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Giles's Game and the Proof Theory of Łukasiewicz Logic

Abstract. In the 1970s, Robin Giles introduced a game combining Lorenzen-style dialogue rules with a simple scheme for betting on the truth of atomic statements, and showed that the existence of winning strategies for the game corresponds to the validity of formulas in Łukasiewicz logic. In this paper, it is shown that 'disjunctive strategies' for Giles's game, combining ordinary strategies for all instances of the game played on the same formula, may be interpreted as derivations in a corresponding proof system. In particular, such strategies mirror derivations in a hypersequent calculus developed in recent work on the proof theory of Łukasiewicz logic.

Keywords: dialogue games, Łukasiewicz logic, many-valued logics, hypersequents.

Introduction

In the 1970s, Robin Giles proposed an account of logical reasoning in physical theories that combines a Lorenzen-style dialogue game with a simple scheme for betting on the truth of atomic statements [16, 18, 1]. The key idea is that atomic statements (represented as propositional variables) refer to positive or negative results of experiments that may have different outcomes when repeated but have a fixed probability of a positive result. The two players, you and I, agree to pay 1€ to the opposing player for each incorrect statement that they make. The payments may vary depending on the results of concrete experiments, but a player wins the game if they *expect* (in the probability-theoretic sense) not to lose money. In general, states of the game consist of statements (formulas built using implication and other standard connectives) made by the players, and moves are based on natural rules for attacking or granting compound statements made by the opposing player.

A remarkable feature of this game, recognized by Giles in [16], is that the existence of a winning strategy for any instance of the game where I start by asserting a formula F and you assert nothing corresponds directly to the validity of F in Łukasiewicz logic Ł. This logic was introduced by Jan Łukasiewicz in the 1930s [23] as the infinite-valued member of a family of many-valued logics first proposed for modelling future contingents. It is currently studied mainly as one of the fundamental t-norm based fuzzy

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logics suitable for formalizing reasoning under vagueness (see e.g., [21]). Indeed, Giles formalized his game also in the context of fuzzy set theory; in particular, the goal of [17] and [19] was to provide a dialogue game based presentation of the 'semantics of fuzzy reasoning'.

The main aim of this paper is to connect Giles's game with recent proof-theoretic presentations of Łukasiewicz logic. Initial attempts at developing proof calculi for Ł made essential use of the cut rule [30], extra syntax such as labelled tableaux or resolution [20, 29, 28, 34], or a reduction to finite-valued logics [2]. The first genuinely analytic (cut-free) Gentzen systems for the logic were presented in [24] both in the framework of sequents and, more naturally, in the framework of hypersequents (multisets of sequents) introduced by Avron in [3] (see also [25]). Interestingly, the rules of the latter system were anticipated partially by Giles himself (with Adamson) who in [1] obtained a sequent calculus for Ł based on the search for winning strategies in his dialogue game. However, the cut rule plays an essential role in this calculus, rendering it unsuitable either as a basis for proof-theoretic investigation or as a convincing interpretation of the game.

Here we show that a cut-free hypersequent calculus, closely related to the system of [24], is obtained for Ł by combining winning strategies into 'disjunctive strategies' for all possible assignments of probabilities. The presentation not only sheds light on the search for strategies in Giles's game but also provides (retrospectively) a convincing semantics for the calculus of [24] for which, unlike standard Gentzen systems, hypersequents are not easily interpreted as formulas of Ł. This relationship between strategies for Giles's game and hypersequent derivations has been mentioned in [6] and [13] but is worked out here for the first time in full detail. Indeed, additional contributions of this paper are a more precise formulation of the game than originally given by Giles, a task requiring certain 'design choices' to be made, and the development of disjunctive strategies for a general class of two-person zero-sum games with perfect information.

1. Giles's Game

Let us start by presenting a slightly generalized version of the game introduced by Giles in [16], consisting of two largely independent building blocks:

(1) Betting for positive results of experiments.

Two players — following Giles, let's call them me and you — agree to pay 1€ to their opponent for every false statement that they make. Let $p_1, p_2, \ldots,$

$q_1, q_2, \ldots$ be propositional variables, representing atomic statements, and let $\bot$ be a propositional constant representing a statement that is always false. We denote by $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$ an *elementary state* of the game, where you assert each p_i in the finite multiset of atomic statements $[p_1, \ldots, p_m]$ and I assert each q_i in the finite multiset $[q_1, \ldots, q_n]$.

Each atomic statement q refers to a (repeatable) *elementary experiment* E_q. By this we mean an experiment resulting in a binary answer ('yes' or 'no'). The statement q may then be read as 'E_q yields a positive result'. To illustrate this idea, consider the elementary state $[p \,\|\, q, q]$. According to the described arrangement, the experiment E_p has to be performed once and the experiment E_q twice. If, e.g., all three results are negative, then I owe you 2€ and you owe me 1€. As long as q is either always true or always false, this setting does not extend beyond the realm of classical logic. The situation becomes more interesting, however, if the experiments can exhibit *dispersion*, i.e., if the same experiment may yield different results when repeated. More formally, for every run of the game, a fixed *risk value* $\langle q \rangle$ in the real unit interval $[0, 1]$ is associated with q. This value $\langle q \rangle$ can be interpreted as the (subjective) probability that E_q yields a negative result. For the atomic statement $\bot$, we let $\langle \bot \rangle = 1$; i.e., the experiment $E_\bot$ always yields a negative answer.

The *risk* associated with a multiset $[q_1, \ldots, q_n]$ of atomic formulas is defined as $\langle q_1, \ldots, q_n \rangle = \sum_{i=1}^{n} \langle q_i \rangle$, where the risk $\langle\rangle$ associated with the empty multiset is defined as 0. Note that, according to standard probability theory, the risk, as defined here, denotes the amount of money that I expect to have to pay to you according to the results of the experiments $E_{q_1}, \ldots, E_{q_n}$ associated with my atomic statements $q_1, \ldots, q_n$. Therefore, with respect to an elementary state $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$, the condition $\langle p_1, \ldots, p_m \rangle \geq \langle q_1, \ldots, q_n \rangle$ expresses that I do not expect any loss (but possibly some gain) when we bet on the truth of the atomic statements asserted by ourselves, as indicated above.

Returning to our example for the elementary state $[p \,\|\, q, q]$, I expect an average loss of 0.5€ if $\langle p \rangle = \langle q \rangle = 0.5$. However, for the risk values $\langle p \rangle = 0.8$ and $\langle q \rangle = 0.3$, my average loss is negative; I expect an average gain of 0.2€.

(2) A dialogue game for the reduction of compound formulas.

Giles follows Paul Lorenzen (see e.g., [22]) in specifying the meaning of connectives via rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

For clarity, let us assume that these formulas are built from propositional variables, the falsity constant (0-ary connective) $\bot$, and the binary connective $\to$. This parsimony is justified (at least partly) by the fact that in Łukasiewicz logic all other connectives can be defined from $\to$ and $\bot$ as follows: $\neg A =_{def} A \to \bot$ (negation), $A \,\&\, B =_{def} \neg(A \to \neg B)$ (strong conjunction), $A \wedge B =_{def} A \,\&\, (A \to B)$ (weak or lattice conjunction), $A \vee B =_{def} ((A \to B) \to B) \wedge ((B \to A) \to A)$ (weak or lattice disjunction). Nevertheless, we will also present a more direct analysis of disjunction and the two different versions of conjunction in Section 3.

The central dialogue rule for implication can be stated as follows:

($R_\to$) If I assert $A \to B$, then whenever you choose to attack this statement by asserting A, I have to assert also B. (And *vice versa*, i.e., for the roles of me and you switched.)

This rule reflects the idea that the meaning of implication is characterized by the principle that asserting 'if A, then B' obliges the assertion also of B if the opponent in a dialogue asserts A. Note, however, that a player may also choose never to attack the opponent's assertion of $A \to B$.

REMARK 1.1. *The presentation of the dialogue game given by Giles in [16] and [18] is not entirely precise. As a consequence, some 'design choices' are needed to obtain a fully formal presentation. Our definitions below of states, moves, runs, and strategies therefore correspond to a particular version of the game. Alternative approaches are discussed at the end of the section.*

Let us denote an arbitrary *dialogue state*, or for short, *d-state*, of the game by $[A_1, \ldots, A_m \,\|\, B_1, \ldots, B_n]$ where $[A_1, \ldots, A_m]$ is the multiset of formulas that are currently asserted by you and $[B_1, \ldots, B_n]$ is the multiset of formulas that are currently asserted by me. An occurrence of a formula F in a d-state is called a *statement*, either by me or you, depending on whether F is on the right or left side of the d-state. (This allows us to distinguish concisely between formulas and formula occurrences.)

A *round* is a transition from one d-state to another *successor d-state* that consists of two *moves*. In each round one of the players is the *initiator* α and the other player is the *respondent* β. The two corresponding moves are:

1. α picks one of the compound statements $A \to B$ asserted by β.
2. α either *attacks* this statement by asserting A or *grants* the formula, which means that α declares never to attack that particular statement of β. In the first case β must respond immediately by asserting B. In the second case no action is required from β.

Since no choice of β is involved in the whole round, both moves count as moves of α. (This will change when we consider disjunctions and strong conjunctions in Section 3 below.)

Each statement can be attacked or granted at most once. Therefore the statement picked by the initiator is removed from the d-state. In denoting strategies as trees, below, we will use *intermediary states* or, for short, *i-states* to reflect the initiator's choice of the statement that gets attacked or granted. Intermediary states are exactly the same as the preceding d-state, except that the statement chosen by the initiator is marked (denoted by underlining).

However, to define games precisely, we need more information. We need to know, for each non-elementary d-state, whose turn it is to initiate the next round. Formally, we define a *regulation* as a function that assigns to each non-elementary d-state either the label $\mathbf{Y}$, for '*you initiate the next round*', or the label $\mathbf{I}$, for '*I initiate the next round*'. These labels indicate that the possible runs of the game are constrained accordingly. A regulation is *consistent* if the label $\mathbf{Y}$ (or $\mathbf{I}$) is only assigned to d-states where such an initiating move is possible, i.e., where there is a compound statement among my (or your) currently asserted formulas. The correct label for an i-state is determined by the label of the immediately preceding d-state: as already mentioned, both moves of a round (referring to implication) are moves of the initiator, and thus the label of an i-state matches that of the preceding d-state.

A *game form* $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a tree of states with the initial d-state $[\Gamma \parallel \Delta]$ as root, where the successor nodes to any state S are the states that result from legal moves at S according to the consistent regulation ρ as explained above. In particular, the leaf nodes of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ are the reachable elementary states. A *game* consists of a game form $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ together with a risk assignment $\langle \cdot \rangle$. A *run* of the game is a branch of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$. In other words, a run consists of a sequence of alternating d-states and i-states, obtained from successive moves as described above, beginning with the initial d-state $[\Gamma \parallel \Delta]$ and ending in an elementary state.

EXAMPLE 1.2. Consider the d-state $[p \to q \parallel a \to b, c \to d]$. If it is my turn to initiate the next round (indicated by the superscript $^{\mathbf{I}}$), then I can only either attack or grant your statement $p \to q$. Accordingly, there are two ways that runs of the game can continue:

$$\begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\underline{p \to q} \,\|\, a \to b, c \to d]^{\mathbf{I}} \\ | \\ [q \,\|\, p, a \to b, c \to d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\underline{p \to q} \,\|\, a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\,\|\, a \to b, c \to d]. \end{array}$$

If it is your turn to move in the same d-state $[p \to q \,\|\, a \to b, c \to d]$, then there are two successor i-states and hence four possible successor d-states depending on (1) which of my two statements you choose, and (2) whether you decide to attack or grant the chosen formula. The corresponding four continuations are:

$$\begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q, a \,\|\, b, c \to d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, c \to d] \end{array}$$

$$\text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} \\ | \\ [p \to q, c \,\|\, a \to b, d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b]. \end{array}$$

Suppose that a run of $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ with risk assignment $\langle \cdot \rangle$ ends with the elementary state $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$. We say that I *win* in that run if I do not have to expect any loss of money resulting from betting on results of the corresponding elementary experiments, more formally: if $\langle p_1, \ldots, p_m \rangle \geq \langle q_1, \ldots, q_n \rangle$. Note that this winning condition refers to *expected* pay-offs and not to any particular payments for results of the experiments associated with the atomic statements of the final elementary state. In particular, $[p \,\|\, p]$ is a *winning state* for me, although it may happen — due to dispersion — that the result of performing experiment E_p is positive for your assertion of p but negative for my assertion of p, meaning that I owe you 1€ according to the agreed betting scheme. In other words, our (and Giles's) definition of 'winning' deliberately ignores concrete results of elementary experiments and only refers to the risk associated with these experiments.

REMARK 1.3. *As mentioned earlier, in order to formulate the game precisely, we have made certain design choices that go beyond the original presentation of Giles. In particular, we have assumed that attacks are answered immediately. The fact that this restriction (forced upon the 'Opponent') does not imply any loss of 'power of the Opponent' is the most non-trivial part of the adequacy proof for intuitionistic logic of Lorenzen's original game. In contrast, this restriction does not affect winning powers in Giles's (more* *symmetric) game: instead of interpreting the formulas occurring in a state as those that are currently asserted, we can just as well interpret them as those formulas that must be asserted either at this or a later time according to the dialogue rules (and thus are immediately available for attacks).*

A second, related issue concerns regulations, formulated here as functions from non-elementary states to **I** *or* **Y**, *just specifying* who *is to play next. Alternatively, as described in [1], a regulation could specify exactly which formula should be attacked or granted next, i.e., a function from non-elementary states to an occurrence of a compound formula in those states.*

2. Strategies

A game *strategy* for a particular player is generally defined as a function from states to states that determines every choice of a legal move by that player but leaves all choices of the other player open. In our context this means that a strategy for me is obtained from a game form by (iteratively from the root) deleting all but one successor of every state labelled **I**. More precisely, a *strategy* (for me) for a game form $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ is a rooted tree of d-states and i-states, fulfilling the following conditions:

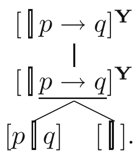
1. The root node is the d-state $[\Gamma \,\|\, \Delta]$.
2. All leaf nodes are elementary d-states.
3. If $\rho([\Pi \,\|\, \Sigma]) = \mathbf{Y}$, then the successor nodes of $[\Pi \,\|\, \Sigma]$ are the i-states that result from marking one of the compound statements in Σ. Such an i-state, $[\Pi \,\|\, \Sigma', \underline{A \to B}]$, where $\Sigma' = \Sigma - [A \to B]$, has the two d-states $[A, \Pi \,\|\, \Sigma', B]$ and $[\Pi \,\|\, \Sigma']$ as successor nodes.
4. If $\rho([\Pi \,\|\, \Sigma]) = \mathbf{I}$, then $[\Pi \,\|\, \Sigma]$ has exactly one i-state of the form $[\underline{A \to B}, \Pi' \,\|\, \Sigma]$, where $\Pi' = \Pi - [A \to B]$, as a successor node. This i-state has either the d-state $[B, \Pi' \,\|\, \Sigma, A]$ or the d-state $[\Pi' \,\|\, \Sigma]$ as its sole successor state.

A strategy for $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ is called a *winning strategy (for me) for a risk assignment* $\langle\cdot\rangle$ if $\langle p_1, \ldots, p_m \rangle \geq \langle q_1, \ldots, q_n \rangle$ holds for each of its leaf nodes $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$.

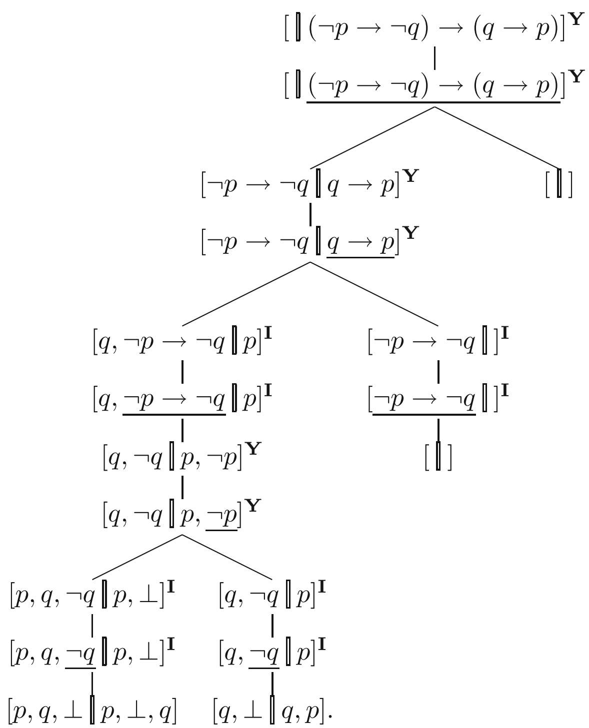
Note that all branching in strategies refers to possible choices by you. There are two different kinds of choices: in a d-state labelled **Y** you first single out one of my compound statements and then decide whether to attack or grant this formula. This is reflected by the i-states: the first choice is represented by the transition from a d-state to an i-state, the second by the transition from an i-state to a d-state.

EXAMPLE 2.1. Consider a game where I initially assert $p \to q$ for some atomic formulas p and q. The game starts with the initial d-state $[\,\parallel p \to q]$: You can either assert p in order to force me to assert q, or explicitly refuse to attack $p \to q$. In the first case, the game ends in the elementary state $[p \parallel q]$; in the second case, the game ends in the elementary state $[\,\parallel\,]$. For a given assignment $\langle \cdot \rangle$ of risk values where $\langle p \rangle \geq \langle q \rangle$, I win the game in both cases. In other words: I have a winning strategy for $p \to q$ for all assignments satisfying $\langle p \rangle \geq \langle q \rangle$.

Note that $[\,\parallel p \to q]$ is the only non-elementary d-state in this game. Therefore there is only one consistent regulation. The corresponding strategy is formally represented by the following tree:



EXAMPLE 2.2. The following, slightly less trivial, example is a strategy for $(\neg p \to \neg q) \to (q \to p)$ for me (recalling that $\neg A =_{def} A \to \bot$) where the regulation is as indicated by the superscripts.



Note that at all four leaf nodes, the winning condition for me is satisfied *independently* of the concrete risk value assignment. Also note that in this particular example, there is only one compound formula to pick at each i-state. Moreover, in many cases, the regulation is determined by the available formulas. E.g., in order to be consistent, the regulation must assign **I** to $[q, \neg p \to \neg q \,\|\, p]$. The d-states $[\neg p \to \neg q \,\|\, q \to p]$ and $[q, \neg q \,\|\, p, \neg p]$, which have you as assigned initiator, could also be initiated by me. However, it is easy to see that these alternative regulations, while leading to different leaf nodes, also result in winning strategies for me that are independent of the given risk values.

3. Other Connectives

So far we have restricted our attention to rules for the implication connective. This is justified somewhat by the fact, to be rediscovered below, that Giles's game corresponds to Łukasiewicz logic, and that in this logic, all other connectives may be defined using just implication and the constant $\bot$. Nevertheless, such an account does not really do justice to the motivation behind the game, namely, to provide an independent account of reasoning. We therefore consider here other connectives, beginning with rules given by Lorenzen for his dialogue game corresponding to intuitionistic logic.

For disjunctive statements, Lorenzen introduced the following rule:

($R_\vee$) If I assert $A \vee B$, then, if attacked, I also have to assert either A or B, where it is my choice which of the two subformulas to assert. (Analogously, if you assert $A \vee B$.)

For conjunctive assertions, Lorenzen proposed a rule similar to ($R_\vee$) but shifting the choice involved to the opponent:

($R_\wedge$) If I assert $A \wedge B$, then, if attacked, I also have to assert either A or B, but it is your choice which of the two subformulas I assert. (Analogously, for the roles of me and you switched.)

Giles followed Lorenzen in using these rules for the decomposition of compound formulas in dialogues. However, in the case of ($R_\wedge$), the following alternative rule may, at first glance, seem to be a more natural analysis of conjunctive assertions:

($R'_\wedge$) If I assert $A \wedge B$, then, if attacked, I also have to assert both A and B. (Analogously, if you assert $A \wedge B$.)

However, in the context of our basic stipulation that exactly 1€ has to be paid to the opponent player for every false assertion made, ($R'_\wedge$) seems problematic. Suppose that I assert $p \wedge q$ and both corresponding experiments, E_p and E_q, yield a negative answer. According to ($R'_\wedge$) this commits me to pay 2€ to you, and not just 1€ as is the case when playing according to rule ($R_\wedge$). To address this concern, we give a player asserting a conjunctive statement the option of hedging their corresponding bet and paying 1€ to the opponent player instead of defending both sub-statements. Employing the 'always false statement' $\bot$, we can formulate the corresponding rule without having to refer to payments directly, using & instead of $\wedge$ to indicate that a different understanding of conjunction is involved:

($R_\&$) If I assert $A \& B$, then, if attacked, I can choose to either assert both A and B, or to assert $\bot$. (Analogously, if you assert $A \& B$.)

Both $\wedge$ and & behave like classical conjunction in a version of Giles's game without dispersion, i.e., where each risk value is either 0 or 1. However, in the more general case, it follows from our results below that ($R_\wedge$) and ($R_\&$) characterize the two different conjunctions — weak conjunction $\wedge$ ('lattice conjunction') and strong conjunction & ('t-norm conjunction'), respectively — of Łukasiewicz logic Ł (see, e.g., [21]).

Remark 3.1. *We have introduced & here as a strong conjunction connective partly because of its importance in the presentation of Łukasiewicz logic as a fuzzy logic. As we will see later, according to this perspective, & is interpreted by the continuous Archimedean t-norm* $\max(0, x + y - 1)$*. It is possible to define rules also for a 'strong disjunction'* $\underline{\vee}$ *corresponding to the continuous t-conorm* $\min(1, x + y)$*. However, these rules are rather unintuitive, reflecting perhaps the strangeness of this function as a form of disjunction.*

Let us now translate these rules into the terminology of the previous section. The general definition of a *round* is as follows:

1. The initiator α picks one of the compound statements asserted by responder β.
2. α either *attacks* or *grants* this statement. Responder β has to respond immediately to each attack, as specified in the dialogue rules above.

Note that in the case of disjunction and conjunction — in contrast to implication — the initiator of a corresponding move can only gain from choosing

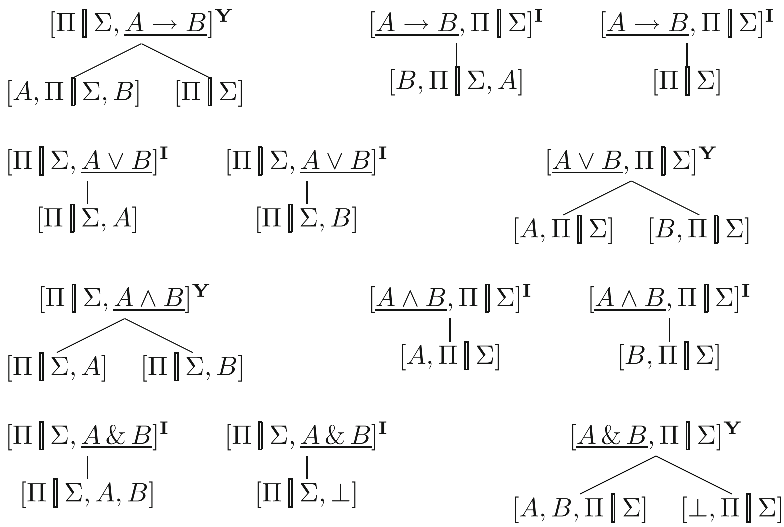
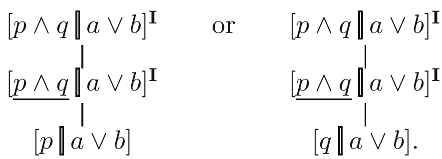


Table 1. Successor d-nodes of i-nodes, according to the rules $(R_{\to})$, $(R_{\vee})$, $(R_{\wedge})$, and $(R_{\&})$.

to attack; i.e., it is never an advantage to grant such a statement. Consequently, we may safely ignore the possibility of granting formulas where the outermost connective is not an implication.

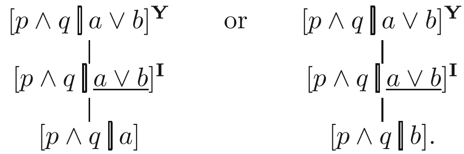
The definition of a strategy is augmented according to the rules $(R_{\vee})$, $(R_{\wedge})$, and $(R_{\&})$. More precisely, in conditions 3 and 4 of the definition of a strategy, the possibilities for successor nodes (d-states) of i-states are as specified in Table 1. Note that for attacks on disjunctions and on strong conjunctions it is the attacked player's turn to move, whereas in the other cases the label of the i-node indicates that the attacking player has to select a move.

Example 3.2. Consider the d-state $[p \wedge q \,\|\, a \vee b]$. Suppose that it is my turn to initiate the next round. Then, since we are ignoring the possibility of granting such statements, I must attack your statement $p \wedge q$ and, in the succeeding move, choose which of p and q to attack. I.e., there are two ways that runs of the game can continue:

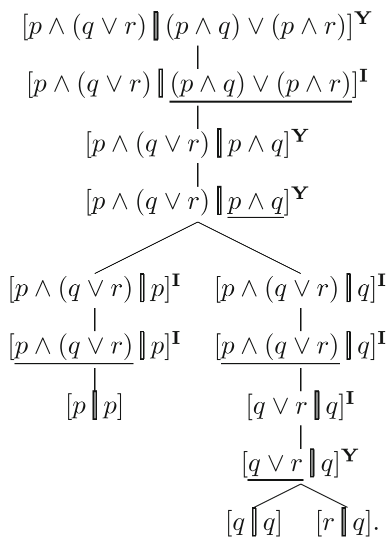


In contrast, suppose that it is your turn to initiate the next round. You must attack my statement $a \vee b$. However, it is then my choice which of a

or b I assert. So the possible continuations are:



Consider now a game with initial state $[p \wedge (q \vee r) \parallel (p \wedge q) \vee (p \wedge r)]$. The following tree is a strategy that respects a given regulation indicated, as before, by the superscripts $^{\mathbf{I}}$ and $^{\mathbf{Y}}$:



Notice that at the leaf nodes $[p \parallel p]$ and $[q \parallel q]$, I win for any risk assignment. But I win at $[r \parallel q]$ only if $\langle r \rangle \geq \langle q \rangle$. I.e., the above tree is a winning strategy for all games where the risk assignment $\langle \cdot \rangle$ satisfies this restriction.

4. Adequacy of Giles's Game for Łukasiewicz Logic

One of the most interesting features of Giles's game is its relationship with the infinite-valued Łukasiewicz logic Ł. The standard semantics for this logic takes the closed interval $[0, 1]$ as a set of truth values with strong conjunction modelled by the nilpotent Archimedean t-norm $x *_{\text{Ł}} y = \max(0, x + y - 1)$, implication by its residuum $x \Rightarrow_{\text{Ł}} y = \min(1, 1 - x + y)$, and $\bot$ by 0. Other connectives $\wedge$ and $\vee$ are modelled by min and max, respectively. However, since all the connectives of Ł can be defined from $\rightarrow$ and $\bot$, as explained in Section 1, we again focus on this restricted language.

A *valuation* $v(\cdot)$ for Ł is a function from the set of formulas into $[0,1]$ that extends an assignment to propositional variables of values in $[0,1]$ by:

$$\begin{array}{rcl} v(\bot) & = & 0 \\ v(A \to B) & = & \min(1, 1 - v(A) + v(B)). \end{array}$$

A formula F is called *valid* in Ł if $v(F) = 1$ for every valuation v.

The usual axiomatization HŁ for Ł in this restricted language consists of the following four axiom schema together with modus ponens:

(Ł1) $A \to (B \to A)$
(Ł2) $(A \to B) \to ((B \to C) \to (A \to C))$
(Ł3) $((A \to B) \to B) \to ((B \to A) \to A)$
(Ł4) $((A \to \bot) \to (B \to \bot)) \to (B \to A)$.

A completeness theorem was given by Wajsberg in the 1930s, but the first published proof, by Rose and Rosser, did not appear until the 1950s.

THEOREM 4.1 ([32]). *A formula F is valid in Ł iff F is derivable in* HŁ.

An algebraic semantics for Ł, the class of *MV-algebras* (usually presented in a different language), was introduced by C. C. Chang in the 1950s [5] and is the focus of a great deal of active research in algebra (see, e.g., [8]).

The key theorem below, connecting validity with winning strategies, was established by Giles in [16]. However, to remain self-contained, we include here our own proof.

THEOREM 4.2. *A formula F is valid in Ł iff I have a winning strategy for the game $\mathbf{G}([\,\|\, F], \rho)$ with any risk assignment $\langle\cdot\rangle$, where ρ is an arbitrary consistent regulation.*

PROOF. Note first that every run of the game $\mathbf{G}([\,\|\, F], \rho)$ with risk assignment $\langle\cdot\rangle$ is finite. For every elementary state $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$, I win if my associated risk (expected loss) $\langle p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n\rangle = \sum_{j=1}^{n} \langle q_j\rangle - \sum_{i=1}^{m} \langle p_i\rangle$ is non-positive. The minimal final risk that I can enforce in any given state S by playing according to an optimal strategy can be calculated by taking into account (1) the maximum over all risks associated with the successor states to S that you can enforce by a move at S, (2) the fact that for any of my choices I can enforce the minimum over the risks at successor states that correspond to my possible moves. Correspondingly, we will show that the notion of my minimal final risk can be extended from elementary states $[p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n]$ to arbitrary states $[\Gamma \,\|\, \Delta]$ such that the following conditions are satisfied:

$$\begin{aligned} \langle \Gamma \,\|\, A \to B, \Delta \rangle &= \max(\langle \Gamma \,\|\, \Delta \rangle, \langle \Gamma, A \,\|\, B, \Delta \rangle) & (1) \\ \langle \Gamma, A \to B \,\|\, \Delta \rangle &= \min(\langle \Gamma \,\|\, \Delta \rangle, \langle \Gamma, B \,\|\, A, \Delta \rangle). & (2) \end{aligned}$$

We have to check that $\langle \cdot \,\|\, \cdot \rangle$ is well-defined, i.e., that conditions (1) and (2) together with the definition of my risk for elementary states can indeed be simultaneously fulfilled and guarantee uniqueness. Moreover, we have to connect risk values and risk assignments with truth values and valuations.

Let us extend the semantics of Ł from formulas to multisets Γ of formulas as follows:

$$v(\Gamma) =_{def} \sum_{F \in \Gamma} v(F).$$

Risk value assignments are placed in one to one correspondence with truth value assignments via the mapping $\langle p \rangle^v = 1 - v(p)$, which extends to:

$$\langle p_1, \ldots, p_m \,\|\, q_1, \ldots, q_n \rangle^v = n - m + v([p_1, \ldots, p_m]) - v([q_1, \ldots, q_n]).$$

Correspondingly, we define the following function for arbitrary states:

$$\langle \Gamma \,\|\, \Delta \rangle^v =_{def} |\Delta| - |\Gamma| + v(\Gamma) - v(\Delta).$$

Note that crucially:

$$v(F) = v([F]) = 1 \;\text{ iff }\; \langle \,\|\, F \rangle^v \leq 0. \qquad (3)$$

We have to show that $\langle \cdot \,\|\, \cdot \rangle^v$ does indeed specify my risk with respect to optimal game strategies. In other words, we have to check that it satisfies conditions (1) and (2). This is established by induction on the complexity of states, i.e., the number of implication symbols they contain. The base case case is immediate. For (1), the corresponding induction step is:

$$\begin{aligned} \langle \Gamma \,\|\, A \to B, \Delta \rangle^v &= |\Delta| + 1 - |\Gamma| + v(\Gamma) - v(\Delta) - v(A \to B) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v + 1 - v(A \to B) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v + 1 - (v(A) \Rightarrow_{Ł} v(B)) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v + 1 - \min(1, 1 - v(A) + v(B)) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v - \min(0, v(B) - v(A)) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v + \max(0, v(A) - v(B)) \\ &= \langle \Gamma \,\|\, \Delta \rangle^v + \max(0, \langle A \,\|\, B \rangle^v) \\ &= \max(\langle \Gamma \,\|\, \Delta \rangle^v, \langle \Gamma, A \,\|\, B, \Delta \rangle^v). \end{aligned}$$

For (2), we have:

$$\begin{array}{rcl}\langle \Gamma, A \to B \,\|\, \Delta\rangle^v & = & |\Delta| - |\Gamma| - 1 + v(\Gamma) + v(A \to B) - v(\Delta)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + v(A \to B)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + (v(A) \Rightarrow_{\text{Ł}} v(B))\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + \min(1, 1 - v(A) + v(B))\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + \min(1, 1 + \langle B \,\|\, A\rangle^v)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + \min(0, \langle B \,\|\, A\rangle^v)\\ & = & \min(\langle \Gamma \,\|\, \Delta\rangle^v, \langle \Gamma, B \,\|\, A, \Delta\rangle^v.\end{array}$$

Note that (1) and (2) hold independently of the order in which compound formulas are decomposed or eliminated. Therefore, for $v(F) = 1$, there is a winning strategy for me for $\mathbf{G}([\,\|\, F], \rho)$ with risk assignment $\langle\cdot\rangle^v$ for *any* consistent regulation ρ. Finally, recall that by (3), F is valid iff $\langle \,\|\, F\rangle^v \leq 0$ for every valuation v. Since this covers *all* possible risk value assignments $\langle\cdot\rangle$, the theorem follows. ■

REMARK 4.3. *It is straightforward to check in a similar way that the game rules for other connectives* $(R_\vee)$, $(R_\wedge)$, *and* $(R_\&)$ *presented in Section 3 match the standard semantic conditions:* $v(A \vee B) = \max(v(A), v(B))$, $v(A \wedge B) = \min(v(A), v(B))$, *and* $v(A \& B) = v(A) *_{\text{Ł}} v(B)$, *respectively. For example, for* $(R_\&)$ *we have to check that the following conditions are satisfied:*

$$\langle \Gamma \,\|\, A \& B, \Delta\rangle = \min(\langle \Gamma \,\|\, \Delta, \bot\rangle, \langle \Gamma \,\|\, \Delta, A, B\rangle) \tag{4}$$

$$\langle \Gamma, A \& B \,\|\, \Delta\rangle = \max(\langle \Gamma, \bot \,\|\, \Delta\rangle, \langle \Gamma, A, B \,\|\, \Delta\rangle). \tag{5}$$

We obtain the following corresponding induction steps:

$$\begin{array}{rcl}\langle \Gamma \,\|\, A \& B, \Delta\rangle^v & = & |\Delta| + 1 - |\Gamma| + v(\Gamma) - v(\Delta) - v(A \& B)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + 1 - v(A \& B)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + 1 - (v(A) *_{\text{Ł}} v(B))\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + 1 - \max(0, v(A) + v(B) - 1)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + \min(1, (1 - v(A)) + (1 - v(B))\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v + \min(1, \langle \,\|\, A, B\rangle^v)\\ & = & \min(\langle \Gamma \,\|\, \Delta, \bot\rangle^v, \langle \Gamma \,\|\, \Delta, A, B\rangle^v)\end{array}$$

$$\begin{array}{rcl}\langle \Gamma, A \& B \,\|\, \Delta\rangle^v & = & |\Delta| - |\Gamma| - 1 + v(\Gamma) + v(A \& B) - v(\Delta)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + v(A \& B)\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + (v(A) *_{\text{Ł}} v(B))\\ & = & \langle \Gamma \,\|\, \Delta\rangle^v - 1 + \max(0, v(A) + v(B) - 1)\\ & = & \max(\langle \Gamma, \bot \,\|\, \Delta\rangle, \langle \Gamma, A, B \,\|\, \Delta\rangle.\end{array}$$

Alternatively, we could consider strong conjunction as defined by $A \,\&\, B =_{def} (A \to (B \to \bot)) \to \bot$*, as already indicated in Section 1.*

5. Disjunctive Strategies

Although Giles explicitly refers to Lorenzen's dialogue game for intuitionistic logic as the source of his own game for Łukasiewicz logic, there is a fundamental difference between the two games. While the winning condition in Lorenzen's game is independent of any semantic considerations, in Giles's game the winning condition depends on the risk values assigned to propositional variables. In general, different risk values call for different strategies. Winning strategies in Lorenzen's game correspond directly to analytic (cut-free) proofs of intuitionistically valid formulas and sequents in a suitable version of Gentzen's sequent calculus for intuitionistic logic. On the other hand, Giles's game is more like a Hintikka-style *evaluation game*, where winning strategies correspond to certifying *truth in a given model*. To connect Giles's game to cut-free proofs in an appropriate calculus for Łukasiewicz logic, we will need to abstract from particular risk assignments and look rather at *disjunctive (winning) strategies* that arise when *disjunctions of states* instead of single game states are considered.

To make the move from strategies to disjunctive strategies more transparent, let us consider the game from a more abstract point of view. In standard game-theoretic terminology, Giles defines an extensive-form two-person zero-sum game with perfect information for each formula and risk assignment. Of course it is essential that the rules and winning conditions are uniform over this infinite family of games. However, since players' preferences are part of the definition of any game, different risk values result in different games even if the initially asserted formulas are the same. This provides the motivation for a switch from states to disjunctive states, which allow strategies for single games to be combined into a structure consisting of appropriate families of such strategies.

Recall that a game without the information about the players' preferences (winning conditions) is called a *game form*, represented as a rooted tree of states endowed with a consistent regulation that specifies which player is to move at any non-elementary state. The outgoing edges at a state register the possible moves of the corresponding player at that state. By adding the players preferences to the leaf nodes (elementary states), we obtain a *game tree* that represents the whole game. In our case it suffices to specify for each leaf, whether it is a winning state for me or for you.

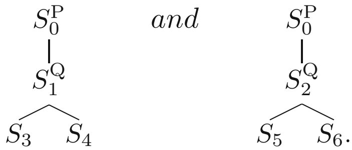
A *strategy* for a player P is a subtree of the game tree where all states that are not labelled by P retain all their successor nodes, and, for each state labelled by P, all but one successor node is removed from the tree. A *winning strategy* for P is a strategy where the winning condition for P is satisfied at all leaf nodes. The definitions in Section 2 of a (winning) strategy for me for Giles's game amount to special instances of these general definitions.

Let us use $D = S_1 \bigvee \ldots \bigvee S_n$ to denote a *state disjunction*. Since the order of its *component states* (ordinary game states) $S_1, \ldots, S_n$ is irrelevant, a disjunctive state may be viewed as a multiset of states. A *disjunctive strategy* for D respecting a regulation ρ is a tree of state disjunctions with root D where the successor nodes are in principle determined in the same way as for ordinary strategies. However, we also allow for the possibility of *duplicating* a component of a state in order to let disjunctive strategies for a player P record different ways of proceeding for P in identical component states. More precisely, there are two kinds of non-leaf nodes $D = S_1 \bigvee \ldots \bigvee S_n$ in a disjunctive strategy for P:

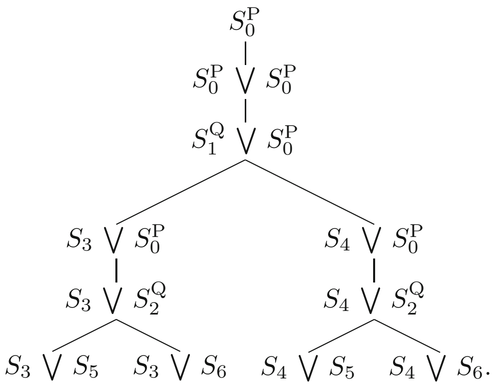
1. *Playing nodes*, focused on some component S_i $(1 \leq i \leq n)$ of D. The successor nodes are like those for S_i in ordinary strategies, except for the presence of additional components (that remain unchanged). I.e., if, according to ρ, it is P's turn to play at S_i, then there is a single successor state disjunction where the component S_i of D is replaced by some S_i' corresponding to some move of P. If the opposing player Q is to move at S_i, then all possible moves of Q determine the successor nodes of D where S_i is replaced by a state obtained by the corresponding move.
2. *Duplicating nodes*, where the single successor node is obtained by duplicating one of the components in D.

Remark 5.1. *Note that, just like ordinary strategies, disjunctive strategies refer to a regulation that tells us which player has to move next at each non-terminal state of a state disjunction. However, we do not associate players with the focus on or duplication of particular component states. An alternative approach could be to introduce 'meta-regulations' that delegate these choices to players. Here, we prefer to keep these choices external to the game itself since they only correspond to different ways of recording possible traversals of the underlying game tree and do not influence the 'winning powers' of players, i.e., the existence or non-existence of winning strategies.*

Example 5.2. To illustrate the notion of a disjunctive strategy, consider a game form where player P has two possible moves at state S_0 leading to successor states S_1 and S_2, respectively. In each of these states, the other player Q has two possible moves, resulting in terminal states S_3, S_4, and S_5, S_6, respectively. This gives P two different (ordinary) strategies:



Such ordinary strategies are disjunctive strategies by definition. However, more interesting is the following combination of the two strategies for P into a single disjunctive strategy for P, where the root is a duplicating node:



Note that our rather abstract presentation of disjunctive strategies does not require that the states of different components refer to the same game. Strictly speaking, different winning conditions give different games even if they share the *game form*. Indeed, this is the case for the particular dialogue game(s) we are interested in here.

Now let us consider a collection of games $\mathcal{W}$ with the same game form (e.g., for Giles's game, obtained by adding risk assignments to the game form). A disjunctive strategy Σ for $\mathcal{W}$ for player P is a *disjunctive winning strategy* for $\mathcal{W}$ if for every game in $\mathcal{W}$, at least one component of each leaf node of Σ is a winning state for P in that game. In particular, a disjunctive winning strategy (for me) for the family of all instances of Giles's game based on $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a disjunctive strategy where in each leaf node for *every* risk assignment $\langle \cdot \rangle$ there is at least one component, i.e., elementary d-state, $[p_1, \ldots, p_m \parallel q_1, \ldots, q_n]$ such that $\langle p_1, \ldots, p_m \rangle \geq \langle q_1, \ldots, q_n \rangle$.

PROPOSITION 5.3. *Let $\mathcal{W}$ be a collection of games based on the same finite game form. Then there exists a disjunctive winning strategy for $\mathcal{W}$ for player* P *iff there is an ordinary winning strategy for* P *for every game in $\mathcal{W}$.*

Proof. For the right-to-left direction, let $\mathcal{S}_{\mathcal{W}}$ be a set of strategies such that for every game in $\mathcal{W}$ some element of $\mathcal{S}_{\mathcal{W}}$ is a winning strategy for P. Since the game form is finite, there are only finitely many different corresponding strategies. Hence $\mathcal{S}_{\mathcal{W}}$ is finite. We can combine its members $\sigma_1, \ldots, \sigma_n$ into a single disjunctive winning strategy as indicated in Example 5.2. More precisely, suppose that we have already constructed a disjunctive winning strategy $\Sigma_{1:i}$ combining the ordinary winning strategies $\sigma_1, \ldots, \sigma_i$ for some $1 \leq i < n$. To obtain a disjunctive strategy that also covers the games for which σ_{i+1} is a winning strategy we may proceed as follows. Add the common initial state S of the games in $\mathcal{W}$ as a new component to each node in $\Sigma_{1:i}$. Add also S as a new root node below the node $S \bigvee S$. Finally, replace each leaf node $S \bigvee E$ with a copy of σ_{i+1} with E added disjunctively to each node.

For the other direction, let Σ be a disjunctive winning strategy for $\mathcal{W}$ for player P. We prove the following slightly more general statement.

- If Σ' is a subtree of Σ with root $D = S_1 \bigvee \ldots \bigvee S_n$, then there exists an ordinary winning strategy for P for every subgame of a game in $\mathcal{W}$ that starts with S_i for some $i \in \{1, \ldots, n\}$.

We proceed by induction on the height of Σ'. For the base case, it suffices to observe that for each game in $\mathcal{W}$, one of the terminal states in D is a winning state for P. For the inductive step, we distinguish two cases.

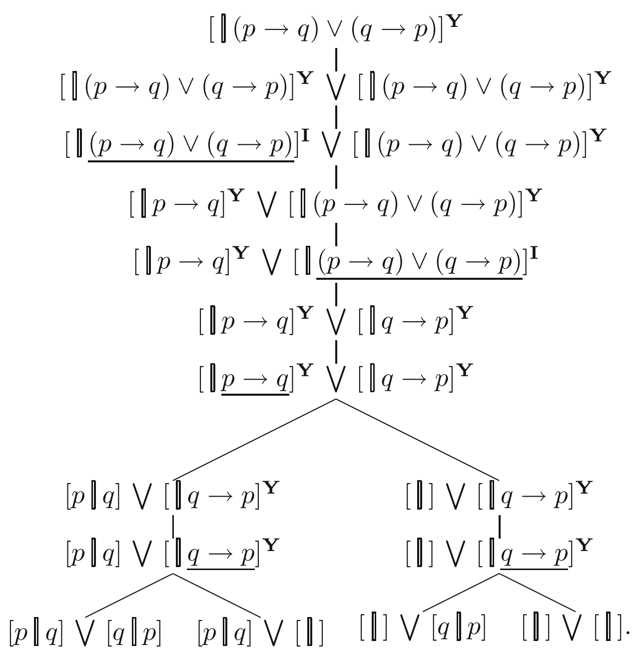
- If the root of Σ' is a duplicating node, then the claim follows immediately from the induction hypothesis.
- If the root $D = S_1 \bigvee \ldots \bigvee S_n$ of Σ' is a playing node focused on S_i, then again the claim follows immediately from the induction hypothesis for those subgames that start with S_j for some $j \neq i$ $(1 \leq i \leq n)$, since those states appear unchanged in the successor nodes to D. For the subgames starting in S_i, recall that ρ determines whether P or some other player initiates the move at S_i. In the former case there is a single successor node D' of D. Applying the induction hypothesis to the subtree with root D' yields the required winning strategy. If another player initiates the move at S_i, then, by definition of a disjunctive strategy, all possible ways to continue the game at this state are recorded in the immediate subtrees below D in Σ'. To construct the required winning strategies for P for the games starting in S_i we combine, using the same move, the relevant winning strategies obtained from applying the induction hypothesis to the immediate subtrees of Σ'. ■

In view of Theorem 4.2, we obtain:

Corollary 5.4. *A formula F is valid in Ł iff I have a disjunctive winning strategy for every instance of Giles's game with initial state $[\,\Vert F]$.*

Note that for any instance F of the Łukasiewicz axiom schema (Ł1)-(Ł3), there exists a single ordinary winning strategy for *any* game with initial state $[\,\Vert F]$, i.e., a strategy for me that is winning for all risk assignments. For (Ł4), there exists an ordinary winning strategy for certain regulations, but for others, a disjunctive winning strategy with duplication is needed. More generally, there are other valid formulas of the logic that require duplication for any regulation.

Example 5.5. It is easy to see that for the initial state $[\,\Vert (p \to q) \vee (q \to p)]$ there is an ordinary strategy for me that is winning for any assignment $\langle\rangle$ such that $\langle p\rangle \leq \langle q\rangle$ and another ordinary winning strategy for any assignment where $\langle q\rangle \leq \langle p\rangle$. However, no single strategy is winning for *all* assignments. Instead, we can combine ordinary strategies into the following disjunctive winning strategy:



6. Hypersequent Calculi

We turn our attention now to the proof theory of Łukasiewicz Logic, and the development of analytic calculi corresponding to Giles's game, again focussing initially on the language restricted to $\rightarrow$ and $\bot$. By way of introduction, let us first consider a calculus – based on the game but unfortunately not analytic – that was proposed already by Giles with Adamson in [1]. Essentially (the formalism is a little different), the authors define a sequent calculus for Ł by considering states $[\Gamma \,\|\, \Delta]$ as sequents, written here as $\Gamma \Rightarrow \Delta$. Intuitively, the logical rules of the calculus proceed by searching for a strategy for $[\Gamma \,\|\, \Delta]$ that is winning for all risk assignments. This does not quite work since there exist valid formulas where no such strategy exists. However, the calculus is made complete by adding a special cut rule. More precisely, the calculus consists of the sequent versions of all winning elementary states as axioms and the following schematic rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \frac{\Gamma, B \Rightarrow A, \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad \frac{\Gamma, A \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}\,.$$

The rules for implication correspond directly to the choices for you and me of whether to attack or grant a formula asserted by the opposing player. The two leftmost rules reflect my choice, while the third rule reflects the fact that I have to be prepared to deal with both your possible choices. Derivations using just these three rules correspond to strategies for me where I have the added privilege of choosing which of the compound formulas asserted by me you should attack or grant. What spoils this nice representation, however, is the presence of the fourth rule. This cut – or better, 'cancellation' – rule can only be interpreted rather unnaturally in the game (as suggested in [1]) by allowing me to add a copy of any formula A to both my and your stock of asserted formulas.

It is shown in [1] that there exists a derivation (a tree of sequents obtained using the rules; see below) of the sequent $\Rightarrow F$ in this calculus iff I have a winning strategy for the game $[\,\|\, F]$ for any regulation and risk assignment. The proof consists of simulating the Hilbert system HŁ presented in Section 4 (i.e., deriving the axioms and modus ponens in the calculus). In fact, the authors comment that they were unable to find a semantic proof of this result. More seriously, the presence of the cut rule (which cannot be eliminated) robs the calculus of its analyticity and much of its virtue as a proof-theoretic presentation of either Ł or the game.

Here we take a different approach. As we have seen in the previous section, a formula F is valid in Ł iff there exists a disjunctive winning strategy for any game with initial state $[\, \| F]$. This suggests representing the search for a disjunctive winning strategy for such games as the search for a proof of F in a suitable formal system. In fact, as will become clear, the appropriate calculus is very closely related to the hypersequent calculus $\mathsf{G}Ł$ defined by Metcalfe, Olivetti, and Gabbay in [24].

Hypersequents were introduced by Avron in [3] as a generalization of Gentzen sequents that takes account of disjunctive or parallel forms of reasoning. Essentially, instead of one sequent as in usual Gentzen systems, there is a collection of sequents that can be worked on at the same time. More precisely, a hypersequent is defined here as a non-empty multiset of ordered pairs of finite multisets of formulas, written:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

or sometimes, for short, as $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$. By taking multisets of formulas and sequents rather than sets or sequences (as used in Avron's original definition [3]), we ensure that the multiplicity but not the order of elements is important.

Formally, a hypersequent can be viewed as just a d-state disjunction where each occurrence of $[\, \| \,]$ is replaced by $\Rightarrow$ and $\bigvee$ is replaced by $\mid$. We retain the duplication of symbols and terminology to distinguish the quite different origins and motivations for these definitions. Bearing this correspondence in mind, however, let us say that an atomic hypersequent $\mathcal{G}$ (i.e., one containing only atomic formulas) is *winning* if for every risk assignment $\langle \cdot \rangle$, there is a member $p_1, \ldots, p_m \Rightarrow q_1, \ldots, q_n$ of $\mathcal{G}$ such that $\langle p_1, \ldots, p_m \rangle \geq \langle q_1, \ldots, q_m \rangle$. Equivalently, as argued in the adequacy proof above (Theorem 4.2), $\mathcal{G}$ is winning iff for every valuation v, there is a member $p_1, \ldots, p_m \Rightarrow q_1, \ldots, q_n$ of $\mathcal{G}$ such that $\sum_{i=1}^m (v(p_i) - 1) \leq \sum_{i=1}^n (v(q_i) - 1)$.

A *hypersequent rule* is a set of ordered pairs consisting of a (possibly empty) set of hypersequents (the premises) and a hypersequent (the conclusion). Such rules are usually presented schematically with A, B standing for formulas, Γ, Δ for multisets of formulas, and $\mathcal{G}$ for an arbitrary hypersequent. A *hypersequent calculus* GL is then just a set of hypersequent rules. A *derivation* in GL of a hypersequent $\mathcal{G}$ from a set of hypersequents Φ is a labelled rooted tree of hypersequents (usually written upside down) where:

1. The root node is $\mathcal{G}$.
2. Each node $\mathcal{H}$ is either in Φ and a leaf node, or has child nodes $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and $\mathcal{H}_1, \ldots, \mathcal{H}_n \,/\, \mathcal{H}$ is an instance of a rule of GL.

Our aim is to show that a disjunctive strategy is essentially, modulo the different terminology, a derivation in a suitable hypersequent calculus. To this end, let us consider some rules reflecting the construction of disjunctive strategies described in the previous section. First, we require the following *external contraction rule* to capture the duplication of game states:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec}).$$

Next, to capture the effect of stepping from a d-state to all possible i-states where a compound statement on the right is marked, we require a 'redundant rule' that simply duplicates the conclusion in the premises:

$$\frac{\mathcal{G} \quad \dots \quad \mathcal{G}}{\mathcal{G}}\ (\textsc{r}).$$

Finally, we introduce rules to deal with attacking and granting implicational formulas. For moves initiated by you, I must deal with the cases where you choose to attack or grant a (marked) formula:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to).$$

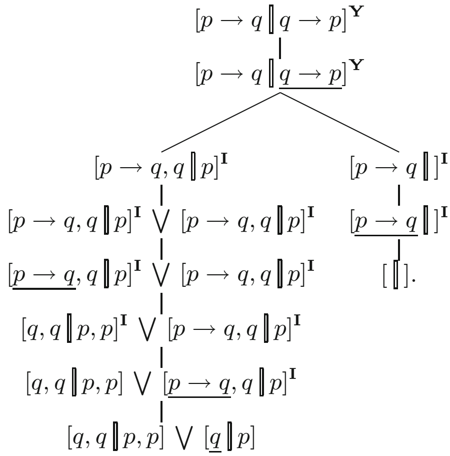
For me, there is the choice of attacking or granting the (marked) formula:

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)_1 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)_2 .$$

However, notice that the construction of a disjunctive strategy, a tree of d-states, corresponds exactly to applications of the rules (EC), (R), $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, and $(\Rightarrow\to)$. Hence, by inspection:

PROPOSITION 6.1. *Every disjunctive strategy for me for* $[\Gamma \parallel \Delta]$ *is (ignoring marking of formulas and the different notation) a hypersequent derivation of* $\Gamma \Rightarrow \Delta$ *from atomic hypersequents using* (EC), (R), $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, *and* $(\Rightarrow\to)$. *Moreover, if the disjunctive strategy is winning, then the atomic hypersequents are winning.*

EXAMPLE 6.2. Consider the following disjunctive strategy and the corresponding hypersequent derivation for the elementary state $[p \to q \parallel q \to p]$:



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{q,q \Rightarrow p,p \mid q \Rightarrow p}{q,q \Rightarrow p,p \mid p \to q, q \Rightarrow p}\,(\to\Rightarrow)_2}{q,q \Rightarrow p,p \mid p \to q, q \Rightarrow p}\,(\textsc{r})}{p \to q, q \Rightarrow p \mid p \to q, q \Rightarrow p}\,(\to\Rightarrow)_1}{p \to q, q \Rightarrow p \mid p \to q, q \Rightarrow p}\,(\textsc{r})}{p \to q, q \Rightarrow p}\,(\textsc{ec}) \quad \dfrac{\dfrac{\Rightarrow}{p \to q \Rightarrow}\,(\to\Rightarrow)_2}{p \to q \Rightarrow}\,(\textsc{r})}{p \to q \Rightarrow q \to p}\,(\Rightarrow\to)}{p \to q \Rightarrow q \to p}\,(\textsc{r}) \quad .$$

However, the opposite direction does not hold. It is not the case that every derivation of $\Gamma \Rightarrow \Delta$ from atomic hypersequents using (EC), (R), $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, and $(\Rightarrow\to)$ is a disjunctive strategy for $[\Gamma \,\|\, \Delta]$. One reason is that applications of the rule $(\Rightarrow\to)$ do not take into account the fact that you could choose to attack *any* of my formulas. To model this directly in the system we would need to adapt the redundant rule (R) to introduce marking of formulas and restrict the implication rules to marked formulas. A further reason is that the hypersequent rules are not restricted by any regulation (although, from the adequacy proof, if there is a disjunctive strategy for one regulation, there is a disjunctive strategy for any regulation).

On the other hand, we can establish the weaker claim that if there is a derivation of $\Gamma \Rightarrow \Delta$ from winning atomic hypersequents using (EC), (R), $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, and $(\Rightarrow\to)$, then there exists, and indeed can be constructed, a disjunctive winning strategy for $[\Gamma \,\|\, \Delta]$ for any consistent regulation ρ. Let us define Ggiles as the hypersequent calculus consisting of (EC), (R), $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, and $(\Rightarrow\to)$ together with (as rules with no premises) all winning atomic hypersequents. The crucial element in the proof is an invertibility lemma, recalling that a rule is invertible for a calculus if whenever the conclusion is derivable in the calculus, then so are the premises.

Observe first that the following rule is derivable in Ggiles using $(\to\Rightarrow)_1$, $(\to\Rightarrow)_2$, and (EC) (i.e., the conclusion is derivable from the premise):

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow).$$

We can also show that certain 'weakening rules', although not derivable in the calculus, are nevertheless admissible in the sense that whenever the premise is derivable, so is the conclusion.

LEMMA 6.3. *The following rules are admissible for* Ggiles*:*

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}\ (\text{W}).$$

PROOF. We prove by induction on the height of a derivation of a hypersequent $\mathcal{G}$ in Ggiles that any hypersequent obtained from $\mathcal{G}$ by adding sequents and formulas on the left of sequents is also derivable in Ggiles.

For the base case, suppose that $\mathcal{G}$ is a winning atomic hypersequent. Let $\mathcal{H}$ be obtained from $\mathcal{G}$ by adding sequents and formulas on the left of sequents. A simple induction shows that $\mathcal{H}$ is derivable from hypersequents $\mathcal{H}_1, \ldots, \mathcal{H}_n$ obtained from $\mathcal{G}$ by adding atomic sequents and atomic formulas on the left. But since $\mathcal{G}$ is winning, it follows immediately that $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are winning. Hence $\mathcal{H}$ is derivable in Ggiles.

For the inductive step, we consider the last rule applied, and the result follows by first applying the induction hypothesis to each premise and then applying the same rule to the hypersequents obtained. ■

LEMMA 6.4. *The rules* $(\to\Rightarrow)$ *and* $(\Rightarrow\to)$ *are invertible for* Ggiles.

PROOF. For $(\Rightarrow\to)$, we establish the more general claim:

If $\mathcal{G} \mid [\Gamma_i \Rightarrow A \to B, \Delta_i]_{i=1}^n$ is derivable in Ggiles, then $\mathcal{G} \mid [\Gamma_i, A \Rightarrow B, \Delta_i]_{i=1}^n$ and $\mathcal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ are derivable in Ggiles.

We proceed by induction on the height of a derivation of $\mathcal{G} \mid [\Gamma_i \Rightarrow A \to B, \Delta_i]_{i=1}^n$. For the base case, the hypersequent is atomic, so $n = 0$ and the claim is trivial. For the induction step, suppose first that the last application of a rule is (EC). Then we apply the induction hypothesis to the premise, followed by an application of (EC), and we are done. Similarly, if the last application of a rule is to any formula other than a distinguished occurrence of $A \to B$, then the claim follows by the induction hypothesis applied to the premises and an application of the same rule. Finally, suppose that the last application of a rule is $(\Rightarrow\to)$ to a distinguished occurrence of $A \to B$. Then without loss of generality, the premises are of the form $\mathcal{G} \mid [\Gamma_i \Rightarrow A \to B, \Delta_i]_{i=2}^n \mid \Gamma_1, A \Rightarrow B, \Delta_1$ and $\mathcal{G} \mid [\Gamma_i \Rightarrow A \to B, \Delta_i]_{i=2}^n \mid \Gamma_1 \Rightarrow \Delta_1$. Applying the induction hypothesis to the first premise, we have that $\mathcal{G} \mid [\Gamma_i, A \Rightarrow B, \Delta_i]_{i=1}^n$ is derivable (the additional fact that $\mathcal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid \Gamma_1, A \Rightarrow B, \Delta_1$ is derivable is not needed). Similarly, applying the induction hypothesis to the second premise, $\mathcal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ is derivable.

For $(\to\Rightarrow)$, the required general claim is:

If $\mathcal{G} \mid [\Gamma_i, A \to B \Rightarrow \Delta_i]_{i=1}^n$ is derivable in Ggiles, then $\mathcal{G} \mid [\Gamma_i, B \Rightarrow A, \Delta_i \mid \Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ is derivable in Ggiles.

The proof is by induction on the height of a derivation of $\mathcal{G} \mid [\Gamma_i, A \to B \Rightarrow \Delta_i]_{i=1}^n$. Again, the base case and the inductive step where the last application of a rule is (EC) or to a formula other than a distinguished occurrence of $A \to B$, are straightforward. Suppose then that the last application of a rule is $(\to\Rightarrow)_1$ or $(\Rightarrow\to)_2$ to a distinguished occurrence of $A \to B$. In the first case, the premise is of the form $\mathcal{G} \mid [\Gamma_i, A \to B \Rightarrow \Delta_i]_{i=2}^n \mid \Gamma_1, B \Rightarrow A, \Delta_1$ and in the second, $\mathcal{G} \mid [\Gamma_i, A \to B \Rightarrow \Delta_i]_{i=2}^n \mid \Gamma_1 \Rightarrow \Delta_1$. By the induction hypothesis, respectively, $\mathcal{G} \mid [\Gamma_i, B \Rightarrow A, \Delta_i \mid \Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid \Gamma_1, B \Rightarrow A, \Delta_1$ or $\mathcal{G} \mid [\Gamma_i, B \Rightarrow A, \Delta_i \mid \Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid \Gamma_1 \Rightarrow \Delta_1$ is derivable. In both cases it follows by the admissibility of (EW) (Lemma 6.3) that $\mathcal{G} \mid [\Gamma_i, B \Rightarrow A, \Delta_i \mid \Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ is derivable. ∎

THEOREM 6.5. $\Gamma \Rightarrow \Delta$ *is derivable in* Ggiles *iff there exists a disjunctive winning strategy for me for* $[\Gamma \,\|\, \Delta]$ *for any consistent regulation* ρ.

PROOF. The right-to-left-direction is a weaker form of Proposition 6.1 which tells us that a disjunctive winning strategy for me for $[\Gamma \,\|\, \Delta]$ respecting any consistent regulation ρ is (modulo the different notation) a derivation of $\Gamma \Rightarrow \Delta$ in Ggiles.

For the opposite direction, we use the invertibility lemma to construct a disjunctive winning strategy for *any* sequent derivable in Ggiles. Let the complexity $|\Gamma \Rightarrow \Delta|$ of a sequent (state) $\Gamma \Rightarrow \Delta$ be the number of implication symbols occurring in it. The complexity of a hypersequent (state disjunction) $\mathcal{G}$ is the multiset of integers $|\mathcal{G}| = [|S| \mid S \in \mathcal{G}]$. We make use of the standard well-ordering of multisets [10]. For multisets α, β of integers: $<_m$ is the transitive closure of $<$, where $\alpha < \beta$ if α is obtained by replacing an element n of β by finitely many (possibly 0) copies of k for some $k < n$.

Claim. If $\mathcal{G}$ is derivable in Ggiles, then there exists a disjunctive winning strategy for me for $\mathcal{G}$ (considered as a state disjunction) for any consistent regulation ρ.

We proceed by induction on $|\mathcal{G}|$ ordered by $<_m$. The base case is immediate since if $\mathcal{G}$ is atomic, then $\mathcal{G}$ is winning and constitutes a disjunctive winning strategy for $\mathcal{G}$ independently of any regulation. For the inductive step, pick any non-atomic member $\Gamma \Rightarrow \Delta$ of $\mathcal{G}$. If the regulation ρ says that it is my turn to play, then choose a compound formula $A \to B$ in Γ. By Lemma 6.4, if $\mathcal{G} = (\mathcal{G}' \mid \Gamma', A \to B \Rightarrow \Delta)$, then $\mathcal{H} = (\mathcal{G}' \mid \Gamma', B \Rightarrow A, \Delta \mid \Gamma' \Rightarrow \Delta)$ is derivable in Ggiles. Since $|\mathcal{H}| <_m |\mathcal{G}|$, by the induction hypothesis, there is a disjunctive winning strategy for me for $\mathcal{G}' \bigvee [\Gamma', B \,\|\, A, \Delta]^? \bigvee [\Gamma' \,\|\, \Delta]^?$ respecting ρ where ? denotes the correct assignment of $\mathbf{Y}$ or $\mathbf{I}$ to states according to ρ. The required disjunctive winning strategy begins with:

$$\begin{array}{c}
\mathcal{G}' \bigvee [\Gamma', A \to B \,\|\, \Delta]^{\mathbf{I}} \\
| \\
\mathcal{G}' \bigvee [\Gamma', A \to B \,\|\, \Delta]^{\mathbf{I}} \bigvee [\Gamma', A \to B \,\|\, \Delta]^{\mathbf{I}} \\
| \\
\mathcal{G}' \bigvee [\Gamma', \underline{A \to B} \,\|\, \Delta]^{\mathbf{I}} \bigvee [\Gamma', A \to B \,\|\, \Delta]^{\mathbf{I}} \\
| \\
\mathcal{G}' \bigvee [\Gamma', B \,\|\, A, \Delta]^{?} \bigvee [\Gamma', A \to B \,\|\, \Delta]^{\mathbf{I}} \\
| \\
\mathcal{G}' \bigvee [\Gamma', B \,\|\, A, \Delta]^{?} \bigvee [\Gamma', \underline{A \to B} \,\|\, \Delta]^{\mathbf{I}} \\
| \\
\mathcal{G}' \bigvee [\Gamma', B \,\|\, A, \Delta]^{?} \bigvee [\Gamma' \,\|\, \Delta]^{?}.
\end{array}$$

Now suppose that the regulation ρ says that is your turn to play in the state $\Gamma \Rightarrow \Delta$ of $\mathcal{G} = (\mathcal{G}' \mid \Gamma \Rightarrow \Delta)$. Let $[A_1 \to B_1, \ldots, A_n \to B_n]$ be the compound formulas in Δ, Δ' the atomic formulas in Δ, and $\Delta_i = \Delta - [A_i \to B_i]$ for $i = 1 \ldots n$. If $\mathcal{G}$ is derivable in Ggiles, then by Lemma 6.4, $\mathcal{G}' \mid \Gamma, A_i \Rightarrow B_i, \Delta_i$ and $\mathcal{G}' \mid \Gamma \Rightarrow \Delta_i$ are derivable in Ggiles for $i = 1 \ldots n$. Hence by the induction hypothesis, there is a disjunctive winning strategy for me for each $\mathcal{G}' \bigvee [\Gamma \,\|\, \Delta_i]^?$ and $\mathcal{G}' \bigvee [\Gamma, A_i \,\|\, B_i, \Delta_i]^?$. We obtain the required disjunctive winning strategy by extending these disjunctive strategies as follows:

$$\begin{array}{c}
\mathcal{G}' \bigvee [\Gamma \,\|\, A_1 \to B_1, \ldots, A_n \to B_n, \Delta']^{\mathbf{Y}} \\
\swarrow \qquad\qquad \searrow \\
\mathcal{G}' \bigvee [\Gamma \,\|\, \underline{A_1 \to B_1}, \Delta_1]^{\mathbf{Y}} \quad \ldots \qquad\qquad \ldots \quad \mathcal{G}' \bigvee [\Gamma \,\|\, \underline{A_n \to B_n}, \Delta_n]^{\mathbf{Y}} \\
\swarrow \quad \searrow \qquad\qquad\qquad\qquad \swarrow \quad \searrow \\
\mathcal{G}' \bigvee [\Gamma, A_1 \,\|\, B_1, \Delta_1]^{?} \quad \mathcal{G}' \bigvee [\Gamma \,\|\, \Delta_1]^{?} \quad \mathcal{G}' \bigvee [\Gamma, A_n \,\|\, B_n, \Delta_n]^{?} \quad \mathcal{G}' \bigvee [\Gamma \,\|\, \Delta_n]^{?}.
\end{array}$$

■

Initial Sequents

$$
\dfrac{}{A \Rightarrow A}\ (\textsc{id}) \qquad \dfrac{}{\Rightarrow}\ (\textsc{emp}) \qquad \dfrac{}{\Gamma, \bot \Rightarrow A}\ (\bot\Rightarrow)
$$

Structural Rules:

$$
\dfrac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ew}) \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec}) \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}\ (\textsc{w})
$$

$$
\dfrac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split}) \qquad \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{mix})
$$

Logical Rules

$$
\dfrac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)_1 \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)
$$

Figure 1. The Hypersequent Calculus GŁ

EXAMPLE 6.6. The following derivation in Ggiles can be read as a winning disjunctive strategy (written upside down and with sequents replacing states) for $[p \to q, p \,\|\, q, q \to p]$ with a regulation stipulating that it is my move first, noting that underlining is added here to emphasize the selection of the formula to be attacked:

$$
\dfrac{
 \dfrac{
  \dfrac{
   \dfrac{
    \dfrac{
     \dfrac{
      \dfrac{
       \dfrac{\dfrac{q,p \Rightarrow p,q \mid p \Rightarrow q \quad q,p \Rightarrow p,q \mid p,q \Rightarrow p,q}{q,p \Rightarrow p,q \mid p \Rightarrow q, \underline{q \to p}}}{q,p \Rightarrow p,q \mid p \Rightarrow q, q \to p}\ (\textsc{r})
       \quad
       \dfrac{\dfrac{q,p,q \Rightarrow p,p,q \mid p \Rightarrow q \quad q,p,q \Rightarrow p,p,q \mid p,q \Rightarrow p,q}{q,p,q \Rightarrow p,p,q, \mid p \Rightarrow q, \underline{q \to p}}}{q,p,q \Rightarrow p,p,q, \mid p \Rightarrow q, q \to p}\ (\textsc{r})
      }{q,p \Rightarrow p,q,\underline{q \to p} \mid p \Rightarrow q, q \to p}\ (\Rightarrow\to)
     }{q,p \Rightarrow p,q,q \to p \mid p \Rightarrow q, q \to p}\ (\textsc{r})
    }{q,p \Rightarrow p,q,q \to p \mid \underline{p \to q}, p \Rightarrow q, q \to p}\ (\to\Rightarrow)_2
   }{q,p \Rightarrow p,q,q \to p \mid p \to q, p \Rightarrow q, q \to p}\ (\textsc{r})
  }{\underline{p \to q}, p \Rightarrow q, q \to p \mid p \to q, p \Rightarrow q, q \to p}\ (\to\Rightarrow)_1
 }{p \to q, p \Rightarrow q, q \to p \mid p \to q, p \Rightarrow q, q \to p}\ (\textsc{r})
}{p \to q, p \Rightarrow q, q \to p}\ (\textsc{ec})
\;.
$$

Our last task is to connect the calculus Ggiles with the calculus GŁ of [24] displayed in Figure 1. Note that in this system, the second implication rule $(\to\Rightarrow)_2$ has become an instance of the 'weakening' rule (W). Also, the redundant rule (R) has been removed for obvious reasons. More significantly, rather than take all winning atomic hypersequents as axioms, a collection of structural rules and very simple axioms are provided.

The following theorem is in fact an immediate corollary of the adequacy proof for Giles's game and the completeness proof for GŁ given in [24]. Nevertheless, we prefer here to give a direct (constructive) proof that connects GŁ to the calculus Ggiles and hence to the game.

THEOREM 6.7. $\Gamma \Rightarrow \Delta$ *is derivable in* GŁ *iff there exists a disjunctive winning strategy for me for* $[\Gamma \,\|\, \Delta]$ *for any consistent regulation* ρ.

PROOF. By Theorem 6.5, it is sufficient to show that a hypersequent $\mathcal{G}$ is derivable in GŁ iff $\mathcal{G}$ is derivable in Ggiles. For the right-to-left-direction, we note that every rule of Ggiles is contained or derivable in GŁ, so it remains to show that each winning atomic hypersequent $\mathcal{G}$ is derivable in GŁ. We proceed by induction on the number k of distinct variables occurring on the left hand side of sequents in $\mathcal{G}$. Suppose that there are none ($k = 0$). Then only $\bot$ occurs on the left hand side of sequents. Since $\mathcal{G}$ satisfies the winning condition, there must be a sequent in $\mathcal{G}$ where the number of occurrences of $\bot$ on the left is greater than or equal to the number of formulas on the right. But then $\mathcal{G}$ is derivable using ($\bot\Rightarrow$), (EW), (W), and (MIX).

For $k > 0$, we pick a variable q occurring on the left of one of the sequents of $\mathcal{G}$. Observe now that we can use the following simplification rule (derivable using (MIX), (EW), and (ID)) backwards repeatedly to obtain a winning atomic hypersequent where q does not occur on both sides of any sequent:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, A \Rightarrow A, \Delta}\ (\textsc{simp}).$$

Moreover, we can also apply (EC) and (SPLIT) backwards to 'multiply' sequents; e.g., using $*$ to denote multiple applications of a rule:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma^k \Rightarrow \Delta^k}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \ldots \mid \Gamma \Rightarrow \Delta}\ (\textsc{split})^*}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec})^*.$$

Carrying out such a procedure for sequents containing q on the left or right, we obtain a winning atomic hypersequent of the form:

$$\mathcal{G}' = (\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j, [q]^\lambda \Rightarrow \Sigma_j]_{j=1}^m)$$

where q does not occur in $\mathcal{G}_0$, Γ_i, Δ_i, Π_j, or Σ_j for $i = 1 \ldots n$ and $j = 1 \ldots m$, and $[q]^\lambda$ denotes $\lambda > 0$ occurrences of q. Now let:

$$\mathcal{H} = (\mathcal{G}_0 \mid [\Gamma_i, \Pi_j \Rightarrow \Sigma_j, \Delta_i]_{i=1\ldots n}^{j=1\ldots m} \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m).$$

Clearly $\mathcal{H}$ contains fewer distinct variables occurring on the left of sequents than $\mathcal{G}'$. Moreover, $\mathcal{G}'$ is derivable from $\mathcal{H}$ as follows (again using $*$ to denote repeated applications of a rule):

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G}_0 \mid [\Gamma_i, \Pi_j \Rightarrow \Sigma_j, \Delta_i]_{i=1\ldots n}^{j=1\ldots m} \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m}{\mathcal{G}_0 \mid [\Gamma_i, \Pi_j, [q]^\lambda \Rightarrow [q]^\lambda, \Sigma_j, \Delta_i]_{i=1\ldots n}^{j=1\ldots m} \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m} (\textsc{simp})^*}{\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j, [q]^\lambda \Rightarrow \Sigma_j]_{j=1}^m \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m} (\textsc{split})^*}{\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j, [q]^\lambda \Rightarrow \Sigma_j]_{j=1}^m \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j, [q]^\lambda \Rightarrow \Sigma_j]_{j=1}^m} (\textsc{w})^*}{\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [q]^\lambda, \Delta_i]_{i=1}^n \mid [\Pi_j, [q]^\lambda \Rightarrow \Sigma_j]_{j=1}^m} (\textsc{ec})^* \quad .$$

Hence it is sufficient to show that $\mathcal{H}$ is winning since then, by the induction hypothesis, $\mathcal{H}$ is derivable in GŁ. Suppose otherwise, i.e., that there exists a falsifying risk assignment $\langle\cdot\rangle$. Define:

$$x = \max(\{\langle\Gamma_i\rangle - \langle\Delta_i\rangle \mid 1 \leq i \leq n\} \cup \{0\})$$
$$y = \min(\{\langle\Sigma_j\rangle - \langle\Pi_j\rangle \mid 1 \leq j \leq m\} \cup \{\lambda\}).$$

We claim that $x < y$. Otherwise there exists i, j such that $\langle\Gamma_i\rangle + \langle\Pi_j\rangle \geq \langle\Sigma_j\rangle + \langle\Delta_i\rangle$, $\langle\Gamma_i\rangle \geq \lambda + \langle\Delta_i\rangle$, or $\langle\Pi_j\rangle \geq \langle\Sigma_j\rangle$, contradicting the fact that $\langle\cdot\rangle$ is a falsifying risk assignment for $\mathcal{H}$. Now let us change the value of $\langle q\rangle$ so that $x < \lambda\langle q\rangle < y$. Then for all $i = 1\ldots n$ and $j = 1\ldots m$, $\langle\Gamma_i\rangle - \langle\Delta_i\rangle < \lambda\langle q\rangle < \langle\Sigma_j\rangle - \langle\Pi_j\rangle$, which gives $\langle\Gamma_i\rangle < \langle\Delta_i \uplus [q]^\lambda\rangle$ and $\langle\Pi_j \uplus [q]^\lambda\rangle < \langle\Sigma_j\rangle$. But $\mathcal{G}'$ is winning, so we obtain the desired contradiction.

For the left-to-right direction, it is easy to see that the rules (EW), (W), (SPLIT), and (MIX) are admissible for Ggiles when restricted to atomic hypersequents. Hence it is sufficient to show that these rules can be pushed upwards over the logical rules in any GŁ-derivation. The cases of (EW), (EC), and (W) are easy. For (MIX) below $(\to\Rightarrow)_1$, we have:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, B \Rightarrow A, \Delta_1}{\mathcal{H} \mid \Gamma_1, A \to B \Rightarrow \Delta_1} (\to\Rightarrow)_1 \quad \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, A \to B, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\textsc{mix}) \qquad \text{becomes}$$

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, B \Rightarrow A, \Delta_1 \quad \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, B, \Gamma_2 \Rightarrow A, \Delta_1, \Delta_2} (\textsc{mix})}{\mathcal{H} \mid \Gamma_1, A \to B, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\to\Rightarrow)_1$$

and for (MIX) below $(\Rightarrow\to)$:

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1, A \Rightarrow B, \Delta_1 \quad \mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1}{\mathcal{H} \mid \Gamma_1 \Rightarrow A \to B, \Delta_1} (\Rightarrow\to) \quad \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A \to B, \Delta_1, \Delta_2} (\textsc{mix}) \qquad \text{becomes}$$

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\textsc{mix}) \quad \dfrac{\mathcal{H} \mid \Gamma_1, A \Rightarrow B, \Delta_1 \quad \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2, A \Rightarrow B, \Delta_1, \Delta_2} (\textsc{mix})}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow A \to B, \Delta_1, \Delta_2} (\Rightarrow\to) \quad .$$

The cases for (SPLIT) follow in a similar fashion. ■

Remark 6.8. *Note that while Theorem 6.7 follows from the adequacy of the game for Ł combined with the completeness and soundness of* GŁ*, our proof is much more informative. In particular, it shows that, without losing completeness for* GŁ*, we need only consider atomic initial sequents and may restrict applications of all structural rules except* (EC) *to atomic hypersequents.*

Example 6.9. The following derivation in GŁ (where $r, r, p, q \Rightarrow r, r, q, p$ is easily derived using (MIX) and (ID)) is an example of a case where the resulting sequent can only be derived if the detour to hypersequents is made.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{r \Rightarrow r}\,(\textsc{id})}{r \Rightarrow r \mid (q \to p) \to r \Rightarrow r}\,(\textsc{ew})
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{r \Rightarrow r}\,(\textsc{id})}{r, p \Rightarrow r, q \mid r \Rightarrow r}\,(\textsc{ew})
\quad
\dfrac{r, r, p, q \Rightarrow r, r, q, p}{r, p \Rightarrow r, q \mid r, q \Rightarrow r, p}\,(\textsc{split})
}{r, p \Rightarrow r, q \mid r \Rightarrow r, q \to p}\,(\Rightarrow\to)
}{r, p \Rightarrow r, q \mid (q \to p) \to r \Rightarrow r}\,(\to\Rightarrow)_1
}{r \Rightarrow r, p \to q \mid (q \to p) \to r \Rightarrow r}\,(\Rightarrow\to)
}{(p \to q) \to r \Rightarrow r \mid (q \to p) \to r \Rightarrow r}\,(\to\Rightarrow)_1
}{(p \to q) \to r, (q \to p) \to r \Rightarrow r \mid (q \to p) \to r \Rightarrow r}\,(\textsc{w})
}{(p \to q) \to r, (q \to p) \to r \Rightarrow r \mid (p \to q) \to r, (q \to p) \to r \Rightarrow r}\,(\textsc{w})
}{(p \to q) \to r, (q \to p) \to r \Rightarrow r}\,(\textsc{ec})
$$

It was shown by Ciabattoni and Metcalfe in [7] that the following cut and cancellation rules are not only admissible for GŁ but can also be eliminated from GŁ + (CUT) and GŁ + (CAN), respectively:

$$
\dfrac{\mathcal{G} \mid \Gamma_1, A \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow A, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\textsc{cut}) \qquad \dfrac{\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\,(\textsc{can}).
$$

Finally, we note that hypersequent rules for other connectives, displayed in Figure 2, can be added to either GŁ or Ggiles. Completeness for the corresponding calculi may either be obtained by extending the preceding proofs or by using the definability of these connectives in terms of $\to$ and $\bot$.

7. Concluding Remarks

Although Giles's account of logical reasoning in physical theories refers only to the infinite-valued Łukasiewicz logic, his game also provides a characterization for each of the finite-valued Łukasiewicz logics. We just restrict the probabilities of positive results of experiments to values in $\{\frac{n-i}{n-1} \mid 1 \leq i \leq n\}$ for some fixed n. In particular, for $n = 2$, no dispersion of results occurs and we obtain a model of classical reasoning as expected. It follows then that the hypersequent calculus Ggiles with all winning atomic hypersequents

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_2 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ (\Rightarrow\wedge)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ (\Rightarrow\vee)_1 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ (\Rightarrow\vee)_2 \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta}\ (\vee\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \bot, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \,\&\, B, \Delta}\ (\Rightarrow\&)_1 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \,\&\, B, \Delta}\ (\Rightarrow\&)_2 \qquad \frac{\mathcal{G} \mid \Gamma, A, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \,\&\, B \Rightarrow \Delta}\ (\&\Rightarrow)$$

Figure 2. Rules for Other Connectives

for the finite-valued game taken as axioms is sound and complete for the corresponding logic.

Perhaps more interesting is the fact that Giles's game becomes adequate for cancellative hoop logic CHL [11] if we disallow experiments that can never succeed, i.e., if only non-zero probabilities are assigned to the positive results of experiments and, as a result, $\bot$ is removed from the language. While the rules for implication stay the same (indeed, the implicational fragments of CHL and Ł coincide), the rules for strong conjunction revert to the original and perhaps more natural rule $(R'_\wedge)$ suggested in Section 3: if I assert $A \,\&\, B$ then, if attacked, I also have to assert both A and B (analogously, if you assert $A \,\&\, B$). A hypersequent calculus for this logic, first presented in [24], is obtained simply by adding the corresponding rules for strong conjunction to the implicational fragment of GŁ.

Connections between variants of Giles's game and other logics are obtained by making changes to both the winning conditions and the underlying structures. In particular, in [6], a connection is briefly described between Giles's game and logical rules in the framework of 'relational hypersequents' that are uniform for the three fundamental t-norm based fuzzy logics: Łukasiewicz logic, Gödel logic, and product logic. The correspondence requires, however, a significant change in the crucial dialogue rule for implication. This variant of the game is discussed in more detail in [13] where also an interpretation of the evaluation of elementary states in terms of supervaluations with respect to precisification spaces is offered. The latter approach is discussed further in [15] where Łukasiewicz logic is enriched by a modal operator modelling 'supertruth' and a corresponding Giles-style characterization is presented.

A different type of relationship between Lorenzen-style dialogue games and hypersequent calculi is described in [12, 14] based on the correspondence between a variant of Gentzen's sequent calculus **LJ** for intuitionistic logic and Lorenzen's original dialogue game. More precisely, it is shown that winning strategies for parallel dialogue games synchronized using different ways to transfer information between dialogues correspond to cut-free proofs in hypersequent calculi for certain intermediate logics, most prominently Gödel logic. It is worth noting that such an interpretation could also be provided for the hypersequent calculi described above for Łukasiewicz logic by allowing several instances of Giles's game to take place in parallel.

All the (related) work mentioned so far has been confined to propositional logics. Giles, however, already in [18], made some interesting remarks on a generalization of his game to first-order logic. Natural dialogue rules can be given, following Lorenzen, for the universal and existential quantifiers. The existence of winning strategies corresponds to truth in a model and is related to the Hintikka-style evaluation game for Ł defined in [9]. However, it is unclear whether these games can be combined to yield a characterization of validity in first-order Łukasiewicz logic similar to the one for propositional Ł using disjunctive strategies. This is not surprising since first-order Ł is known to be non-axiomatizable [33], indeed Π_2-complete [31]. Nevertheless, there remains open the possibility of providing an alternative characterization, using perhaps an infinitary rule, or of investigating interesting fragments, as pursued from a proof-theoretic perspective in [4].

Finally, we remark that Mundici in [26, 27] has achieved a quite different game-theoretic characterization of Łukasiewicz logic based on Ulam's game with lies. Truth value assignments in n-valued Łukasiewicz logic are related to states of knowledge of a Questioner that are determined by binary (yes/no) answers of a Responder who may lie up to n times. Łukasiewicz's truth functions then model corresponding operations on knowledges states. In particular, valid formulas correspond to information that is confirmed by any knowledge state. Mundici's approach has triggered interesting research on the connections between many-valued logics and communication protocols with fixed bounds of error. However, possible relationships with Giles's game and with proof theory remain unexplored.

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