Intrinsic Properties of Quantum Systems

P. Hájíček · J. Tolar

Abstract A new realist interpretation of quantum mechanics is introduced. Quantum systems are shown to have two kinds of properties: the usual ones described by values of quantum observables, which are called extrinsic, and those that can be attributed to individual quantum systems without violating standard quantum mechanics, which are called intrinsic. The intrinsic properties are classified into structural and conditional. A systematic and self-consistent account is given. Much more statements become meaningful than any version of Copenhagen interpretation would allow. A new approach to classical properties and measurement problem is suggested. A quantum definition of classical states is proposed.

1 Introduction

Quantum mechanics does not seem to be fully understood even after about eighty years of its very successful existence and a lot of work is being done on its interpretation or modification today (e.g., [1, 2]). The present paper describes an approach to its conceptual foundation from a new point of view.

P. Hájíček
Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland
e-mail: hajicek@itp.unibe.ch

J. Tolar (✉)
Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic
e-mail: jiri.tolar@fjfi.cvut.cz

Accounts of the conceptual structure of quantum mechanics usually start from a 'minimal' interpretative framework on which adherents of different interpretations can agree. For example in [3], this is referred to as 'pragmatic approach' to quantum theory and a review of it is called 'The Rules of Quantum Theory', pp. 67–75. In [4], a similar review is called 'Rules of Quantum Mechanics', pp. 42–56. From d'Espagnat review, we leave out Rules 7–10 that concern the so-called ideal measurements (in our language, ideal measurements are preparations). Isham' Rules 1–4 and d'Espagnat Rules 1–6 are practically equivalent and where they differ, they are compatible. We call their logical union *standard quantum mechanics*.

An interpretation of quantum mechanics is defined as those hypotheses that are added to standard quantum mechanics.[1] For example, Copenhagen interpretation adds the hypothesis that quantum systems cannot posses real properties or even that quantum systems do not exist. Many-world interpretation adds that all values that a measurement can give really exist in different branches of the world, etc.

Standard quantum mechanics can be characterised as a set of rules allowing the computation of probabilities for the outcomes of registrations which follow specific preparations. The preparations and registrations work only with classical systems and classical properties, the existence of which is not denied. The question is, however, left open whether quantum system really exist and can posses real properties.[2] In dependence of how an interpretation answers this question, it is classified as realist or anti-realist. Some criteria are formulated in [3], (p. 68): For an anti-realist interpretation:

> The notion of an individual physical system 'having' or 'possessing' of all its physical quantities is *inappropriate* in the context of quantum theory.

For a realist interpretation:

> It *is* appropriate in quantum theory to say that an individual system possesses values of its physical quantities. In this context, 'appropriate' signifies that propositions of this type can be handled using standard propositional logic.

To understand this text, one has first to know what are 'physical quantities of an individual quantum system'. For practically all physicists, these are the values of observables (examples are [3–5, 8]). For a realists, it remains then only to ask what can be a maximal set of *determinate* (but perhaps unknown) properties for a system, S, say, in a state ρ, the so-called determinate set. The assumption that they are determinate must not lead to any contradictions that might follow from the standard quantum mechanics and possible measurements of other observables in the determinate set as well as from some further desirable conditions (for details, see [8]). Examples of such contradiction can be obtained from the Kocher-Specker theorem (see, e.g., [5]). Important is that the absence of contradiction is only required for measurements of

---

[1]If the hypotheses modify standard quantum theory as in, e.g., pilot-wave theory by Bohm and de Broglie [2], then we do not call the result quantum mechanics.

[2]In Isham's and d'Espagnat account of standard quantum theory, the existence of quantum systems seem to be assumed. But this is not what everybody can accept (see the discussion in [5–7]: they define quantum systems just as equivalence classes of preparations or registrations).

observables inside the determinate set and measurements of other observables can be ignored. The resulting mathematical problem has been solved (see [8]). All sets of determinate properties are necessarily restricted in the sense that they never contain all observables that are in principle measurable on S in ρ (at least for Hilbert spaces that have more than two dimensions).

The standpoint of the present paper is very different. We consider a value of observable O that can be measured on a quantum system S as a physical quantity of composite system $S + A$, where A is the apparatus that measures O. For us, it is, therefore, not a physical quantity of individual system S and we call it *extrinsic property* of S. Probabilities can be interpreted epistemically only as concerning the composite system. The so-called determinate properties are then mostly the extrinsic ones. Extrinsic properties of S have an important but only intermediate role: from them, something about genuine physical quantities of S can be inferred. We call the part of standard quantum mechanics that concerns extrinsic properties *phenomenology of observation*. A precise account is Sect. 2.

The main new idea of the present paper is a proposal of what the quantities of individual quantum systems are. It is very different from what is taught at universities and what is believed by experts who know all existing literature. We start from two principles. First, our reality condition is stronger than that for determinate properties: it requires that the attribution does not lead to contradictions with *any* possible measurement that could be performed on S in ρ according to the basic rules of standard quantum mechanics. Hence, most determinate properties violate our reality condition. Second, we necessarily consider properties that are different from values of observables. They are more sophisticated and need not have their values in $\mathbb{R}^n$. If we find such a quantity, we accept the hypothesis that does ascribe it to individual quantum systems and call such properties *intrinsic*.[3] In Sect. 3, it is shown that there are two kinds of intrinsic properties, structural and conditional, and that there is plenty of them.

Many physicists are aware of objectivity of properties such as mass and charge but our notion of *structural properties* includes much more quantities. A structural property of any quantum system is, e.g., the form of its Hamiltonian operator. Clearly, it is a specific feature different from system to system and it is amenable to an exact mathematical description even if not by a quantity that takes on numerical values. According to our opinion, the ultimate aim of all quantum measurements (such as the scattering experiments in CERN) is to determine the structural properties of real quantum systems (such as parameters of the standard model). *Conditional* are those properties that can be given to quantum systems by preparations. For example, Dirac ([9], p. 46) and von Neumann ([10], p. 253) added to the standard quantum mechanics the interpretation of eigenstates as the only special case in which a value of an observable is determinate before measurement. But our notion of conditional properties goes far beyond this.[4] Thus, the basic ones are summarized and mathematically

---

[3]We take the standpoint that such ontological hypotheses are meaningful if they have some relation to observation. An ontological hypothesis is allowed if its consequences are not disproved by existing evidence and if it is logically compatible with other physical theories.

[4]The analysis of experiments in the so-called *quantum mechanics on phase space* [12] introduces the notion of property that generalizes single values of observables to probability distributions of such values.

expressed by state operators, while the more advanced ones extend the information given by state operators. Existence of such advanced properties has been established by our previous paper [11].

Our paper is the first systematic, self-consistent and complete account of structural and conditional properties. It makes the ontological hypotheses of objective existence of these properties and recognizes that the hypotheses form an interpretation of quantum mechanics which is different from any other known one (versions of Copenhagen, Everett, etc.). The interpretation may be called realist because it satisfies Isham's criterion.

As mentioned above, standard quantum theory works directly only with classical systems and classical properties. In particular, the measuring apparatuses are classical systems. In this context, another classification criterion of anti-realist versus realist interpretation emerges. Measurement is considered as a fundamental notion of the theory and classical systems as different from quantum systems in the former and classical systems are considered as quantum systems with special quantum properties and measurement processes as some kind of quantum processes that represent nothing fundamentally new in the latter (see, e.g., Isham, p. 68).

Section 4 turns to this part of our interpretation. We conjecture that the dichotomy of quantum and classical worlds can be replaced by the difference between extrinsic and intrinsic properties of quantum systems. In particular, average values of observables in prepared states are intrinsic. This enables us to use methods of statistical physics to construct quantum models of some classical properties. An example explaining this idea is carefully described in Appendix. A tentative generalisation to the classical properties of a macroscopic body that determine its complete classical state is given: they can be chosen as intrinsic average values and small variances of certain operators of the underlying quantum system. Recently, there has been some progress concerning quantum models of further classical properties [13].

Our interpretation also gives some non-trivial hints of how the measurement problem is to be approached. On the one hand, corresponding to its notion of classical systems and classical properties, the assumption that measuring apparatuses are exclusively classical (that is non quantum) has to be abandoned. On the other, von Neumann model of measuring apparatus, where readings of an apparatus are eigenvalues of an operator, is incompatible with it.

2 Phenomenology of Observation

This section briefly describes what part of standard quantum theory belongs to the phenomenology of observation. To shorten the exposition, we shall base it on the accounts of standard quantum mechanics as given in [3, 4]. The Rules will be referred to just by their numbers.

At the beginning of any measurement stands what is usually called *preparation*. The name is somewhat misleading. What is meant is a set of conditions that can be

---

Some of these properties are similar to some of our conditional properties. However, quantum mechanics on phase space is a modification rather than an interpretation of quantum mechanics. It postulates the existence of informationally complete measurements. Our approach is different.

described in classical terms, to which an individual quantum system has been subject and that determine its state. This can, but need not, include some human activity in laboratory. For example, we can know that a quantum system inside the Sun is the plasma with a given composition and that its classical conditions are certain temperature and pressure. The description of classical conditions is important in order that the same preparation can be recognized or reproduced. Thus, a series of repeated experiments is feasible, and the set of individual quantum systems obtained by repeating the same preparation is called *ensemble*. Clearly, the notion of ensemble is in many aspects closely connected to that of preparation. Isham's Rules 1 and 2 and d'Espagnat's Rules 1, 2 concern states.

At the end of any quantum measurement there is what is often called a *registration*. It is an interaction of an individual quantum system in a specific state with a classical system, the *measuring apparatus* that determines a value of a set of commuting observables. About representation of observables by operators is the Rule 3 by Isham and Rules 4, 5 and 6 by d'Espagnat.[5]

Rule 4 by Isham and Rule 3 by d'Espagnat concerning dynamics are considered as a part of standard quantum mechanics but not of the phenomenology. All mathematics that is associated with the included Rules, such as the theory of Hilbert spaces and self-adjoint operators, are considered as part of the phenomenology.

Let us introduce the word 'property' in order to have a general notion of observable characteristics concerning quantum systems. For instance, the values of observables in quantum mechanics are properties. We define:

Definition 1 Extrinsic properties of quantum system S are those values of observables[6] pertaining to S that are not uniquely determined by the preparation of S.

The extrinsic properties are not real properties of quantum systems in the following sense: the assumption that an extrinsic property P of a quantum system S as measured by an apparatus A is possessed by S independently of, or already before, their registration, leads to contradictions with other possible measurements on S. An example is the well-known double-slit experiment (see also [4], Sect. 4.3).

---

[5]A more general mathematical object can be mentioned as representing registrations, the *positive operator valued measure* (POVM). However, any registration represented by a POVM of a system S is nothing but a registration associated with a suitable observable of an extended system, $S + S'$, S' being the so-called ancilla [5]. Thus, conceptually, POVM belong to extrinsic properties because of both measuring apparatus and ancilla.

[6]More generally, extrinsic properties can be described as linear subspaces in the Hilbert space of the system. They represent the mathematical counterpart of the so-called YES-NO experiments [14, 15]. The set of linear subspaces admits the usual operations on conjunction (linear hull), disjunction (intersection) and negation (orthogonal complement), but the resulting orthocomplemented lattice is not a Boolean lattice [16]. As it is well known, the set of 'classical' properties of a single system forms a Boolean lattice (of subsets of classical phase space). If we pretend that the extrinsic properties of a quantum system are properties of a well-defined single system, then we are lead to abandon the ordinary logic and introduce the so-called *quantum logic*. But this pretence is against all logic because the extrinsic properties are properties of many different systems each consisting of the quantum system plus some apparatus.

3 The Intrinsic Properties of Quantum Systems

A property P can be ascribed directly to a quantum system S if consequences of S possessing P do not contradict results of any measurement that can be carried out on S (even very difficult measurements so as to be practically not feasible). Let us define:

Definition 2 Let S be a quantum system and P a property that can be directly ascribed to S alone so that the assumption of S really possessing P does not lead to contradiction with any measurements that can in principle be done on S according to the rules of standard quantum mechanics. Then P is called intrinsic property.

For understanding the notion, it is important to discriminate between attributing a property to *an individual quantum system* on the one hand and directly measuring the property on *an individual quantum system* on the other. Many structural properties, such as cross sections or branching ratios, are obtained only after many measurements, of rather different (extrinsic) properties, on many copies of a system. The same is true for conditional properties such as a state operator. Still, no formal-logical problem arises if one wants to attribute them to individual systems.

3.1 Structural Properties

First, we turn to those intrinsic properties that are easy: nobody would seriously deny that they can be ascribed to quantum systems. They are also the most important properties of quantum systems in the sense that the ultimate aim of experimenters is to determine them.

Quantum systems can be classified into equivalence classes with the beautiful property that the structures of each two different systems of the same class are absolutely identical. Examples: electrons, protons, hydrogen atoms, etc. Let us define:

Definition 3 All properties that are uniquely determined by the class of quantum system are called structural.

The first among the structural properties is the *composition of a quantum system*. Experience and practice lead to ideas about what such a composition can be. For example, in the non-relativistic case, there must be a definite number[7] of some particles with definite masses, spins and charges. For a relativistic case, there are analogous rules: we have fields of certain (bare) masses, spins and charges. For example, the non-relativistic model of hydrogen atom consists of two particles, proton and electron, that have certain masses, spins and charges.

The next step is to determine the quantum observables that can be measured on the system. For example, each particle contributes to the observables by three coordinates and three momenta. Thus, in the hydrogen case, there will be (in addition to

---

[7]There are non-relativistic systems, in which some particle numbers are variable, such as those of quasi-particles in solid state physics. Of course, these particle numbers do not belong to the structure of the systems and they are not intrinsic but extrinsic properties in our conception.

other observables) six coordinates and six components of momenta. The *set of observables that can be measured on a given system* is its intrinsic structural property. This information is different from that about the values of these observables.[8] The algebras of observables contain for example information about superselection observables (which form the centre of the algebra), so these observables are also structural properties.

The composition and the observables of a system are used to set up the Hamiltonian of non-relativistic or the action functional for relativistic systems. The *form of the Hamiltonian or the action* are mathematical expressions of the structure and thus intrinsic properties.[9]

Using the Hamiltonian or the action, we can write down the dynamical laws: the Schrödinger equation or the path-integral formula. Hence, the dynamical part of standard quantum theory (Rule 4 by Isham, Rule 3 by d'Espagnat) is included into the structural properties of our interpretation. From the dynamical laws, other important intrinsic properties can be calculated, for example the spectrum of the hydrogen atom. The spectrum is clearly a structural intrinsic property of the hydrogen atom that can be ascribed to the system itself independently of any measurement. This will not lead to any contradictions with other measurements or ideas of quantum mechanics. We can recognize the system with the help of its intrinsic properties. For example, if we detect light from somewhere deep in the Universe and find the Balmer series in its spectrum, then we know that there is hydrogen there. The numbers such as cross sections, branching ratios etc. are further examples of structural intrinsic properties. Moreover, the Hamiltonian contains information about all symmetries of the system, because these are represented by the operators that commute with the Hamiltonian. Thus, symmetries are structural properties.

Next, it seems that many-particle systems may have structural properties that are not found in small quantum systems. An example is provided by molecules of the deoxyribonucleic acid. The number of their structures grows (roughly) exponentially with the number of the four kinds of constituents because possible orderings of the constituents define different structures. It is clearly wrong to say that we know all kinds of structural properties of macroscopic systems and investigations in this direction might be useful. For example, rich intrinsic properties of large systems might imply the observed properties of the universe and so enable a new approach to quantum cosmology without need of bizarre theories such as many world interpretation [3].

---

[8]More precisely, the set of observables can be embedded in a so-called C^*-algebra that represents a part of the physical structure of the system, see [17], Vol. 3. Thus, it is an intrinsic property of it. Moreover, such algebras have representations on a Hilbert spaces. A representation defines the Hilbert space of the system. Of course, for systems with finite number of degrees of freedom, the Hilbert space representation is uniquely defined (up to unitary equivalence) by the algebra, so it does not contain any further information on an independent structure of the system, but the algebras of relativistic fields possess many inequivalent representations of which only few are physical, corresponding to different phases of the system. A physical representation is then clearly an independent structural intrinsic property of the field.

[9]The energy of a system S that can be measured by suitable apparatus A is an observable. The value of energy obtained on S by A is a 'beable', it is not an intrinsic property of S but that of the composite system $S + A$. The three notions of measured energy value, energy measurements and the form of Hamiltonian are related to each other but they are clearly not identical.

These steps form the everyday practice of quantum mechanics. An application of quantum mechanics starts with a proposal of a model for the quantum system under study. This is done by specifying its structural properties. For each system, we can attempt different possible models, calculate the extrinsic properties of each and compare with the experimental evidence gained in a number of quantum measurements. In this way, the models can be confirmed or disproved. The sets of intrinsic and extrinsic properties are model dependent. What is relevant is that every quantum model exhibits both intrinsic and extrinsic properties.

3.2 Conditional Properties

Encouraged by the triviality of the assumption that structural properties are intrinsic, we start to look for some intrinsic properties that can have different values for one and the same class of quantum systems. Let us define:

Definition 4 A property is conditional if its value is uniquely determined by a preparation according to the rules of the standard quantum mechanics. The 'value' is the value of the mathematical expression that describes the property and it may be more general than just a real number. No registration is necessary to establish such a property but a correct registration cannot disprove its value; in some cases, registrations can confirm the value.

This can best be explained by examples. Suppose that a system S has been prepared in the eigenstate $|o\rangle$ of an observable O with the eigenvalue o. Now, think: could any conceivable registration made on S thus prepared in $|o\rangle$ contradict the assumption that S possesses the value o of O? The standard rules of quantum mechanics clearly say no. More generally, any quantum state ρ that has been prepared for the system S is a property of S; its value ρ (i.e., a positive self-adjoint operator with trace 1) can only be confirmed by registrations following the preparation.

Next consider a particle S with spin $1/2$. The state with spin projection to the z-axis equal to $\hbar/2$ can be prepared. Then, no contradictions can result from the assumption that S with this value of σ_z really exists. Thus, the value $\hbar/2$ of σ_z is one example of a conditional property. Of course, S does not possess any value of σ_x after such preparation; this would only be brought about by a corresponding measurement and is not uniquely determined by the preparation. Hence, it is an extrinsic property. On the other hand, the average (also called expectation or mean) value of σ_x in the prepared state has a well-known value defined by the state and hence it is another example of a conditional property. (Averages will play a key role in the definition of classical properties.)

One could try to object that there has been the preparation, this is a 'kind of measurement' and the property depends on this 'measurement'. Moreover, the preparation has used an apparatus A, say, and the property seems therefore to be a property of the system $S + A$ and not S alone. However, these objections concern clearly also the structural properties: an apparatus that prepares a beam of electrons is different from that for a beam of protons. Moreover, they could be also raised in Newton mechanics: giving a snooker ball momentum p requires a careful action of the cue. Still, nobody questions the existence of the momentum p on the ball alone after the poke.

Any preparation defines a specific state of the quantum system. States can be pure ones or mixtures and we describe them generally by state operators. There is a measure of how restrictive and special the preparation process is, namely the entropy.[10] It is a function of state and thus an important conditional property.

More advanced examples of conditional properties concern mixtures. In [11], we have shown that, in some cases, ρ does not contain all information available by registrations concerning the prepared ensemble. One can consider the ensemble of particles as defining a state of a single particle, of two particles, etc., and the information involved in the preparation can thus be described by a set of state operators. Such a set is the mathematical description of the property, which is clearly intrinsic, and of conditional character. It can be confirmed by registration.

An example of conditional property that cannot always be confirmed by registration is the difference between proper and improper mixtures (for definitions, see [4]). Suppose that a physicist prepares states $|1\rangle, \ldots, |n\rangle$ of a quantum system S and mixes them with frequencies $c_1, \ldots, c_n$ so that the resulting state of S can be described by state operator

$$\rho = \sum_{k=1}^{n} c_k |k\rangle \langle k|. \tag{1}$$

This is a proper mixture and the particular decomposition (1) of ρ is a conditional property, that is a real property of the prepared ensemble. In particular, we can assume that the system *really is* always in *either* of the states $|1\rangle, \ldots, |n\rangle$ with respective probabilities $c_1, \ldots, c_n$. The decomposition (1) is not the unique decomposition that the state operator ρ admits but it is the one that is uniquely determined by the preparation. According to the standard quantum mechanics, which we adhere to, two different decompositions of the same state operator cannot be distinguished by any registrations. This however represents no embarrassment for us: we do not adhere to the positivist maxim that there is only what is measured. It is also clear that any time evolution of such a proper mixture is the proper mixture of the evolved states with the same probabilities and determines a well-defined decomposition at each time. On the other hand, given an improper-mixture state σ of S then no individual system S can in general be assumed to *really be* in any of the component states of any decomposition of σ. An example is the Einstein-Podolsky-Rosen experiment [5].

What are possible conditional properties of a given quantum system? Clearly, all state operators that can be prepared (some limitations are due to superselection rules) belong to them. Other properties are either derived from the state operators (such as the average of an observable) or added to state operators (such as the decomposition describing a proper mixture). It seems, that state operators are also universal in the following sense. Even if we do not know what the source or origin of a system is (for example the protons coming in cosmic radiation), the assumption that it is described by some state operator does not lead to any contradictions.

All examples that have been listed show that the intrinsic and extrinsic properties are physically inseparably entangled with each other. Even the definition of intrinsic

---

[10]The term 'entropy' always means the von Neumann entropy in this paper.

properties uses the notion of registration of extrinsic properties: an intrinsic property can be ascribed to the system alone without giving rise to contradictions with the results of all possible registrations. Similarly, extrinsic properties cannot be defined without the notion of a measuring apparatus with its classical properties, which are a kind of intrinsic properties in our point of view (see the next section). Thus, e.g., the notion is clearly untenable that the intrinsic properties can be explained purely in terms of the extrinsic ones. Still, both kinds of properties are logically clearly distinguished, and we conjecture that the physical in-and-extrinsic tangle does not lead to any logical contradictions.

4 Classical Properties

There is a lot of systems around us that behave as classical physics prescribes, at least to a good approximation. We would like to have quantum models of such systems. The features that are most difficult to reproduce are summarized in the so-called principle of *macroscopic realism* [18], but as formulated by Leggett it is too strong for our needs. Let us modify the principle as follows:

1. A macroscopic system which has available to it two or more *distinct classical states* is at any given time in a definite one of those states.
2. It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics.

Here, we have just replaced Leggett's 'macroscopically distinct (quantum) states' by 'distinct classical states', and we call the resulting principle *modified macroscopic realism*. Of course, if the classical states include pure quantum states, point 1 of the macroscopic realism violates the principle of superposition. Then, one has to assume that some as yet unknown phenomena exist at the macroscopic level which are not compatible with standard quantum mechanics (see, e.g., [18] and the references therein). However, no such phenomena have been observed. We ought therefore to suggest how our modified macroscopic realism could be derived from quantum mechanics, at least in principle.

Observe that the modified macroscopic realism as it stands cannot be obtained from the decoherence theory, at least in its present form. For that e.g. the word 'is' in point 1 had to be replaced by 'appears to be' (see [19]). The 'appears' would undermine our quantum realism. We are optimistic because it seems that our interpretation provides some new tools. Of course, this does not mean that the argument is circular but only that the output does not contradict the input.

Hence, let us assume that all physical systems are quantum systems. More precisely, there is one level of description (approximative model of some aspects of a real system) of a classical system S_c and of its classical properties for which quantum theory is not needed, namely the classical description, and for which the modified principle of macroscopic realism is valid. In addition, every classical system S_c can also be understood as a quantum system S_q underlying S_c such that the classical properties of S_c are some intrinsic properties of S_q. This follows from our definition of intrinsic properties and from the modified macroscopic realism. Namely, any classical state of S_c is defined by values of some classical properties. As it must also be a

state of S_q, the assumed reality of the state requires that these classical properties are intrinsic properties of S_q.

The quantum description of S_q consists of the following points. (1) The composition of S_q must be defined. (2) The algebra of observables that can be measured on S_q is to be determined. As any observable is measurable only by a classical apparatus, the existence of such apparatuses must also be assumed from the very beginning. Quantum description of S_q will thus always contain some classical elements. This does not mean that classicality has been smuggled in because, in our approach, classical properties are specific quantum ones. (3) A Hamiltonian operator or an action functional of the system must be set up. Finally, the known classical properties $P_1, P_2, \ldots, P_K$ of S_c must be listed and each derived as an intrinsic property of S_q from the three sets of assumptions above. This is a self-consistent framework for a non-trivial problem.

There are intrinsic properties of S_q that are not classical properties of S_c, e.g., the set of all quantum observables measurable on S_q. Hence, classical properties must be some specific intrinsic properties and the question is, which.

To begin with, let us consider the so-called semi-classical (or WKB) approximation. This includes the observation that, for a number of systems, the average values in special states of a number of quantum observables follow classical (say, Newton mechanics) trajectories. This is surely a good start because, as we have seen in Sect. 3.2, in some cases, average values can be considered as conditional properties. Moreover, everything what we can measure on classical systems has a form of average value and its variance. This is evident from the description of any classical experiment. How are these averages and variances related first to the relevant classical theory and, second, to the averages and variances of quantum operators?

As the first question is concerned, it is often assumed that improvements in measuring techniques will in principle, in some limit, lead to zero variance. This is in agreement with the classical theory such as Newton mechanics. It predicts that the trajectories are completely sharp if the initial data are so, and does not put any limit on the accuracy with which the initial state can be prepared. The point of view adopted here is different (it is originally due to Exner [20], p. 669, and Born [21]): some part of the variances can never be removed and the classical theories are only approximative models.

The second question contains two different problems. First, if we measure several times the position of the Moon on its trajectory around the Earth, then the variance in the results is surely not connected to our preparing the system of Earth and Moon these many times to get the desired ensemble. But the classical systems are robust in the sense that most classical measurements practically do not disturb them (point 2 of modified macroscopic realism). Thus, one can assume that the values we obtain by several measurements on one and the same system could equivalently be obtained if the measurements were performed on several identically prepared systems. The hypothesis is therefore plausible that some intrinsic properties we are looking for are averages with small variances associated with preparations under identical relevant conditions. If the variance of a given average value is sufficiently small, it can be and is usually viewed as a property of each individual element of the ensemble. We conjecture that this is the way classical systems come to possessing their properties just from the classical experimental point of view. Second, there are classical properties

that cannot be viewed as intrinsic averages of quantum operators but are structural or different conditional properties. Some examples will be given later.

Next, there is a restriction on quantum models of classical properties: they cannot be averages with small variance that are defined by pure states such as coherent ones. Not only are pure states readily linearly superposed but any quantum registration (a generalized measurement: positive operator valued measure) that were to find the parameters of a coherent state would strongly change the state. The only available hint of what classical properties may be comes from thermodynamics. Indeed, statistical physics is a successful method of deriving macroscopic properties from microscopic ones. Moreover, the notions of structural and conditional properties enable a cleaner formulation of quantum statistical physics. The following is a brief sketch of a specific example from the thermodynamic-equilibrium theory.

Let S be a (non-relativistic) quantum system with number of particles comparable to Avogadro number. We call such systems *macroscopic*. Let its structure be described by a Hamiltonian H. Imagine that S is prepared in all possible quantum states (not necessarily by humans in laboratories). Consider only those of these states that have a fixed average value $\bar{E}$ of internal energy. A well-defined average value is a conditional property that exists for each of the prepared states and hence the imaginary selection (without need of any additional registration) is legitimate. Let us call this subset of prepared states $\bar{E}$-*ensemble*.

Next, let the state $\rho_{\bar{E}}$ be defined by the requirement that it maximizes the entropy under the condition that the average internal energy has the value $\bar{E}$. This is known as the Gibbs state of S. The state is purely mathematical because no preparation process for it has been specified. The central conjecture of statistical physics reads: *For macroscopic systems, important statistical properties of $\bar{E}$-ensemble coincide to a very good approximation with the corresponding statistical properties of $\rho_{\bar{E}}$.* Claims, equivalent to this conjecture can to a large extent be derived from quantum mechanics ([17], Vol. 4), in the thermodynamic limit. Bayesian approach [22] to probability and entropy is also helpful. (The thermodynamic limit is, of course, not a physical condition but a mathematical method of how the structural property of being macroscopic can be brought into play.)

What are the 'important' statistical properties above? Some of them are average values and variances of a very small but definite subset $\mathcal{T}_S$ of the algebra of all quantum observables of S. Clearly, these are conditional properties because they are determined by the prepared states from $\bar{E}$-ensemble. The observables from $\mathcal{T}_S$ are extensive quantities associated with some of the ordinary thermodynamic variables. For instance, consider a gas in a vessel of volume V. The operator of internal energy $\mathbf{E}$ of the gas (the Hamiltonian in the rest frame) belongs to $\mathcal{T}_S$. We can also choose some small but macroscopic partial volume δV at a specific position within V and consider the particle number $\delta \mathbf{N}$ inside δV. Operator $\delta \mathbf{N}$ can be constructed from the projectors on the position eigenstates of all particles in S. The energy $\delta \mathbf{E}$ inside δV can be constructed as a coarse-grained operator (see [23]) because the exact energy operator does not commute with operators of particle positions. It seems that all quantum observables from $\mathcal{T}_S$ are macroscopic in the sense that they have a coarse-grained character or concern many particles.

Other thermodynamic variables are not average values of quantum observables. Examples are structural quantities such as the total mass and particle number of

a macroscopic body or state quantities such as the maximal value of entropy and the corresponding temperature (the Lagrange multiplier that appears naturally in the problem of maximization of entropy).

The average values of observables from $\mathcal{T}_S$ determine a thermodynamic state of the system. Let us consider such a state as an example of a classical state (appearing in point 1 of the Principle) of the quantum system S. For example, the internal energy and the volume determine the state of a simple ideal gas. Thus, one macroscopic state is compatible with a huge number of microscopic (quantum) states of S. It is very important to understand that a macroscopic state of S is conceptually different from any microscopic state of it, and that there are no linear superpositions of macroscopic states. The sets of average values of operators from $\mathcal{T}_S$ do not form a linear space that could lead to a definition of state superposition of a fixed system: addition of extensive quantities entails addition of the corresponding systems.

It can be shown that the observables from $\mathcal{T}_S$ have negligible relative variances in the Gibbs state. (The property that they are extensive plays an important role.) Thus, the average values of the observables can be given individual meaning: each individual system from $\bar{E}$-ensemble *possesses* a value of the observables within certain limits. Is such an average already a classical property satisfying the requirements of the modified macroscopic realism? Point 1 is satisfied by construction. Point 2 is just plausible as yet: the influence of measurement can still be large as concerns the microstate but it can change it to another microstate that is compatible with the original macrostate and so it need not change the macrostate. Clearly, statistical physics in our interpretation is the quantum theory of at least some macroscopic properties.

The discussion above motivates a general definition of the classical state making it analogous to the thermodynamic state as follows.

Definition 5 Let the state of classical system S_c be described by the set of n numbers $\{A_1, \ldots, A_n\}$ that represent values of some classical observables with variances $\{\Delta A_1, \ldots, \Delta A_n\}$. Let the corresponding quantum system S_q contain in its algebra a set of n observables $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ that correspond to the classical ones. Then the quantum counterpart of the classical state $\{A_1, \ldots, A_n\}$ is the set of all quantum states ρ such that

$$\mathrm{Tr}(\rho \mathbf{a}_k) = A_k, \quad \sqrt{\mathrm{Tr}(\rho \mathbf{a}_k^2) - [\mathrm{Tr}(\rho \mathbf{a}_k)]^2} = \Delta A_k.$$

Let us call the quantities in $A_1, \ldots, A_n$ state coordinates.[11] For example, the classical state of a mass point in mechanics can be described by three coordinates Q_k and three momenta P_k, and the operator algebra of its quantum analogue contains the corresponding operators $\mathbf{q}_k$ and $\mathbf{p}_k$. Of course, we can define a sensible classical state only for macroscopic systems so that their classical states contain huge numbers of quantum states and in this way much less information than their quantum states. Each of the many quantum states satisfying the above equations can be viewed as representing one and the same classical state.

---

[11] They are not uniquely determined by the classical system and we assume that the choice can be done so that all state coordinates are averages of quantum operators.

Classical states defined in this way can be understood as equivalence classes: two quantum states are equivalent, if the state coordinates have the same averages and variances in them. For such classes, one can try to define a superposition operation by forming superpositions of vectorial representative of the states: let $|a\rangle \in \{\rho\}$ and $|b\rangle \in \{\sigma\}$, where $\{\rho\}$ denotes the class with representative ρ, then

$$\{c|a\rangle + d|b\rangle\} := c\{\rho\} + d\{\sigma\}.$$

However, we find that there are often more vectorial representations in each class and that superposition of another pair does not lead to the same class. $|a'\rangle \in \{\rho\}$ and $|b'\rangle \in \{\sigma\}$ are other such vectors then

$$\{c|a\rangle + d|b\rangle\} = \{c|a'\rangle + d|b'\rangle\}$$

can hold only in few exceptional cases.

To summarize the main points of our theory of classical properties, let us first compare it with some well-known approaches to the problem. Thus, we mention the quantum decoherence theory [19, 24], the theories based on coarse-grained operators [5, 25, 26], the Coleman-Hepp theory [27–31] and its modifications [32]. At the present time, the problem does not seem to be solved in a satisfactory way, the shortcoming of the above theories being well known [4, 33, 34]. Our approach is free of these shortcomings.

It starts at the idea that all classical properties of a macroscopic system S in a quantum state ρ are certain *intrinsic* quantum properties of S in ρ. Then, first, intrinsic properties are quantum properties of all quantum systems and there is no question about how they emerge in quantum mechanics. This avoids e.g. the artificial construction in the Coleman-Hepp approach. The new point is that they are considered as, and proved to be, objective in our paper. Hence, second, they could in principle serve as classical properties because they can satisfy the principle of modified macroscopic realism. This avoids the problems of both the quantum-decoherence and the coarse-grained theory that assume values of quantum observables to be real. This, as analysed in [8], can be done only for restricted classes of observables, all other measurements being forbidden. Third, we conjecture that certain macroscopic quantum systems possess intrinsic properties that can model all their classical properties. Hence, classical states and properties defined in the present paper are available only for some quantum systems and the relation between classical and quantum states is not one-to-one but one-to-many. This is different from other approaches such as Wigner-Weyl-Moyal scheme, quantum-mechanics-on phase-space theory or coherent state approach. Finally, our modelling or construction of classical properties uses the way analogous to that of statistical physics. Thus, Definition 5 starts a project of modelling classical properties of quantum systems including the internal (thermodynamic) and external (mechanical) properties. A full derivation including the complete list of all assumptions is described with the help of an example in Appendix. Models of classical mechanics are constructed in [13].

An important piece of our interpretation is the existence of classical macroscopic apparatuses that are needed for the phenomenology of observation. Some necessary conditions such apparatuses must satisfy not only in order that the phenomenology

works but also that our realist interpretation has a reliable basis are summarised in the modified principle of macroscopic realism.

The problem to construct a quantum model of registration process is the most difficult one in the field of conceptual foundation. A quantum explanation of classical properties is only a part of the problem. There is much activity in this field. The references given above deal also with the measurement problem. No satisfactory solution seems to be known.

One cause of the difficulties may be the model of measuring apparatus that has been proposed by von Neumann and by Jauch [10, 35]. The key assumption of the model is that the values shown by the apparatus are some of its extrinsic properties. For example, the pointer states are eigenstates of some quantum operator. Let us briefly describe it for the case of quantities with discrete values (continuous quantities would need a slightly different approach).

Suppose quantum system $\mathcal{S}$ is prepared in initial state $|\mathcal{S}1\rangle$ and the observable to be measured, $\mathbf{a}$, has eigenvalues a_k and eigenstates $|a_k\rangle$. We can write

$$|\mathcal{S}1\rangle = \sum_k c_k |a_k\rangle.$$

The apparatus that makes the registration is quantum system $\mathcal{A}$ in initial state $|\mathcal{A}1\rangle$ and its pointer observable $\mathbf{A}$ has eigenvalues A_k and eigenstates $|A_k\rangle$.

The next assumption is that there is an interaction between the two systems that leads to unitary evolution

$$|\mathcal{S}1\rangle \otimes |\mathcal{A}1\rangle \mapsto \sum_k c_k |a_k\rangle \otimes |A_k\rangle.$$

If we trace out $\mathcal{S}$, we obtain the final state of the apparatus,

$$\sum_k |c_k|^2 |A_k\rangle\langle A_k|. \tag{2}$$

This a mixture of the eigenstates $|A_k\rangle\langle A_k|$ of the apparatus with the 'correct' probabilities $|c_k|^2$.

The problem is that (2) is not a proper mixture. Nothing prevents us to view the process above as a preparation of the apparatus in the state (2) and this preparation does not contain any steps that would bring $\mathcal{A}$ into any of the states $|A_k\rangle$ during each individual measurement.[12] The usual way out is to employ another apparatus $\mathcal{B}$, which is non-quantum so that an interaction between $\mathcal{A}$ and $\mathcal{B}$ can bring $\mathcal{A}$ always into one of the states $|A_k\rangle$.

Clearly, the model contradicts the experience. One apparatus is sufficient, it is a system with classical properties and the outcome of any individual measurement is represented by a definite change of a classical property of the apparatus. Moreover, the model is incompatible with our interpretation. The extrinsic property that represents the apparatus readings had to be replaced by an intrinsic one. We have to use

---

[12]In the decoherence theory, another component, the environment, is added at the beginning and traced out at the end. The result is again an improper mixture and the problem remains exactly the same.

our theory of classical properties as described in the previous section. To construct a model of such an apparatus is a problem that will be addressed in a separate paper.

To summarize: Our interpretation suggests a new approach to quantum theory of classical properties and of measurement because it allows quantum systems to have also properties that are not extrinsic.

Appendix: Quantum Model of Classical Property

The purpose of Appendix is to construct a quantum model of a classical property, the length of a body, as an average value with a small variance. No original calculation is to be expected, but simple and well known ideas are carefully interpreted according to the lines described in Sect. 4. This entails that, first, the quantum structure of the system must be defined, second, the basic intrinsic properties such as the spectrum calculated, and, third, some intrinsic properties derived that satisfy our definition of classical property.

.1 Composition, Hamiltonian and Spectrum

We shall consider a linear chain of N identical particles of mass μ distributed along the x-axis with the Hamiltonian

$$H = \frac{1}{2\mu} \sum_{n=1}^{N} p_n^2 + \frac{\kappa^2}{2} \sum_{n=2}^{N} (x_n - x_{n-1} - \xi)^2, \tag{3}$$

involving only nearest neighbour elastic forces. Here x_n is the position, p_n the momentum of the n-th particle, κ the oscillator strength and ξ the equilibrium interparticle distance. The parameters μ, κ and ξ are intrinsic properties (the last two defining the potential function).

This kind of chain seems to be different from most that are studied in literature: the positions of the chain particles are dynamical variables so that the chain can move as a whole. However, the chain can still be solved by methods that are described in [36–38].

First, we find the variables u_n and q_n that diagonalize the Hamiltonian describing the so-called normal modes. The transformation is

$$x_n = \sum_{m=0}^{N-1} Y_n^m u_m + \left(n - \frac{N+1}{2}\right)\xi, \tag{4}$$

and

$$p_n = \sum_{m=0}^{N-1} Y_n^m q_m, \tag{5}$$

where the mode index m runs through $0, 1, \ldots, N-1$ and Y_n^m is an orthogonal matrix; for even m,

$$Y_n^m = A(m) \cos\left[\frac{\pi m}{N}\left(n - \frac{N+1}{2}\right)\right], \tag{6}$$

while for odd m,

$$Y_n^m = A(m) \sin\left[\frac{\pi m}{N}\left(n - \frac{N+1}{2}\right)\right] \tag{7}$$

and the normalization factors are given by

$$A(0) = \frac{1}{\sqrt{N}}, \qquad A(m) = \sqrt{\frac{2}{N}}, \quad m > 0. \tag{8}$$

To show that u_n and q_n do represent normal modes, we substitute (4) and (5) into (3) and obtain, after some calculation,

$$H = \frac{1}{2\mu}\sum_{m=0}^{N-1} q_m^2 + \frac{\mu}{2}\sum_{m=0}^{N-1} \omega_m^2 u_m^2,$$

which is indeed diagonal. The mode frequencies are

$$\omega_m = \frac{2\kappa}{\sqrt{\mu}}\sin\frac{m}{N}\frac{\pi}{2}. \tag{9}$$

Consider the terms with $m = 0$. We have $\omega_0 = 0$, and $Y_n^0 = 1/\sqrt{N}$. Hence,

$$u_0 = \sum_{n=1}^{N}\frac{1}{\sqrt{N}}x_n, \qquad q_0 = \sum_{n=1}^{N}\frac{1}{\sqrt{N}}p_n,$$

so that

$$u_0 = \sqrt{N}X, \qquad q_0 = \frac{1}{\sqrt{N}}P,$$

where X is the centre-of-mass coordinate of the chain and P is its total momentum. The 'zero' terms in the Hamiltonian then reduce to

$$\frac{1}{2M}P^2$$

with $M = N\mu$ being the total mass. Thus, the 'zero mode' describes a straight, uniform motion of the chain as a whole. The other modes are harmonic oscillators called 'phonons' with eigenfrequencies ω_m, $m = 1, 2, \ldots, N - 1$. The spectrum of our system is built from the mode frequencies by the formula

$$E = \sum_{m=1}^{N-1} v_m \hbar \omega_m, \tag{10}$$

where $\{v_m\}$ is an $(N - 1)$-tuple of non-negative integers—phonon occupation numbers.

At this stage, a new and important assumption must be done. We imagine that all states ρ of the modes $m = 1, \ldots, N - 1$ are prepared that have the same average internal energy $\bar{E}$,

$$\mathrm{Tr}\left[\rho\left(H - \frac{P^2}{2M}\right)\right] = \bar{E}.$$

We further assume that it is done in a perfectly random way, i.e., all other conditions or bias are to be excluded. Hence, the resulting mixture must maximize the entropy. In this way, the maximum of entropy does not represent an additional condition but rather the absence of any. The resulting state $\rho_{\bar{E}}$ is the Gibbs state of the internal degrees of freedom. The conditions that define the preparation of Gibbs state are objective and need not have to do with human laboratory activity.

The internal energy has itself a very small relative variance in the Gibbs state; this need not be assumed from the start. Thus, it is a classical property. All other classical internal properties will turn out to be functions of the classical internal energy. Hence, for the internal degrees of freedom, $\bar{E}$ forms itself a complete set of state coordinates introduced in Sect. 4. The mathematics associated with the maximum entropy principle is variational calculus. The condition of fixed averaged energy is included with the help of Lagrange multiplier denoted by λ. It becomes a function $\lambda(\bar{E})$ for the resulting state. As it is well known, $\lambda(\bar{E})$ has to do with temperature.

The phonons of one species are excitation levels of a harmonic oscillator, so we have

$$u_m = \sqrt{\frac{\hbar}{2\mu\omega_m}}(a_m + a_m^\dagger),$$

where a_m is the annihilation operator for the m-th species. The diagonal matrix elements between the energy eigenstates $|v_m\rangle$ that we shall need then are

$$\langle v_m|u_m|v_m\rangle = 0, \qquad \langle v_m|u_m^2|v_m\rangle = \frac{\hbar}{2\mu\omega_m}(2v_m + 1). \tag{11}$$

For our system, the phonons of each species form statistically independent subsystems, hence the average of an operator concerning only one species in the Gibbs state $\rho_{\bar{E}}$ of the total system equals the average in the Gibbs state for the one species. Such a Gibbs state operator for the m-th species has the form

$$\rho_m = \sum_{v_m=0}^{\infty} |v_m\rangle p_{v_m}^{(m)} \langle v_m|,$$

where

$$p_{v_m}^{(m)} = Z_m^{-1}\exp(-\hbar\lambda\omega_m v_m)$$

and Z_m is the partition function for the m-th species

$$Z_m(\lambda) = \sum_{v_m=0}^{\infty} e^{-\lambda\hbar\omega_m v_m} = \frac{1}{1 - e^{-\lambda\hbar\omega_m}}. \tag{12}$$

.2 The length of the body

The classical property that will be defined and calculated in our quantum model is the average length of the body. Let us define the length operator by

$$L = x_N - x_1. \tag{13}$$

It can be expressed in terms of modes u_m using (4),

$$L = (N-1)\xi + \sum_{m=0}^{N-1} (Y_N^m - Y_1^m) u_m.$$

The differences on the right-hand side are non-zero only for odd values of m, and equal then to $-2Y_1^m$. We easily find, using (7) and (8):

$$L = (N-1)\xi - \sqrt{\frac{8}{N}} \sum_{m=1}^{[N/2]} (-1)^m \cos\left(\frac{2m-1}{N}\frac{\pi}{2}\right) u_{2m-1}. \tag{14}$$

The average length is obtained inserting (11),

$$\langle L \rangle_{\bar{E}} = (N-1)\xi. \tag{15}$$

It is a function of intrinsic properties N, ξ and $\bar{E}$.

Equation (14) is an important result. It shows that contributions to the length are more or less evenly distributed over all odd modes. The even distribution will lead to a very small variance of L in Gibbs states. Let us give the proof that the relative variance of the length is indeed small. The proof is not trivial because the distribution is not constant. The relative variance is

$$\frac{\Delta L}{\langle L \rangle_{\bar{E}}} = \frac{\sqrt{\langle L^2 \rangle_{\bar{E}} - \langle L \rangle_{\bar{E}}^2}}{\langle L \rangle_{\bar{E}}}.$$

To estimate the variance ΔL to the leading order for large N, we start with

$$\langle L^2 \rangle_{\bar{E}} = (N-1)^2 \xi^2$$
$$+ \frac{8}{N} \sum_{m=1}^{[N/2]} \sum_{n=1}^{[N/2]} (-1)^{m+n} \cos\left(\frac{2m-1}{N}\frac{\pi}{2}\right) \cos\left(\frac{2n-1}{N}\frac{\pi}{2}\right)$$
$$\times \langle u_{2m-1} u_{2n-1} \rangle_{\bar{E}}.$$

Since

$$\langle u_{2m-1} u_{2n-1} \rangle_{\bar{E}} = \delta_{mn} \langle u_{2m-1}^2 \rangle_{\bar{E}},$$

the above formula leads to

$$\langle L^2 \rangle_{\bar{E}} - \langle L \rangle_{\bar{E}}^2 = \frac{8}{N} \sum_{m=1}^{[N/2]} \cos^2\left(\frac{2m-1}{N}\frac{\pi}{2}\right) \langle u_{2m-1}^2 \rangle_{\bar{E}},$$

where

$$\langle u_{2m-1}^2 \rangle_{\bar{E}} = \frac{1}{Z_{2m-1}} \sum_{v_{2m-1}=0}^{\infty} \frac{\hbar}{2\mu\omega_{2m-1}} (2v_{2m-1} + 1) \exp(-\lambda\hbar\omega_{2m-1}v_{2m-1}).$$

Introducing dimensionless quantities

$$x_m = \sin\left(\frac{2m-1}{N}\frac{\pi}{2}\right), \qquad \gamma = \frac{2\hbar\kappa\lambda}{\sqrt{\mu}},$$

we can substitute $\omega_{2m-1} = (2\kappa/\sqrt{\mu})x_m$ and obtain the intermediate result

$$\langle L^2 \rangle_{\bar{E}} - \langle L \rangle_{\bar{E}}^2 = \frac{2}{N}\frac{\hbar}{\kappa\sqrt{\mu}} \sum_{m=1}^{[N/2]} \frac{1-x_m^2}{x_m} \frac{1+e^{-\gamma x_m}}{1-e^{-\gamma x_m}}.$$

In order to extract the leading term for large N, we note that

$$x_m - x_{m-1} = \frac{\pi}{N}\cos\frac{2m-1}{N}\frac{\pi}{2} + O(N^{-2}).$$

Then we can write

$$\langle L^2 \rangle_{\bar{E}} - \langle L \rangle_{\bar{E}}^2 \approx \frac{2}{\pi}\frac{\hbar}{\kappa\sqrt{\mu}} \sum_{m=1}^{[N/2]} (x_m - x_{m-1})f(x_m),$$

where

$$f(x) = \frac{\sqrt{1-x^2}}{x}\frac{1+e^{-\gamma x}}{1-e^{-\gamma x}}.$$

By inspection, f is a decreasing function of x in the interval $(0, 1)$ diverging to plus infinity at $x \to 0+$ and going through zero at $x = 1$. The leading term at $x \to 0+$ is

$$f(x) = \frac{2}{\gamma x^2}[1 + O(x)].$$

The block diagram of the sum now shows that

$$\sum_{m=1}^{[N/2]} (x_m - x_{m-1})f(x_m) < 2x_1 f(x_1) + \int_{x_1}^1 dx\, f(x).$$

The dependence of the integral on its lower bound can be approximated by

$$\int_{x_1}^1 dx\, f(x) = \text{const} + \frac{2}{\gamma x_1}[1 + O(x_1)].$$

Thus, the leading term in the sum is $6/\gamma x_1 \approx 12N/\gamma\pi$ so that the leading term in $\langle L^2 \rangle_{\bar{E}} - \langle L \rangle_{\bar{E}}^2$ is $(12/\pi^2\lambda\kappa^2)N$. We obtain the final result valid for large N

$$\frac{\Delta L}{\langle L \rangle_{\bar{E}}} \approx \frac{2\sqrt{3}}{\pi\kappa\xi\sqrt{\lambda}}\frac{1}{\sqrt{N}}. \tag{16}$$

Thus, the small relative variance for large N need not be assumed from the start. The only assumptions are values of some structural properties and that an average value of energy is fixed. In the sense explained in Sect. 4, the length is then a classical property of our model body. We have obtained even more information: the internal-energy independence of the length (in this model, the dependence is trivial). This is an objective relation that can be in principle tested by measurements.

Similar results can be obtained for further thermodynamic properties such as elasticity coefficient, specific heat etc. They all are well known to have small variances in Gibbs states. The reason is that the contributions to these quantities are homogeneously distributed over the normal modes and the modes are mechanically and statistically independent. Further important classical properties are the mechanical ones: centre of mass and total momentum. In fact, these quantities can be chosen as the rest of the state coordinates for the whole chain. The contributions to them are perfectly homogeneously distributed over all atoms, not modes: the bulk motion is mechanically and statistically independent of all other modes and so its variances will not be small in Gibbs states. Still, generalized statistical methods can be applied to it. This is done in a separate paper [13].

The last remark is that the thermodynamic equilibrium can settle down starting from an arbitrary state only if some weak but non-zero interaction exists between the phonons. This can easily be arranged so that the influence of the interaction on our result is negligible.

References

1. Griffiths, R.B.: Consistent Quantum Theory. Cambridge University Press, Cambridge (2002)
2. Nikolic, H.: Quantum mechanics: myths and facts. arXiv:quant-ph/0609163
3. Isham, C.J.: Lectures on Quantum Theory. Mathematical and Structural Foundations. Imperial College Press, London (1995)
4. d'Espagnat, B.: Veiled Reality. Addison–Wesley, Reading (1995)
5. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer Academic, Dordrecht (1995)
6. Ludwig, G.: Foundations of Quantum Mechanics. Springer, Berlin (1983)
7. Kraus, K.: States, Effects, and Operations. Lecture Notes in Physics, vol. 190. Springer, Berlin (1983)
8. Bub, J.: Interpreting the Quantum World. Cambridge University Press, Cambridge (1997)
9. Dirac, P.A.M.: Quantum Mechanics, 4th edn. Clarendon Press, Oxford (1958)
10. von Neumann, J.: Mathematical Foundation of Quantum Mechanics. Princeton University Press, Princeton (1955)

11. Tolar, J., Hájíček, P.: Phys. Lett. A **353**, 19 (2006)
12. Schroeck, F.E.: Quantum Mechanics on Phase Space. Kluwer Academic, Dordrecht (1996)
13. Hájíček, P.: Quantum model of classical mechanics: maximum entropy packets. arXiv:0901.0436
14. Piron, C.: Foundations of Quantum Physics. Benjamin, Reading (1976)
15. Piron, C.: Found. Phys. **2**, 287 (1972)
16. Birkhoff, G., von Neumann, J.: Ann. Math. **37**, 823 (1936)
17. Thirring, W.: Lehrbuch der Mathematischen Physik. Springer, Berlin (1980)
18. Leggett, A.J.: J. Phys.: Condens. Matter **14**, R415 (2002)
19. Giulini, D., Joos, E., Kiefer, C., Kupsch, J., Stamatescu, I.-O., Zeh, H.D.: Decoherence and the Appearance of Classical World in Quantum Theory. Springer, Berlin (1996)
20. Exner, F.: Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften. Deuticke, Leipzig (1922)
21. Born, M.: Phys. Bl. **11**, 49 (1955)
22. Jaynes, E.T.: Probability Theory. The Logic of Science. Cambridge University Press, Cambridge (2003)
23. van Kampen, N.G.: Physica A **194**, 542 (1993)
24. Zurek, W.H.: Rev. Mod. Phys. **75**, 715 (2003)
25. Poulin, D.: Phys. Rev. A **71**, 022102 (2005)
26. Kofler, J., Brukner, Č.: Phys. Rev. Lett. **99**, 180403 (2007)
27. Hepp, K.: Helv. Phys. Acta **45**, 237 (1972)
28. Bell, J.S.: Helv. Phys. Acta **48**, 93 (1975)
29. Bóna, P.: Acta Phys. Slov. **23**, 149 (1973)
30. Bóna, P.: Acta Phys. Slov. **25**, 3 (1975)
31. Bóna, P.: Acta Phys. Slov. **27**, 101 (1977)
32. Sewell, G.L.: Quantum Mechanics and its Emergent Macrophysics. Princeton University Press, Princeton (2002)
33. Bell, J.S.: Against 'measurement'. In: Miller, A.I. (ed.) Sixty Two Years of Uncertainty. Plenum, New York (1990)
34. Wallace, D.: The quantum measurement problem: state of play. arXiv:0712.0149v1 [quant-ph]
35. Jauch, J.M.: Helv. Phys. Acta **37**, 293 (1964)
36. Kittel, C.: Introduction to Solid State Physics. Wiley, New York (1976)
37. Rutherford, D.E.: Proc. R. Soc. (Edinb.), Ser. A **62**, 229 (1947)
38. Rutherford, D.E.: Proc. R. Soc. (Edinb.), Ser. A **63**, 232 (1951)