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On Modal μ-Calculus and Gödel-Löb Logic

Abstract. We show that the modal μ-calculus over GL collapses to the modal fragment by showing that the fixpoint formula is reached after two iterations and answer to a question posed by van Benthem in [4]. Further, we introduce the modal $\mu^{\sim}$-calculus by allowing fixpoint constructors for any formula where the fixpoint variable appears guarded but not necessarily positive and show that this calculus over GL collapses to the modal fragment, too. The latter result allows us a new proof of the de Jongh, Sambin Theorem and provides a simple algorithm to construct the fixpoint formula.

Keywords: Fixpoint, Modal μ-Calculus, Gödel-Löb Logic.

1. Introduction

The modal μ-calculus is an extension of propositional modal logic, with least and greatest fixpoint operators. The term "μ-calculus" and the idea of extending modal logic with fixpoints appeared for the first time in the paper of Scott and De Bakker [11] and was further developed by others. Nowadays, the term "modal μ-calculus" stands for the formal system introduced by Kozen [10]. The standard semantics of the modal μ-calculus is given by transition systems. As usual, formulae are interpreted as subsets of a system, the set of states where the property expressed by the formula holds. Many natural properties such as "there is an infinite path" can be expressed by a modal μ-formula. Indeed, it is a powerful logic of programs subsuming dynamic and temporal logics like PDL, PLTL, CTL and CTL*. We refer to Bradfield and Stirling's tutorial article [8] or Stirling's book [12] for a thorough introduction to this system.

Gödel-Löb logic, GL, is used to investigate what arithmetical theories can express in a restricted language about their provability predicates. As a modal logic, provability logic has been studied since the early seventies, and has had important applications in the foundations of mathematics. Beside the arithmetical interpretation there is also a semantics given, as for almost all modal logics, by transition systems. The class of all transitive and upward well-founded systems forms a complete semantics for GL.

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Fixpoints and fixpoint theorems play an important role in GL. The most famous one, the *existence of a fixpoint* for guarded formulae was proved by de Jongh and Sambin independently (c.f., [15]). Even though it is formulated and proved by strictly modal methods, the fixpoint theorem still has great arithmetical significance. The *uniqueness of the fixpoint* was proved later by Bernardi, de Jongh and Sambin independently (c.f., [15]).

Since the modal μ-calculus is a general framework to study fixpoints in modal logic, studying the modal μ-calculus over GL is a promising work. This has been done by van Benthem in [4] and Visser in [17]. Both authors establish, by using the de Jongh, Sambin fixpoint theorem, that the modal μ-calculus over GL collapses to its modal fragment. But since they use the already known fixpoint theorem in order to establish this collapse in [4] van Benthem writes:

> "Our ... analysis does not explain why provability fixed-points are *explicitly definable* in the modal base language. Indeed, the general reason seems unknown."

In this paper we answer this question. More precisely, we prove the collapse of the modal μ-calculus over GL without using the de Jongh, Sambin Theorem by showing that fixpoints are reached after two iterations of well-named fixpoint formulae.

Fixpoint theorems in GL hold also for modal formulae where the variable appears guarded but not necessarily positively and, from this point of view, this first result is not completely satisfactory since modal μ-calculus allows fixpoint constructors only for syntactically positive formulae. Therefore, we also introduce the modal $\mu^{\sim}$-calculus which allows fixpoint constructors for formulae where the fixpoint variable appears guarded. As can be done also for the standard μ-calculus we define the semantics by way of games, in this case only over transitive and upward well-founded transition systems and, by using game-theoretical, we show that the modal $\mu^{\sim}$-calculus collapses to the modal fragment by providing an explicit syntactical translation of the modal $\mu^{\sim}$-calculus into GL which preserves logical equivalence. As a corollary of the collapse, we obtain a new version of the de Jongh, Sambin Fixpoint Theorem with a simple algorithm which shows how the fixpoint can be computed. In this sense we give an answer to a generalisation of van Benthem's question. Summing up, the modal $\mu^{\sim}$-calculus allows us to apply techniques similar as those known from the standard μ-calculus to GL and could be regarded as a starting point for further studies in this direction.

Both the collapse of the modal μ-calculus over GL and the one of the $\mu^{\sim}$-calculus over the same class of models are proved by using techniques and results from [2].

In the next section we repeat the preliminaries and some results which are already known. In Section 3 we analyse the modal μ-calculus over GL and show that it collapses to the modal fragment. In the last section we introduce the modal $\mu^{\sim}$-calculus and show a collapse to the modal fragment. The result is then used to provide a new proof of the uniqueness theorem of Bernardi, de Jongh and Sambin and of the existence theorem of de Jongh, Sambin. For the last one we also give a simple algorithm which shows how the fixpoint can be computed.

2. Preliminaries

2.1. Gödel-Löb Logic GL

We start from an infinite countable set Prop of *propositional variables*. Then the collection $\mathcal{L}_{\mathsf{GL}}$ of GL-*formulae* is given by:

$$\varphi ::= p \mid \ \sim p \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathsf{Prop}$. If all propositional variables occurring in φ are in $\mathsf{P} \subseteq \mathsf{Prop}$, we write $\varphi \in \mathcal{L}_{\mathsf{GL}}(\mathsf{P})$. If ψ is a subformula of φ, we write $\psi \leq \varphi$. We write $\psi < \varphi$ when ψ is a proper subformula. $\mathsf{sub}(\varphi)$ is the set of all subformulae of φ. The formula $\neg\varphi$ is defined by using de Morgan dualities for boolean connectives and the modal dualities for $\Diamond$ and $\Box$ and the law of double negation. As usual, we introduce implication $\varphi \to \psi$ as $\neg\varphi \vee \psi$ and equivalence $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\varphi \to \psi)$. We say that $p \in \mathsf{Prop}$ is *guarded* in φ if $p \leq \varphi$ and all occurrences of p are in the scope of a modal operator.

The axioms and inference rules below give a deduction system for GL. As usual we write $\mathsf{GL} \vdash \varphi$ if there is a derivation of φ in the system presented below.

Axioms: All classical propositional tautologies, the *Distribution Axiom* from modal logic

$$(\Box(\alpha \to \beta) \wedge \Box\alpha) \to \Box\beta$$

and the *Löb Axiom*

$$\Box(\Box\alpha \to \alpha) \to \Box\alpha.$$

Inference Rules: Beside the classical *Modus Ponens*

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

we have the *Necessitation Rule*

$$\frac{\alpha}{\Box\alpha}.$$

As for all modal logics the semantics of GL is given by transition systems. A *transition system* $\mathcal{T}$ is of the form $(\mathsf{S}, \rightarrow^{\mathcal{T}}, \lambda^{\mathcal{T}})$ where S is a set of *states*, $\rightarrow^{\mathcal{T}}$ is a binary relation on S called the *accessibility relation* and $\lambda : \mathsf{Prop} \rightarrow \wp(\mathsf{S})$ is a *valuation* for all propositional variables. A transition system $\mathcal{T}$ with a distinguished state s is called a *pointed transition system* and denoted by $(\mathcal{T}, s)$. $\mathbb{T}$ denotes the class of all pointed transition systems. The accessibility relation is called *upward well-founded* if there is no infinite chain of the form

$$s_0 \rightarrow^{\mathcal{T}} s_1 \rightarrow^{\mathcal{T}} s_2 \rightarrow \ldots.$$

By $\mathbb{T}^{\text{wft}}$ we denote the subclass of pointed transition systems such that the accessibility relation is transitive and upward well-founded.

Given a transition system $\mathcal{T}$, the *denotation of* φ *in* $\mathcal{T}$, $\|\varphi\|_{\mathcal{T}}$, that is, the set of states satisfying a formula φ is defined inductively on the structure of φ. For all transition systems we set

- $\|p\|_{\mathcal{T}} = \lambda(p)$ and $\| \sim p\|_{\mathcal{T}} = \mathsf{S} \setminus \lambda(p)$ for all $p \in \mathsf{Prop}$,
- $\|\alpha \wedge \beta\|_{\mathcal{T}} = \|\alpha\|_{\mathcal{T}} \cap \|\beta\|_{\mathcal{T}}$,
- $\|\alpha \vee \beta\|_{\mathcal{T}} = \|\alpha\|_{\mathcal{T}} \cup \|\beta\|_{\mathcal{T}}$,
- $\|\Box\alpha\|_{\mathcal{T}} = \{s \in \mathsf{S} \mid \forall t((s \rightarrow^{\mathcal{T}} t) \Rightarrow t \in \|\alpha\|_{\mathcal{T}})\}$, and
- $\|\Diamond\alpha\|_{\mathcal{T}} = \{s \in \mathsf{S} \mid \exists t((s \rightarrow^{\mathcal{T}} t) \wedge t \in \|\alpha\|_{\mathcal{T}})\}$.

We say that a pointed transition system $(\mathcal{T}, s)$ is a model of a GL-formula if and only if $s \in \|\varphi\|_{\mathcal{T}}$. If all pointed transition systems $(\mathcal{T}, s) \in \mathbb{T}^{\text{wft}}$ are a model of φ then we write $\mathsf{GL} \models \varphi$. A proof of the next theorem can be found in [6].

THEOREM 2.1. *For all* GL*-formulae* φ *we have that*

$$\mathsf{GL} \vdash \varphi \quad \textit{if and only if} \quad \mathsf{GL} \models \varphi.$$

2.2. The modal μ-calculus

The *language of the modal* μ*-calculus* results by adding greatest and least fixpoint operators to propositional modal logic. More precisely, given an infinite countable set Prop of *propositional variables*, the collection $\mathcal{L}_\mu$ of *modal* μ*-formulae* (or simply μ-formulae) is defined as follows:

$$\varphi ::= p \mid \ \sim p \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi$$

where $p, x \in \mathsf{Prop}$ and x occurs only positively in $\eta x.\varphi$ ($\eta \in \{\nu, \mu\}$), that is, $\sim x$ is not a subformula of φ.

The fixpoint operators μ and ν can syntactically be viewed as quantifiers. Therefore we use the standard terminology and notations as for quantifiers and, for instance, $\mathsf{free}(\varphi)$ denotes the set of all propositional variables occurring free in φ and $\mathsf{bound}(\varphi)$ those occurring bound. Further, we define $\mathsf{var}(\varphi) = \mathsf{free}(\varphi) \cup \mathsf{bound}(\varphi)$. If ψ is a subformula of φ, we write $\psi \leq \varphi$. We write $\psi < \varphi$ when ψ is a proper subformula. $\mathsf{sub}(\varphi)$ is the set of all subformulae of φ. We write $\varphi \in \mathcal{L}_\mu(\mathsf{P})$ if it holds that $\mathsf{free}(\varphi) \subseteq \mathsf{P}$, with $\mathsf{P} \subseteq \mathsf{Prop}$.

The negation $\neg\varphi$ of a μ-formula φ is defined inductively as for GL-formulae and, further, for μ and ν we use

$$\neg\mu x.\varphi(x) \equiv \nu x.\neg\varphi(x)[x/\neg x] \quad \text{and} \quad \neg\nu x.\varphi(x) \equiv \mu x.\neg\varphi(x)[x/\neg x].$$

Given a μ-formula φ, for all sets of bound variables $X \subseteq \mathsf{bound}(\varphi)$, the formula $\varphi^{\mathsf{free}(X)}$ is obtained from φ by eliminating all fixpoint operators binding a variable $x \in X$ but leaving the previously bound variables x as a free occurrences. When $X = \mathsf{bound}(\varphi)$, sometimes we simply write φ^{free}.

Let $\varphi \in \mathcal{L}_\mu(\mathsf{P})$. An *alternating μ-chain* in φ of length k is a sequence

$$\varphi \geq \mu x_0.\psi_0 > \nu x_1.\psi_1 > \cdots > \mu/\nu_{k-1}.\psi_{k-1}$$

where for every $i < k-1$ the variable x_i is free in every ψ such that $\psi_i \geq \psi \geq \psi_{i+1}$. The maximum length of an alternating μ-chain in φ is denoted by $\max^\mu(\varphi)$. ν-chains and $\max^\nu(\varphi)$ are defined analogously. The *alternation depth* of a μ-formula φ, denoted by $\mathsf{ad}(\varphi)$, is the maximum of $\max^\mu(\varphi)$ and $\max^\nu(\varphi)$. If φ is a purely modal formula, we set $\mathsf{ad}(\varphi) = 0$.

The axioms and inference rules below define the deduction system Koz. As usual we write $\mathsf{Koz} \vdash \varphi$ if there is a derivation of φ in the system presented below.

Axioms: All classical propositional tautologies and the *Distribution Axiom* from modal logic and the *fixpoint axioms*

$$\eta x.\varphi(x) \leftrightarrow \varphi(\eta x.\varphi(x)), \quad \eta \in \{\mu, \nu\}.$$

Inference Rules: Beside the classical *Modus Ponens* and the *Necessitation Rule* we have the *Induction Rule*

$$\frac{\varphi \to \alpha[x/\varphi]}{\varphi \to \nu x.\alpha}.$$

It can be shown that the fixpoint axioms can be replaced by the following weaker axioms:

$$\nu x.\varphi(x) \to \varphi(\nu x.\varphi(x)) \quad \text{and} \quad \mu x.\varphi(x) \leftarrow \varphi(\mu x.\varphi(x)).$$

We say that a variable $x \in \mathsf{bound}(\varphi)$ is *well-bound* in φ if no two distinct occurrences of fixpoint operators in φ bind x, and x occurs only once in φ. A propositional variable p is *guarded* in a formula $\varphi \in \mathcal{L}_\mu$ if every occurrence of p in φ is in the scope of a modal operator. A formula φ of $\mathcal{L}_\mu$ is said to be *well-named* if every $x \in \mathsf{bound}(\varphi)$ is guarded and well-bounded in φ. For all well-named φ, if x is bound in φ then there is exactly one subformula occurrence $\eta x.\delta \leq \varphi$ which bounds x, this formula is denoted by φ_x. In Lemma 2.4 we will see that any μ-formula φ is equivalent to a well-named formula $\mathsf{wn}(\varphi)$, therefore, if nothing else mentioned, we assume that all formulae are well-named.

Let $\varphi(x)$ be a μ-formula. If x is free and occurs only positively in φ, then we define $\varphi^n(x)$ for all n inductively such that $\varphi^1(x) = \varphi(x)$ and such that

$$\varphi^{k+1}(x) \equiv \varphi[x/\varphi^k(x)].$$

$\varphi^n(\top)$ and $\varphi^n(\bot)$ are obtained by substituting x with $\top$ or $\bot$ respectively.

The *rank*, $\mathsf{rank}(\varphi)$, of a formula φ is an ordinal number defined inductively as follows:

- $\mathsf{rank}(p) = \mathsf{rank}(\sim p) = 1$
- $\mathsf{rank}(\triangle\, \alpha) = \mathsf{rank}(\alpha) + 1$ where $\triangle \in \{\Box, \Diamond\}$
- $\mathsf{rank}(\alpha \circ \beta) = \mathsf{max}\{\mathsf{rank}(\alpha), \mathsf{rank}(\beta)\} + 1$ where $\circ \in \{\wedge, \vee\}$
- $\mathsf{rank}(\eta x.\alpha) = \mathsf{sup}\{\mathsf{rank}(\alpha^n(x)) + 1 \;\; ; \;\; n \in \mathbb{N}\}$ where $\eta \in \{\nu, \mu\}$.

In the joint work with Krähenbühl [3] (see also [1]) one of the authors shows that the algorithm for rank terminates. Further, it is an easy exercise to show that for all formulae φ we have that $\mathsf{rank}(\varphi) = \mathsf{rank}(\neg\varphi)$. The next lemma shows that well-naming iterated formulae which are already well-named does not affect the rank.

Lemma 2.2. *For all well-named formulae φ such that x appears only positively and all $n \in \mathbb{N}$ we have that*

$$\mathsf{rank}(\varphi^n(\top)) = \mathsf{rank}(\mathsf{wn}(\varphi^n(\top))).$$

Similarly for $\bot$.

Proof. The result follows from the fact that since φ is well-named the equivalent well-named formula is given by simply renaming bound variables in $\varphi^n(\bot)$. This can be verified by showing by induction on n that there are no free occurrences of a variable x in $\varphi^n(\top)$ which becomes bound in $\varphi(\varphi^n(\top))$. ■

As for GL the semantics of modal μ-calculus is given by transition systems. In order to define the denotation to fixpoint formulae let λ be a valuation, p a propositional variable and S' a subset of states S; we set for all propositional variables p'

$$\lambda[p \mapsto \mathsf{S}'](p') = \begin{cases} \mathsf{S}' & \text{if } p' = p, \\ \lambda(p') & \text{otherwise.} \end{cases}$$

Given a transition system $\mathcal{T} = (\mathsf{S}, \to^{\mathcal{T}}, \lambda^{\mathcal{T}})$, then $\mathcal{T}[p \mapsto \mathsf{S}']$ denotes the transition system $(\mathsf{S}, \to^{\mathcal{T}}, \lambda^{\mathcal{T}}[p \mapsto \mathsf{S}'])$. Given a transition system $\mathcal{T}$, the *denotation of φ in $\mathcal{T}$*, $\|\varphi\|_{\mathcal{T}}$, that is, the set of states satisfying a formula φ is defined inductively on the structure of φ as it was for GL and, in addition we set

- $\|\nu x.\alpha\|_{\mathcal{T}} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\alpha(x)\|_{\mathcal{T}[x \mapsto \mathsf{S}']}\}$, and
- $\|\mu x.\alpha\|_{\mathcal{T}} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha(x)\|_{\mathcal{T}[x \mapsto \mathsf{S}']} \subseteq \mathsf{S}'\}$.

For a formula $\varphi(x)$ and set of states $\mathsf{S}' \subseteq \mathsf{S}$ we sometimes write $\|\varphi(\mathsf{S}')\|_{\mathcal{T}}$ instead of $\|\varphi(x)\|_{\mathcal{T}[x \mapsto \mathsf{S}']}$. When clear from the context we use $\|\varphi(x)\|_{\mathcal{T}}$ for the function

$$\|\varphi(x)\|_{\mathcal{T}} : \begin{cases} \wp(\mathsf{S}) \to \wp(\mathsf{S}) \\ \mathsf{S}' \mapsto \|\varphi(\mathsf{S}')\|_{\mathcal{T}}. \end{cases}$$

By the Tarski-Knaster Theorem, c.f. [16], $\|\nu x.\alpha(x)\|_{\mathcal{T}}$ is the greatest fixpoint and $\|\mu x.\alpha(x)\|_{\mathcal{T}}$ the least fixpoint of the operator $\|\alpha(x)\|_{\mathcal{T}}$. Also for the modal μ-calculus we have a completeness theorem, due to Walukiewicz.

THEOREM 2.3 ([18]). *For all μ-formulae φ we have that*

$$\models \varphi \quad \textit{if and only if} \quad \mathsf{Koz} \vdash \varphi.$$

The next lemma states some basic properties of denotations.

LEMMA 2.4. *For all transition systems $\mathcal{T} = (\mathsf{S}, \to^{\mathcal{T}}, \lambda^{\mathcal{T}})$ and all formulae φ we have that*

1. $\|\neg\varphi\|_{\mathcal{T}} = \mathsf{S} \setminus \|\varphi\|_{\mathcal{T}}$,
2. $\|\eta x.\eta y.\varphi(x,y)\|_{\mathcal{T}} = \|\eta x.\varphi(x,x)\|_{\mathcal{T}}$, *where* $\eta \in \{\mu, \nu\}$,
3. $\|\nu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\top)\|_{\mathcal{T}}$, *if x is not guarded,*
4. $\|\mu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\bot)\|_{\mathcal{T}}$, *if x is not guarded.*
5. *There is a well-named formula* $\mathsf{wn}(\varphi)$ *such that* $\|\varphi\|_{\mathcal{T}} = \|\mathsf{wn}(\varphi)\|_{\mathcal{T}}$.

Proof. Part 1 to part 4 are classical properties of the modal μ-calculus. Part 5 is a straightforward consequence of parts 2 to 4. ∎

From now on, we assume that wn is a function associating to every formula φ a well-named formula $\mathsf{wn}(\varphi)$ and by $\mathcal{L}_\mu^{\mathsf{wn}}$ we denote the set of all well-named μ-formulae.

2.3. Embedding GL into the modal μ-calculus

In this subsection we define an embedding t from GL into the modal μ-calculus. First, we define the function $()^* : \mathcal{L}_{\mathsf{GL}}(P) \to \mathcal{L}_\mu(P)$ recursively on the structure of the formula such that

- $(p)^* \equiv p$ and $(\sim p)^* \equiv \sim p$,
- $(\alpha \wedge \beta)^* \equiv (\alpha)^* \wedge (\beta)^*$ and $(\alpha \vee \beta)^* \equiv (\alpha)^* \vee (\beta)^*$
- $(\Box\alpha)^* \equiv \nu x.\Box(x \wedge (\alpha)^*)$, and
- $(\Diamond\alpha)^* \equiv \mu x.\Diamond(x \vee (\alpha)^*)$.

The embedding $t : \mathcal{L}_{\mathsf{GL}}(P) \to \mathcal{L}_\mu(P)$ is now defined as

$$t(\varphi) \equiv \mu x.\Box x \to (\varphi)^*.$$

The following theorem is due to van Benthem [4]. It shows that GL which semantically lives on transitive and upward well-founded transition systems can be translated into the modal μ-calculus over arbitrary transition systems. For the first equivalence van Benthem provides a syntactical proof without using completeness results.

Theorem 2.5 ([4]). *For all formulae $\varphi \in \mathcal{L}_{\mathsf{GL}}$ we have that*

$$(\mathsf{GL} \vdash \varphi \quad \Leftrightarrow \quad \mathsf{Koz} \vdash t(\varphi)) \quad \textit{and} \quad (\models_{\mathsf{GL}} \varphi \quad \Leftrightarrow \quad \models t(\varphi)).$$

2.4. Parity games

Let V be a set. By V^* we denote the set of finite sequences on V, and by V^+ we denote the set of nonempty sequences. Finally, by V^ω we denote the set of infinite sequences over V.

A *game* $\mathcal{G}$ is defined in terms of an arena A and a winning condition W. In our case an arena is simply a bi-partite graph $A = \langle V_0, V_1, E\rangle$, where $V_0 \cap V_1 = \emptyset$ and the edge relation, or set of moves, is $E \subseteq (V_0 \cup V_1) \times (V_0 \cup V_1)$. Let $V = V_1 \cup V_2$ be the set of vertices, or positions, of the arena. Given two vertices $a, b \in V$, we say that b is a successor of a, if $(a, b) \in E$. The set of all successors of a is sometimes denoted by aE or $E(a)$. We say that b is reachable from a if there are $a_1, \dots, a_n \in V$ such that $a_1 = a$, $a_n = b$ and for every $0 < i < n$, $a_{i+1} \in a_i E$.

A *play* in the arena A can be finite or infinite. In the former case, the play is a non empty finite path $\pi = a_1 \dots a_n \in V^+$ such that for every $0 < i < n$, $a_{i+1} \in a_i E$ and $a_n E = \emptyset$. In the last case, the play consists in an infinite path $\pi = a_1 \dots a_n \dots \in V^\omega$ with $a_{i+1} \in a_i E$ for every $i > 0$. Thus a finite or infinite play in a game can be seen as the trace of a token moved on the arena by two Players, Player 0 and Player 1, in such a way that if the token is in position $a \in V_i$, then Player i has to choose a successor of a where to move the token.

The set of *winning conditions* W is a subset of V^ω. Thus, given a game $\mathcal{G} = (A, W)$ a play π is winning for Player 0 iff

1. if π is finite, then the last position a_n of the play is in V_1,
2. if π is infinite, then it must be a member of W.

A play is winning for Player 1 if it is not winning for Player 0. In this framework we are interested in what is called a *parity winning condition.* That is, given a set of vertices V, we assume a colouring or ranking function $\Omega : V \to \omega$ such that $\Omega[V]$ is bounded. Then, the set W of winning conditions is defined as the set of all infinite sequences π such that the greatest priority appearing infinitely often in $\Omega(\pi)$ is even.

Let A be an arena. A strategy for Player i is simply a function $\sigma_i : V^* V_i \to V$, with $i = 0, 1$. A prefix $a_1 \dots a_n$ of a play is said to be *compatible* or *consistent* with σ_i iff for every j with $1 \leq j < n$ and $a_j \in V_i$, it holds that $\sigma_i(a_1 \dots a_j) = a_{j+1}$. A finite or infinite play is compatible or consistent with σ_i if each of its prefixes which is in $V^* V_i$ is compatible with σ_i. The strategy σ_i is said to be a *winning strategy* for Player i on W if every play consistent with σ_i is winning for Player i. A position $a \in V$ is winning for Player i in the parity game $\mathcal{G}$ iff there is a strategy σ for Player i such that every play compatible with σ which starts from a is winning for Player i. A winning strategy σ is called *memoryless* if $\sigma(a_1 \dots a_n) = \sigma(b_1 \dots b_n)$, when $a_n = b_n$. For parity games we have a memoryless determinacy result.

THEOREM 2.6 ([9, 14]). *In a parity game, one of the Players has a memoryless winning strategy from each vertex.*

Having in mind this theorem, in the sequel we assume that all winning strategies are memoryless, that is, a winning strategy in a parity games for Player 0 is a function $\sigma : V_0 \to V$, analogously for Player 1.

2.5. Evaluation games for the modal μ-calculus

In this subsection we will see, given $\varphi \in \mathcal{L}_\mu$ and a pointed transition system $(\mathcal{T}, s_0)$ with $\mathcal{T} = (\mathsf{S}, \to^{\mathcal{T}}, \lambda^{\mathcal{T}})$, how to determine the corresponding parity game $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$, called also the *evaluation game of* φ *over* $(\mathcal{T}, s_0)$.

The arena of $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$ is the triple $\langle V_0, V_1, E\rangle$ which is defined recursively such that

$$\langle \varphi, s_0\rangle \in V$$

(remember that $V = V_0 \cup V_1$) and such that if $\langle \psi, s\rangle \in V$ then we distinguish the following cases:

- If $\psi \equiv (\sim)p$ and $p \in \mathsf{free}(\varphi)$. In this case we set $E\langle \psi, s\rangle = \emptyset$ and

$$\langle \psi, s\rangle \in V_1 \text{ iff } \begin{cases} s \in \lambda^{\mathcal{T}}(\psi) & \text{if } \psi \equiv p \\ s \notin \lambda^{\mathcal{T}}(\psi) & \text{if } \psi \equiv \sim p. \end{cases}$$

- If $\psi \equiv x$ and $x \in \mathsf{bound}(\varphi)$. Given $\langle \varphi_x, s\rangle \in V$, in this case we set

$$(\langle \psi, s\rangle, \langle \varphi_x, s\rangle) \in E$$

  and we have

$$\langle \psi, s\rangle \in V_0 \text{ iff } x \text{ is a } \mu\text{-variable.}$$

- If $\psi \equiv \alpha \wedge \beta$. In this case we have $\langle \psi, s\rangle \in V_1$ and

$$(\langle \psi, s\rangle, \langle \alpha, s\rangle) \in E \text{ and } (\langle \psi, s\rangle, \langle \beta, s\rangle) \in E$$

- If $\psi \equiv \alpha \vee \beta$. In this case we have $\langle \psi, s\rangle \in V_0$ and

$$(\langle \psi, s\rangle, \langle \alpha, s\rangle) \in E \text{ and } (\langle \psi, s\rangle, \langle \beta, s\rangle) \in E$$

- If $\psi \equiv \Box\alpha$. In this case we have $\langle \psi, s\rangle \in V_1$ and

$$(\langle \psi, s\rangle, \langle \alpha, s'\rangle) \in E \text{ for all } s' \text{ such that } s \to^{\mathcal{T}} s'.$$

- If $\psi \equiv \Diamond\alpha$. In this case we have $\langle \psi, s\rangle \in V_0$ and

$$(\langle \psi, s\rangle, \langle \alpha, s'\rangle) \in E \text{ for all } s' \text{ such that } s \to^{\mathcal{T}} s'.$$

- If $\psi \equiv \nu x.\alpha$. In this case we have $\langle \psi, s\rangle \in V_1$ and

$$(\langle \psi, s\rangle, \langle \alpha, s\rangle) \in E.$$

- If $\psi \equiv \mu x.\alpha$. In this case we have $\langle \psi, s\rangle \in V_0$ and

$$(\langle \psi, s\rangle, \langle \alpha, s\rangle) \in E.$$

We complete the definition of the parity game $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$ by defining the partial priority function $\Omega : V \to \omega$. The function is only defined on states of the form $\langle \eta x.\delta, s\rangle \in V$, where $\eta \in \{\mu, \nu\}$. In this case we have that:

$$\Omega(\langle \psi, s\rangle) = \begin{cases} \mathsf{ad}(\eta x.\delta) & \text{if } \eta = \mu \text{ and } \mathsf{ad}(\eta x.\delta) \text{ is odd, or} \\ & \quad \eta = \nu \text{ and } \mathsf{ad}(\eta x.\delta) \text{ is even;} \\ \mathsf{ad}(\eta x.\delta) - 1 & \text{if } \eta = \mu \text{ and } \mathsf{ad}(\eta x.\delta) \text{ is even, or} \\ & \quad \eta = \nu \text{ and } \mathsf{ad}(\eta x.\delta) \text{ is odd.} \end{cases}$$

Remember that if the play π is finite, Player 0 wins iff the last vertex of the play belongs to V_1, and if the play π is infinite, Player 0 wins iff the greatest priority appearing infinitely often is even.

THEOREM 2.7 ([13]). *$(\mathcal{T}, s) \in \|\varphi\|$ iff Player 0 has a winning strategy for $\mathcal{E}(\varphi, (\mathcal{T}, s))$.*

This result can be seen as the "game-theoretical version" of what is usually called the *Fundamental Theorem* of the semantic of the modal μ-calculus.

The next lemma verifies that over upward well-founded transition systems, least fixpoints and greatest fixpoints coincide.

LEMMA 2.8. *Let $\mathcal{T}$ be an upward well-founded transition system. Then, for every $\varphi(x) \in \mathcal{L}_\mu$ such that x is guarded and positive it holds that*

$$\|\mu x.\varphi(x)\|_{\mathcal{T}} = \|\nu x.\varphi(x)\|_{\mathcal{T}}.$$

PROOF. Note, that in an evaluation game there are no infinite regeneration of x since then we would have an infinite chain of the form

$$s_0 \to^{\mathcal{T}} s_1 \to^{\mathcal{T}} s_2 \ldots .$$

Therefore, a winning play for $\nu x.\varphi$ is also a winning play for $\mu x.\varphi$. With Theorem 2.7 we get the result. ■

3. The modal μ-calculus over GL

In this section we show that the expressivity of the modal μ-calculus over GL, that is, over transitive and upward well-founded transition systems, is the same as the one of the modal base language. In this sense we answer to van Benthem's question cited in the introduction.

In [2] the authors showed that over transitive transition systems every μ-formula is equivalent to a μ-formula without alternation of fixpoint operators. Moreover, they showed that under certain conditions a fixpoint operator can be eliminated by regenerating the formula:

Theorem 3.1 ([2]). *Let $\mathcal{T}$ be a transitive transition system, and let $\varphi(x)$ be a well-named μ-formula such that $x \in \mathsf{free}(\varphi)$ and occurs only once. Then*

1. *If x is in the scope of a $\Box$ in $\nu x.\varphi(x)$ then*

$$\|\nu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi^2(\top)\|_{\mathcal{T}}.$$

2. *If x is in the scope of a $\Diamond$ in $\mu x.\varphi(x)$ then*

$$\|\mu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi^2(\bot)\|_{\mathcal{T}}.$$

Definition 3.2. The translation $\tau : \mathcal{L}_{\mu}^{\mathsf{wn}}(P) \to \mathcal{L}_{\mathsf{GL}}(P)$ is defined recursively on the rank of the formula such that $\tau((\sim)p) \equiv (\sim)p$, such that τ distributes over boolean and modal connectives and such that for all $\eta \in \{\mu, \nu\}$ we have

$$\tau(\eta x.\varphi) = \begin{cases} \tau(\mathsf{wn}(\varphi^2(\top))) & x \text{ is in the scope of a } \Box \text{ in } \varphi, \\ \tau(\mathsf{wn}(\varphi^2(\bot))) & \text{else.} \end{cases}$$

Obviously, by first well-naming a formula and then applying τ we get a translation from $\mathcal{L}_{\mu}(P)$ to $\mathcal{L}_{\mathsf{GL}}(P)$.

Given the fact that over well-founded transition systems greatest and least fixpoint coincide, this result gives us the collapse of the modal μ-calculus over GL into its modal fragment.

Theorem 3.3. *On transitive and upward well-founded transition systems we have that the following holds for every $\varphi \in \mathcal{L}_{\mu}$:*

$$\|\varphi\|_{\mathcal{T}} = \|\tau(\mathsf{wn}(\varphi))\|_{\mathcal{T}}.$$

Proof. By Lemma 2.4 we can assume that φ is well-named. The proof is by induction on $\mathsf{rank}(\varphi)$. The base case and the case where $\mathsf{rank}(\varphi)$ is a successor ordinal are straightforward. If $\mathsf{rank}(\varphi)$ is a limit ordinal then φ is of the form $\eta x.\alpha$ ($\eta \in \{\mu, \nu\}$). Assume that φ is of the form $\nu x.\varphi$. If x is in the scope of a $\Box$ in φ then the induction step follows from Theorem 3.1.1. Else, x is only in the scope of some $\Diamond$ in φ. In this case by Lemma 2.8 we have that

$$\|\nu x.\varphi\|_{\mathcal{T}} = \|\mu x.\varphi\|_{\mathcal{T}}$$

and by applying Theorem 3.1.2 we get the induction step. The case where φ is of the form $\mu x.\varphi$ is shown by analogous arguments. ■

4. The modal $\mu^{\sim}$-calculus

In this section we introduce a new language, called the modal $\mu^{\sim}$-calculus, which, in some sense, can be seen as an extension of the guarded fragment of the modal μ-calculus. The main novelty is that we allow the μ-operator to bind negative (and guarded) occurrences of propositional variables. Therefore, the modal $\mu^{\sim}$-calculus allows us to refer explicitly, that is, in a μ-calculus style, to fixpoints of guarded formulae. For example, the fixpoint of the "equation" $p \leftrightarrow \alpha(p)$ where $\alpha(x)$ is a guarded formula can be directly denoted as $\mu x.\alpha(x)$. As it can be done for the modal μ-calculus the semantics of the modal $\mu^{\sim}$-calculus is defined by way of games over transitive and upward well-founded transition systems. We provide an explicit syntactical translation of the modal $\mu^{\sim}$-calculus into GL which preserves logical equivalence. As a corollary of the collapse, we obtain a new version of the de Jongh, Sambin Fixpoint Theorem. The modal $\mu^{\sim}$-calculus could be seen as a starting point for the application of tools of the standard μ-calculus, as for example games, to GL.

4.1. Basic notions and results

The *language of the modal $\mu^{\sim}$-calculus*, $\mathcal{L}_{\mu^{\sim}}$, is almost the same as the one for the modal μ-calculus with the only difference that we allow fixpoint constructors also when the bound variable is appearing negatively, that is, *modal $\mu^{\sim}$-formulae* (or simply $\mu^{\sim}$-formulae) are defined as follows:

$$\varphi ::= p \mid \ {\sim}p \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi$$

where $p, x \in \mathsf{Prop}$ and where x appears guarded in φ. All syntactical notions, such as $\mathcal{L}_{\mu^{\sim}}(\mathsf{P})$, bound variable, rank of a formula, φ_x, $\varphi^{\mathsf{free}(X)}$ etc., are defined as for the modal μ-calculus. Without loss of generality, we always suppose that $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$.

We say that a $\mu^{\sim}$-formula φ is in *normal form* if $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$ and if for all subformulae of φ of the form $\mu x.\mu y.\alpha$ we have that

- α is not of the form $\mu z.\beta$, and
- x occurs only negatively in α and y has only positive occurrences in α.

For the substitution, if $x \in \mathsf{free}(\varphi)$, then $\varphi[x/\psi]$ is given by substituting $\neg\psi$ to every negative occurrence $\sim x$ and by substituting ψ to every positive occurrence x. As for the modal μ-calculus negation is defined by using de Morgan laws and the duality of $\Box$ and $\Diamond$, in addition we set

$$\neg\mu x.\alpha \equiv \mu x.\neg\alpha[x/\neg x].$$

The last equivalence can be rather surprising at a first look. It is motivated by the fact that the modal $\mu^{\sim}$-calculus will be interpreted over upward well-founded models, where least and greatest fixpoint coincide.

Note that, for every $y \in \mathsf{bound}(\mu x.\alpha)$, y is negative in $\mu x.\alpha$ if and only if y is negative in $\neg \mu x.\alpha$.

The semantics for the modal $\mu^{\sim}$-calculus over GL is given by evaluation games on pointed upward well-founded and transitive transition systems. These evaluation games are similar to the ones for the modal μ-calculus. Let $\varphi \in \mathcal{L}_{\mu^{\sim}}$ and $(\mathcal{T}, s) \in \mathbb{T}^{\mathrm{wft}}$.

- First we construct recursively the two arenas $\langle V_0^+, V_1^+, E^+\rangle$ from φ and $(\mathcal{T}, s)$ and $\langle V_0^-, V_1^-, E^-\rangle$ from $\neg\varphi$ and $(\mathcal{T}, s)$ as it is done for the modal μ-calculus, then, for each vertex of the form $\langle {\sim} x, t\rangle$ which was generated in the recursion defining the arena $\langle V_0^+, V_1^+, E^+\rangle$ we add the condition

  $$\langle {\sim} x, t\rangle \in V_0^+ \text{ and } E^+(\langle {\sim} x, t\rangle) = \emptyset,$$

  and if it was generated in the recursion defining the arena $\langle V_0^-, V_1^-, E^-\rangle$ we set

  $$\langle {\sim} x, t\rangle \in V_0^- \text{ and } E^-(\langle {\sim} x, t\rangle) = \emptyset.$$

- Then the arena of $\mathcal{E}(\varphi, (\mathcal{T}, s))$ is the triple $\langle V_0, V_1, E\rangle$ defined by taking the disjoint union of the two arenas, with the following modification:
  - For every vertex of the form $\langle {\sim} x, t\rangle$ where $x \in \mathsf{bound}(\varphi)$ we set

  $$E(\langle \sim x, t\rangle) = \begin{cases} \{\langle \neg\varphi_x, t\rangle\} \subseteq V^- & \text{if } \langle {\sim} x, t\rangle \in V_0^+ \\ \{\langle \varphi_x, t\rangle\} \subseteq V^+ & \text{if } \langle {\sim} x, t\rangle \in V_0^-. \end{cases}$$

Since we are on upward well-founded models and that all regenerated variables are guarded, all plays are finite. Therefore, we have that Player 0 wins if and only if the last vertex of the play belongs to Player 1. Since therefore we do not have to care about priorities the definition of evaluation game for $\mu^{\sim}$-formulae is admissible and well-defined[1].

We say that a pointed upward well-founded transitive transition system $(\mathcal{T}, s)$ is a model of a $\mu^{\sim}$-formula if and only if Player 0 has a winning strategy in $\mathcal{E}(\varphi, (\mathcal{T}, s))$. Further, we define

$$\|\varphi\|_{\mathcal{T}}^{\mathcal{W}} = \{s \in \mathsf{S} \mid (\mathcal{T}, s) \text{ is a model of } \varphi\}.$$

[1]Note that on non well-founded models a play can be infinite. Thus, since in this kind of plays it is possible that Player 0 and Player 1 "switch" their roles infinitely often, it is not clear how to extend our game-theoretical approach also to non well-founded models by adding a natural and uniform (parity) winning conditions for infinite plays.

By $\|\varphi\|^{\mathcal{W}}$ we denote the class of all upward well-founded and transitive models of φ, that is, all pointed transition systems $(\mathcal{T}, s)$, transitive and upward well-founded, such that $s \in \|\varphi\|^{\mathcal{W}}$.

EXAMPLE 4.1. Consider the formula $\mu x.\Diamond \sim x$. This formula says that Player 0 can always force the number of the states visited in a play to be even. Because the considered models are transitive, this implies that the formula says that the root of the models has at least one accessible state.

The next lemma states some basic properties of denotation.

LEMMA 4.2. *For all transition systems $\mathcal{T} = (\mathsf{S}, \rightarrow^{\mathcal{T}}, \lambda^{\mathcal{T}})$ and all $\mu^{\sim}$-formulae $\mu x.\varphi$ we have that*

1. $\|\mu x.\mu y.\varphi(x, y)\|^{\mathcal{W}}_{\mathcal{T}} = \|\mu x.\varphi(x, x)\|^{\mathcal{W}}_{\mathcal{T}}$*,*
2. $\|\mu x_1 \ldots \mu x_n.\varphi\|^{\mathcal{W}}_{\mathcal{T}} = \|\mu x_{p(1)} \ldots \mu x_{p(n)}.\varphi\|^{\mathcal{W}}_{\mathcal{T}}$*, where p is any permutation over $\{1, \ldots, n\}$,*
3. *There is a well-named formula $\mathsf{wn}(\varphi)$ such that $\|\varphi\|^{\mathcal{W}}_{\mathcal{T}} = \|\mathsf{wn}(\varphi)\|^{\mathcal{W}}_{\mathcal{T}}$,*
4. *There is a formula $\mathsf{nf}(\varphi)$ in normal form such that $\|\varphi\|_{\mathcal{T}} = \|\mathsf{nf}(\varphi)\|_{\mathcal{T}}$.*

PROOF. Part 1 is by definition of the evaluation game for the modal $\mu^{\sim}$-calculus. Part 2 is proved by an easy induction on the length of the prefix. Part 3 is a straightforward consequence of part 1. Part 4 is a straightforward consequence of part 1 and part 2. ■

The next lemma shows that over upward well-founded models, the positively bounded fragment of the modal $\mu^{\sim}$-calculus coincides with the guarded fragment of the standard modal μ-calculus.

LEMMA 4.3. *Let $\varphi \in \mathcal{L}_{\mu} \cap \mathcal{L}_{\mu^{\sim}}$. Then for every upward-well founded model $\mathcal{T}$*

$$\|\varphi\|^{\mathcal{W}}_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}}.$$

PROOF. This follows by applying Theorem 2.7 to the fact that, for every $\varphi \in \mathcal{L}_{\mu} \cap \mathcal{L}_{\mu^{\sim}}$, the evaluation games for $\mathcal{L}_{\mu^{\sim}}$ and the evaluation games for the μ-calculus coincide over upward well-founded models. ■

The next lemma shows that negation behaves as expected.

LEMMA 4.4. *Let φ be a $\mu^{\sim}$-formula and $\mathcal{T} = (\mathsf{S}, \rightarrow^{\mathcal{T}}, \lambda^{\mathcal{T}})$ an upward well-founded transition system. We have that*

$$\|\neg\varphi\|^{\mathcal{W}}_{\mathcal{T}} = \mathsf{S} \setminus \|\varphi\|^{\mathcal{W}}_{\mathcal{T}}.$$

Proof. Consider the evaluation game $\mathcal{E}(\varphi, (\mathcal{T}, s))$ where Player 0 starts to play as Player 1 and vice versa. Clearly Player 0 (resp. Player 1) has a winning strategy in this modified game iff she has a winning strategy in $\mathcal{E}(\neg\varphi, (\mathcal{T}, s))$. From this fact we get the claim. ∎

The next lemma shows that in the modal $\mu^{\sim}$-calculus formulae of the form $\mu x.\varphi$ indeed define a fixpoint.

Lemma 4.5. *For every $\mu x.\varphi \in \mathcal{L}_{\mu^{\sim}}$ and every upward well-founded transition system $\mathcal{T}$ it holds that*

$$\|\mu x.\varphi\|_{\mathcal{T}}^{\mathcal{W}} = \|\varphi[x/\mu x.\varphi]\|_{\mathcal{T}}^{\mathcal{W}}.$$

Proof. This result follows straightforwardly by definition of the evaluation game for the modal $\mu^{\sim}$-calculus. ∎

4.2. The unicity of fixpoints

Let $\mathcal{T}$ be a upward well-founded and transitive transition system and φ a $\mu^{\sim}$-formula. Consider an arbitrary (memoryless) strategy σ for Player 0, not necessarily winning. We define the *restriction of* $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$ *on* σ, denoted by $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$, as follows:

- The set of positions $V|_\sigma$ of the restriction is given by all nodes which are the positions of some play compatible with σ starting from position $\langle\varphi, s_0\rangle$,
- The arena of $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$ is the triple $\langle V_0|_\sigma, V_1|_\sigma, E|_\sigma\rangle$ where:
  1. $V_0|_\sigma = \emptyset$,
  2. $V_1|_\sigma = V|_\sigma$,
  3. if $\langle\psi, s\rangle \in V|_\sigma \cap V_1$ then $E|_\sigma(\langle\psi, s\rangle) = E(\langle\psi, s\rangle)$, and
  4. if $\langle\psi, s\rangle \in V|_\sigma \cap V_0$ then $E|_\sigma(\langle\psi, s\rangle) = \{\sigma(\langle\psi, s\rangle)\}$.

We have that in $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$ the only Player who can move is Player 1. This can be done because the moves for Player 0 are already completely determined by the (memoryless) strategy σ. Clearly, any play in $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$ is a play in $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$ compatible with σ. We say that a play π in $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$ is winning for Player 0 if and only if the play π is winning for Player 0 in $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$. If σ is a winning strategy for Player 0 then any play in $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$ is winning for Player 0.

Definition 4.6. Let $\mathcal{T}$ be a upward well-founded transitive transition system, φ a $\mu^{\sim}$-formula and σ any strategy for Player 0 in the parity game $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$. Then, for every position $\langle \psi, s\rangle$ of $\mathcal{E}|_\sigma(\varphi, (\mathcal{T}, s_0))$, we define a measure $d(\langle \psi, s\rangle)$:

$$d(\langle \psi, s\rangle) = \begin{cases} 0 & \text{if} \quad E|_\sigma(\langle \psi, s\rangle) = \emptyset \\ \sup\{d(\langle \psi, s'\rangle) + 1 : \langle \psi, s'\rangle \in E|_\sigma(\langle \psi, s\rangle)\} & \text{otherwise.} \end{cases}$$

Note that, since $\mathcal{T}$ is upward well-founded, there cannot be an infinite chain of the form $\langle a_0, a_1, a_2, \dots\rangle$ such that for every $i \geq 0, \langle a_i, a_{i+1}\rangle \in E|_\sigma$. Therefore for all evaluation games $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$ and all vertices $\langle \psi, v\rangle$ in the arena $d(\langle \psi, v\rangle)$ is a well-defined ordinal number, such that if a vertex $\langle \alpha, v'\rangle$ is reachable from a vertex $\langle \beta, v''\rangle$ then we have that $d(\langle \alpha, v'\rangle) < d(\langle \beta, v''\rangle)$.

The next theorem shows that a fixpoint formula $\mu x.\alpha(x)$ in the modal $\mu^{\sim}$-calculus defines a fixpoint, as proved in Lemma 4.5, and that any other fixpoint of a formula $\alpha(x)$ is identical to $\mu x.\alpha(x)$. In this sense it is an existence and uniqueness theorem, and it is the central result of the section.

Theorem 4.7. *Let $\varphi(x) \in \mathcal{L}_{\mu^{\sim}}$ and $x \in \mathsf{free}(\varphi)$ a guarded variable. Let $\mathcal{T}$ be a upward well-founded transitive transition system. Then, for every $A \subseteq \mathsf{S}$ we have*

$$\|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} = A \text{ if and only if } A = \|\mu x.\varphi(x)\|^{\mathcal{W}}_{\mathcal{T}}.$$

Proof. The implication from right to left follows from Lemma 4.5. In order to prove the implication from left to right, suppose $\|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} = A$. From the definition of evaluation game we straightforwardly can derive the inclusion $A \subseteq \|\mu x.\varphi(x)\|^{\mathcal{W}}_{\mathcal{T}}$. For the other inclusion, suppose that $s \in \|\mu x.\varphi(x)\|^{\mathcal{W}}_{\mathcal{T}}$. We have that Player 0 has a winning strategy σ in $\mathcal{E}(\mu x.\varphi, (\mathcal{T}, s))$. Consider the restricted evaluation game $\mathcal{E}|_\sigma(\mu x.\varphi, (\mathcal{T}, s))$. For each vertex $\langle \alpha, v\rangle$ in $\mathcal{E}|_\sigma(\mu x.\varphi, (\mathcal{T}, s))$, we have that $d(\langle \alpha, v\rangle)$ is a well-defined measure. Clearly, with the following claim we finish the proof.

Claim: For all vertices of the form $\langle \alpha, s'\rangle$ in $\mathcal{E}|_\sigma(\mu x.\varphi, (\mathcal{T}, s))$ if $\alpha = \mu x.\varphi$ then $s' \in A$, and, if $\alpha = \mu x.\neg\varphi$ then $s' \notin A$.

The proof of the claim is by induction on d. Since $d(\langle \mu x.\varphi, s'\rangle) > 0$ and $d(\langle \mu x.\neg\varphi, s'\rangle) > 0$ the induction base is trivial. For the induction step assume first that we have a vertex $\langle \mu x.\varphi, s'\rangle$ in $\mathcal{E}|_\sigma(\mu x.\varphi, (\mathcal{T}, s))$. We distinguish two cases:

1. If from $\langle \mu x.\varphi, s'\rangle$ there is no a reachable vertex of the form $\langle \mu x.\varphi, s''\rangle$ or $\langle \mu x.\neg\varphi, s''\rangle$ then we have that $s' \in \|\varphi(A')\|^{\mathcal{W}}_{\mathcal{T}}$ for all sets of states A' and, therefore, we also have $s' \in \|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}}$. Since by assumption $\|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} = A$ we proved the claim.

2. Otherwise we distinguish two subcases given by the first vertex reached which is of either the form $\langle \mu x.\varphi, s'' \rangle$ or $\langle \mu x.\neg\varphi, s'' \rangle$.
   (a) If the first vertex reached of such kind is $\langle \mu x.\varphi, s'' \rangle$, then, since we have that $d(\langle \mu x.\varphi, s' \rangle) > d(\langle \mu x.\varphi, s'' \rangle)$, by induction hypothesis we get $s'' \in A$.
   (b) If the first vertex reached of such kind is $\langle \mu x.\neg\varphi, s'' \rangle$, then, since we have that $d(\langle \mu x.\varphi, s' \rangle) > d(\langle \mu x.\neg\varphi, s'' \rangle)$, by induction hypothesis we get $s'' \notin A$.

   Therefore, for each play consistent with σ starting from $\langle \mu x.\varphi, s' \rangle$ it holds that if it reaches first a vertex of the form $\langle \mu x.\varphi, s'' \rangle$ (or equivalently of the form $\langle x, s'' \rangle$) we have that $s'' \in A$, and, if it reaches first a vertex of the form $\langle \mu x.\neg\varphi, s'' \rangle$ (or equivalently of the form $\langle \sim x, s'' \rangle$) we have that $s'' \notin A$. But it can easily be seen that this implies $s' \in \|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}}$. Since by assumption $\|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} \subseteq A$ we finish the induction step for the case $\alpha = \mu x.\varphi(x)$.

The induction step for a vertex of the form $\langle \mu x.\neg\varphi, s' \rangle$ is verified in the same way by using the fact that by Lemma 4.4 we have that $\|\neg\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} = S \setminus \|\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}}$ and, therefore, by assumption that $\|\neg\varphi(A)\|^{\mathcal{W}}_{\mathcal{T}} = S \setminus A$. ■

Corollary 4.8. *Let φ and ψ be two $\mu^{\sim}$-formulae. If for all upward well-founded transitive transition system $\mathcal{T}$ we have that $\|\psi\|^{\mathcal{W}}_{\mathcal{T}} = \|\varphi\|^{\mathcal{W}}_{\mathcal{T}}$ then for all variables x and all $\mathcal{T}$ we have that*

$$\|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}} = \|\mu x.\varphi\|^{\mathcal{W}}_{\mathcal{T}}$$

Proof. By the "if" direction of Theorem 4.7 we have that

$$\|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}} = \|\psi\|^{\mathcal{W}}_{\mathcal{T}[x \mapsto \|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}}]}$$

and with the premise of the corollary we get

$$\|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}} = \|\varphi\|^{\mathcal{W}}_{\mathcal{T}[x \mapsto \|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}}]}.$$

Applying the "only if" direction of Theorem 4.7 we obtain that

$$\|\mu x.\psi\|^{\mathcal{W}}_{\mathcal{T}} = \|\mu x.\varphi\|^{\mathcal{W}}_{\mathcal{T}}.$$

■

The next theorem provides a new proof of Bernardi, de Jongh, Sambin Theorem (c.f. Chapter 8 in [6] or [15]) using our results on the modal $\mu^{\sim}$-calculus.

THEOREM 4.9. *Let $\varphi(x) \in \mathcal{L}_{\mathsf{GL}}$, where x is guarded. We have that*

$$\mathsf{GL} \vdash \Box^s(p \leftrightarrow \varphi(p)) \wedge \Box^s(q \leftrightarrow \varphi(q)) \rightarrow (p \leftrightarrow q)$$

where $\Box^s\varphi :\equiv \Box\varphi \wedge \varphi$.

PROOF. By Theorem 2.5 it is enough to show that for all $\varphi(x) \in \mathcal{L}_{\mathsf{GL}}$ it holds that

$$\models \mu x.\Box x \rightarrow (((\Box^s)(p \leftrightarrow \varphi(p)) \wedge (\Box^s)(q \leftrightarrow \varphi(q))) \rightarrow (p \leftrightarrow q))^*.$$

And this can be done by showing for all $\varphi(x) \in \mathcal{L}_{\mathsf{GL}}$ that the following formula is valid for all transition systems

$$\mu x.\Box x \rightarrow (p \leftrightarrow \varphi(p) \wedge q \leftrightarrow \varphi(q) \wedge \Box^*(q \leftrightarrow \varphi(q)) \wedge \Box^*(q \leftrightarrow \varphi(q)) \rightarrow (p \leftrightarrow q)) \tag{1}$$

where $\Box^*\gamma \equiv \nu x.\Box(x \wedge \gamma)$.

Assume, that we have

$$s \in \|\mu x.\Box x \wedge p \leftrightarrow \varphi(p) \wedge q \leftrightarrow \varphi(q) \wedge \Box^*(q \leftrightarrow \varphi(q)) \wedge \Box^*(q \leftrightarrow \varphi(q))\|_{\mathcal{T}}.$$

Then, $(\mathcal{T}, s)$ is well-founded and we have for s and for all reachable states s' from s that $q \leftrightarrow \varphi(q)$ and $p \leftrightarrow \varphi(p)$. Therefore, if we assume that $\mathcal{T}$ consists of s and all reachable states from s, which is an admissible assumption, we get that we have

$$\lambda^{\mathcal{T}}(p) = \|\varphi(\lambda^{\mathcal{T}}(p))\|^{\mathcal{W}}_{\mathcal{T}} \quad \text{and} \quad \lambda^{\mathcal{T}}(q) = \|\varphi(\lambda^{\mathcal{T}}(q))\|^{\mathcal{W}}_{\mathcal{T}}.$$

By Theorem 4.7 we get that

$$\lambda^{\mathcal{T}}(p) = \|\mu x.\varphi(x)\|^{\mathcal{W}}_{\mathcal{T}} \quad \text{and} \quad \lambda^{\mathcal{T}}(q) = \|\mu x.\varphi(x)\|^{\mathcal{W}}_{\mathcal{T}}.$$

and therefore we obtain that

$$s \in \|p \leftrightarrow q\|_{\mathcal{T}}.$$

We have shown Equation 1 and finished the proof. ∎

4.3. Collapsing the modal $\mu^{\sim}$-calculus

In this subsection we provide an explicit syntactical translation of the modal μ-calculus into GL which preserves logical equivalence. As a corollary, we obtain a new proof of the de Jongh, Sambin Fixpoint Theorem which provides an explicit construction of the fixpoint formula based on the syntactical translation defining the collapse.

First of all, remember that, by Lemma 4.2.4, we can suppose that every $\mu^{\sim}$-formula is in normal form.

Lemma 4.10. *Let $\alpha(x)$ be a modal formula such that x appears only negatively and guarded. Then, for every $\mathcal{T} \in \mathbb{T}^{wft}$ we have that*

$$\|\mu x.(\alpha[x/\alpha(x)])\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu x.\alpha(x)\|_{\mathcal{T}}^{\mathcal{W}}.$$

Proof. Let A be $\|\mu x.\alpha(x)\|_{\mathcal{T}}^{\mathcal{W}}$. By the "if" direction of Theorem 4.7 we have that $\|\alpha(A)\|_{\mathcal{T}}^{\mathcal{W}} = A$. We can iterate this equivalence twice and get

$$\|\alpha[x/\alpha(A)]\|_{\mathcal{T}}^{\mathcal{W}} = A.$$

Applying the "only if" direction of Theorem 4.7 gives us

$$\|\mu x.((\alpha[x/\alpha(x)])\|_{\mathcal{T}}^{\mathcal{W}} = A$$

and therefore the proof of this lemma. ∎

Note that, if $x \in \mathsf{bound}(\mu x.\alpha)$ appears only negatively, then x occurs only positively in $\mu x.(\alpha[x/\alpha(x)])$.

Everything is now set up in order to prove that the modal $\mu^{\sim}$-calculus over GL collapses to its modal fragment.

Definition 4.11. The syntactical translation $\mathcal{I} : \mathcal{L}_{\mu^{\sim}} \to \mathcal{L}_{\mathsf{GL}}$ uses the translation τ from $\mathcal{L}_{\mu}$ to $\mathcal{L}_{\mathsf{GL}}$ of Definition 3.2. It is defined recursively as follows:

- $\mathcal{I}(p) = p$ and $\mathcal{I}(\sim p) = \sim p$.
- $\mathcal{I}(\bot) = \bot$ and $\mathcal{I}(\top) = \top$.
- $\mathcal{I}(\alpha \circ \beta) = \mathcal{I}(\alpha) \circ \mathcal{I}(\beta)$, where $\circ \in \{\wedge, \vee\}$.
- $\mathcal{I}(\triangle\, \beta) = \triangle\, \mathcal{I}(\beta)$, where $\triangle \in \{\Box, \Diamond\}$.
- Assume that $\mathsf{nf}(\mu x.\mathcal{I}(\alpha(x)))$ is of the form $\mu z.\mu y.\hat{\alpha}(z, y)$. We set

  $$\mathcal{I}(\mu x.\alpha) = \tau(\mathsf{wn}(\mu z.\beta(z))),$$

  where $\beta(z) \equiv \tau(\mathsf{wn}(\mu y.\hat{\alpha}(z, y)))[z/\tau(\mathsf{wn}(\mu y.\hat{\alpha}(z, y)))]$).

Lemma 4.12. *The translation $\mathcal{I}$ is well-defined and, moreover, if*

$$\varphi \in \mathcal{L}_{\mu^{\sim}}(P) \textit{ then } \mathcal{I}(\varphi) \in \mathcal{L}_{\mathsf{GL}}(P).$$

Proof. By induction on the structure of the formula. The only critical case is when $\varphi \equiv \mu x.\alpha$. By induction hypothesis, $\mathcal{I}(\alpha(x)) \in \mathcal{L}_{\mathsf{GL}}$. Therefore $\hat{\alpha}(y, z) \in \mathcal{L}_{\mathsf{GL}}$. By definition of normal form, z occurs only negatively and y occurs only positively in $\mu z.\mu y.\hat{\alpha}(y, z)$. Thus, $\mu y.\hat{\alpha}(y, z) \in \mathcal{L}_{\mu}$. This implies that $\mathsf{wn}(\mu y.\hat{\alpha}(y, z))$ is well-defined and by Theorem 3.3 that

$\tau(\mathsf{wn}(\mu y.\hat{\alpha}(y,z))) \in \mathcal{L}_{\mathsf{GL}}$. Note that y occurs only positively in $\hat{\alpha}(y,z)$ and $\mathsf{wn}(\mu y.\hat{\alpha}(y,z))$ is given by duplicating and renaming y. Therefore, it follows that z occurs only positively in

$$\tau(\mathsf{wn}(\mu y.\hat{\alpha}(z,y)))[z/\tau(\mathsf{wn}(\mu y.\hat{\alpha}(z,y)))]).$$

This implies that $\mu z.\beta(z) \in \mathcal{L}_{\mu}$ and therefore that $\mathsf{wn}(\mu z.\beta(z))$ is well-defined. Thence by Theorem 3.3 we have that $\tau(\mathsf{wn}(\mu z.\beta(z))) \in \mathcal{L}_{\mathsf{GL}}$. ■

THEOREM 4.13. *Let $\varphi \in \mathcal{L}_{\mu^{\sim}}$. On upward well-founded and transitive transition systems $\mathcal{T}$ we have that*

$$\|\varphi\|_{\mathcal{T}}^{\mathcal{W}} = \|\mathcal{I}(\varphi)\|_{\mathcal{T}}^{\mathcal{W}}.$$

PROOF. The proof goes by induction on $\mathsf{rank}(\varphi)$. If $\mathsf{rank}(\varphi) = 1$ or $\mathsf{rank}(\varphi)$ is a successor ordinal the induction step is straightforward. If $\mathsf{rank}(\varphi)$ is a limit ordinal then φ is of the form $\mu x.\alpha$. In this case by Lemma 4.2 we have that

$$\|\mu x.\mathcal{I}(\alpha)\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\mu y.\hat{\alpha}(z,y)\|_{\mathcal{T}}^{\mathcal{W}}.$$

Since by induction hypothesis we have that $\|\mathcal{I}(\alpha)\|_{\mathcal{T}}^{\mathcal{W}} = \|\alpha\|_{\mathcal{T}}^{\mathcal{W}}$, with Corollary 4.8 we get that

$$\|\mu x.\mathcal{I}(\alpha)\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu x.\alpha\|_{\mathcal{T}}^{\mathcal{W}}$$

and therefore that

$$\|\mu x.\alpha\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\mu y.\hat{\alpha}(z,y)\|_{\mathcal{T}}^{\mathcal{W}}. \tag{2}$$

Since by Lemma 4.12 and by construction of normal forms, $\hat{\alpha}$ is a modal formula we have that $\mu y.\hat{\alpha} \in \mathcal{L}_{\mu}$. With Theorem 3.3 and Lemma 4.2 we get that for all upward well-founded and transitive $\mathcal{T}$ we have that

$$\|\mu y.\hat{\alpha}\|_{\mathcal{T}}^{\mathcal{W}} = \|\tau(\mathsf{wn}(\mu y.\hat{\alpha}))\|_{\mathcal{T}}^{\mathcal{W}}.$$

By Corollary 4.8 it holds that

$$\|\mu z.\mu y.\hat{\alpha}\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\tau(\mathsf{wn}(\mu y.\hat{\alpha}))\|_{\mathcal{T}}^{\mathcal{W}}$$

and with Equation 2 that

$$\|\mu x.\alpha\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\tau(\mathsf{wn}(\mu y.\hat{\alpha}))\|_{\mathcal{T}}^{\mathcal{W}}. \tag{3}$$

Remember that y occurs only positively and z only negatively in $\hat{\alpha}$. Moreover $\mathsf{wn}(\mu y.\hat{\alpha}(y,z))$ is obtained by multiplying and renaming y. Therefore, since z appears only negatively in $\mu y.\alpha(y,z)$ it appears only negatively in $\mathsf{wn}(\mu y.\hat{\alpha}(y,z))$, too. Now, note that by definition of τ we are "regenerating" the formula only on positive occurrences and, therefore, we have that z appears only negatively in $\tau(\mu y.\hat{\alpha})$, too. By Lemma 4.10 it holds that

$$\|\mu z.(\tau(\mathsf{wn}(\mu y.\hat{\alpha})))\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\big(\tau(\mathsf{wn}(\mu y.\hat{\alpha}))[z/\tau(\mathsf{wn}(\mu y.\hat{\alpha}))](z)\big)\|_{\mathcal{T}}^{\mathcal{W}}.$$

With Equation 3 we get

$$\|\mu x.\alpha\|_{\mathcal{T}}^{\mathcal{W}} = \|\mu z.\big(\tau(\mathsf{wn}(\mu y.\hat{\alpha}))[z/\tau(\mathsf{wn}(\mu y.\hat{\alpha}))](z)\big)\|_{\mathcal{T}}^{\mathcal{W}}.$$

By Lemma 4.2 and Theorem 3.3 we finish the induction step. ■

The last theorem of the paper is a new version of the de Jongh, Sambin Fixpoint Theorem. Our version provides an explicit construction of the fixpoint formula based on the definition of $\mathcal{I}$.

THEOREM 4.14. *Let $\varphi(x) \in \mathcal{L}_{\mathsf{GL}}(\mathsf{P})$, where x is guarded. We have that*

$$\mathsf{GL} \vdash \mathcal{I}(\mu x.\varphi) \leftrightarrow \varphi(\mathcal{I}(\mu x.\varphi)).$$

Further if $\varphi \in \mathcal{L}_{\mathsf{GL}}(\mathsf{P})$ then we have that $\mathcal{I}(\mu x.\varphi) \in \mathcal{L}_{\mathsf{GL}}(\mathsf{P} \setminus \{x\})$.

PROOF. The fact that $\mathcal{I}(\mu x.\varphi) \in \mathcal{L}_{\mathsf{GL}}(\mathsf{P} \setminus \{x\})$ follows from Lemma 4.12. For the provable equivalence, we show that $\mathsf{GL} \models \mathcal{I}(\mu x.\varphi) \leftrightarrow \varphi(\mathcal{I}(\mu x.\varphi))$. The proof then follows by Theorem 2.1. Let $\mathcal{T} \in \mathbb{T}^{\mathrm{wft}}$. We have

$$\begin{array}{rcll}
\|\mathcal{I}(\mu x.\varphi(x))\|_{\mathcal{T}} & = & \|\mu x.\varphi(x)\|_{\mathcal{T}}^{\mathcal{W}} & \text{Lemma 4.3 and Theorem 4.13} \\
& = & \|\varphi(\mu x.\varphi(x))\|_{\mathcal{T}}^{\mathcal{W}} & \text{Lemma 4.5} \\
& = & \|\varphi(x)\|_{\mathcal{T}[x\mapsto\|\mu x.\varphi\|_{\mathcal{T}}^{\mathcal{W}}]}^{\mathcal{W}} & \text{Definiton of evaluation game} \\
& = & \|\varphi(x)\|_{\mathcal{T}[x\mapsto\|\mathcal{I}(\mu x.\varphi)\|_{\mathcal{T}}^{\mathcal{W}}]}^{\mathcal{W}} & \text{Theorem 4.13} \\
& = & \|\varphi(x)\|_{\mathcal{T}[x\mapsto\|\mathcal{I}(\mu x.\varphi)\|_{\mathcal{T}}]} & \text{Lemma 4.3} \\
& = & \|\varphi(\mathcal{I}(\mu x.\varphi))\|_{\mathcal{T}} & \text{Definition of denotation}
\end{array}$$
■

We end with two examples where we apply our translation in order to find a solution of a modal equation.

EXAMPLE 4.15. Consider the modal equation $x \leftrightarrow \neg\Box x$. This is the same as

$$x \leftrightarrow \Diamond \sim x. \tag{4}$$

By Theorem 4.14 the $\mu^{\sim}$-formula $\mu x.\Diamond \sim x$ is the solution of Equation 4. By definition of $\mathcal{I}$ we have that

$$\mathcal{I}(\mu x.\Diamond \sim x) = \tau(\mu x.\Diamond\neg\Diamond \sim x) = \tau(\mu x.\Diamond\Box x) = \Diamond\Box\Diamond\Box\top.$$

Note, that on upward-well-founded transitive transition system $\mathcal{T}$, it holds that $\|\Diamond\Box\Diamond\Box\top\|_{\mathcal{T}} = \|\neg\Box\bot\|_{\mathcal{T}}$.

Example 4.16. Consider the modal equation $x \leftrightarrow (\Box(x \to q) \to \Box \sim x)$. This is the same as

$$x \leftrightarrow \Diamond(x\wedge \sim q) \vee \Box \sim x. \tag{5}$$

By Theorem 4.14 the formula $\mathcal{I}(\mu x.\Diamond(x\wedge \sim q) \vee \Box \sim x)$ is a solution of Equation 5. Let's trace the construction of the fixpoint given by Definition 4.11:

We have that

$$\hat{\alpha} \equiv \Diamond(x\wedge \sim q) \vee \Box \sim y$$

and that

$$\tau(\mu x.\hat{\alpha}) \equiv \Diamond((\Diamond(\bot\wedge \sim q) \vee \Box \sim y)\wedge \sim q) \vee \Box \sim y.$$

The formula $\tau(\mu x.\hat{\alpha})$ can be simplified by using the following equivalence

$$\|\tau(\mu x.\hat{\alpha})\|_{\mathcal{T}} = \|\Diamond(\Box \sim y\wedge \sim q) \vee \Box \sim y\|_{\mathcal{T}}.$$

Now, we calculate $\beta(y)$ of Definition 4.11 by using the simplified $\tau(\mu x.\hat{\alpha})$ above and get

$$\beta(y) \equiv \Diamond(\Box\neg(\Diamond(\Box \sim y\wedge \sim q) \vee \Box \sim y)\wedge \sim q) \vee \Box\neg(\Diamond(\Box \sim y\wedge \sim q) \vee \Box \sim y).$$

By definition of negation, we get

$$\beta(y) \equiv \Diamond(\Box(\Box(\Diamond y \vee q) \wedge \Diamond y)\wedge \sim q) \vee \Box(\Box(\Diamond y \vee q) \wedge \Diamond y).$$

Note that the following semantical equivalences hold

- $\|\Diamond(\Box(\Box(\Diamond y \vee q) \wedge \Diamond y)\wedge \sim q)\|_{\mathcal{T}} = \|\Diamond(\Box\bot\wedge \sim q)\|_{\mathcal{T}}$, and
- $\|\Box(\Box(\Diamond y \vee q) \wedge \Diamond y)\|_{\mathcal{T}} = \|\Box\bot\|_{\mathcal{T}}$.

Therefore, we get

$$\|\mu y.\beta(y)\|_{\mathcal{T}} = \|\Diamond(\Box\bot\wedge \sim q) \vee \Box\bot\|_{\mathcal{T}} = \|\Box(\Box\bot \to q) \to \Box\bot\|_{\mathcal{T}}.$$

Since $\mathcal{I}(\mu x.\Diamond(x\wedge \sim q) \vee \Box \sim x) \equiv \tau(\mathsf{wn}(\mu y.\beta(y)))$ it follows that the formula $\Box(\Box\bot \to q) \to \Box\bot$ is a solution of Equation 5.

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