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An interpolation theorem for proper holomorphic embeddings

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Abstract Given a Stein manifold x of dimension $n > 1$, a discrete sequence $\{a_j\} \subset X$, and a discrete sequence $\{b_j\} \subset \mathbb{C}^m$ where $m \geq N = \left[\frac{3n}{2}\right] + 1$, there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^m$ satisfying $f(a_j) = b_j$ for every $j = 1, 2, \ldots$

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1 Introduction

It is known that every Stein manifold of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$, and this N is the smallest possible by the examples of Forster [9]. The corresponding embedding theorem

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with N replaced by $N' = \left[\frac{3n+1}{2}\right] + 1$ was proved by Eliashberg and Gromov in [6] following an earlier announcement in [18]. For even values of $n \in \mathbb{N}$ we have $N = N'$ and hence their result is the best possible; for odd values of n the optimal result was obtained by Schürmann [25], also for Stein spaces with bounded embedding dimension. A key ingredient in the known proofs of these results is the homotopy principle for holomorphic sections of elliptic submersions over Stein manifolds [14,17].

In this paper we prove the following embedding theorem with interpolation on discrete sequences; for Stein spaces see Theorem 3.1.

Theorem 1.1 *Let X be a Stein manifold of dimension $n > 1$, and let $\{a_j\}_{j\in\mathbb{N}} \subset X$ and $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^m$ be discrete sequences without repetitions. If $m \geq N = \left[\frac{3n}{2}\right] + 1$ then there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^m$ satisfying*

$$f(a_j) = b_j \qquad (j = 1, 2, \ldots). \tag{1.1}$$

This result is optimal in all dimensions $n > 1$ in view of Forster's examples [9]. For even values of $n \in \mathbb{N}$ Theorem 1.1 coincides with a result of Prezelj to the effect that the conclusion holds with N replaced by $N' = \left[\frac{3n+1}{2}\right] + 1$ (Theorem 1.1 (a) in [23]). Our methods also give a different proof of Prezelj's result to the effect that, under the assumptions of Theorem 1.1 and with $m \geq \left[\frac{3n+1}{2}\right]$, there exists a proper holomorphic immersion $f\colon X \to \mathbb{C}^m$ satisfying (1.1); see Theorem 1.1 (b) in [23].

Prezelj obtained her results by carefully elaborating the constructions of Eliashberg and Gromov [6] and Schürmann [25]. It is not clear whether the method from [23] could be improved so as to give the optimal result also for odd values of n. We prove Theorem 1.1 by combining the known embedding theorems with methods of the theory of holomorphic automorphisms of Euclidean spaces.

If we increase the target dimension to $N \geq 2 \dim X + 1$ then it is possible to extend any proper holomorphic embedding $Y \hookrightarrow \mathbb{C}^N$ from an arbitrary closed complex submanifold $Y \subset X$ (not only a discrete set!) to a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^N$ [1,3,22].

Before proceeding, we recall that a discrete sequence $\{a_j\}_{j\in\mathbb{N}}$ in $\mathbb{C}^N$ is said to be *tame* in the sense of Rosay and Rudin [24] if there exists a holomorphic automorphism of $\mathbb{C}^N$ which maps a_j to the point $e_j = (j, 0, \ldots, 0)$ for $j = 1, 2, \ldots$ Several criteria for tameness can be found in [24]; for example, a sequence contained in a proper affine complex subspace of $\mathbb{C}^N$ is tame.

Theorem 1.1 follows directly from the following two results. The first one is seen by an inspection of the proofs in [6] and [25] (see Sect. 3 below). The second one is the main new result of this paper; it has been proposed in [4], and it improves the result of [21].

All sequences are assumed to be without repetition.

Theorem 1.2 (Eliashberg–Gromov–Schürmann) *Given a Stein manifold X of dimension $n > 1$ and a discrete sequence $\{a_j\}_{j\in\mathbb{N}} \subset X$, there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$ such that the sequence $\{f(a_j)\}_{j\in\mathbb{N}}$ is tame in $\mathbb{C}^N$. There also exists a proper holomorphic immersion $f\colon X \to \mathbb{C}^{[(3n+1)/2]}$ with the same property.*

Theorem 1.3 *Let $N > 1$, let X be a closed, proper complex subvariety of $\mathbb{C}^N$, and let $\{a_j\}_{j\in\mathbb{N}} \subset X$ be a discrete sequence which is tame in $\mathbb{C}^N$. For every discrete sequence $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^N$ there exist a domain $\Omega \subset \mathbb{C}^N$ containing X and a biholomorphic map $\Phi\colon \Omega \to \mathbb{C}^N$ onto $\mathbb{C}^N$ such that $\Phi(a_j) = b_j$ for $j = 1, 2, \ldots$*

Thus $X \to \Phi(X) \subset \mathbb{C}^N$ is another embedding of X into $\mathbb{C}^N$ which interpolates the given sequences. In addition one can prescribe finite order jets of $\Phi(X)$ at all points of the sequence which belong to the regular locus of the subvariety (Sect. 2). Note that Ω in Theorem 1.3 is a *Fatou-Bieberbach domain*. The fact that $\Phi(X)$ can be made to contain a given discrete sequence $\{b_j\} \subset \mathbb{C}^N$, but without matching points, had been proved (for complex lines $\mathbb{C} \hookrightarrow \mathbb{C}^2$) in [12], and in general in [10]. Not surprisingly, the interpolation is considerably more difficult to achieve.

Since any discrete sequence contained in a proper *algebraic* subvariety of $\mathbb{C}^N$ is tame [24], Theorem 1.3 applies to all discrete sequences $\{a_j\} \subset X$, $\{b_j\} \subset \mathbb{C}^N$ when X is contained in a proper algebraic subvariety of $\mathbb{C}^N$.

Example 2.4 below shows that Theorem 1.3 fails in general for non-tame sequences $\{a_j\}$. The following problem of embedding with interpolation for a given Stein manifold whose embedding dimension is lower than the general dimension N from Theorem 1.2 therefore remains open.

Problem 1.4 Let X be a Stein manifold (or a Stein space) which admits a proper holomorphic embedding into $\mathbb{C}^m$ for some $m \in \mathbb{N}$. Given discrete sequences $\{a_j\}_{j\in\mathbb{N}} \subset X$ and $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^m$ without repetitions, does there exist a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^m$ satisfying the interpolation condition (1.1)?

Since any discrete sequence in $\mathbb{C}^N = \mathbb{C}^N \times \{0\} \subset \mathbb{C}^{N+1}$ is tame in $\mathbb{C}^{N+1}$ [24], Theorem 1.3 implies the following improvement of Proposition 2.7 from [21] (adding only one extra dimension instead of two).

Corollary 1.5 *Let X be a Stein space which admits a proper holomorphic embedding into $\mathbb{C}^N$. If $m \geq N + 1$ then for any two discrete sequences $\{a_j\}_{j\in\mathbb{N}} \subset X$ and $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^m$ without repetitions there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^m$ satisfying* (1.1).

The case $\dim X = 1$, i.e., when X is an open Riemann surface, is absent from the statement and discussion of Theorem 1.1. The standard method fails when trying to embed such X into $\mathbb{C}^2$ (it embeds into $\mathbb{C}^3$, also with interpolation on discrete sets [1,3,21]). For results in this direction see the survey [11] and the recent papers of Fornæss Wold [7,8] who showed in particular that every finitely connected planar domain embeds in $\mathbb{C}^2$, thereby extending the result of Globevnik and Stensønes [16].

Problem 1.6 For which open Riemann surfaces X is Problem 1.4 solvable with $m = 2$? Is it solvable for every finitely connected planar domain?

Only two examples come to mind: X an algebraic curve in $\mathbb{C}^2$ when the result follows by applying Theorem 1.3, and X the unit disc when the interpolation theorem is due to Globevnik [15].

2 Proof of Theorem 1.3

We shall use the theory of holomorphic automorphisms of $\mathbb{C}^N$. The precise result which we shall need is the following.

Theorem 2.1 ([5], Theorems 1.1 and 1.2) *Assume that $N > 1$, $\{a_j\}$ and $\{a'_j\}$ are tame sequences in $\mathbb{C}^N$, $K \subset \mathbb{C}^N$ is a compact, polynomially convex set contained in $\mathbb{C}^N \backslash \{a_j\}$, and g is a holomorphic automorphism of $\mathbb{C}^N$ such that $g(K) \subset \mathbb{C}^N \backslash \{a'_j\}$. Then for every $\epsilon > 0$ there exists a holomorphic automorphism ϕ of $\mathbb{C}^N$ satisfying $\phi(a_j) = a'_j$ $(j = 1, 2, \ldots)$, $\sup_{z \in K} |\phi(z) - g(z)| < \epsilon$, and $\sup_{w \in g(K)} |\phi^{-1}(w) - g^{-1}(w)| < \epsilon$. In addition one can prescribe finite order jets of ϕ at the points $\{a_j\}$, and one can choose ϕ to exactly match g up to a prescribed finite order at finitely many points of K.*

The statement concerning the approximation of g^{-1} on $g(K)$ is a consequence of the approximation of g on a slightly larger polynomially convex set containing K in its interior, provided that $\epsilon > 0$ is sufficiently small.

The proof of Theorem 2.1 in [5] relies upon the developments in [2,13] and especially [10]. We shall use the special case of Theorem 2.1 when g is the identity map and the sequence $\{a'_j\}$ differs from $\{a_j\}$ only in finitely many terms. (Any modification of a tame sequence on finitely many terms is again tame.) The following lemma will provide the key step.

Lemma 2.2 *Let $\{a_j\} \subset X \subset \mathbb{C}^N$ and $\{b_j\} \subset \mathbb{C}^N$ satisfy the hypotheses of Theorem 1.3. Let $B \subset B' \subset \mathbb{C}^N$ be closed balls and $L = X \cap B'$. Assume that all points of the $\{b_j\}$ sequence which belong to $B \cup L$ coincide with the corresponding points of the $\{a_j\}$ sequence, and all remaining points of the $\{a_j\}$ sequence are contained in $X \backslash L$. Given $\epsilon > 0$ and a compact set $K \subset X$, there exist a ball $B'' \subset \mathbb{C}^N$ containing B' (B'' may be chosen as large as desired), a compact polynomially convex set $M \subset X$ with $L \cup K \subset M$, and a holomorphic automorphism θ of $\mathbb{C}^N$ satisfying the following properties*:

(i) $|\theta(z) - z| < \epsilon$ *for all* $z \in B \cup L$,
(ii) *if* $a_j \in M$ *for some index* j *then* $\theta(a_j) = b_j \in B''$,
(iii) *if* $b_j \in B' \backslash (B \cup L)$ *for some* j *then* $a_j \in M$ *and* $\theta(a_j) = b_j$,

(iv) $\theta(M) \subset \mathrm{Int}B''$, *and*
(v) *if* $a_j \in X\backslash M$ *for some* j *then* $\theta(a_j) \in \mathbb{C}^N\backslash B''$.

Remark 2.3 If θ satisfies the conclusion of Lemma 2.2 then the set

$$L' = \{z \in X \colon \theta(z) \in B''\}$$

contains M (and hence $K \cup L$), and $L'\backslash M$ does not contain any points of the $\{a_j\}$ sequence (since the θ-image of any point $a_j \in X\backslash M$ lies outside of B'' according to (v)).

Proof An automorphism θ of $\mathbb{C}^N$ with the required properties will be constructed in two steps, $\theta = \psi \circ \phi$.

Since $X \cap B \subset L \subset X$ and the sets B and L are polynomially convex, $B \cup L$ is also polynomially convex (see e.g., Lemma 6.5 in [10], p. 111).

By applying a preliminary automorphism of $\mathbb{C}^N$ which is very close to the identity map on $B \cup L$ we may assume that X does not contain any points of the $\{b_j\}$ sequence, except those which coincide with the corresponding points $a_j \in X$. The same procedure will be repeated whenever necessary during later stages of the construction without mentioning it again.

Choose a pair of compact, polynomially convex neighborhoods $D_0 \subset D \subset \mathbb{C}^N$ of $B \cup L$, with $D_0 \subset \mathrm{Int}D$, such that D does not contain any additional points of the $\{a_j\}$ or the $\{b_j\}$ sequence. Choose $\epsilon_0 > 0$ so small that

$$\mathrm{dist}(B \cup L, \mathbb{C}^N\backslash D_0) > \epsilon_0, \quad \mathrm{dist}(D_0, \mathbb{C}^N\backslash D) > \epsilon_0.$$

By decreasing $\epsilon > 0$ if necessary we may assume $0 < \epsilon < \epsilon_0$.

Choose a compact polynomially convex set $M \subset X$ containing $K \cup (X \cap D)$ (and hence the set L), and also containing all those points of the $\{a_j\}$ sequence for which the corresponding point b_j is contained in the ball B'. (Of course M may also contain some additional points of the $\{a_j\}$ sequence for which $b_j \in \mathbb{C}^N\backslash B'$.) Theorem 2.1 furnishes an automorphism ϕ of $\mathbb{C}^N$ satisfying the following:

(a) $\sup_{z\in D} |\phi(z) - z| < \frac{\epsilon}{2}$ and $\sup_{z\in D} |\phi^{-1}(z) - z| < \frac{\epsilon}{2}$,
(b) $\phi(a_j) = b_j$ for all $a_j \in M$, and
(c) $\phi(a_j) = a_j$ for all $a_j \in X\backslash M$.

Condition (a) and the choice of ϵ imply $\phi(D_0) \subset D$ and $\phi(\mathbb{C}^N\backslash D) \cap D_0 = \emptyset$, and the latter condition also implies $\phi(X) \cap D_0 \subset \phi(M)$. Since the sets $\phi(M)$ and D_0 are polynomially convex, their union $\phi(M) \cup D_0$ is also polynomially convex (Lemma 6.5 in [10]).

Choose a large ball $B'' \subset \mathbb{C}^N$ containing $\phi(M) \cup B'$. Theorem 2.1 furnishes an automorphism ψ of $\mathbb{C}^N$ satisfying the following:

(a') $|\psi(z) - z| < \frac{\epsilon}{2}$ when $z \in \phi(M) \cup D_0$,
(b') $\psi(\phi(a_j)) = \phi(a_j) = b_j$ for all $a_j \in M$, and
(c') $\psi(a_j) \in \mathbb{C}^N\backslash B''$ for all $a_j \in X\backslash M$.

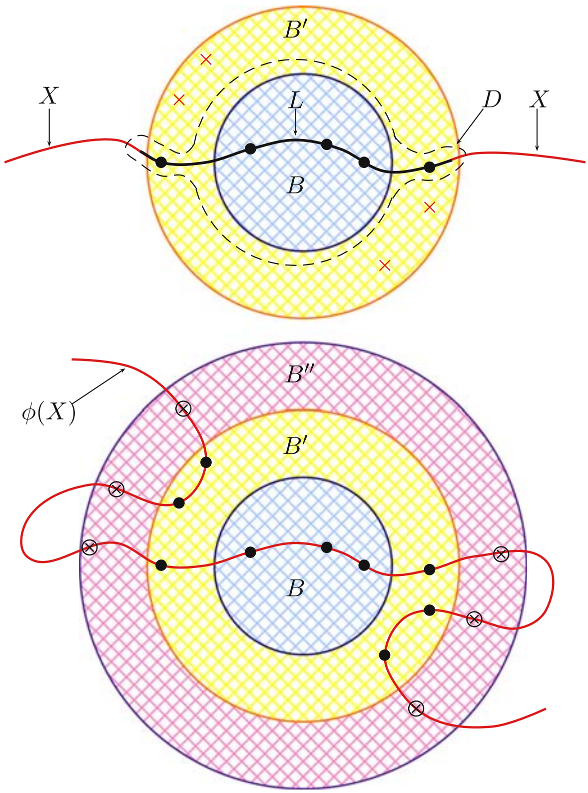

Fig. 1 The proof of Lemma 2.2

We may also require that ψ fixes all points $\phi(a_j) \in \phi(X)\backslash B''$. It is immediate that $\theta = \psi \circ \phi$ satisfies the conclusion of Lemma 2.2. □

The scheme of proof of Lemma 2.2 is illustrated in Fig. 1. The first drawing shows the initial situation; the thick dots on X indicate the points $b_j \in B \cup L$ which agree with the corresponding points a_j, while the crosses indicate the remaining points $b_j \in B'$ which will be matched with the images of a_j by applying the automorphism ϕ. The second drawing shows the situation after the application of ϕ: The large black dots in $\phi(X) \cap B'$ indicate the points $b_j = \phi(a_j) \in B'$, while the crossed dots on the subvariety $\phi(X)$ inside the set $B''\backslash B'$ will be expelled from the ball B'' by the next automorphism ψ.

Proof of Theorem 1.3 Choose an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ by compact sets. Fix a number ϵ with $0 < \epsilon < 1$. We shall inductively construct the following:

(a) A sequence of holomorphic automorphisms Φ_k of $\mathbb{C}^N$ ($k \in \mathbb{N}$),
(b) An exhaustion $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} L_j = X$ by compact, polynomially convex sets,
(c) A sequence of balls $B_1 \subset B_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} B_j = \mathbb{C}^N$ centered at $0 \in \mathbb{C}^N$ whose radii satisfy $r_{k+1} > r_k + 1$ for $k = 1, 2, \ldots$,

such that the following hold for all $k = 1, 2, \ldots$ (conditions (iv) and (v) are vacuous for $k = 1$):

(i) $\Phi_k(L_k) = \Phi_k(X) \cap B_{k+1}$,
(ii) if $a_j \in L_k$ for some j then $\Phi_k(a_j) = b_j$,
(iii) if $b_j \in \Phi_k(L_k) \cup B_k$ for some j then $a_j \in L_k$ and $\Phi_k(a_j) = b_j$,
(iv) $L_{k-1} \cup K_{k-1} \subset \mathrm{Int} L_k$,
(v) $|\Phi_k(z) - \Phi_{k-1}(z)| < \epsilon\, 2^{-k}$ for all $z \in B_{k-1} \cup L_{k-1}$.

To begin we set $B_0 = \emptyset$ and choose a pair of balls $B_1 \subset B_2 \subset \mathbb{C}^N$ whose radii satisfy $r_2 \geq r_1 + 1$. Theorem 2.1 furnishes an automorphism Φ_1 of $\mathbb{C}^N$ such that $\Phi_1(a_j) = b_j$ for all those (finitely many) indices j for which $b_j \in B_2$, and $\Phi_1(a_j) \in \mathbb{C}^N \backslash B_2$ for the remaining indices j. (Of course we only need to move finitely many points of the $\{a_j\}$ sequence.) Setting $L_1 = \{z \in X : \Phi_1(z) \in B_2\}$, the properties (i), (ii) and (iii) are satisfied for $k = 1$ and the remaining two properties (iv), (v) are void.

Assume inductively that we have already found sets $L_1, \ldots, L_k \subset X$, balls $B_1, \ldots, B_{k+1} \subset \mathbb{C}^N$ and automorphisms $\Phi_1, \ldots, \Phi_k$ such that (i)–(v) hold up to index k. We now apply Lemma 2.2 with $B = B_k$, $B' = B_{k+1}$, X replaced by $X_k = \Phi_k(X) \subset \mathbb{C}^N$, and $L = \Phi_k(L_k) \subset X_k$. This gives us a compact polynomially convex set $M = M_k \subset X_k$ containing $\Phi_k(K_k \cup L_k)$, an automorphism $\theta = \theta_k$ of $\mathbb{C}^N$, and a ball $B'' = B_{k+2} \subset \mathbb{C}^N$ of radius $r_{k+2} \geq r_{k+1} + 1$ such that the conclusion of Lemma 2.2 holds. In particular, $\theta_k(M_k) \subset B_{k+2}$, the interpolation condition is satisfied for all points $b_j \in \theta_k(M_k) \cup B_{k+1}$, and the remaining points in the sequence $\{\Phi_k(a_j)\}_{j \in \mathbb{N}}$ are sent by θ_k out of the ball B_{k+2}. Setting

$$\Phi_{k+1} = \theta_k \circ \Phi_k, \quad L_{k+1} = \{z \in X : \Phi_{k+1}(z) \in B_{k+2}\}$$

one easily checks that the properties (i)–(v) hold for the index $k + 1$ as well. (Note that L_{k+1} corresponds to the set L' in Remark 2.3). The induction may now continue.

Let Ω consist of all points $z \in \mathbb{C}^N$ for which the sequence $\{\Phi_k(z)\}_{k \in \mathbb{N}}$ remains bounded. Proposition 5.2 in [10] (p. 108) implies that $\lim_{k \to \infty} \Phi_k = \Phi$ exists on Ω, the convergence is uniform on compacts in Ω, and $\Phi : \Omega \to \mathbb{C}^N$ is a biholomorphic map of Ω onto $\mathbb{C}^N$ (a Fatou-Bieberbach map). In fact, $\Omega = \bigcup_{k=1}^{\infty} \Phi_k^{-1}(B_k)$ (Proposition 5.1 in [10]). From (v) we see that $X \subset \Omega$, and properties (ii), (iii) imply that $\Phi(a_j) = b_j$ for all $j = 1, 2, \ldots$ This completes the proof of Theorem 1.3.

Example 2.4 We show that Theorem 1.3 is not valid in general if $\{a_j\}$ is a non-tame sequence in $\mathbb{C}^N$. Choose a sequence $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^N$ whose complement $\mathbb{C}^N \backslash \{a_j\}_{j \in \mathbb{N}}$ is Eisenman N-hyperbolic [20,24]. As already mentioned in the introduction, any complex subvariety $X \subset \mathbb{C}^N$ can be embedded in $\mathbb{C}^N$ so that its image contains a given sequence [10], and hence we may assume that $\{a_j\}_{j \in \mathbb{N}} \subset X$. Assume that Theorem 1.3 holds, i.e., there is a biholomorphic map $\Phi : \Omega \to \mathbb{C}^N$ from a domain $\Omega \subset \mathbb{C}^N$ containing X onto $\mathbb{C}^N$ satisfying $\Phi(a_j) = b_j$ for all $j = 1, 2, \ldots$ The set $\Omega \backslash \{a_j\}_{j \in \mathbb{N}}$, being contained in $\mathbb{C}^N \backslash \{a_j\}_{j \in \mathbb{N}}$, is Eisenman

N-hyperbolic, and hence its Φ-image $\mathbb{C}^N\backslash\{b_j\}_{j\in\mathbb{N}}$ is Eisenman N-hyperbolic as well. But this is not true in general, for instance if the sequence $\{b_j\}_{j\in\mathbb{N}}$ is tame in $\mathbb{C}^N$.

3 Embedding Stein spaces with interpolation

We begin by indicating how Theorem 1.2 is obtained from Schürmann's proof in [25].

One begins by choosing a sufficiently generic almost proper holomorphic map $b\colon X\to\mathbb{C}^n$ with $n=\dim X$; this means that there are sequences of compact special analytic polyhedra $K_1\subset K_2\subset\cdots\subset\bigcup_{j\in\mathbb{N}}K_j=X$ and polydiscs $P_1\subset P_2\subset\cdots\subset\bigcup_{j\in\mathbb{N}}P_j=\mathbb{C}^n$ such that $b|_{K_j}\colon K_j\to P_j$ is a proper map sending the boundary ∂K_j to ∂P_j for every $j=1,2,\ldots$ Such maps were first constructed by Bishop [3] where the reader can find more details; another source is Chapter VII in [19].

For a fixed b as above one then constructs a holomorphic map $g\colon X\to\mathbb{C}^{N-n}$ such that $f=(b,g)\colon X\hookrightarrow\mathbb{C}^N$ is a proper holomorphic embedding. The map g is obtained as the limit $g=\lim_{k\to\infty}g_k$ where the map $g_k\colon X\to\mathbb{C}^{N-n}$ accomplishes the job on K_k and it approximates g_{k-1} uniformly on K_{k-1}. The map g has three tasks: to insure properness (this is done by choosing $|g_k|$ sufficiently large on $K_k\backslash K_{k-1}$), to eliminate the kernel of the differential of b, and to separate pairs of points which are not separated by b. Such map can be found by the 'elimination of singularities' method, due to Eliashberg and Gromov [6], which proceeds by a finite induction over strata in a suitable stratification of X. When extending the map from one stratum to the next one uses the h-principle for sections of elliptic submersions [14,17]. For the present purposes it is not necessary to understand this method completely, and we refer the reader to [6] and [25] for further details.

Suppose now that $\{a_j\}$ is a discrete sequence in X. It is possible to choose the exhaustion of X by special analytic polyhedra K_k as above such that $K_k\backslash K_{k-1}$ contains at most one point of the sequence for each k. Call this point a_k. When constructing the map g_k (which fulfills the relevant conditions on K_k) it now suffices to require that the modulus of the last component of the point $g_k(a_k)$ is sufficiently large; it was already observed in [23,25] that this condition is easily built into the construction. In this way we can achieve that the last components of the sequence $\{g(a_j)\}_{j\in\mathbb{N}}$ form a discrete sequence (without repetitions) in $\mathbb{C}$. It follows from standard methods (see e.g., [24]) that the sequence $f(a_j)=(b(a_j),g(a_j))\in\mathbb{C}^N$ is then tame. This proves Theorem 1.2.

Essentially the same proof applies if X is a (reduced) Stein space with singularities and with bounded embedding dimension [25]. Let $\mathrm{Embdim}_x X$ denote the local embedding dimension of X at x, that is, the smallest integer such that the germ of X at x embeds as a local closed complex subvariety of the Euclidean space of that dimension. Assume that

$$q=\mathrm{Embdim}\,X:=\sup_{x\in X}\mathrm{Embdim}_x X<+\infty.$$

Let $n(k)$ denote the dimension of the analytic set of points in X at which X has embedding dimension at least k. Set

$$b'(X) = \max\{k + \lceil n(k)/2 \rceil : k = 0, \dots, q\}.$$

With this notation we have the following result, extending Theorem 1.1.

Theorem 3.1 *Let $n > 1$ and let X be an n-dimensional Stein space of finite embedding dimension. Let $m \geq N = \max\{\left\lceil \frac{3n}{2} \right\rceil + 1, b'(X)\}$. Given discrete sequences $\{a_j\} \subset X$ and $\{b_j\} \subset \mathbb{C}^m$ without repetitions, there exists a proper holomorphic embedding $f \colon X \hookrightarrow \mathbb{C}^m$ satisfying $f(a_j) = b_j$ for $j = 1, 2, \dots$*

Theorem 3.1 is proved in the same way as Theorem 1.1 by first embedding X into $\mathbb{C}^m$ so that $\{a_j\}$ is mapped to a tame sequence in $\mathbb{C}^m$ (this is accomplished by the modification of the proof in [25] described above), and subsequently applying Theorem 1.3.

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