\mathfrak{A}[R, \alpha, a, b, c]$ is deterministic in the following sense: from $\mathfrak{A}[R, \alpha, a, b, c]$ we can conclude that exactly one of the clauses (1)–(22) of the previous definition is satisfied for these α, a, b and c.

For any $\mathcal{L}$ formula $B[\alpha, a, b, c]$ with at most the indicated free variables we write $\mathfrak{A}[B, \alpha, a, b, c]$ for the $\mathcal{L}$ formula resulting by replacing each occurrence of an atomic formula of the form $R(\alpha, r, s, t)$ in $\mathfrak{A}[R, \alpha, a, b, c]$ by $B[\alpha, r, s, t]$. The following theorem is a special case of "Definition by Σ Recursion" as developed in Barwise [4].

Theorem 8 *There exists a Σ formula $B[\alpha, a, b, c]$ of $\mathcal{L}$ with at most α, a, b and c free so that $\mathsf{KP}\omega$ proves*

$$(\Sigma\text{-}\mathsf{Rec}/\mathfrak{A}) \qquad B[\alpha, a, b, c] \leftrightarrow \mathfrak{A}[B, \alpha, a, b, c].$$

Any such a formula $B[\alpha, a, b, c]$ may be used to describe the αth level of the interpretation of the OST application $(ab \simeq c)$. Accordingly, we proceed as follows.

Definition 9 *Let $B_{\mathfrak{A}}[\alpha, a, b, c]$ be a Σ formula of $\mathcal{L}$ associated to the operator form $\mathfrak{A}[R, \alpha, a, b, c]$ according to (Σ-$\mathsf{Rec}/\mathfrak{A}$) of the previous theorem. Then we define*

$$\begin{aligned} B^{<\alpha}_{\mathfrak{A}}[a, b, c] &:= (\exists \beta < \alpha) B_{\mathfrak{A}}[\beta, a, b, c], \\ Ap_{\mathfrak{A}}[a, b, c] &:= \exists \alpha B_{\mathfrak{A}}[\alpha, a, b, c]. \end{aligned}$$

We continue with showing that $Ap_{\mathfrak{A}}[a,b,c]$ is functional in its third argument. The next lemma takes care of the only critical case in the proof of this property and motivates the rather complicated clause (22) of Definition 7 above.

Lemma 10 *We can prove in* $\mathsf{KP}\omega$ *that*

$$B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a] \wedge B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b] \;\rightarrow\; \alpha = \beta \wedge a = b.$$

PROOF. We work informally in $\mathsf{KP}\omega$ and assume, without loss of generality, that $\alpha \leq \beta$. From the left hand side of the claimed assertion we obtain:

$$(1) \qquad a \in L_\alpha \wedge b \in L_\beta,$$

$$(2) \qquad B_{\mathfrak{A}}^{<\alpha}[f, a, \widehat{\top}] \wedge B_{\mathfrak{A}}^{<\beta}[f, b, \widehat{\top}],$$

$$(3) \qquad (\forall x \in L_\alpha)(x <_L a \rightarrow \neg B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]),$$

$$(4) \qquad (\forall x \in L_\beta)(x <_L b \rightarrow \neg B_{\mathfrak{A}}^{<\beta}[f, x, \widehat{\top}]),$$

$$(5) \qquad (\forall \gamma < \alpha)(\forall x \in L_\gamma)\neg B_{\mathfrak{A}}^{<\gamma}[f, x, \widehat{\top}]),$$

$$(6) \qquad (\forall \gamma < \beta)(\forall x \in L_\gamma)\neg B_{\mathfrak{A}}^{<\gamma}[f, x, \widehat{\top}]).$$

From (1), (2), (5) and (6) we conclude $\alpha = \beta$. But then (1) – (4) immediately imply that the sets a and b have to be identical as well. □

Lemma 11 *We can prove in* $\mathsf{KP}\omega$*:*

1. $B_{\mathfrak{A}}^{<\alpha}[a, b, u] \wedge B_{\mathfrak{A}}^{<\alpha}[a, b, v] \;\rightarrow\; u = v.$
2. $Ap_{\mathfrak{A}}[a, b, u] \wedge Ap_{\mathfrak{A}}[a, b, v] \;\rightarrow\; u = v.$

PROOF. Since the previous lemma is at our disposal, the first assertion is easily proved by induction on α. The second assertion is a straightforward consequence of the first. □

The embedding of OST into $\mathsf{KP}\omega + (V{=}L)$ first requires to deal with the terms of $\mathcal{L}^\circ$. This is achieved by associating to each term t of $\mathcal{L}^\circ$ a formula $[\![t]\!]_{\mathfrak{A}}(u)$ of $\mathcal{L}$ expressing that u is the value of t under the interpretation of the OST-application via the Σ formula $Ap_{\mathfrak{A}}$.

Definition 12 *For each* $\mathcal{L}^\circ$ *term* t *we introduce an* $\mathcal{L}$ *formula* $[\![t]\!]_{\mathfrak{A}}(u)$*, with* u *not occurring in* t*, which is inductively defined as follows:*

1. *If t is a variable or the constant ω, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(t = u)$.*
2. *If t is another constant, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(\widehat{t} = u)$.*
3. *If t is the term (rs), then we set*

$$[\![t]\!]_{\mathfrak{A}}(u) \;:=\; \exists x \exists y([\![r]\!]_{\mathfrak{A}}(x) \,\wedge\, [\![s]\!]_{\mathfrak{A}}(y) \,\wedge\, Ap_{\mathfrak{A}}[x,y,u]).$$

Observe that for every term t of $\mathcal{L}^\circ$ its translation $[\![t]\!]_{\mathfrak{A}}(u)$ is a Σ formula of $\mathcal{L}$. By this treatment of the terms of $\mathcal{L}^\circ$, the translation of arbitrary formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}$ is predetermined.

Definition 13 *The translation of an $\mathcal{L}^\circ$ formula A into the $\mathcal{L}$ formula A^* is inductively defined as follows:*

1. *For the atomic formulas of $\mathcal{L}^\circ$ we stipulate*

$$\begin{aligned}
(t{\downarrow})^* &\;:=\; \exists x [\![t]\!]_{\mathfrak{A}}(x),\\
(s \in t)^* &\;:=\; \exists x \exists y([\![s]\!]_{\mathfrak{A}}(x) \,\wedge\, [\![t]\!]_{\mathfrak{A}}(y) \,\wedge\, x \in y),\\
(s = t)^* &\;:=\; \exists x \exists y([\![s]\!]_{\mathfrak{A}}(x) \,\wedge\, [\![t]\!]_{\mathfrak{A}}(y) \,\wedge\, x = y).
\end{aligned}$$

2. *If A is a formula $\neg B$, then A^* is $\neg B^*$.*
3. *If A is a formula $(B \Diamond C)$ for $\Diamond$ being the binary junctor $\vee$ or $\wedge$, then A^* is $(B^* \Diamond C^*)$.*
4. *If A is a formula $(\exists x \in t)B[x]$, then*

$$A^* \;:=\; \exists y([\![t]\!]_{\mathfrak{A}}(y) \,\wedge\, (\exists x \in y)B^*[x]).$$

5. *If A is a formula $(\forall x \in t)B[x]$, then*

$$A^* \;:=\; \forall y([\![t]\!]_{\mathfrak{A}}(y) \,\rightarrow\, (\forall x \in y)B^*[x]).$$

6. *If A is a formula $QxB[x]$ for a quantifier Q, then A^* is $QxB^*[x]$.*

It is an easy exercise to check that the translations of the axioms of the logic of partial terms are provable in $\mathsf{KP}\omega + (V{=}L)$. The following lemma states the same for all the mathematical axioms of OST.

Lemma 14 *For every axiom A of OST we have*

$$\mathsf{KP}\omega + (V{=}L) \;\vdash\; A^*.$$

Proof. All basic set-theoretic axioms of OST are not affected by this translation and are available in $\mathsf{KP}\omega + (V{=}L)$ as well. Regarding all other axioms of OST, the definition of $\mathfrak{A}[R, \alpha, a, b, c]$ has been tailored so that this lemma goes through. This is more or less trivial for all applicative axioms and the logical operations axioms (1) – (4). To handle the remaining axioms, i.e. bounded existential quantification and all operational set-theoretic axioms, we work informally in $\mathsf{KP}\omega + (V{=}L)$ and treat them separately.

1. Bounded existential quantification. Its premise $(f : a \to \mathbb{B})$, as a formulation in OST, translates into

$$(\forall x \in a)(Ap_{\mathfrak{A}}[f, x, \widehat{\top}] \vee Ap_{\mathfrak{A}}[f, x, \widehat{\bot}]), \tag{1}$$

and by Σ reflection there must be an ordinal α such that

$$(\forall x \in a)(B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}] \;\vee\; B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\bot}]). \tag{2}$$

A first consequence of this assertion and Lemma 11 is that

$$(\forall x \in a)(Ap_{\mathfrak{A}}[f, x, \widehat{\top}] \;\leftrightarrow\; B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]), \tag{3}$$

and in view of Definition 7 – in fact its clauses (16) and (17) – assertion (2) also implies

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \;\vee\; \mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\bot}], \tag{4}$$

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \;\leftrightarrow\; (\exists x \in a) B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]. \tag{5}$$

Together with Theorem 8 and Lemma 11 we now conclude from (3) – (5) that

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \;\vee\; Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\bot}], \tag{6}$$

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \;\leftrightarrow\; (\exists x \in a) Ap_{\mathfrak{A}}[f, x, \widehat{\top}]. \tag{7}$$

But now the previous lines (6) and (7) mean nothing other than

$$[\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\top}) \;\vee\; [\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\bot}), \tag{8}$$

$$[\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\top}) \;\leftrightarrow\; (\exists x \in a) Ap_{\mathfrak{A}}[f, x, \widehat{\top}]. \tag{9}$$

Hence we have shown in $\mathsf{KP}\omega + (V{=}L)$ that (1) implies (8) and (9). However, this implication is the translation of the OST axiom about bounded existential quantification, which is thus proved in $\mathsf{KP}\omega + (V{=}L)$.

2. Operational separation for definite operations. As in the previous case we deduce from the translation of the left hand side ($f : a \to \mathbb{B}$) of the respective axiom of OST that

$$(\forall x \in a)(Ap_{\mathfrak{A}}[f, x, \widehat{\top}] \vee Ap_{\mathfrak{A}}[f, x, \widehat{\bot}]). \tag{10}$$

By Σ reflection there exists a set b such that

$$(\forall x \in a)(Ap^b_{\mathfrak{A}}[f, x, \widehat{\top}] \vee Ap^b_{\mathfrak{A}}[f, x, \widehat{\bot}]), \tag{11}$$

and, using Δ_0 separation, we can introduce a set c satisfying

$$\forall x(x \in c \;\leftrightarrow\; x \in a \wedge Ap^b_{\mathfrak{A}}[f, x, \widehat{\top}]). \tag{12}$$

In the next step we select an ordinal α which is so that a, b and c belong to L_α. Having done that, it is easily checked that (11), (12), Σ persistence and Lemma 11 yield

$$c \;=\; \{x \in a : B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\top}]\} \;=\; \{x \in a : Ap_{\mathfrak{A}}[f, x, \widehat{\top}]\}, \tag{13}$$

$$(\forall x \in a)(B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\top}] \vee B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\bot}]). \tag{14}$$

Looking back at Definition 7 – clause (19) – we see that (13) and (14) imply

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbb{S}}, f \rangle, a, c].$$

Making use of Theorem 8 once more, it is immediately clear that the previous assertion leads to

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbb{S}}, f \rangle, a, c]. \tag{15}$$

Now we recollect Lemma 11 and deduce from (13) and (15) that

$$\exists y [\![\mathbb{S}(f, a)]\!]_{\mathfrak{A}}(y), \tag{16}$$

$$\forall x(\exists y([\![\mathbb{S}(f, a)]\!]_{\mathfrak{A}}(y) \wedge x \in y) \;\leftrightarrow\; x \in a \wedge Ap_{\mathfrak{A}}[f, x, \widehat{\top}]). \tag{17}$$

These two statements corresponds to (the translation of) the conclusion of separation for definite operations. As we have just seen, (10) implies (16) and (17), provably in $\mathsf{KP}\omega{+}(V{=}L)$. Hence also the OST axiom about operational separation is established in $\mathsf{KP}\omega + (V{=}L)$.

3. Operational replacement. The premise of such an OST axiom is of the form ($f : a \to \mathbb{V}$), and so its translation into $\mathcal{L}$ gives

$$(\forall x \in a)\exists y Ap_{\mathfrak{A}}[f, x, y]. \tag{18}$$

Hence, because of Σ reflection, there is a set b satisfying

$$(\forall x \in a)(\exists y \in b) Ap^b_{\mathfrak{A}}[f, x, y]. \tag{19}$$

We apply Δ_0 separation to find a set c such that

$$\forall y (y \in c \leftrightarrow y \in b \wedge (\exists x \in a) Ap^b_{\mathfrak{A}}[f, x, y]) \tag{20}$$

and afterwards select some ordinal α big enough for a, b and c being elements of L_α. Because of Σ persistency and Lemma 11 we can deduce from (19) and (20) that

$$c = \{y \in b : (\exists x \in a) B^{<\alpha}_{\mathfrak{A}}[f, x, y]\} = \{y : (\exists x \in a) Ap_{\mathfrak{A}}[f, x, y]\}, \tag{21}$$

$$(\forall x \in a)(\exists y \in c) B^{<\alpha}_{\mathfrak{A}}[f, x, y] \wedge (\forall y \in c)(\exists x \in a) B^{<\alpha}_{\mathfrak{A}}[f, x, y]. \tag{22}$$

By clause (20) of the form of the operator form $\mathfrak{A}$, which has been introduced in Definition 7, we immediately obtain

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbb{R}}, f \rangle, a, c]$$

from (22). As above, by means of Theorem 8, the previous assertion yields

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbb{R}}, f \rangle, a, c]. \tag{23}$$

To finish this case, it only remains to verify that, in view of Lemma 11, assertions (21) and (23) give us

$$\exists z [\![\mathbb{R}(f, a)]\!]_{\mathfrak{A}}(z), \tag{24}$$

$$\forall y (\exists z ([\![\mathbb{R}(f, a)]\!]_{\mathfrak{A}}(z) \wedge y \in z) \leftrightarrow (\exists x \in a) Ap_{\mathfrak{A}}[f, x, y]). \tag{25}$$

As in the previous two cases we have thus shown that $\mathsf{KP}\omega + (V{=}L)$ proves an implication, namely the implication from (17) to (24) and (25). Since that is the translation of the OST axiom about operational reflection, $\mathsf{KP}\omega + (V{=}L)$ is able to deal with this principle as well.

4. Operational choice. To deal with that, we start off from the OST statement $\exists x (fx = \top)$, which translates into $\mathcal{L}$ as

$$\exists x Ap_{\mathfrak{A}}[f, x, \widehat{\top}]. \tag{26}$$

Since $\in$-induction is available in $\mathsf{KP}\omega + (V{=}L)$, statement (26) implies that there is a least ordinal α such that

$$(\exists x \in L_\alpha) B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\top}]. \tag{27}$$

In a next step we exploit the fact that $<_L$ well-orders the universe so that (27) allows us to pick the least set a with respect to $<_L$ satisfying

$$a \in L_\alpha \wedge B_{\mathfrak{A}}^{<\alpha}[f, a, \widehat{\top}]. \tag{28}$$

According to clause (22) of the definition of the operator form $\mathfrak{A}$, see Definition 7, we therefore have

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \widehat{\mathbb{C}}, f, a]$$

from which a final application of Theorem 8 leads to

$$Ap_{\mathfrak{A}}[\widehat{\mathbb{C}}, f, a]. \tag{29}$$

Trivially, (28) also implies

$$Ap_{\mathfrak{A}}[f, a, \widehat{\top}] \tag{30}$$

for that a. Therefore (29) and (30) can be turned into

$$\exists x([\![\mathbb{C}f]\!]_{\mathfrak{A}}(x) \wedge Ap_{\mathfrak{A}}[f, x, \widehat{\top}]). \tag{31}$$

To sum up, the implication from (26) to (31), i.e. the translation of operational choice into $\mathcal{L}$, can be verified in $\mathsf{KP}\omega + (V{=}L)$. This completes the proof of our lemma. □

Theorem 15 *The theory* OST *can be embedded into* $\mathsf{KP}\omega + (V{=}L)$*; i.e. for all formulas* A *of* $\mathcal{L}^\circ$ *we have*

$$\mathsf{OST} \vdash A \quad \Longrightarrow \quad \mathsf{KP}\omega + (V{=}L) \vdash A^*.$$

Proof. This theorem is a simple consequence of the previous lemma since the theory $\mathsf{KP}\omega + (V{=}L)$ is clearly closed under all rules of inference available in OST. □

It is well-known that $\mathsf{KP}\omega + (V{=}L)$ is a conservative extension of $\mathsf{KP}\omega$ for absolute formulas. If we combine this result with Theorem 6 and Theorem 15, we obtain the following corollary, which settles the question of the consistency strength of OST.

Corollary 16 *The theory* OST *is conservative over* $\mathsf{KP}\omega$ *for absolute formulas. In particular,* OST *and* $\mathsf{KP}\omega$ *are equiconsistent.*

4 The consistency strength of $\mathsf{OST}(\mathbb{P})$

As it will turn out, $\mathsf{OST}(\mathbb{P})$ is closely related to the theory $\mathsf{KP}(\mathcal{P})$ of so-called power admissible sets. It is formulated in the language $\mathcal{L}(\mathcal{P})$ which is obtained from $\mathcal{L}$ by adding the new binary relation symbol $\mathcal{P}$. The formulas of $\mathcal{L}(\mathcal{P})$ are defined as the formulas of $\mathcal{L}$, but with expressions of the form $\mathcal{P}(a, b)$ permitted as atomic formulas as well.

The $\Delta_0(\mathcal{P})$ formulas are those formulas of $\mathcal{L}(\mathcal{P})$ which do not contain unbounded quantifiers, and also the notions of $\Sigma(\mathcal{P})$, $\Pi(\mathcal{P})$ and $\Delta(\mathcal{P})$ formulas are the obvious generalizations of Σ, Π and Δ formulas, respectively; in particular, each $\mathcal{P}(a, b)$ is $\Delta_0(\mathcal{P})$. It is then only a matter of routine, by exploiting the constant $\mathbb{P}$ and axiom ($\mathbb{P}$), to ascertain the following analogue of Lemma 1 for the system $\mathsf{OST}(\mathbb{P})$.

Lemma 17 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every $\Delta_0(\mathcal{P})$ formula $A[\vec{u}]$ of $\mathcal{L}^\circ(\mathbb{P})$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ(\mathbb{P})$ term t_A such that $\mathsf{OST}(\mathbb{P})$ proves*

$$t_A\!\downarrow \;\wedge\; (t_A : \mathbb{V}^n \to \mathbb{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t_A(\vec{x}) = \top).$$

The theory $\mathsf{KP}(\mathcal{P})$ is the following extension of $\mathsf{KP}\omega$: (i) it encompasses the axioms extensionality, pair, union and infinity; (ii) $\in$-induction is formulated for arbitrary $\mathcal{L}(\mathcal{P})$ formulas; (iii) we have $\Delta_0(\mathcal{P})$ separation and $\Delta_0(\mathcal{P})$ collection; (iv) finally, the new axiom ($\mathcal{P}$) provides the meaning of the relation symbol $\mathcal{P}$,

$$(\mathcal{P}) \qquad \forall x \exists y \mathcal{P}(x, y) \;\wedge\; \forall x \forall y(\mathcal{P}(x, y) \;\leftrightarrow\; \forall z(z \in y \leftrightarrow z \subset x)).$$

It simply formalizes that $\mathcal{P}$ is the graph of the power set function, acting on the whole universe of sets. This axiom ($\mathcal{P}$) is reminiscent of the operational power set axiom ($\mathbb{P}$), and so the next embedding result should not come as a surprise.

Theorem 18 *Modulo translating the atomic formulas $\mathcal{P}(a, b)$ as $(\mathbb{P}a = b)$, the theory $\mathsf{KP}(\mathcal{P}) + (\mathsf{AC})$ is contained in $\mathsf{OST}(\mathbb{P})$.*

Proof. The $\mathsf{KP}(\mathcal{P})$ versions of all axioms of $\mathsf{KP}(\mathcal{P}) + (\mathsf{AC})$ are proved in $\mathsf{OST}(\mathbb{P})$ analogously to the proof of Theorem 6; the translation of ($\mathcal{P}$) is a trivial consequence of ($\mathbb{P}$). $\square$

We establish an upper bound for $\mathsf{OST}(\mathbb{P})$ by an easy modification of the argument in the previous section. Again we include the axiom of constructibility ($V{=}L$). Then we extend the disjunction in Definition 7 by a clause taking care of the constant $\mathbb{P}$.

Definition 19 *We choose* $\mathfrak{C}[R, \alpha, a, b, c]$ *to be the* $\Delta(\mathcal{P})$ *formula of* $\mathcal{L}(\mathcal{P}, R)$ *defined as*

$$\mathfrak{C}[R, \alpha, a, b, c] \quad := \quad c \in L_\alpha \ \wedge \ (\mathfrak{B}[R, \alpha, a, b, c] \ \vee \ (a = \widehat{\mathbb{P}} \wedge \mathcal{P}(b, c))),$$

where $\mathfrak{B}[R, \alpha, a, b, c]$ *is the formula introduced in Definition 7.*

In $\mathsf{KP}(\mathcal{P})$ we have $\Sigma(\mathcal{P})$ recursion. Completely in the line of the previous section we apply it now, of course, to the operator form $\mathfrak{C}[R, \alpha, a, b, c]$, yielding the following analogue of Theorem 8.

Theorem 20 *There exists a* $\Sigma(\mathcal{P})$ *formula* $B[\alpha, a, b, c]$ *of* $\mathcal{L}(\mathcal{P}, R)$ *with at most* α, a, b *and* c *free so that* $\mathsf{KP}(\mathcal{P})$ *proves*

$$(\Sigma(\mathcal{P})\text{-}\mathsf{Rec}/\mathfrak{C}) \qquad\qquad B[\alpha, a, b, c] \ \leftrightarrow \ \mathfrak{C}[B, \alpha, a, b, c].$$

Naturally, each $\Sigma(\mathcal{P})$ formula $B[\alpha, a, b, c]$ fulfilling this recursion equation $(\Sigma(\mathcal{P})\text{-}\mathsf{Rec}/\mathfrak{C})$ is now a possible candidate for interpreting the $\mathsf{OST}(\mathbb{P})$ application $(ab \simeq c)$.

Definition 21 *Let* $B_{\mathfrak{C}}[\alpha, a, b, c]$ *be a* $\Sigma(\mathcal{P})$ *formula of* $\mathcal{L}(\mathcal{P})$ *associated to the operator form* $\mathfrak{C}[R, \alpha, a, b, c]$ *according to* $(\Sigma(\mathcal{P})\text{-}\mathsf{Rec}/\mathfrak{C})$ *of the previous theorem. Then we define*

$$Ap_{\mathfrak{C}}[a, b, c] \ := \ \exists \alpha B_{\mathfrak{C}}[\alpha, a, b, c].$$

It only remains to proceed as in the previous section, but with $Ap_{\mathfrak{A}}[a, b, c]$ replaced by $Ap_{\mathfrak{C}}[a, b, c]$. The analogues of Lemma 10 and Lemma 11 are proved as earlier, and then, for each $\mathcal{L}^\circ(\mathbb{P})$ term t, an $\mathcal{L}(\mathcal{P})$ formula $[\![t]\!]_{\mathfrak{C}}(u)$ is introduced, saying that u is the value of the term t under the interpretation of the $\mathsf{OST}(\mathbb{P})$ application via $Ap_{\mathfrak{C}}$. Finally, following the pattern of Definition 13 and based on these $[\![t]\!]_{\mathfrak{C}}(u)$, each $\mathcal{L}^\circ(\mathbb{P})$ formula A is canonically translated into a formula $A^\sharp$ of $\mathcal{L}(\mathcal{P})$.

Theorem 22 *The theory* $\mathsf{OST}(\mathbb{P})$ *can be embedded into* $\mathsf{KP}(\mathcal{P}) + (V{=}L)$*; i.e. for all formulas* A *of* $\mathcal{L}^\circ(\mathbb{P})$ *we have*

$$\mathsf{OST}(\mathbb{P}) \vdash A \quad \Longrightarrow \quad \mathsf{KP}(\mathcal{P}) + (V{=}L) \vdash A^\sharp.$$

Proof. Recalling Lemma 14, which trivially carries over from OST and $\mathsf{KP}\omega + (V{=}L)$ to $\mathsf{OST}(\mathbb{P})$ and $\mathsf{KP}(\mathcal{P}) + (V{=}L)$, only the axiom $(\mathbb{P})$ about operational power set has to be taken care of. So pick a set a. By the axiom $(\mathcal{P})$ of $\mathsf{KP}(\mathcal{P})$ we know that there exists a set b such that $\mathcal{P}(a, b)$ and $\forall z(z \in b \leftrightarrow z \subset a)$. Aside from that, the axiom $(V{=}L)$ provides for an ordinal α for which $b \in L_\alpha$. According to Definition 19, Theorem 20 and Definition 21 we therefore have

$$B_{\mathfrak{C}}[\alpha, \widehat{\mathbb{P}}, a, b] \qquad \text{and} \qquad Ap_{\mathfrak{C}}[\widehat{\mathbb{P}}, a, b].$$

Therefore $\widehat{\mathbb{P}}$ codes a total operation from the collection of all sets to sets which maps a set to its power set, as desired. □

Unfortunately, the combination of Theorem 18 and Theorem 22 does not completely settle the question about the consistency strength of $\mathsf{OST}(\mathbb{P})$ yet. So far we have an interesting lower and an interesting upper bound, but it still has to be determined what the relationship between $\mathsf{KP}(\mathcal{P})$ and $\mathsf{KP}(\mathcal{P}) + (V{=}L)$ is.

5 A conservative extension of ZFC

The purpose of this section is to identity an OST-like operational set theory which is a conservative extension of ZFC, thus answering a question raised in Feferman [11] and the following discussion. To do so, we begin with extending the language $\mathcal{L}^\circ(\mathbb{P})$ to the new language $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ resulting from the addition of the new constant $\mathbf{E}$.

The role of $\mathbf{E}$ is to act as the unbounded analogue of the constant $\mathbf{e}$, which deals with bounded existential quantification. Therefore, the meaning of $\mathbf{E}$ is given by the axiom

$$(\mathbf{E}) \qquad (f : \mathbb{V} \to \mathbb{B}) \;\to\; (\mathbf{E}(f) \in \mathbb{B} \,\wedge\, (\mathbf{E}(f) = \top \;\leftrightarrow\; \exists x(fx = \top))).$$

Then $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ is the theory which consists of all axioms of OST, now formulated for all $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ formulas, plus the power set axiom $(\mathbb{P})$ and the axiom $(\mathbf{E})$ about unbounded existential quantification. However, $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ is stronger than ZFC, and its proof-theoretic analysis will be carried out in a forthcoming publication.

In this article we concentrate ourselves on the subsystem $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ which is obtained from $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ by restricting the schema of $\in$-induction for arbitrary formulas to $\in$-induction for sets. As the following lemma shows, $\in$-induction is provable in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ for total operations from $\mathbb{V}$ to $\mathbb{B}$.

Lemma 23 *In* $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ *we can prove that*

$$(f : \mathbb{V} \to \mathbb{B}) \wedge \forall x((\forall y \in x)(fy = \top) \to (fx = \top)) \to \; \forall x(fx = \top).$$

PROOF. We show the contraposition and assume that $(f : \mathbb{V} \to \mathbb{B})$ and that there exists a set a with the property $(fa = \bot)$. By separation for definite operations we can introduce the set

$$b \;:=\; \{a\} \cup \{x \in TC(a) : fx = \bot\},$$

where $TC(a)$ is written for the transitive closure of a; the existence of transitive closures is evident in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$. Now apply $\in$-induction to this non-empty b. As a result, we are provided with an $\in$-minimal element c of b, i.e.

$$fc = \bot \wedge (\forall y \in c)(fy = \top).$$

The existence of such a set c is exactly what was needed for completing the proof of this lemma. □

As we will see in the following, $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ contains ZFC and can be reduced to ZFL, i.e. to $\mathsf{ZF} + (V = L)$. Consequently, $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ is a conservative extension of ZFC.

As the *pure* formulas of $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ we denote those $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ formulas which do not contain the function symbol $\circ$ or the relation symbol $\downarrow$. That means that the pure $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ formulas are the usual set-theoretic formulas in which the constants of $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ may occur as additional parameters. Since in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ the constant $\mathbf{E}$ is available, Lemma 1 can be straightforwardly extended to pure formulas.

Lemma 24 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every pure formula $A[\vec{u}]$ of $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ term t_A such that $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ yields*

$$t_A\downarrow \;\wedge\; (t_A : \mathbb{V}^n \to \mathbb{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t_A(\vec{x}) = \top).$$

The embedding of ZFC into $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ is now a matter of routine: extensionality, pair, union and infinity are obvious; separation, replacement and $\in$-induction of ZF can be dealt with in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$, in view of Lemma 24, by operational separation, operational replacement and Lemma 23, respectively. Therefore we have the following theorem.

Theorem 25 *The theory ZFC is contained in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$.*

This establishes the lower proof-theoretic bound of $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$. The reduction of $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ to ZFL is more complicated. We achieve this by (i) interpreting $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ into the auxiliary theory ZFL^r_Ω and (ii) reducing ZFL^r_Ω to ZFL.

6 The auxiliary system ZFL^r_Ω

Our next steps are similar to the approach taken in Jäger [17] and Feferman and Jäger [13]. In these articles an extension PA^r_Ω of Peano arithmetic PA is introduced which is tailored for a sufficiently careful treatment of inductive definitions over the natural numbers and strong enough to interpret the non-constructive μ-operator of the basic theory $\mathsf{BON}(\mu)$ of operations and numbers. Now we replace the collection of the natural numbers by the universe of all sets and extend ZFL – instead of PA – to the system ZFL^r_Ω for dealing with inductive definitions over the sets.

As in Section 3 we pick an n-ary relation symbol R which does not belong to the language $\mathcal{L}$ and write $\mathcal{L}(R)$ for the extension of $\mathcal{L}$ by R. An $\mathcal{L}(R)$ formula which contains at most $a_1, \dots, a_n$ free is called an n-ary operator form, and we let $\mathfrak{F}[R, a_1, \dots, a_n]$ range over such forms.

Based on a model $\mathcal{M}$ of ZFL with universe $|\mathcal{M}|$, any n-ary operator form $\mathfrak{F}[R, \vec{a}]$ gives rise to subsets $I^\alpha_\mathfrak{F}$ of $|\mathcal{M}|^n$ generated inductively for all ordinals α (not only those belonging to $|\mathcal{M}|$) by

$$I^{<\alpha}_\mathfrak{F} \;:=\; \bigcup_{\beta<\alpha} I^\beta_\mathfrak{F} \quad \text{and} \quad I^\alpha_\mathfrak{F} \;:=\; \{\langle \vec{x} \rangle \in |\mathcal{M}|^n : \mathcal{M} \models \mathfrak{F}[I^{<\alpha}_\mathfrak{F}, \vec{x}]\}.$$

These sets $I^\alpha_\mathfrak{F}$ are the *stages* of the inductive definition induced by $\mathfrak{F}[R, \vec{a}]$, relative to $\mathcal{M}$; for many models $\mathcal{M}$, operator forms $\mathfrak{F}[R, \vec{a}]$ and ordinals α the $I^\alpha_\mathfrak{F}$ are not elements of $|\mathcal{M}|$. We now enrich ZFL so that we can speak about such stages.

The theory ZFL^r_Ω is formulated in the language $\mathcal{L}_\Omega$ which extends $\mathcal{L}$ by adding a new sort of so called stage variables $\rho, \sigma, \tau, \dots$ (possibly with subscripts) as well as new binary relation symbols $\prec$ and $\mathrel{\mathring{=}}$ for the less and equality relation for stage variables, respectively. Moreover, $\mathcal{L}_\Omega$ includes an $(n+1)$-ary relation symbol $Q_\mathfrak{F}$ for each operator form $\mathfrak{F}[R, a_1, \dots, a_n]$. The set terms of $\mathcal{L}_\Omega$ are the set terms of $\mathcal{L}$, and the stage terms of $\mathcal{L}_\Omega$ are the stage variables. The atomic formulas of $\mathcal{L}_\Omega$ are the atomic formulas of $\mathcal{L}$ plus all expressions $(\sigma \prec \tau)$, $(\sigma \mathrel{\mathring{=}} \tau)$ and $Q_\mathfrak{F}(\sigma, \vec{s})$ for each n-ary operator form $\mathfrak{F}[R, \vec{a}]$. Usually we write $Q^\sigma_\mathfrak{F}(\vec{s})$ instead of $Q_\mathfrak{F}(\sigma, \vec{s})$.

The formulas $(A, B, C, A_1, B_1, C_1, \dots)$ of $\mathcal{L}_\Omega$ are generated from these atoms by closure under negation, conjunction and disjunction, bounded and unbounded quantification over sets, bounded stage quantification $(\exists \sigma \prec \tau)$ and $(\forall \sigma \prec \tau)$ as well as unbounded stage quantification $\exists \sigma$ and $\forall \sigma$. The Δ^Ω_0 formulas are those $\mathcal{L}_\Omega$ formulas that do not contain unbounded stage quantifiers. An $\mathcal{L}_\Omega$ formula A is is called Σ^Ω if all positive occurrences of unbounded stage quantifiers in A are existential and all negative occurrences of unbounded stage quantifiers in A are universal; it is called Π^Ω if all positive occurrences of unbounded stage quantifiers in A are universal and all negative occurrences of unbounded stage quantifiers in A are existential. Further, we write A^σ to denote the $\mathcal{L}_\Omega$ formula which is obtained from A by replacing all unbounded stage quantifiers $Q\tau$ in A by bounded stage quantifiers $(Q\tau \prec \sigma)$. Additional abbreviations are

$$Q_{\mathfrak{F}}^{\prec\sigma}(\vec{s}) \ := \ (\exists\tau \prec \sigma) Q_{\mathfrak{F}}^{\tau}(\vec{s}) \qquad \text{and} \qquad Q_{\mathfrak{F}}(\vec{s}) \ := \ \exists\sigma Q_{\mathfrak{F}}^{\sigma}(\vec{s}).$$

Clearly, any formula of $\mathcal{L}$ is a (trivial) Δ_0^Ω formula, and A^σ is Δ_0^Ω for any $\mathcal{L}_\Omega$ formula A.

The theory ZFL_Ω^r is formulated in classical two sorted predicate logic with equality in both sorts; in addition it contains as non-logical axioms all ZFL-axioms of the language $\mathcal{L}$, axioms about stage variables and operator forms, Σ^Ω reflection plus separation, replacement and $\prec$-induction for Δ_0^Ω formulas.

I. ZFL-axioms. All axioms of the theory ZFL formulated in the language $\mathcal{L}$; they do not refer to stage variables or relation symbols associated to operator forms.

II. Linearity axioms. For all stage variables ρ, σ and τ:

$$\sigma \not\prec \sigma \ \wedge \ (\rho \prec \sigma \wedge \sigma \prec \tau \ \rightarrow \ \rho \prec \tau) \ \wedge \ (\sigma \prec \tau \ \vee \ \sigma \mathrel{\mathring{=}} \tau \ \vee \ \tau \prec \sigma).$$

III. Operator axioms. For all operator forms $\mathfrak{F}[R, \vec{u}]$ and all set terms $\vec{s}$:

$$Q_{\mathfrak{F}}^{\sigma}(\vec{s}) \ \leftrightarrow \ \mathfrak{F}[Q_{\mathfrak{F}}^{\prec\sigma}, \vec{s}].$$

IV. Σ^Ω reflection. For all Σ^Ω formulas A:

(Σ^Ω-Ref) $$A \ \rightarrow \ \exists\sigma A^\sigma.$$

V. Δ_0^Ω Separation. For all Δ_0^Ω formulas $A[u]$ and all set terms s:

(Δ_0^Ω-Sep) $$\exists x (x = \{y \in s : A[y]\}).$$

VI. Δ_0^Ω Replacement. For all Δ_0^Ω formulas $A[u, v]$ and all set terms s:

(Δ_0^Ω-Rep) $$(\forall x \in s)\exists! y A[x, y] \ \rightarrow \ \exists z \forall y (y \in z \ \leftrightarrow \ (\exists x \in s) A[x, y]).$$

VII. Δ_0^Ω induction along $\prec$. For all Δ_0^Ω formulas $A[u]$:

$$(\Delta_0^\Omega\text{-}\mathsf{I}_\prec) \qquad \forall\sigma((\forall\tau \prec \sigma)A[\tau] \to A[\sigma]) \to \forall\sigma A[\sigma].$$

It is important to observe that the stage variables do not belong to the collection of sets; they constitute a different entity which is used to "enumerate" the stages of the inductive definition associated to each operator form. However, in the form of Δ_0^Ω separation and Δ_0^Ω replacement they can nevertheless help to constitute new sets in a carefully restricted way. The theory ZFL_Ω^r is restricted in the sense that the axioms in groups V, VI and VII are restricted to Δ_0^Ω formulas.

6.1 Interpreting $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ into ZFL_Ω^r

Before introducing a specific ternary operator form, which will be the crucial step in modelling $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ within ZFL_Ω^r, we fix a further set $\widehat{\mathbf{E}}$ as code for the constant $\mathbf{E}$, making sure that no conflicts arise in connection with the coding machinery introduced in Section 3.

Definition 26 *The operator form $\mathfrak{F}[R,a,b,c]$ is defined to be the disjunction of the following clauses:*

(1) $a = \widehat{\mathsf{k}} \wedge c = \langle\widehat{\mathsf{k}}, b\rangle$,

(2) $Tup_2(a) \wedge (a)_1 = \widehat{\mathsf{k}} \wedge (a)_2 = c$,

(3) $a = \widehat{\mathsf{s}} \wedge c = \langle\widehat{\mathsf{s}}, b\rangle$,

(4) $Tup_2(a) \wedge (a)_1 = \widehat{\mathsf{s}} \wedge c = \langle\widehat{\mathsf{s}}, (a)_2, b\rangle$,

(5) $Tup_3(a) \wedge (a)_1 = \widehat{\mathsf{s}} \wedge \exists x\exists y(R((a)_2, b, x) \wedge R((a)_3, b, y) \wedge R(x, y, c))$,

(6) $a = \widehat{\mathbf{el}} \wedge c = \langle\widehat{\mathbf{el}}, b\rangle$,

(7) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{el}} \wedge (a)_2 \in b \wedge c = \widehat{\top}$,

(8) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{el}} \wedge (a)_2 \notin b \wedge c = \widehat{\bot}$,

(9) $a = \widehat{\mathbf{non}} \wedge b = \widehat{\top} \wedge c = \widehat{\bot}$,

(10) $a = \widehat{\mathbf{non}} \wedge b = \widehat{\bot} \wedge c = \widehat{\top}$,

(11) $a = \widehat{\mathbf{dis}} \wedge c = \langle\widehat{\mathbf{dis}}, b\rangle$,

(12) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{dis}} \wedge (a)_2 = \widehat{\top} \wedge c = \widehat{\top}$,

(13) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbf{dis}} \;\wedge\; (a)_2 = \widehat{\bot} \;\wedge\; b = \widehat{\top} \;\wedge\; c = \widehat{\top}$,

(14) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbf{dis}} \;\wedge\; (a)_2 = \widehat{\bot} \;\wedge\; b = \widehat{\bot} \;\wedge\; c = \widehat{\bot}$,

(15) $a = \widehat{\mathbf{e}} \;\wedge\; c = \langle \widehat{\mathbf{e}}, b \rangle$,

(16) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbf{e}} \;\wedge\; (\exists x \in b) R((a)_2, x, \widehat{\top}) \;\wedge\; c = \widehat{\top}$,

(17) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbf{e}} \;\wedge\; (\forall x \in b) R((a)_2, x, \widehat{\bot}) \;\wedge\; c = \widehat{\bot}$,

(18) $a = \widehat{\mathbb{S}} \;\wedge\; c = \langle \widehat{\mathbb{S}}, b \rangle$,

(19) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbb{S}} \;\wedge\; (\forall x \in b)(R((a)_2, x, \widehat{\top}) \;\vee\; R((a)_2, x, \widehat{\bot})) \;\wedge$

$\forall x (x \in c \;\leftrightarrow\; x \in b \wedge R((a)_2, x, \widehat{\top}))$,

(20) $a = \widehat{\mathbb{R}} \;\wedge\; c = \langle \widehat{\mathbb{R}}, b \rangle$,

(21) $Tup_2(a) \;\wedge\; (a)_1 = \widehat{\mathbb{R}} \;\wedge\; (\forall x \in b)(\exists y \in c) R((a)_2, x, y) \;\wedge$

$(\forall y \in c)(\exists x \in b) R((a)_2, x, y)$,

(22) $a = \widehat{\mathbb{C}} \;\wedge\; R(b, c, \widehat{\top}) \;\wedge\; \forall x (x <_L c \;\rightarrow\; \neg R(b, x, \widehat{\top})) \;\wedge\; \forall x \neg R(\widehat{\mathbb{C}}, b, x)$,

(23) $a = \widehat{\mathbb{P}} \;\wedge\; \forall x (x \in c \;\leftrightarrow\; x \subset b)$,

(24) $a = \widehat{\mathbf{E}} \;\wedge\; \exists x R(b, x, \widehat{\top}) \;\wedge\; c = \widehat{\top}$,

(25) $a = \widehat{\mathbf{E}} \;\wedge\; \forall x R(b, x, \widehat{\bot}) \;\wedge\; c = \widehat{\bot}$.

A first observation is concerned with properties of the formulas $Q^{\sigma}_{\mathfrak{F}}(a, b, c)$ and $Q_{\mathfrak{F}}(a, b, c)$, which are induced by the operator form $\mathfrak{F}[R, a, b, c]$, and states their functionality.

Lemma 27 *We can prove in* ZFL^r_Ω*:*

1. $Q^{\prec\sigma}_{\mathfrak{F}}(f, a, u) \wedge Q^{\prec\sigma}_{\mathfrak{F}}(f, a, v) \;\rightarrow\; u = v$.
2. $Q_{\mathfrak{F}}(f, a, u) \wedge Q_{\mathfrak{F}}(f, a, v) \;\rightarrow\; u = v$.

PROOF. The first assertion is proved by Δ^Ω_0 induction on σ. All details are similar to (even simpler than) those of the proofs of Lemma 10 and Lemma 11 and are left to the reader. The second assertion is an immediate consequence of the first. □

The desired interpretation is obtained by following Section 3 again, this time with $Ap_{\mathfrak{A}}[a, b, c]$ replaced by $Q_{\mathfrak{F}}(a, b, c)$. In parallel to Definition 12 an $\mathcal{L}_\Omega$

formula $[\![t]\!]_{\mathfrak{F}}(u)$ is assigned to any $\mathcal{L}^\circ(\mathbf{E},\mathbb{P})$ term t, saying that u is the value of the term t under the interpretation of the $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ application via $Q_{\mathfrak{F}}$. And in parallel to Definition 13, employing these $[\![t]\!]_{\mathfrak{F}}(u)$, each $\mathcal{L}^\circ(\mathbf{E},\mathbb{P})$ formula A is translated into a formula $A^\Diamond$ of $\mathcal{L}_\Omega$ in the obvious way. Please keep in mind that A and $A^\Diamond$ are identical in the case that A is an $\mathcal{L}$ formula.

Theorem 28 *The theory $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ can be embedded into ZFL^r_Ω; i.e. for all formulas A of $\mathcal{L}^\circ(\mathbf{E},\mathbb{P})$ we have*

$$\mathsf{OST}^r(\mathbf{E},\mathbb{P}) \vdash A \quad \Longrightarrow \quad \mathsf{ZFL}^r_\Omega \vdash A^\Diamond.$$

PROOF. The theory ZFL^r_Ω clearly validates all logical axioms of $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ and is closed under all rules of inference of $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$. Hence we can concentrate ourselves on the interpretation of the non-logical axioms of $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$. The treatment of the applicative axioms and the basic set-theoretic axioms with $\in$-induction restricted to sets is unproblematic. The logical operations axioms can be treated (with minor modifications) as in the proof of Lemma 14, and we turn to the remaining axioms and work informally in ZFL^r_Ω.

1. Operational separation for definite operations. From the left hand side $(f : a \to \mathbb{B})$ of such an axiom we obtain that

$$(\forall x \in a)(Q_{\mathfrak{F}}(f,x,\widehat{\top}) \vee Q_{\mathfrak{F}}(f,x,\widehat{\bot})),$$

and by Σ^Ω reflection there exists a σ such that

$$(1) \qquad (\forall x \in a)(Q^{\prec\sigma}_{\mathfrak{F}}(f,x,\widehat{\top}) \vee Q^{\prec\sigma}_{\mathfrak{F}}(f,x,\widehat{\bot})).$$

In view of Δ^Ω_0 separation we therefore have a set b satisfying

$$(2) \qquad \forall x(x \in b \leftrightarrow x \in a \wedge Q^{\prec\sigma}_{\mathfrak{F}}(f,x,\widehat{\top})).$$

Because of Lemma 27 and (1) for this b we also have

$$(3) \qquad \forall x(x \in b \leftrightarrow x \in a \wedge Q_{\mathfrak{F}}(f,x,\widehat{\top})).$$

Clause (19) of Definition 26, together with (1) and (2), yields $Q^\sigma_{\mathfrak{F}}(\langle\widehat{\mathbb{S}},f\rangle,a,b)$, leading directly to

$$(4) \qquad Q_{\mathfrak{F}}(\langle\widehat{\mathbb{S}},f\rangle,a,b).$$

But then lines (3) and (4) ensure that

$$\exists y[\![\mathbb{S}(f,a)]\!]_{\mathfrak{F}}(y) \;\wedge\; \forall x(\exists y([\![\mathbb{S}(f,a)]\!]_{\mathfrak{F}}(y) \wedge x \in y) \leftrightarrow x \in a \wedge Q_{\mathfrak{F}}(f,x,\widehat{\top})).$$

This is the translation of the right hand side of our axiom about operational separation for definite operations, which is herewith established in ZFL^r_Ω.

2. Operational replacement. Any such axiom has a premise of the form $(f : a \to \mathbb{V})$ which translates into

$$(\forall x \in a)\exists y Q_{\mathfrak{F}}(f, x, y),$$

and therefore Σ^Ω reflection provides a σ such that

$$(\forall x \in a)\exists y Q_{\mathfrak{F}}^{\prec\sigma}(f, x, y). \tag{5}$$

Because of Lemma 27 we thus even have

$$(\forall x \in a)\exists! y Q_{\mathfrak{F}}^{\prec\sigma}(f, x, y).$$

Hence, by Δ_0^Ω replacement there exists a set b for which

$$\forall y(y \in b \leftrightarrow (\exists x \in a)Q_{\mathfrak{F}}^{\prec\sigma}(f, x, y)), \tag{6}$$

and, as in the previous case, Lemma 27 and (5) imply

$$\forall y(y \in b \leftrightarrow (\exists x \in a)Q_{\mathfrak{F}}(f, x, y)). \tag{7}$$

Note that by clause (21) of Definition 26, (5) and (6) it follows

$$Q_{\mathfrak{F}}(\langle \widehat{\mathbb{R}}, f\rangle, a, b). \tag{8}$$

Finally, lines (7) and (8) immediately lead to

$$\exists z [\![\mathbb{R}(f, a)]\!]_{\mathfrak{F}}(z) \;\wedge\; \forall y(\exists z([\![\mathbb{R}(f, a)]\!]_{\mathfrak{F}}(z) \wedge y \in z) \;\leftrightarrow\; (\exists x \in a)Q_{\mathfrak{F}}(f, x, y)).$$

This shows that operational replacement holds in ZFL^r_Ω because the previous line is the translation of the conclusion of the respective axiom.

3. Operational choice. In this case we have a premise $\exists x(fx = \top)$ which translates into $\exists x Q_{\mathfrak{F}}(f, x, \widehat{\top})$, i.e. into

$$\exists \sigma \exists x Q_{\mathfrak{F}}^{\sigma}(f, x, \widehat{\top}).$$

By Δ_0^Ω induction along $\prec$ there exists a $\prec$-minimal τ such that

$$\exists x Q_{\mathfrak{F}}^{\prec\tau}(f, x, \widehat{\top}),$$

and, by $(V = L)$, there exists an ordinal α such that

$$(\exists x \in L_\alpha)Q_{\mathfrak{F}}^{\prec\tau}(f, x, \widehat{\top}).$$

Now we use Δ_0^Ω separation to introduce the set $\{x \in L_\alpha : Q_{\mathfrak{F}}^{\prec\tau}(f,x,\widehat{\top})\}$ and select that element a of this set which is least with respect to the well-ordering $<_L$ of the universe. From this choice of a we see that

$$Q_{\mathfrak{F}}^{\sigma}(\widehat{\mathbb{C}}, f, a).$$

and it is easily checked, as before, that a is the appropriate value of $\mathbb{C}f$. Therefore, the translation of operational choice is also provable in ZFL_Ω^r.

4. Operational power set. The interpretation of this is straightforward and can be omitted.

5. Unbounded existential quantification. The premise $(f : \mathbb{V} \to \mathbb{B})$ of such an axiom translates into

$$\forall x(Q_{\mathfrak{F}}(f,x,\widehat{\top}) \vee Q_{\mathfrak{F}}(f,x,\widehat{\bot})),$$

and Σ^Ω reflection provides a σ such that

$$\forall x(Q_{\mathfrak{F}}^{\prec\sigma}(f,x,\widehat{\top}) \vee Q_{\mathfrak{F}}^{\prec\sigma}(f,x,\widehat{\bot})). \tag{9}$$

By Lemma 27 this assertion implies

$$\forall x(Q_{\mathfrak{F}}(f,x,\widehat{\top}) \leftrightarrow Q_{\mathfrak{F}}^{\prec\sigma}(f,x,\widehat{\top})). \tag{10}$$

Furthermore, it is easily seen that clauses (24) and (25) of Definition 26 and assertion (9) yield

$$Q_{\mathfrak{F}}^{\sigma}(\widehat{\mathbf{E}}, f, \widehat{\top}) \vee Q_{\mathfrak{F}}^{\sigma}(\widehat{\mathbf{E}}, f, \widehat{\bot}) \quad \text{and} \quad Q_{\mathfrak{F}}^{\sigma}(\widehat{\mathbf{E}}, f, \widehat{\top}) \leftrightarrow \exists x Q_{\mathfrak{F}}^{\prec\sigma}(f,x,\widehat{\top}).$$

By the definition of $Q_{\mathfrak{F}}$, line (10) and Lemma 27 we conclude

$$Q_{\mathfrak{F}}(\widehat{\mathbf{E}}, f, \widehat{\top}) \vee Q_{\mathfrak{F}}(\widehat{\mathbf{E}}, f, \widehat{\bot}) \quad \text{and} \quad Q_{\mathfrak{F}}(\widehat{\mathbf{E}}, f, \widehat{\top}) \leftrightarrow \exists x Q_{\mathfrak{F}}(f,x,\widehat{\top}).$$

This allows us to deduce

$$[\![\mathbf{E}(f)]\!]_{\mathfrak{F}}(\widehat{\top}) \vee [\![\mathbf{E}(f)]\!]_{\mathfrak{F}}(\widehat{\bot}) \quad \text{and} \quad [\![\mathbf{E}(f)]\!]_{\mathfrak{F}}(\widehat{\top}) \leftrightarrow \exists x Q_{\mathfrak{F}}(f,x,\widehat{\top})$$

and verifies (the translation of) the conclusion of our axiom about unbounded existential quantification, finishing the proof of our theorem. $\square$

6.2 Reducing ZFL^r_Ω to ZFL

It remains to reduce our auxiliary theory ZFL^r_Ω to ZFL. To this end we introduce an auxiliary system S_Ω which is a Gentzen style reformulation of ZFL^r_Ω. The capital Greek letters $\Theta, \Phi, \Psi, \ldots$ (possibly with subscripts) denote finite sequences of $\mathcal{L}_\Omega$ formulas, and sequents are formal expressions of the form $\Phi \supset \Psi$. We write $\Phi[\vec{\sigma}] \supset \Psi[\vec{\sigma}]$ to express that all formulas in Φ and Ψ are of the form $A[\vec{\sigma}]$. The collection of all Σ^Ω and Π^Ω formulas is denoted by ∇^Ω.

S_Ω is an extension of the classical Gentzen sequent calculus *LK* (cf., e.g., Girard [14] or Takeuti [24]) by additional axioms and rules of inference which take care of the non-logical axioms of ZFL^r_Ω. The axioms and rules of S_Ω can be grouped as follows.

I. Axioms. For all Δ^Ω_0 formulas A and all axioms B of ZFL^r_Ω which belong to ∇^Ω:

$$A \supset A \qquad \text{and} \qquad \supset B.$$

II. Structural rules. The structural rules of S_Ω consist of the usual weakening, exchange and contraction rules.

III. Propositional rules. The propositional rules of S_Ω consist of the usual rules for introducing the propositional connectives on the left and right hand sides of sequents.

IV. Quantifier rules. Formulated for existential quantifiers; the corresponding rules for universal quantifiers must also be included. By $(\star)$ we mark those rules where the designated free variables are not to occur in the conclusion:

$$\frac{\Phi \;\supset\; \Psi,\, A[s]}{\Phi \;\supset\; \Psi,\, \exists x A[x]}\,, \qquad \frac{\Phi,\, A[u] \;\supset\; \Psi}{\Phi,\, \exists x A[x] \;\supset\; \Psi}\,(\star),$$

$$\frac{\Phi \;\supset\; \Psi,\, A[\sigma]}{\Phi \;\supset\; \Psi,\, \exists \tau A[\tau]}\,, \qquad \frac{\Phi,\, A[\sigma] \;\supset\; \Psi}{\Phi,\, \exists \tau A[\tau] \;\supset\; \Psi}\,(\star),$$

$$\frac{\Phi \;\supset\; \Psi,\, \rho \prec \sigma \land A[\rho]}{\Phi \;\supset\; \Psi,\, (\exists \tau \prec \sigma) A[\tau]}\,, \qquad \frac{\Phi,\, \rho \prec \sigma \land A[\rho] \;\supset\; \Psi}{\Phi,\, (\exists \tau \prec \sigma) A[\tau] \;\supset\; \Psi}\,(\star).$$

V. Σ^Ω reflection rules. For all Σ^Ω formulas A and all stage variables σ which are not free in A:

$$\frac{\Phi \;\supset\; \Psi,\, A}{\Phi \;\supset\; \Psi, \exists \sigma A^\sigma}\,.$$

VI. Δ_0^Ω induction rules along $\prec$. For all Δ_0^Ω formulas $A[\sigma]$:

$$\frac{\Phi \ \supset \ \Psi, \forall\sigma((\forall\tau \prec \sigma)A[\tau] \to A[\sigma])}{\Phi \ \supset \ \Psi, \forall\sigma A[\sigma]}.$$

VII. Cuts. For all $\mathcal{L}_\Omega$ formulas A:

$$\frac{\Phi \ \supset \ \Psi, A \qquad \Phi, A \ \supset \ \Psi}{\Phi \ \supset \ \Psi}.$$

For any natural number n the notion $\mathsf{S}_\Omega \vdash^n \Phi \supset \Psi$ is used to express that the sequent $\Phi \supset \Psi$ is provable in S_Ω by a proof of depth less than or equal to n; we write $\mathsf{S}_\Omega \vdash_*^n \Phi \supset \Psi$ if $\Phi \supset \Psi$ is provable in S_Ω by a proof of depth less than or equal to n so that all its cut formulas belong to ∇^Ω. In addition, $\mathsf{S}_\Omega \vdash \Phi \supset \Psi$ and $\mathsf{S}_\Omega \vdash_* \Phi \supset \Psi$ mean that there exists a natural number n so that $\mathsf{S}_\Omega \vdash^n \Phi \supset \Psi$ and $\mathsf{S}_\Omega \vdash_*^n \Phi \supset \Psi$, respectively.

One readily notes that the main formulas of all axioms and rules of the system S_Ω belong to ∇^Ω. Therefore, following the lines of Jäger [17], where a conceptually related system G_Ω is considered, and applying standard techniques of proof theory as presented, for example, in Girard [14], Schütte [23] or Takeuti [24], we obtain the following weak cut elimination theorem for S_Ω.

Theorem 29 (Weak cut elimination for S_Ω) *For all sequents $\Phi \supset \Psi$ we have that*

$$\mathsf{S}_\Omega \vdash \Phi \supset \Psi \quad \Longrightarrow \quad \mathsf{S}_\Omega \vdash_* \Phi \supset \Psi.$$

Of course, the axioms and rules of S_Ω are tailored so that the ZFL_Ω^r can be embedded into S_Ω in a straightforward manner: the ZFL-axioms, the linearity axioms, the operator axioms, the axioms about Δ_0^Ω separation and Δ_0^Ω replacement are axioms of S_Ω; the Σ^Ω reflection axioms of ZFL_Ω^r are proved in S_Ω by means of the rules for Σ^Ω reflection, and the instances of Δ_0^Ω induction along $\prec$ can be derived in S_Ω by making use of the corresponding rules. Hence we have the following theorem.

Theorem 30 *If the $\mathcal{L}_\Omega$ formula A is provable in ZFL_Ω^r, then there exists a natural number n such that*

$$\mathsf{S}_\Omega \vdash^n \ \supset A.$$

Combining Theorem 29 and Theorem 30 we obtain the following corollary. It implies, in particular, that every formula A from ∇^Ω provable in ZFL_Ω^r has a proof tree in S_Ω which consists of formulas from ∇^Ω only.

Corollary 31 *If the $\mathcal{L}_\Omega$ formula A is provable in ZFL^r_Ω, then there exists a natural number n such that*

$$\mathsf{S}_\Omega \vdash^n_* \supset A.$$

Our next aim is to reduce the ∇^Ω fragment of S_Ω to ZFL. For this purpose we first introduce for all operator forms $\mathfrak{F}[R, \vec{a}]$ of $\mathcal{L}(R)$ and all natural numbers n the $\mathcal{L}_\Omega$ formulas $J^{<n}_\mathfrak{F}(\vec{a})$ and $J^n_\mathfrak{F}(\vec{a})$; they are defined by simultaneous induction on n as follows:

$$J^{<n}_\mathfrak{F}(\vec{a}) \;:=\; \bigvee_{m<n} J^m_\mathfrak{F}(\vec{a}) \quad \text{and} \quad J^n_\mathfrak{F}(\vec{a}) \;:=\; \mathfrak{F}[J^{<n}_\mathfrak{F}, \vec{a}].$$

Also, given a natural number n, we write $\bar{n}$ for the finite von Neuman ordinal corresponding to n. For a formula A from ∇^Ω we use the notation $A\langle\vec{\sigma}\rangle$ to express that all its free stage variables belong to the list $\vec{\sigma}$; the analogous convention is employed for ∇^Ω sequents.

Definition 32 *Let $\vec{\sigma}$ be a finite string of stage variables, $\vec{p}$ a finite string of natural numbers of the same length and n a natural number. For any formula $A\langle\vec{\sigma}\rangle$ from ∇^Ω the $\mathcal{L}$ formula $A^{(n)}\langle\vec{p}\rangle$ is inductively defined as follows:*

1. *If $A\langle\vec{\sigma}\rangle$ is an atomic $\mathcal{L}$ formula, then $A^{(n)}\langle\vec{p}\rangle := A[\vec{\sigma}]$.*
2. *If $A\langle\vec{\sigma}\rangle$ is a formula $(\sigma_i \prec \sigma_j)$, then $A^{(n)}\langle\vec{p}\rangle := (\bar{p}_i < \bar{p}_j)$.*
3. *If $A\langle\vec{\sigma}\rangle$ is a formula $(\sigma_i \mathrel{\mathring{=}} \sigma_j)$, then $A^{(n)}\langle\vec{p}\rangle := (\bar{p}_i = \bar{p}_j)$.*
4. *If $A\langle\vec{\sigma}\rangle$ is a formula $Q^{\sigma_i}_\mathfrak{F}(\vec{s})$, then $A^{(n)}\langle\vec{p}\rangle := J^{p_i}_\mathfrak{F}(\vec{s})$.*
5. *If $A\langle\vec{\sigma}\rangle$ is a formula $\neg B\langle\vec{\sigma}\rangle$, then $A^{(n)}\langle\vec{p}\rangle := \neg B^{(n)}\langle\vec{p}\rangle$.*
6. *If $A\langle\vec{\sigma}\rangle$ is a formula $(B\langle\vec{\sigma}\rangle \diamond C\langle\vec{\sigma}\rangle)$ for $\diamond$ being the binary junctor $\vee$ or $\wedge$, then*
$$A^{(n)}\langle\vec{p}\rangle := (B^{(n)}\langle\vec{p}\rangle \diamond C^{(n)}\langle\vec{p}\rangle).$$
7. *If $A\langle\vec{\sigma}\rangle$ is a formula $(Qx \in s)B\langle\vec{\sigma}\rangle$ or $QxB\langle\vec{\sigma}\rangle$ for a quantifier Q, then*
$$A^{(n)}\langle\vec{p}\rangle := (Qx \in s)B^{(n)}\langle\vec{p}\rangle \quad \text{or} \quad A^{(n)}\langle\vec{p}\rangle := QxB^{(n)}\langle\vec{p}\rangle.$$
8. *If $A\langle\vec{\sigma}\rangle$ is a formula $(\exists\tau \prec \sigma_i)B\langle\tau, \vec{\sigma}\rangle$, then*
$$A^{(n)}\langle\vec{p}\rangle := \bigvee_{k<p_i} B^{(n)}\langle k, \vec{p}\rangle.$$

9. *If $A\langle\vec{\sigma}\rangle$ is a formula $(\forall\tau \prec \sigma_i)B\langle\tau,\vec{\sigma}\rangle$, then*

$$A^{(n)}\langle\vec{p}\rangle := \bigwedge_{k<p_i} B^{(n)}\langle k,\vec{p}\rangle.$$

10. *If $A\langle\vec{\sigma}\rangle$ is a formula $\exists\tau B\langle\tau,\vec{\sigma}\rangle$, then $A^{(n)}\langle\vec{p}\rangle := \bigvee_{k<n} B^{(n)}\langle k,\vec{p}\rangle$.*

11. *If $A\langle\vec{\sigma}\rangle$ is a formula $\forall\tau B\langle\tau,\vec{\sigma}\rangle$, then $A^{(n)}\langle\vec{p}\rangle := \bigwedge_{k<n} B^{(n)}\langle k,\vec{p}\rangle$.*

We easily convince ourselves that the translation $A^{(n)}\langle\vec{m}\rangle$ of an $\mathcal{L}$ formula $A\langle\vec{\sigma}\rangle$ is identical to $A\langle\vec{\sigma}\rangle$. In order to extend the previous translation to sequents of formulas from ∇^Ω we need some further notation. If Φ is a finite sequence of formulas from ∇^Ω, we write Φ^+ for the set of all Σ^Ω formulas which occur in Φ and Φ^- for the set of all formulas from Φ which do not belong to Φ^+. Hence all elements of Φ^- are Π^Ω formulas, and each element of Φ either belongs to Φ^+ or to Φ^-.

Definition 33 *Let $\vec{\sigma}$ be a finite string of stage variables, $\vec{p}$ a finite string of natural numbers of the same length and m, n any natural numbers. For any sequent $(\Phi \supset \Psi)\langle\vec{\sigma}\rangle$ of formulas from ∇^Ω we define $(\Phi \supset \Psi)^{(m,n)}\langle\vec{p}\rangle$ to be the $\mathcal{L}$ formula*

$$\bigvee_{A\langle\vec{\sigma}\rangle\in\Phi^+} (\neg A)^{(m)}\langle\vec{p}\rangle \;\vee\; \bigvee_{A\langle\vec{\sigma}\rangle\in\Phi^-} (\neg A)^{(n)}\langle\vec{p}\rangle \;\vee\; \bigvee_{A\langle\vec{\sigma}\rangle\in\Psi^+} A^{(n)}\langle\vec{p}\rangle \;\vee\; \bigvee_{A\langle\vec{\sigma}\rangle\in\Psi^-} A^{(m)}\langle\vec{p}\rangle\,.$$

The following theorem provides the desired reduction of the ∇^Ω fragment of S_Ω to ZFL. It is based on an asymmetric treatment of the existential and universal stage quantifiers in the ∇^Ω sequents.

Theorem 34 (Reduction theorem) *Let $\vec{\sigma}$ be a finite string of stage variables. Then for all sequents $(\Phi \supset \Psi)\langle\vec{\sigma}\rangle$ of formulas from ∇^Ω, all natural numbers m, n and all finite strings $\vec{p}$ of natural numbers of the same lengths as $\vec{\sigma}$ such that $\vec{p} < m$ we have*

$$\mathsf{S}_\Omega \vdash^n_* (\Phi \supset \Psi)\langle\vec{\sigma}\rangle \quad\Longrightarrow\quad \mathsf{ZFL} \vdash (\Phi \supset \Psi)^{(m,m+2^n)}\langle\vec{p}\rangle.$$

Proof. If the sequent $(\Phi \supset \Psi)\langle\vec{\sigma}\rangle$ is of the form B where B is an instance of Δ^Ω_0 separation or Δ^Ω_0 replacement, then it translates into an instance of separation or replacement for $\mathcal{L}$ formulas and is therefore provable in ZFL. In all other essential aspects the proof of this theorem is a simple adaptation of the proof of the corresponding theorem in Jäger [17], and there is no point in outlining all details here. $\square$

Corollary 35 *If A is an $\mathcal{L}$ formula provable in* $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$*, then A is already provable in* ZFL.

Proof. If A is provable in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$, then Theorem 28 implies the provability of A in ZFL^r_Ω. Hence, according to Corollary 31, there exists a natural number n for which $\mathsf{S}_\Omega \vdash^n_* \supset A$. It only remains to apply the previous reduction theorem to deduce our assertion. □

Let us summarize what we have obtained: according to Theorem 25 the theory ZFC is contained in $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$, and, conversely, by the corollary above, $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ is reducible to ZFL with respect to all $\mathcal{L}$ formulas. Furthermore, a standard result in set theory states the conservativity of ZFL over ZFC for absolute formulas, thus implying the equiconsistency of $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ and ZFC.

Corollary 36 *The theory* $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ *is conservative over* ZFC *for absolute formulas. In particular,* $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ *and* ZFC *are equiconsistent.*

This finishes this article. In subsequent publications we will deal with the non-restricted version $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ of $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ and various extensions of operational set theory by reflection principles.

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