Econ Theory (2009) 40:509–528
DOI 10.1007/s00199-008-0383-3

EXPOSITA NOTE

# Optimal auction with resale—a characterization of the conditions

**Tymofiy Mylovanov · Thomas Tröger**

Received: 16 November 2006 / Accepted: 20 May 2008 / Published online: 3 June 2008


**Abstract** Zheng has proposed a seller-optimal auction for (asymmetric) independent-private-value environments where inter-bidder resale is possible. Zheng's construction requires novel assumptions—Resale Monotonicity, Transitivity, and Invariance—on the bidders' value distribution profile. The only known examples of distribution profiles satisfying these assumptions in environments with three or more bidders are uniform distributions. Using inverse virtual valuation functions as a novel tool, we characterize the set of distribution profiles satisfying Zheng's assumptions. Our characterization result shows that the assumptions, while being strong, are satisfied by many non-uniform distribution profiles. Hence, Zheng's result applies more generally than one may have thought before. A crucial step in our analysis is to show that Invariance implies Resale Monotonicity and Transitivity.

**Keywords** Independent private values · Optimal auction · Resale · Inverse virtual valuation function

**JEL Classification** D44

We thank Rod Garratt, Benny Moldovanu, Georg Nöldeke, and Charles Zheng for helpful comments. Financial Support by the German Science Foundation (DFG) through SFB/TR 15 "Governance and the Efficiency of Economic Systems" is gratefully acknowledged.

T. Mylovanov
Department of Economics, Penn State University, University Park, PA 16802, USA
e-mail: mylovanov@gmail.com

T. Tröger (✉)
Department of Economics, University of Bonn, Adenauerallee 24–26, 53113 Bonn, Germany
e-mail: ttroeger@uni-bonn.de

## 1 Introduction

The optimal sales mechanism for a seller who faces privately informed buyers typically leads to a *biased* allocation: with positive probability, the seller allocates the good to a buyer whose valuation is not the highest in the market (Myerson 1981).[1] Yet, in spite of the bias inherent in the Myerson allocation, by assumption the winning buyer cannot attempt to resell the good. Although no resale is reasonable in some environments, such an assumption is less compelling in many others. Many types of goods, from art and wine to radio spectrum, can be, and are, resold.[2]

Resale opportunities create an environment where the allocation implemented by the initial seller's mechanism is *intermediate*: any allocation bias will be reduced by further sales transactions. Moreover, the anticipation of resale changes the buyers' incentives in the initial mechanism, potentially decreasing the initial seller's profit.

Zheng (2002) demonstrates that in *some* environments it is possible for a seller who cannot prohibit resale to obtain the *same* profit as when she can prohibit resale.[3] The seller (as well as every re-seller) uses a mechanism that *exaggerates* the allocation bias beyond what would be optimal in the absence of resale; a buyer who wins will partly undo the bias by reselling the good so that the final allocation is the Myerson allocation. Thus, by the revenue equivalence theorem, the initial seller obtains the same profit as when she can prohibit resale.

The purpose of our paper is to obtain a precise characterization of the environments in which Zheng's scheme works; from the previous literature, the only known examples of such environments feature uniformly distributed buyer valuations. We consider buyers with valuations that are smoothly and stochastically independently distributed with increasing hazard rates (Zheng's Hazard Rate assumption), and the distributions are, in a stochastic sense, ranked across buyers (Zheng's Uniform Bias assumption).[4] We focus on the three novel assumptions that are required for Zheng's scheme: Resale Monotonicity (RM), Transitivity (TR), and Invariance (IV). We characterize the set of profiles of probability distributions (c.d.f.s) for which the novel assumptions hold.[5]

Our results imply that in environments with two buyers Zheng's assumptions are relatively weak. For example, they are satisfied whenever the stochastically higher-ranked buyer's valuation has a weakly decreasing density. In environments with three or more buyers, Zheng shows that the assumptions hold for uniform distributions;

[1] This type of inefficiency is absent if the buyers are ex-ante symmetric (Riley and Samuelson 1981).

[2] In some environments, a motive for resale is that a re-seller has access to a bigger portion of the market than the initial seller. See Bose and Deltas (2007) for an analysis of exclusive dealing in such environments.

[3] Zheng models a sequential mechanism selection game. Other recent contributions to the growing literature on mechanism design with limited commitment include Bester and Strausz (2001); Calzolari and Pavan (2006), and Skreta (2006).

[4] Lebrun (2005) relaxes Uniform Bias in the two-buyer case, and shows that a generalized version of Zheng's scheme still works.

[5] Zheng's assumptions appear to be the "natural" ones. However, a general proof that Zheng's novel assumptions are *necessary* for achieving the Myerson allocation in *any* equilibrium of his game is not available. Lebrun (2005, Appendix 9) provides a proof for a subset of intermediate allocation rules in the two-buyer case.

our characterization result implies that the assumptions are rather strong, but include many non-uniform distributions.

What are the difficulties underlying the design of an optimal sales mechanism when resale cannot be prevented? To get the Myerson allocation as the final allocation, the seller must implement an intermediate allocation that anticipates two potential conflicts between any current owner of the good and any future owners: first, the conflict about *whether* to resell or keep the good, and, second, the conflict about *to whom* to resell.

Zheng's assumption Resale Monotonicity (RM) addresses the conflict of two successive owners about whether to resell or keep the good. Suppose the intermediate allocation is biased such that, through optimal resale from one buyer to another, the final allocation is consistent with the Myerson allocation. Assumption RM requires that the intermediate allocation has a monotonicity property: the probability that a buyer obtains the good is weakly increasing in her valuation. This guarantees that the scheme of intermediate and final allocations is incentive compatible.

Zheng's assumption Invariance (IV) addresses the conflict of successive owners about to whom to resell. Suppose that the current owner sets up an intermediate allocation such that each buyer resells (rather than keeps) the good if and only if the current owner wants her to resell according to the intended final Myerson allocation. This intermediate allocation rule induces certain posterior beliefs about the buyers' valuations. In general, given the updated beliefs the next owner wants a different allocation bias than the current owner. Assumption IV requires the distributions to be such that successive owners want the same allocation bias.[6]

In environments with two buyers, implementing the Myerson allocation through resale is relatively easy because one of the two fundamental conflicts between successive owners is absent: there can be no conflict about to whom to resell. The initial seller must only align the first buyer's incentives about whether or not to resell. Accordingly, assumptions TR and IV are empty and RM is the main assumption. Our characterization result shows that RM is satisfied, for example, if the stochastically higher-ranked buyer's valuation has a weakly decreasing density or if both buyers' decumulative distributions have a power form. There also exist examples where RM fails (see footnote 10).

In environments with three or more buyers, IV is the main assumption; we show that it implies RM and TR. Index the buyers in the order of increasing stochastic rank. For any given distributions of buyer 1 and buyer $n$, and any profile of nested supports, our characterization result shows that there exists *at most one* profile of distributions for the buyers 2 to $n-1$ such that IV is satisfied. Such a profile exists if and only if the distribution of buyer $n$ has a weakly decreasing density. In particular, Zheng's assumptions are satisfied for many non-uniform distributions.

Our characterization result shows that the supports together with buyer $n$'s c.d.f. fully determine the c.d.f.s of all other buyers except buyer 1's. In this sense, Zheng's assumptions are strong.

[6] Zheng's third novel assumption, Transitivity (TR), is intended to exclude the possibility of cycles of resale transactions. We show that TR is implied by IV and, hence, is not of independent importance.

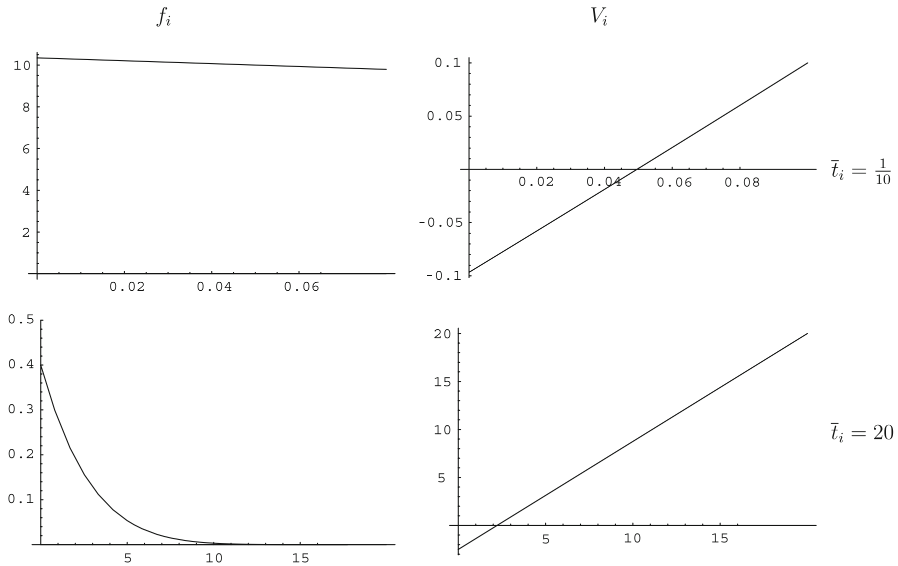


**Fig. 1** Density $f_i$ ($2 \leq i \leq n-1$) and virtual valuation function $V_i$ implied by Zheng's assumptions, for different values of $\bar{t}_i$, where $F_n(t) = 1 - ((20-t)/20)^8$ on $T_n = [0, 20]$, and $T_i = [0, \bar{t}_i]$. If $\bar{t}_i = 1/10$, then $f_i$ is approximately constant (i.e., $F_i$ is uniform). If $\bar{t}_i = 20$, then $F_i = F_n$

The implications of Zheng's assumptions are particularly striking if the highest possible valuation is the same for all buyers: there will be at most one resale transaction on the equilibrium path, and the c.d.f.s for buyers 2 to $n-1$ must be *affine transformations* of buyer $n$'s c.d.f. If, in addition, all buyers have the same lowest valuation, then the c.d.f.s for buyers 2 to $n$ are identical.

In cases where the highest possible valuation differs across buyers, closed-form solutions for c.d.f.s can be obtained only in exceptional cases, and there can be up to $n-1$ resale transactions. Intuition about the implications of Zheng's assumptions can nevertheless be obtained from our characterization. Specifically, keeping buyer $n$'s c.d.f. fixed we can picture how buyer $i$'s ($2 \leq i \leq n-1$) implied c.d.f. depends on the support $T_i$ of buyer $i$'s c.d.f.. If $T_i$ equals the support for buyer $n$, then buyer $i$'s c.d.f. is identical to buyer $n$'s. As $T_i$ shrinks to a smaller interval, buyer $i$'s c.d.f. starts becoming different from buyer $n$'s, and, if $T_i$ is sufficiently small, becomes approximately uniform (cf. Fig. 1).

Our main technical contribution is the use of inverse virtual valuation functions as a tool of analysis. According to Myerson (1981), to each possible buyer valuation a *virtual* valuation is assigned, which depends on the distribution of the buyer's valuation. A seller optimally allocates her good not to the buyer with the highest valuation, but to the buyer with the highest virtual valuation.[7] The inverse relationship that maps virtual valuations into actual valuations is the buyer's inverse virtual valuation function

[7] A buyer's virtual valuation can be interpreted as the marginal revenue of a discriminating monopolist when selling to the buyer (Bulow and Roberts 1989).

(IVVF). Our characterization result formulates Zheng's novel assumptions purely in terms of IVVFs.

Working with IVVFs is natural because Zheng's assumptions relate the buyers' distribution functions at points where the buyers' virtual valuations coincide, while the actual valuations are different. Formulating such assumptions in terms of IVVFs makes virtual valuation become the independent variable, and thus allows us to transform the assumptions into differential inequalities and equations that can be solved with standard methods. Once the IVVFs are determined, the distributions of the buyers' valuations can be computed using straightforward methods.

## 2 Outline of the results

We begin by introducing Zheng's assumption Hazard Rate (HR), which is a standard smoothness and hazard-rate monotonicity condition, and by introducing notation concerning virtual valuation functions. By HR, the inverse virtual valuation function (IVVF) exists for every buyer. Formulae about how a buyer's IVVF is related to her c.d.f., density, and hazard rate are provided. Next we introduce Zheng's assumption Uniform Bias (UB), which simplifies the analysis by introducing an unambiguous stochastic ranking of the buyers. HR and UB are maintained throughout the analysis.

We recall Zheng's notion of a bid inflation function and derive simple properties. Bid inflation functions play a crucial role, because they define the seller-intended degree of bias in an intermediate allocation towards any buyer relative to any higher-ranked buyer.

Next Resale Monotonicity (RM), the first of Zheng's novel assumptions, is recalled. RM requires that bid inflation functions are weakly increasing. Proposition 1 reformulates this monotonicity property in terms of densities, hazard rates, and IVVFs, with virtual valuation as the independent variable. By expressing densities and hazard rates in terms of virtual valuation functions, we use Proposition 1 to reformulate RM purely in terms of IVVFs, as a set of differential inequalities (Corollary 1). In environments with two buyers, RM is equivalent to one differential inequality. We show how solving this inequality and back-translating IVVFs into c.d.f.s leads to insights about the implications of RM.

Turning to environments with three or more buyers, Zheng's assumptions Transitivity (TR) and Invariance (IV) become relevant. Proposition 2 reformulates IV in terms of densities and IVVFs, with virtual valuation as the independent variable. Combining this result with Proposition 1, it is easy to see that IV implies RM (Corollary 2).

We next introduce "outbidding" as a binary relation on the set of buyers that is defined in terms of the bid inflation functions. We use this relation to define a stronger and simplified version TR* of Zheng's assumption TR. Corollary 3 shows that IV implies TR* and hence TR.

By expressing densities in terms of virtual valuation functions, we use Proposition 2 to reformulate IV purely in terms of inverse virtual valuation functions, as a set of differential equations, supplemented by inequalities (Corollary 4). By solving the equations we obtain the formulae that Zheng's assumptions imply for the IVVFs (Proposition 3). We draw some qualitative conclusions from these formulae. Back-translating these formulae in terms of the buyers' c.d.f.s is possible in two extreme cases: if the highest possible valuation is the same for all buyers except buyer 1 (Corollary 5), and if the support of a buyer's c.d.f. is small compared to the support of buyer $n$ (Corollary 6). In general, closed-form solutions for c.d.f.s cannot be obtained; we discuss a parametric class of distributions (Corollary 7) and determine c.d.f.s numerically (Fig. 1).

Proofs are relegated to the Appendix.

## 3 Results

We reiterate only those aspects of Zheng's model that are needed to state and analyze his assumptions. Consider an independent-private-value auction environment with $n \geq 2$ buyers. The distribution (c.d.f.) for the valuation of buyer $i = 1, \ldots, n$ is denoted $F_i$ with support $T_i$.

Assumption 1 (HR) of Zheng consists of standard elements and needs no further discussion.

**Assumption 1** (Hazard rate) For each player $i$, the support $T_i$ of $F_i$ is convex and bounded from below. If $T_i$ is a non-degenerate interval, the density function $f_i$ is positive and continuous on $T_i$ and differentiable in its interior, and $(1 - F_i(t_i))/f_i(t_i)$ is a weakly decreasing function of $t_i$ on $T_i$.

We add the assumptions that for all $i$, the support $T_i$ is non-degenerate and bounded, the derivative $f_i'$ exists at the boundary of $T_i$, and $f_i'$ is continuous on $T_i$. Let $\underline{t}_i = \min T_i$ and $\bar{t}_i = \max T_i$. Define the hazard rate $\lambda_i(t_i) = f_i(t_i)/(1 - F_i(t_i))$ for all $t_i < \bar{t}_i$.

The virtual valuation functions $V_i$ $(i = 1, \ldots, n)$ are defined by $V_i(t_i) = t_i - (1 - F_i(t_i))/f_i(t_i)$ $(t_i \in T_i)$. Given the above assumptions, the derivative $V_i'$ exists and is continuous and $\geq 1$. Moreover,

$$V_i(T_i) = [V_i(\underline{t}_i), \bar{t}_i] \quad (i = 1, \ldots, n). \tag{1}$$

The inverse virtual valuation function (IVVF) $V_i^{-1}$ is well-defined on $V_i(T_i)$. The derivative $(V_i^{-1})'$ is continuous and

$$\forall v_i \in V_i(T_i) : \ (V_i^{-1})'(v) \in (0, 1]. \tag{2}$$

A straightforward computation shows that

$$f_i \text{ weakly decreasing} \Leftrightarrow (V_i^{-1})' \geq 1/2. \tag{3}$$

Because [see, e.g., Krishna (2002, p. 255)],

$$F_i(t) = 1 - \exp\left(-\int_{\underline{t}_i}^{t} \lambda_i(t')\mathrm{d}t'\right), \tag{4}$$

and $\lambda_i(t) = 1/(t - V_i(t))$ for all $t \in [\underline{t}_i, \bar{t}_i)$,

$$f_i(V_i^{-1}(v)) = \frac{1}{V_i^{-1}(v) - v} \mathrm{e}^{-\int_{\underline{t}_i}^{V_i^{-1}(v)} \frac{1}{t' - V_i(t')} \mathrm{d}t'} \quad (v \in [V_i(\underline{t}_i), \bar{t}_i)). \tag{5}$$

Moreover, by definition of $\lambda_i$,

$$\lambda_i(V_i^{-1}(v)) = \frac{1}{V_i^{-1}(v) - v} \quad (v \in [V_i(\underline{t}_i), \bar{t}_i))). \tag{6}$$

Using (5) and (6), we can determine the marginal effect of a change of a buyer's virtual valuation on her logarithmic hazard rate and logarithmic density,

$$\frac{\mathrm{d}}{\mathrm{d}v} \ln \lambda_i(V_i^{-1}(v)) = \frac{1 - (V_i^{-1})'(v)}{V_i^{-1}(v) - v} \quad (v \in [V_i(\underline{t}_i), \bar{t}_i)), \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}v} \ln f_i(V_i^{-1}(v)) = \frac{1 - 2(V_i^{-1})'(v)}{V_i^{-1}(v) - v} \quad (v \in [V_i(\underline{t}_i), \bar{t}_i)). \tag{8}$$

Assumption 2 (UB) of Zheng states that the buyers $i = 1, \ldots, n$ can be ranked in terms of the support $T_i$ and of the virtual valuation function $V_i$. Observe that Assumption 2 is equivalent to hazard rate dominance if $T_1 = \cdots = T_n$.[8]

**Assumption 2** (Uniform bias) For all $i, j = 1, \ldots, n$, if $i < j$ then $T_i \subseteq T_j$ and $V_i(x) \geq V_j(x)$ for all $x \in T_i$.

By (1) and UB,

$$\forall i, j = 1, \ldots, n: \quad \text{if } i < j \text{ then } V_i(T_i) \subseteq V_j(T_j). \tag{9}$$

For all $i$ and $x \in (\underline{t}_i, \bar{t}_i]$, let

$$V_i^x(t_i) = t_i - \frac{F_i(x) - F_i(t_i)}{f_i(t_i)} \quad (t_i \in [\underline{t}_i, x])$$

denote the virtual valuation function for buyer $i$ given the information that her type belongs to the interval $[\underline{t}_i, x]$. For all $i < j$, Zheng (2002, p. 2210) defines *bid-inflation functions* $\beta_{ij} : T_i \to T_j$ implicitly by[9]

$$V_j^{\beta_{ij}(t_i)}(V_j^{-1}(V_i(t_i))) = t_i. \tag{10}$$

[8] Hazard rate dominance is a stronger requirement than stochastic dominance and a weaker requirement than likelihood ratio dominance [see, e.g., Krishna (2002, Appendix B)].

[9] Observe that the functions $\beta_{ij}$, are, in general, well-defined only if UB holds. Lebrun (2005) relaxes UB and extends the definition of $\beta_{ij}$. Accordingly, he obtains a generalized version of RM.

The $\beta_{ij}$ functions play a central role. If type $t_i$ of buyer $i$ believes that buyer $j$'s type belongs to $[\underline{t}_j, \beta_{ij}(t_i)]$, then she optimally resells to the buyer-$j$ types in $[V_j^{-1}(V_i(t_i)), \bar{t}_j]$, so that buyer $i$'s resale decision is aligned with the initial seller's intended final Myerson (1981) allocation. To rewrite (10) more explicitly, subtract $V_i(t_i)$ on the right-hand side and $V_j(V_j^{-1}(V_i(t_i)))$ on the left-hand side. This yields

$$\frac{1 - F_j(\beta_{ij}(t_i))}{f_j(V_j^{-1}(V_i(t_i)))} = t_i - V_i(t_i). \tag{11}$$

Hence, $\beta_{ij}$ is continuous and

$$\beta_{ij}(\bar{t}_i) = \bar{t}_j. \tag{12}$$

Moreover, from (10),

$$\beta_{ij}(t_i) \geq V_j^{-1}(V_i(t_i)) \quad (t_i \in T_i). \tag{13}$$

To guarantee incentive compatibility of the allocation scheme, Zheng makes Assumption 3 (RM).

**Assumption 3** (Resale monotonicity) For all $i, j = 1, \ldots, n$, if $i < j$ then $\beta_{ij}$ is weakly increasing.

Because (10) relates the buyers' distributions at points where they tie with their virtual valuations, RM can be expressed most transparently by using IVVFs.

**Proposition 1** *Suppose that HR and UB hold. Then RM holds if and only if, for all $i < j$,*

$$\frac{f_j(V_j^{-1}(v))}{\lambda_i(V_i^{-1}(v))} \quad \text{is weakly decreasing for all } v \in [V_i(\underline{t}_i), \bar{t}_i). \tag{14}$$

That is, at any point where buyers $i$ and $j$ tie with their virtual valuations, the ratio of buyer $j$'s density and buyer $i$'s hazard rate must not increase as the virtual valuation increases. For instance, RM is satisfied if the densities of buyers' 2 to $n$ are weakly decreasing (given HR). To obtain a computationally useful reformulation of (14), one takes the logarithm so that quantities referring to buyer $i$ become additively separated from quantities referring to buyer $j$ and requires that the derivative is non-positive. This yields a characterization of RM in terms of differential inequalities (15) involving IVVFs.

**Corollary 1** *Suppose that HR and UB hold. Then RM holds if and only if, for all $i < j$,*

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \frac{1 - (V_i^{-1})'(v)}{V_i^{-1}(v) - v} \geq \frac{1 - 2(V_j^{-1})'(v)}{V_j^{-1}(v) - v}. \tag{15}$$

It is not possible to simplify (15) by using additional properties of virtual valuation functions, because there are essentially no additional properties: any continuously differentiable function defined on an interval $[\underline{t}_i, \check{t}]$ ($\check{t} < \bar{t}_i$) with derivative not smaller than 1 and values below the identity function can be extended to the virtual valuation function of some c.d.f. $F_i$ satisfying HR (to see this, use, e.g., Krishna 2002, p. 255).

In environments with two buyers, the assumptions assumed in Zheng (2002) are HR, UB, and RM. Corollary 1 can then be used to compute all c.d.f. profiles satisfying Zheng's assumptions: for any given c.d.f. of buyer $n = j = 2$, (15) provides a linear differential inequality for the IVVF of buyer $i = 1$, from which buyer 1's possible c.d.f.s can be computed. Observe that, if buyer $n = 2$ has a weakly decreasing density, then the left-hand side of (15) with $i = 1$ is $\geq 0$ by (2), and the right-hand side of (15) with $j = 2$ is $\leq 0$ by (3); hence, RM is satisfied.[10]

To give another 2-buyer example, suppose that the buyers' decumulative distribution functions have a power form, that is, there exist numbers $b_1, b_2 > 0$ such that

$$F_i(t) = 1 - \left(\frac{\bar{t}_i - t}{\bar{t}_i - \underline{t}_i}\right)^{b_i}. \tag{16}$$

In this case, virtual valuation functions are linear. HR is satisfied, and UB is satisfied if and only if $T_1 \subseteq T_2$, $b_1 \leq b_2$ and $\underline{t}_1(b_2 - b_1) \geq \bar{t}_2 b_1 - \bar{t}_1 b_2$. Condition (15) reduces to $\bar{t}_j \geq \bar{t}_i - b_i(\bar{t}_i - v)$ for all $v \in [V_i(\underline{t}_i), \bar{t}_i)$. Hence, RM is implied by UB.

For environments with $n \geq 3$ buyers, Zheng (2002) makes the additional assumptions Transitivity (TR) and Invariance (IV). Let $\beta_{ij}^{-1}(t_j) = \inf\{t_i \in T_i \mid \beta_{ij}(t_i) \geq t_j\}$ for $t_j \leq \bar{t}_j$.

**Assumption 4** (Transitivity) If buyer $i$ is ranked before buyer $j$ and $j$ is ranked before buyer $k$ ($i < j < k$), then for any $t_j$ less than or equal to the supremum of the range of $\beta_{ij}$, $\beta_{ik}(\beta_{ij}^{-1}(t_j)) \geq V_k^{-1}(V_j(t_j))$.

**Assumption 5** (Invariance) For all $w = 1, \ldots, n$, and $i, j > w$, if $t_i \leq \beta_{wi}(t_w)$ and $t_j \leq \beta_{wj}(t_w)$, then[11]
$V_i(t_i) \geq$ (resp. $=$)$V_j(t_j)$ implies $f_i(V_i^{-1}(V_w(t_w)))/f_i(t_i) \geq$ (resp. $=$)$f_j(V_j^{-1}(V_w(t_w)))/f_j(t_j)$.

To understand IV, suppose the initial seller sells to buyer $w$ rather than buyers $i$ or $j$ if and only if $t_i \leq \beta_{wi}(t_w)$ and $t_j \leq \beta_{wj}(t_w)$. By definition of the bid inflation rules, this yields an intermediate allocation such that buyer $w$ resells to $i$ or $j$ if and only if

[10] Things are less straightforward if buyer 2's density is not weakly increasing. The right-hand side of (15) is then $> 0$ for some $v = \check{v}$. Assumption RM can still hold (for example, when both buyers have the same c.d.f. $F_1 = F_2$). However, one can always find buyer-1 c.d.f.s (with the same support as the buyer-2 c.d.f.) such that RM is violated. The proof works by constructing buyer 1's c.d.f. such that the left-hand side of (15) equals 0 at $v = \check{v}$ (Mylovanov and Tröger 2005).

[11] Zheng's paper contains a typo in Assumption 5 that is corrected here. He requires that "…> … implies …> …", but this is not needed and obviously is not meant because it would be violated by his own Example 3.

the initial seller wants her to do so according to the intended final Myerson allocation. IV with $i \neq j$ is equivalent to the requirement that $V_i^{\beta_{wi}(t_w)}(t_i) - V_j^{\beta_{wj}(t_w)}(t_j)$ has the same sign as $V_i(t_i) - V_j(t_j)$, so that buyer $w$'s optimal decision *to whom* to resell is also aligned with the initial seller's intentions (Zheng 2002, p. 2217).[12]

IV can be expressed most transparently in terms of virtual valuations. IV implies that the ratio of any two buyers' densities, except buyer 1's, it constant across all points where the buyers tie with their virtual valuations.

**Proposition 2** *Suppose that $n \geq 3$ and HR and UB hold. Then IV holds if and only if for $i, j \geq 2$ there exist constants $c_{ij} > 0$ such that*

$$\forall v_i \in V_i(T_i),\ v_j \in V_j(T_j): \quad v_i \geq (resp. =) v_j \Rightarrow \frac{f_i(V_i^{-1}(v_i))}{f_j(V_j^{-1}(v_j))} \leq (resp. =) c_{ij}. \quad (17)$$

*Moreover, IV implies that the densities $f_i$ $(i \geq 2)$ are weakly decreasing.*

To see that IV implies RM, suppose first that $i = 1$. Then (14) holds because $f_j$ is weakly decreasing by Proposition 2, and $\lambda_i$ is weakly increasing by HR. If $i \geq 2$,

$$\frac{f_j(V_j^{-1}(v))}{\lambda_i(V_i^{-1}(v))} = \underbrace{\frac{f_j(V_j^{-1}(v))}{f_i(V_i^{-1}(v))}}_{=1/c_{ij}} \underbrace{(1 - F_i(V_i^{-1}(v)))}_{\text{decreasing}}.$$

Hence, in essence, IV implies RM because a buyer's hazard rate is a growing multiple of the buyer's density. We have shown the following.

**Corollary 2** *Suppose that $n \geq 3$ and HR, UB, and IV hold. Then RM is satisfied.*

The next result shows that IV implies TR. In fact, IV implies a condition TR* that is stronger than TR and is easier to interpret. For any $i < j$, say that $i$ *outbids* $j$ if $\beta_{ij}(t_i) \geq t_j$ and that $j$ outbids $i$ if "$\leq$" holds. Condition TR* requires that, for any type profile, outbidding is a transitive binary relation on the set of buyers.

Condition TR* allows a particularly simple interpretation of the winner-selection rule in Zheng (2002): the seller assigns the good to a buyer who outbids all other buyers; by TR*, such a buyer always exists.

**Corollary 3** *Suppose that $n \geq 3$ and HR, UB, and IV hold. Then TR*, and hence, TR, is satisfied.*

To obtain a computationally useful reformulation of IV, we take the same approach as towards Corollary 1: taking the logarithm and then the derivative. We obtain a set of differential equations and inequalities involving IVVFs. From this reformulation

[12] It appears that, in line with this observation, Zheng (2002) uses IV only with $i \neq j$. To prove Corollary 2 below, we also use IV with $i = j$. While this rounds up our presentation, it is not needed for our main characterization formula (20).

one sees (18) that IV requires the expressions (8) to be identical for all buyers except buyer 1. The inequality part in the definition of IV is captured in (19), which by (3) is equivalent to the requirement that all c.d.f.s except buyer 1's have a weakly decreasing density.

**Corollary 4** *Suppose that $n \geq 3$ and HR and UB hold. Then IV holds if and only if for all $j > i \geq 2$,*

$$\forall v \in [V_i(\underline{t}_i), \overline{t}_i) : \quad \frac{1 - 2(V_i^{-1})'(v)}{V_i^{-1}(v) - v} = \frac{1 - 2(V_j^{-1})'(v)}{V_j^{-1}(v) - v}, \tag{18}$$

$$(V_i^{-1})' \geq 1/2. \tag{19}$$

Corollaries 2–4 suggest a procedure to construct systematically all c.d.f. profiles that satisfy Zheng's assumptions if $n \geq 3$. One begins with an arbitrary profile of nested supports for the buyers' c.d.f.s, and with any c.d.f. for buyer $n$ that is consistent with HR and satisfies $(V_n^{-1})' \geq 1/2$ [this inequality is necessary for IV, see (19)]. Then one solves the differential equations (18) with $j = n$ to compute IVVFs for buyers 2 to $n-1$ (the solutions will depend on the chosen supports). The corresponding c.d.f.s can be calculated from (5). Finally, one chooses for buyer 1 any c.d.f. that is consistent with HR and UB. Proposition 3 shows that this procedure works.

The following result provides a complete characterization of the set of c.d.f. profiles satisfying HR, UB, RM, TR, and IV. In particular, we give the solutions (20) to the differential equations (18).

**Proposition 3** *Let $n \geq 3$. Let $T_1 \subseteq \cdots \subseteq T_n$ be any compact intervals. Let $F_n$ be any c.d.f. that is consistent with HR and satisfies $(V_n^{-1})' \geq 1/2$.*

*Then there exists a unique profile $(F_2, \ldots, F_{n-1})$ such that HR, UB, RM, TR, and IV hold for $(F_1, \ldots, F_n)$, where $F_1$ is any c.d.f. that is consistent with HR and UB.*

*For all $j > i \geq 2$ and $v \in V_i(T_i)$,*

$$V_i^{-1}(v) = V_j^{-1}(v) - \sqrt{(V_j^{-1}(v) - v)(V_j^{-1}(\overline{t}_i) - \overline{t}_i)}\; e^{-\frac{1}{2}\int_{V_j^{-1}(v)}^{V_j^{-1}(\overline{t}_i)} \frac{1}{t' - V_j(t')} \mathrm{d}t'}. \tag{20}$$

Several qualitative conclusions can be drawn from Proposition 3. First, the lowest possible value $\underline{t}_i$ does not appear in (20). Hence, the virtual valuation function implied for buyer $i$ for a lower value of $\underline{t}_i$ is an extension of the virtual valuation function implied for buyer $i$ for a higher value of $\underline{t}_i$. Thus, c.d.f. $F_i$ implied for buyer $i$ for the lower value of $\underline{t}_i$ is an extension of an affine transformation of the c.d.f. implied for buyer $i$ for the larger value of $\underline{t}_i$. Second, the lowest possible value $\underline{t}_j$ does not appear in (20). Hence, $F_i$ is independent of $\underline{t}_j$. Third, $F_i$ is independent of the shape of $F_j$ above the point $\hat{t} = V_j^{-1}(\overline{t}_i)$. In particular, $F_i$, is independent of $\overline{t}_j$, as long as the probability mass $F_j(\hat{t})$ remains constant. The shape above $\hat{t}$ does not matter because these types of buyer $j$ are so high that even the highest type of buyer $i$ optimally sells to them.

The corollary below describes the c.d.f. profiles satisfying Zheng's assumptions if the largest possible valuation is the same for the buyers 2 to $n$: the c.d.f.s of buyers 2 to $n-1$ must be affine transformations of the c.d.f. of buyer $n$. We omit the straightforward proof.

**Corollary 5** *Let $n \geq 3$. Suppose that HR holds and $\bar{t}_2 = \cdots = \bar{t}_n$.*
*Then UB, RM, TR, and IV are satisfied if and only if* (i) $T_1 \subseteq T_2$ *and* $\underline{t}_2 \geq \cdots \geq \underline{t}_n$, (ii) $V_1(t_1) \geq V_2(t_1)$ *for all* $t_1 \in T_1$, (iii) *the density* $f_n$ *is weakly decreasing, and* (iv)

$$\forall i \geq 2: \; F_i(t) = \frac{F_n(t) - F_n(\underline{t}_i)}{1 - F_n(\underline{t}_i)} \quad (t \in T_i). \tag{21}$$

In general, it can take up to $n$ subsequent sale transactions on Zheng's (2002) equilibrium path until the final allocation is obtained.[13] Corollary 5 implies that there will be at most two transactions if the largest possible valuation is the same for the buyers 2 to $n$. From (21) it follows that whenever buyer $i \geq 2$ has a larger valuation than buyer $j \geq 2$, then $i$'s virtual valuation is also larger than $j$'s. Given this, there will be no resale trading between buyers 2 to $n$ on the equilibrium path—the final allocation is obtained by the initial auction or by a resale transaction from buyer 1 to one of the other buyers.

To obtain more insight into the implications of Proposition 3, we consider a fixed $F_n$ and ask how the implied c.d.f. $F_i$ $(2 \leq i \leq n-1)$ varies if the support $T_i$ varies. If $T_i = T_n$, then, as in Corollary 5, $F_i$ must be identical to $F_n$. As $T_i$ shrinks to a smaller interval, $F_i$ starts becoming different from $F_n$ and, if $T_i$ is sufficiently small, becomes approximately uniform. This is shown in the following result.

**Corollary 6** *Let $n \geq 3$. Suppose that HR, UB, RM, TR, and IV are satisfied. Let $2 \leq i \leq n-1$ and $x \in [\underline{t}_n, \bar{t}_n)$. As $\bar{t}_i \to x$ and $\underline{t}_i \to x$, the c.d.f. $F_i$ converges uniformly to the uniform distribution on $T_i$.*

It is interesting to contrast Proposition 3 with Zheng (2002, Example 3), where it is shown that Zheng's assumptions are satisfied if every buyer's c.d.f. is uniform (on a possibly different interval for each buyer). Proposition 3 reveals that if the c.d.f. for buyer $n$ is uniform, then the assumptions are satisfied if and only if the c.d.f.s for buyers 2 to $n-1$ are uniform as well (with nested supports), while buyer 1 may have any c.d.f. that is consistent with HR and UB.

Finally, let us consider a parametric class of distributions. For any $b > 0$, consider, as in (16), the c.d.f.

$$F_n(t) = 1 - \left( \frac{\bar{t}_n - t}{\bar{t}_n - \underline{t}_n} \right)^b. \tag{22}$$

[13] For an example, let $n = 3$, $F_i$ uniform $(1 \leq i \leq 3)$, $\bar{t}_2 = \bar{t}_1 + 4\epsilon$, and $\bar{t}_3 = \bar{t}_1 + 12\epsilon$ for some small $\epsilon > 0$. On the equilibrium path, the sequence of successive owners initial seller $\rightarrow$ buyer 1 $\rightarrow$ buyer 2 $\rightarrow$ buyer 3 occurs, for example, if $t_1 \in (\bar{t}_1 - \epsilon, \bar{t}_1)$, $t_2 \in (\bar{t}_1 + 2\epsilon, \bar{t}_1 + 3\epsilon)$, and $t_3 \in (\bar{t}_1 + 7\epsilon, \bar{t}_1 + 10\epsilon)$. To verify this, use the formulas following Corollary 5.1 in Zheng (2002).

The corresponding virtual valuation function $V_n$ is linear, with slope $1 + 1/b$. Corollary 7 shows, perhaps surprisingly, that Zheng's assumptions imply that the virtual valuation function $V_i$ $(2 \leq i \leq n-1)$ is non-linear, except if $F_n$ is uniform (that is, if $b = 1$). Hence, it appears that a closed-form solution for the virtual valuation function of buyer $i$, and, hence, for the value distribution $F_i$, exists only in exceptional cases.[14] We omit the straightforward proof of the following result.

**Corollary 7** *Let $n \geq 3$. Suppose that HR, UB, RM, TR, and IV are satisfied. If $F_n$ is given by (22), then $b \geq 1$, and, for all $2 \leq i \leq n-1$,*

$$V_i^{-1}(v) = v + \frac{1}{1+b}\left(\bar{t}_n - v - \frac{(\bar{t}_n - \bar{t}_i)^{\frac{b+1}{2}}}{(\bar{t}_n - v)^{\frac{b-1}{2}}}\right).$$

Figure 1 shows numerically computed instances of Corollary 7. This illustrates the gradual transformation of $F_i$ from being identical to $F_n$ if $T_i = T_n$ to being uniform if $T_i$ is small. Note that $V_i$ is linear only if $\bar{t}_i = 20 = \bar{t}_n$; if $\bar{t}_i = 1/10$, then $V_i$ is approximately linear, with slope 2, implying that $f_i$ is approximately constant.

## 4 Conclusion

Our results delineate the extent to which a seller's optimal allocation à la Myerson (1981) can be implemented through resale using Zheng's (2002) construction. As such, our findings are important for any model that uses an optimal sales mechanism as a building block. Implementing the optimal allocation through resale is fairly easy in two-buyer environments, but is rather difficult in environments with three or more buyers.

The use of IVVFs may be considered the main technical contribution of this paper. The Myerson allocation relates buyers' c.d.f.s at points where their virtual valuations coincide, so that it is natural to take virtual valuation as the independent variable and apply IVVFs.

Elsewhere (Garratt et al. 2007), another advantage of IVVFs is highlighted. A buyer's IVVF applied to a seller's valuation yields the lowest buyer valuation that the seller optimally sells to. Hence, if the seller's valuation is her private information, then the probability of selling can be conveniently expressed using an IVVF. Auctions with resale, where the resale seller is, in general, privately informed about her valuation, is one natural class of examples. Garratt et al. (2007) use IVVFs together with the envelope theorem in order to obtain convenient equilibrium payoff formulas in the context of an English auction where the resale seller uses an optimal auction. They derive their main collusion result by using, in particular, a first-order Taylor approximation of an IVVF.

The two fundamental conflicts of successive owners of a good—about whether to resell and to whom to resell—are relevant for any market with resale. Hence, conditions

[14] A closed-form solution for $V_i(t)$ can be obtained if $b \in \{1, 2, 3, 5, 7\}$, by using the well-known solution formulas for polynomials up to degree 4.

related to Zheng's assumptions can be expected to come up in future work on resale.[15] Our characterization may prove to be a useful guide to understanding such related conditions as well.

## Appendix

*Proof of Proposition 1* Using the variable $v = V_i(t_i)$ in (11),

$$\frac{1 - F_j(\beta_{ij}(V_i^{-1}(v)))}{f_j(V_j^{-1}(v))} = V_i^{-1}(v) - v = \frac{1}{\lambda_i(V_i^{-1}(v))}.$$

The claim follows because both $F_j$ and $V_i^{-1}$ are strictly increasing functions. □

*Proof of Corollary 1*. Taking the logarithm in (14) yields that $\beta_{ij}$ is weakly increasing if and only if

$$\ln f_j(V_j^{-1}(v)) - \ln \lambda_i(V_i^{-1}(v)) \quad \text{is weakly decreasing for all } v \in [V_i(\underline{t}_i), \overline{t}_i).$$

Because a continuously differentiable function is weakly decreasing if and only if its derivative is non-positive, $\beta_{ij}$ is weakly decreasing if and only if

$$\frac{\mathrm{d}}{\mathrm{d}v} \ln \lambda_i(V_i^{-1}(v)) \geq \frac{\mathrm{d}}{\mathrm{d}v} \ln f_j(V_j^{-1}(v)) \quad (v \in [V_i(\underline{t}_i), \overline{t}_i)).$$

Using (7) and (8), the proof is complete. □

*Proof of Proposition 2*. "only if": Define $t_i = V_i^{-1}(v_i)$ and $t_j = V_j^{-1}(v_j)$. By (12), $t_i \leq \overline{t}_i = \beta_{1i}(\overline{t}_1)$, and $t_j \leq \overline{t}_j = \beta_{1j}(\overline{t}_1)$. Hence, using IV with $w = 1$ and $t_w = \overline{t}_1$,

$$\frac{f_i(V_i^{-1}(v_i))}{f_j(V_j^{-1}(v_j))} = \frac{f_i(t_i)}{f_j(t_j)} \leq (\text{resp.} =) \frac{f_i(V_i^{-1}(V_1(\overline{t}_1)))}{f_j(V_j^{-1}(V_1(\overline{t}_1)))} =: c_{ij}.$$

"if": Consider $i, j > w \geq 1$ and $t_i \leq \beta_{wi}(t_w)$, $t_j \leq \beta_{wj}(t_w)$ such that $V_i(t_i) \geq (\text{resp.} =) V_j(t_j)$.

Using (17) with $v_i = v_j = V_w(t_w)$,

$$c_{ij} = \frac{f_i(V_i^{-1}(V_w(t_w)))}{f_j(V_j^{-1}(V_w(t_w)))}. \tag{23}$$

[15] For example, Lebrun (2005) considers the design of personalized entry fees in a second-price auction with resale. Under an assumption similar to RM, Lebrun constructs an equilibrium in mixed strategies that implements the same allocation as Zheng. Lebrun (2005) relaxes UB. As long as UB is satisfied, his assumption (2005, Corollary 7) is essentially equivalent to RM (cf. footnote 9). Assumptions TR and IV play no role because Lebrun's analysis is restricted to two buyers. Garratt et al. (2007) use Zheng's assumptions (as well as the current paper's characterization) in their analysis of collusion in English auctions with resale.

Using (17) with $v_i = V_i(t_i)$ and $v_j = V_j(t_j)$,

$$\frac{f_i(t_i)}{f_j(t_j)} \leq (\text{resp.} =) c_{ij} \stackrel{(23)}{=} \frac{f_i(V_i^{-1}(V_w(t_w)))}{f_j(V_j^{-1}(V_w(t_w)))}.$$

This completes the proof.

To show the "Moreover" part, use (17) with $i = j \geq 2$ and $v_i = v_j$, one sees that $c_{ii} = 1$. Hence, (17) implies that $f_i(V_i^{-1}(\cdot))$ is weakly decreasing, which implies the same property for $f_i$. □

*Proof of Corollary 3.* Observe that, by (11), for any buyers $i < j$, buyer $i$ outbids $j$ if and only if

$$1 - F_j(t_j) \geq (t_i - V_i(t_i)) f_j(V_j^{-1}(V_i(t_i))),$$

or, equivalently,[16]

$$\frac{f_j(t_j)}{\lambda_j(t_j)} \geq \frac{f_j(V_j^{-1}(V_i(t_i)))}{\lambda_i(t_i)}. \tag{24}$$

By the same reasoning, $j$ outbids $i$ if and only if the reverse inequality of (24) holds.

Suppose now that $i < j$ and $i < k$, where $i$ outbids $j$ and $k$ outbids $i$. Then, (24) holds and

$$\frac{f_k(t_k)}{\lambda_k(t_k)} \leq \frac{f_k(V_k^{-1}(V_i(t_i)))}{\lambda_i(t_i)}. \tag{25}$$

Dividing (25) through (24) yields

$$\frac{f_k(t_k)}{\lambda_k(t_k)} \frac{\lambda_j(t_j)}{f_j(t_j)} \leq \frac{f_k(V_k^{-1}(V_i(t_i)))}{f_j(V_j^{-1}(V_i(t_i)))} \stackrel{\text{Proposition 2}}{=} \frac{f_k(V_k^{-1}(V_j(t_j)))}{f_j(V_j^{-1}(V_j(t_j)))} = \frac{f_k(V_k^{-1}(V_j(t_j)))}{f_j(t_j)}.$$

Cancelling $f_j(t_j)$ on both sides yields that $k$ outbids $j$. The remaining arguments towards showing that outbidding is a transitive relation are analogous.

To show TR, define $t_i = \beta_{ij}^{-1}(t_j)$ and $t_k = \beta_{ik}(t_i)$. By TR*, we can conclude that $\beta_{jk}(t_j) \leq t_k$, showing TR by (13). □

*Proof of Corollary 4.* "only if": Let $j > i \geq 2$. Taking the logarithm on the r.h.s. of (17), there exist constants $C_{ij}$ such that for all $v_i \in V_i(T_i)$ and $v_j \in V_j(T_j)$,

$$v_i \geq (\text{resp.} =) v_j \Rightarrow \ln(f_i(V_i^{-1}(v_i))) \leq (\text{resp.} =) \ln(f_j(V_j^{-1}(v_j))) + C_{ij}. \tag{26}$$

[16] Recall that the hazard rates are defined only if $t_i < \bar{t}_i$ and $t_j < \bar{t}_j$; we assume this in the following; by continuity of the bid inflation rules, the proof extends to the upper ends of the supports.

Using (26) with $v = v_i = v_j$,

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i): \ \ln(f_i(V_i^{-1}(v))) = \ln(f_j(V_j^{-1}(v))) + C_{ij}. \tag{27}$$

Taking derivatives in (27) and using (8), we obtain (18). Inequalities (19) follow from (3) and the "Moreover" in Proposition 2.

"if": By (18) and (8), there exist constants $C_{ij}$ such that (27) holds for $j > i \geq 2$. Applying the exponential function to (27) yields

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i): \ \frac{f_i(V_i^{-1}(v))}{f_j(V_j^{-1}(v))} = c_{ij}, \tag{28}$$

where we define $c_{ij} = \exp C_{ij}$. For all $i \geq 2$, define $c_{ii} = 1$. For all $i > j \geq 2$, define $c_{ij} = 1/c_{ji}$. By (28), for all $i, j \geq 2$,

$$\forall v \in V_i(T_i) \cap V_j(T_j): \ \frac{f_i(V_i^{-1}(v))}{f_j(V_j^{-1}(v))} = c_{ij}. \tag{29}$$

Using (19) and (3), $f_i(V_i^{-1}(\cdot))$ and $f_j(V_j^{-1}(\cdot))$ are weakly decreasing. Together with (29) this implies (17), and IV follows from Proposition 2. □

The proof of Proposition 3 relies on

**Lemma 1** *Let $l \in \{2, \dots, n-1\}$. Let $F_{l+1}$ denote a c.d.f. that is consistent with HR and has a weakly decreasing density. Let $V_{l+1}$ denote the corresponding virtual valuation function.*

*Then, for any interval $[\underline{t}, \bar{t}] \subseteq [V_{l+1}(\underline{t}_{l+1}), \bar{t}_{l+1}]$, there exists a unique continuously differentiable function $g$ on $[\underline{t}, \bar{t}]$ such that*

$$g \leq V_{l+1}^{-1}, \tag{30}$$

$$g(\bar{t}) = \bar{t}, \tag{31}$$

*and*

$$\forall v \in [\underline{t}, \bar{t}): \ \frac{2g'(v) - 1}{g(v) - v} = \frac{2(V_{l+1}^{-1})'(v) - 1}{V_{l+1}^{-1}(v) - v} =: h_{l+1}(v). \tag{32}$$

*Proof of Lemma 1.* Suppose that $\bar{t} < \bar{t}_{l+1}$. Because $h_{l+1}$ is continuous at $v = \bar{t}$, standard results for differential equations [see, e.g., Walter (1998, p. 62)] show the existence of a unique $g$ satisfying (31) and (32). It remains to show (30). From (32),

$$\forall v \in [\underline{t}, \bar{t}): \ \text{if } g(v) > V_{l+1}^{-1}(v) \text{ then } g'(v) \geq (V_{l+1}^{-1})'(v).$$

Hence, if $g(\hat{v}) > V_{l+1}^{-1}(\hat{v})$ for some $\hat{v} < \bar{t}$, then $g(\bar{t}) > V_{l+1}^{-1}(\bar{t})$. On the other hand, $V_{l+1}^{-1}(\bar{t}) \geq \bar{t} = g(\bar{t})$ by definition of $V_{l+1}$, a contradiction. Thus, (30).

Now suppose that $\bar{t} = \bar{t}_{l+1}$. Because $h_{l+1}(v) \to \infty$ as $v \to \bar{t}$, standard uniqueness results for differential equations do not apply. However, $g = V_{l+1}^{-1}$ obviously satisfies (30)–(32). Let $g = k$ denote another function satisfying the same conditions.

Multiplying (32) by $g(v) - v$ and subtracting the resulting expression with $g = k$ from the resulting expression with $g = V_{l+1}^{-1}$ yields the homogeneous linear equation $2m'(v) = m(v)h_{l+1}(v)$ for $m := V_{l+1}^{-1} - k$. Hence,

$$k(v) = \alpha\, \mathrm{e}^{\int_{\underline{t}}^{v} \frac{h_{l+1}(w)}{2} \mathrm{d}w} + V_{l+1}^{-1}(v), \tag{33}$$

for some $\alpha \in \mathbb{R}$. Because $k \le V_{l+1}^{-1}$ by (30), we have $\alpha \le 0$. By (33),

$$k'(v) = \alpha \frac{h_{l+1}(v)}{2}\, \mathrm{e}^{\int_{\underline{t}}^{v} \frac{h_{l+1}(w)}{2} \mathrm{d}w} + (V_{l+1}^{-1})'(v). \tag{34}$$

By (3), $h_{l+1}(v) \ge 0$. Hence, (34) implies $k'(v) \le (V_{l+1}^{-1})'(v)$. Together with $k(\bar{t}) = \bar{t} = V_{l+1}^{-1}(\bar{t})$ this implies $k \ge V_{l+1}^{-1}$. Hence, $k = V_{l+1}^{-1}$. □

*Proof of Proposition 3.* By Corollaries 2 and 3, we can ignore RM and TR throughout the proof.

"Existence": We show the existence of $F_2, \ldots, F_{n-1}$ by proving inductively, for all $l < n$, claim

($*l$) There exists a profile $F_{l+1}, \ldots, F_n$ that is consistent with HR and UB, (18) holds for all $j > i \ge l + 1$, and $(V_i^{-1})' \ge 1/2$ for all $i \ge l + 1$.

Claim ($*(n-1)$) holds by assumption. Suppose that ($*l$) holds for some $l \in \{2, \ldots, n-1\}$. Let $V_{l+1}$ denote the virtual valuation function for $F_{l+1}$. By Lemma 1, there exists a continuously differentiable function $g$ on $[V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]$ such that (30)–(32) hold with $\underline{t} = V_{l+1}(\underline{t}_{l+1})$ and $\bar{t} = \bar{t}_l$.

Recall from (2) and (3) that

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l] : \ \frac{1}{2} \le (V_{l+1}^{-1})'(v) \le 1. \tag{35}$$

Consider

$$A = \arg \min_{v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]} g(v) - v.$$

If $\hat{v} < \bar{t}_l$ and $\hat{v} \in A$, then $g'(\hat{v}) - 1 \ge 0$ from the first-order conditions, hence $g(\hat{v}) - \hat{v} > 0$ by (32) and (35), in contradiction with (31). We conclude that $A = \{\bar{t}_l\}$. Hence,

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l) : \ g(v) > v. \tag{36}$$

From (32), (35), and (36),

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l] : \; g'(v) \geq \frac{1}{2}. \tag{37}$$

In particular, we can define the inverse

$$g^{-1} : \; [g(V_{l+1}(\underline{t}_{l+1})), \bar{t}_l] \to [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]. \tag{38}$$

From (30) we obtain $g(V_{l+1}(\underline{t}_{l+1})) \leq \underline{t}_{l+1} \leq \underline{t}_l$. Hence, $g^{-1}$ exists on $[\underline{t}_l, \bar{t}_l]$. Moreover, by (36), $t' - g^{-1}(t') > 0$ for all $t' \in [\underline{t}_l, \bar{t}_l)$. Thus, we can define

$$F_l(t) = \begin{cases} 1 - \mathrm{e}^{-\int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t'} & \text{if } t \in [\underline{t}_l, \bar{t}_l), \\ 1 & \text{if } t = \bar{t}_l. \end{cases} \tag{39}$$

The function $F_l$ is continuous at $\bar{t}_l$ because, for all $t < \bar{t}_l$,

$$\begin{aligned} \int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t' &= \int_{g^{-1}(\underline{t}_l)}^{g^{-1}(t)} \frac{g'(w)}{g(w) - w} \mathrm{d}w \\ &> \int_{g^{-1}(\underline{t}_l)}^{g^{-1}(t)} \frac{g'(w) - 1}{g(w) - w} \mathrm{d}w \\ &= \ln(t - g^{-1}(t)) - \ln(\underline{t}_l - g^{-1}(\underline{t}_l)) \\ &\quad \to_{t \to \bar{t}_l} \infty \quad \text{because } \bar{t}_l - g^{-1}(\bar{t}_l) = 0. \end{aligned}$$

From (39) we obtain on $[\underline{t}_l, \bar{t}_l)$ the continuously differentiable density

$$f_l(t) = \frac{1}{t - g^{-1}(t)} \mathrm{e}^{-\int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t'} \quad (t \in [\underline{t}_l, \bar{t}_l)). \tag{40}$$

Using (5) with $i = l + 1$ and (40), Eq. (32) implies that the derivative $(\ln f_l(g(v)))' = \ln(f_{l+1}(V_{l+1}^{-1}(v)))'$ for all $v \in [g^{-1}(\underline{t}_l), \bar{t}_l)$. Hence,

$$\exists c > 0 \; \forall v \in [g^{-1}(\underline{t}_l), \bar{t}_l) : \; f_l(g(v)) = c f_{l+1}(V_{l+1}^{-1}(v)). \tag{41}$$

Because $f_{l+1}(V_{l+1}^{-1}(g^{-1}(\cdot)))$ is continuously differentiable on $[\underline{t}_l, \bar{t}_l]$, (41) shows that $f_l$ extends continuously differentiable to the point $t = \bar{t}_l$.

Define $V_l = g^{-1}|_{T_l}$. Using (30), Eq. (32) implies $g' \leq (V_{l+1}^{-1})'$. Hence, $g' \leq 1$ by (35). Hence, $V_l' \geq 1$, which implies that $(1 - F_l(\cdot))/f_l(\cdot)$ is weakly decreasing. In summary, $F_l$ is consistent with HR. By (37), $(V_l^{-1})' \geq 1/2$. By (30), $F_l$ is consistent with UB. By (32), Eq. (18) holds with $i = l$ and $j = l + 1$. By induction, (18) holds for all $j \geq i = l$. This completes the proof of claim (*$(j-1)$).

"Uniqueness": Let $l \in \{2, \ldots, n-1\}$. Consider a c.d.f. $F_{l+1}$ that is consistent with HR. Let $V_{l+1}$ denote the corresponding virtual valuation function. Consider two c.d.f.s $F$ and $\check{F}$ for buyer $l$ with support $T_l$ that are consistent with HR, UB, and IV. Denote by $V$ and $\check{V}$ the corresponding virtual valuation functions. Without loss of generality, $V(\underline{t}_l) \leq \check{V}(\underline{t}_l)$. By Corollary 4 and the uniqueness statement in Lemma 1,

$$\forall v \in [\check{V}(\underline{t}_l), \bar{t}_l]: \ V^{-1}(v) = \check{V}^{-1}(v). \tag{42}$$

Applying (42) at $v = \check{V}(\underline{t}_l)$ yields $V^{-1}(\check{V}(\underline{t}_l)) = \check{V}^{-1}(\check{V}(\underline{t}_l)) = \underline{t}_l$, hence $\check{V}(\underline{t}_l) = V(\underline{t}_l)$. This together with (42) implies $V = \check{V}$ and thus $F = \check{F}$, completing the uniqueness proof.

It remains to verify (20). Let $g$ denote a function on $[V_i(\underline{t}_i), \bar{t}_i]$ that equals the r.h.s. of (20). It is straightforward to check that $g$ satisfies (30)–(32) with $l = j-1$, $\underline{t} = V_i(\underline{t}_i)$, and $\bar{t} = \bar{t}_i$. Hence, $g = V_i^{-1}$ on $[V_i(\underline{t}_i), \bar{t}_i]$ by Corollary 4 and the uniqueness statement in Lemma 1. □

*Proof of Corollary 6.* Let the function $h(v, \bar{t}_i)$ denote the right-hand side of (20) with $j = n$. Then $h(\bar{t}_i, \bar{t}_i) = \bar{t}_i$. Let $h_1 = \partial h / \partial v$. A straightforward computation shows that $h_1$ is continuous on $V_n(T_n) \times T_n$, and $h_1(x, x) = \frac{1}{2}$. Let $\epsilon \in (0, 1/6)$. For all $\underline{t}_i$ and $\bar{t}_i$ sufficiently close to $x$,

$$\frac{1}{2} - \epsilon < h_1(v, \bar{t}_i) < \frac{1}{2} + \epsilon \quad \text{if } \bar{t}_i - 3(\bar{t}_i - \underline{t}_i) \leq v \leq \bar{t}_i. \tag{43}$$

From (20), $V_i^{-1}(v) = h(v, \bar{t}_i)$ for all $v \in [V_i(\underline{t}_i), \bar{t}_i]$. Hence, for such $v$ we have $h(v, \bar{t}_i) \geq \underline{t}_i$.

Suppose that $V_i(\underline{t}_i) \leq \hat{v} := \bar{t}_i - 3(\bar{t}_i - \underline{t}_i)$. Then $h(\hat{v}, \bar{t}_i) \geq \underline{t}_i$, implying

$$\bar{t}_i - \underline{t}_i \geq h(\bar{t}_i, \bar{t}_i) - h(\hat{v}, \bar{t}_i) \overset{(43)}{>} (\frac{1}{2} - \epsilon)(\bar{t}_i - \hat{v}) \geq \frac{\bar{t}_i - \hat{v}}{3},$$

a contradiction to the definition of $\hat{v}$. Hence, $V_i(\underline{t}_i) > \hat{v}$. Using this together with (43),

$$\frac{1}{2} - \epsilon < (V_i^{-1})'(v) < \frac{1}{2} + \epsilon \quad \text{for all } v \in V_i(T_i).$$

Hence, given any $\epsilon > 0$, if $\bar{t}_i$ and $\underline{t}_i$ are sufficiently close to $x$, then

$$2 - \epsilon < V_i'(t) < 2 + \epsilon \quad \text{for all } t \in T_i.$$

Using this together with $V_i(\bar{t}_i) = \bar{t}_i$, we obtain lower and upper bounds for $V_i$,

$$\bar{t}_i - (2-\epsilon)(\bar{t} - t) \geq V_i(t) \geq \bar{t}_i - (2+\epsilon)(\bar{t} - t) \quad \text{for all } t \in T_i.$$

Hence, using (4),

$$\mathrm{e}^{\frac{1}{1-\epsilon} \int_{\underline{t}_i}^{t} \frac{1}{\bar{t}_i - t'} \mathrm{d}t'} \leq 1 - F_i(t) \leq \mathrm{e}^{\frac{1}{1+\epsilon} \int_{\underline{t}_i}^{t} \frac{1}{\bar{t}_i - t'} \mathrm{d}t'} \quad \text{for all } t \in T_i,$$

or, equivalently,

$$\left(\frac{\bar{t}_i - t}{\bar{t}_i - \underline{t}_i}\right)^{\frac{1}{1-\epsilon}} \leq 1 - F_i(t) \leq \left(\frac{\bar{t}_i - t}{\bar{t}_i - \underline{t}_i}\right)^{\frac{1}{1+\epsilon}} \quad \text{for all } t \in T_i. \tag{44}$$

Observe that

$$\lim_{y \to 1} \max_{x \in [0,1]} \mid x^y - x \mid = 0.$$

Hence,

$$\lim_{\epsilon \to 0} \max_{t \in T_i} \left| \left(\frac{\bar{t}_i - t}{\bar{t}_i - \underline{t}_i}\right)^{\frac{1}{1 \pm \epsilon}} - \frac{\bar{t}_i - t}{\bar{t}_i - \underline{t}_i} \right| = 0.$$

Applying this to (44) shows that

$$\lim_{\underline{t}_i \to x,\ \bar{t}_i \to x} \max_{t \in T_i} \left| F_i(t) - \frac{t - \underline{t}_i}{\bar{t}_i - \underline{t}_i} \right| = 0,$$

as was to be shown. □

## References

Bester, H., Strausz, R.: Contracting with imperfect commitment and the revelation principle: the single agent case. Econometrica **69**, 1077–1098 (2001)
Bose, S., Deltas, G.: Exclusive versus non-exclusive dealing in auctions with resale. Econ Theory **31**, 1–17 (2007)
Bulow, J., Roberts, J.: The simple economics of optimal auctions. J Polit Econ **97**, 1060–1090 (1989)
Calzolari, G., Pavan, A.: Monopoly with resale. RAND J Econ **37**, 362–375 (2006)
Garratt, R., Tröger, T., Zheng, C.: Collusion via Resale. Working paper, Iowa State University (2007)
Krishna, V.: Auction Theory. London: Academic Press (2002)
Lebrun, B.: Optimality and the second-price auction with resale. Discussion paper, York University (2005)
Myerson, R.: Optimal auction design. Math Oper Res **6**, 58–73 (1981)
Mylovanov, T., Tröger, T.: A note on Zheng's conditions for implementing an optimal auction with resale. Bonn Econ discussion paper 29/2005 (2005)
Riley, J., Samuelson, W.: Optimal auctions. Am Econ Rev **71**, 381–392 (1981)
Skreta, V.: Sequentially optimal mechanisms. Rev Econ Stud **73**, 1085–1111 (2006)
Walter, W.: Ordinary Differential Equations. New York: Springer (1998)
Zheng, C.: Optimal auction with resale. Econometrica **70**, 2197–2224 (2002)