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Embedding some Riemann surfaces into $\mathbb{C}^2$ with interpolation

Frank Kutzschebauch · Erik Løw · Erlend Fornæss Wold

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Abstract We prove that several types of open Riemann surfaces, including the finitely connected planar domains, embed properly into $\mathbb{C}^2$ such that the values on any given discrete sequence can be arbitrarily prescribed.

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1 Introduction

It is known that every Stein manifold of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$, and this N is the smallest possible due to an example of Forster [9]. The corresponding embedding result with N replaced by $N' = \left[\frac{3n+1}{2}\right] + 1$ was announced by Eliashberg and Gromov in 1970 [13] and proved in 1992 [2]. For even values of $n \in \mathbb{N}$ we have $N = N'$ and hence the result of Eliashberg and Gromov is the best possible. For odd n we have $N' = N + 1$, and in this case the optimal result was obtained by Schürmann [17], also for Stein spaces with bounded embedding dimension. A key ingredient in these results is the homotopy principle for holomorphic sections of elliptic submersions over Stein manifolds [4,12].

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F. Kutzschebauch (✉)
Mathematisches Institut, Universität Bern, Sidlerstr. 5, 3012 Bern, Switzerland
e-mail: frank.kutzschebauch@math.unibe.ch

E. Løw · E. F. Wold
Matematisk Institutt, Universitet i Oslo, Postboks 1053, Blindern, 0316 Oslo, Norway
e-mail: elow@math.uio.no

E. F. Wold
e-mail: erlendfw@math.uio.no

Combining the known embedding results and the theory of holomorphic automorphisms of $\mathbb{C}^N$ Forstneric, Ivarsson, Prezelj and the first author [5] proved the above mentioned embedding results with additional interpolation on discrete sequences.

In the case $n = 1$ the above mentioned methods do not apply. It is still an open problem whether every open Riemann surface embeds properly into $\mathbb{C}^2$. Recently the third author achieved results concerning that problem, and in this paper we prove the corresponding results with interpolation on discrete sequences, thus solving the second part of Problem 1.6 in [5]:

Let X be a Riemann surface. We say that X *embeds into* $\mathbb{C}^2$ *with interpolation* if the following holds for all discrete sequences $\{a_j\} \subset X$ and $\{b_j\} \subset \mathbb{C}^2$ without repetition: There exists a proper holomorphic embedding $f : X \hookrightarrow \mathbb{C}^2$ with $f(a_j) = b_j$ for $j = 1, 2, \ldots$

Theorem 1 *If X is one of the following Riemann surfaces then X embeds into $\mathbb{C}^2$ with interpolation*:

(1) *A finitely connected planar domain.*
(2) *A finitely connected planar domain with a regularly convergent sequence of points removed.*
(3) *A domain in a torus with at most two complementary components.*
(4) *A finitely connected subset of a torus whose complementary components do not reduce to points.*
(5) *A Riemann surface whose double is hyperelliptic.*
(6) *A smoothly bounded Riemann surface in $\mathbb{C}^2$ with a single boundary component.*

This list includes all instances of open Riemann surfaces we are aware of admitting proper holomorphic embeddings into $\mathbb{C}^2$. We note that embeddings of hyperelliptic Riemann surfaces were obtained by Forstnerič and Černe in [1].

Our method of proof follows the idea of the third author in [7] to embed X as a Riemann surface in $\mathbb{C}^2$ with unbounded boundary components and then construct a Fatou Bieberbach domain whose intersection with the closure is exactly X. We construct the Fatou-Bieberbach domain not as a basin of attraction, but as the set where a certain sequence of holomorphic automorphisms of $\mathbb{C}^2$ converges. The main ingredient is a version of Lemma 1 in [6]. We tried to formalize the ingredients in the proof and formulate a more general technical theorem (Theorem 2) which implies Theorem 1.

At each step of the inductive construction we take care of the additional interpolation condition in the same clever way as in [5].

The above theorem has already been proved in the special case that X is the unit disc by Globevnik in [10] and in the special case that X is an algebraic curve in $\mathbb{C}^2$ (e.g. $X = \mathbb{C}\setminus\{\text{finitely many points}\}$) by Forstneric, Ivarsson, Prezelj and the first author in [5].

More results on embedding with interpolation can be found in [14].

2 Proof of theorem 1

We shall use the theory of holomorphic automorphisms of $\mathbb{C}^N$.

Let $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ denote the projection onto the first coordinate, Δ_R denotes the open disc of radius R in $\mathbb{C}$, $\overline{\Delta}_R$ its closure, B_R is the ball of radius R in $\mathbb{C}^2$ and $\overline{B}_R$ its closure.

Definition 2.1 Given finitely many disjoint smooth real curves in $\mathbb{C}^2$ without self intersection $\Gamma_i = \{\gamma_i(t) : t \in [0, \infty) \text{ or } t \in (-\infty, \infty)\}$ $i = 1, 2, \ldots, m$, $\Gamma = \bigcup_{i=1}^m \Gamma_i$, and a countable subset $E \subset \mathbb{C}^2 \setminus \Gamma$, which is discrete in $\mathbb{C}^2 \setminus \Gamma$. We say that (Γ, E) has the *nice projection property* if there is a holomorphic automorphism $\alpha \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$ of $\mathbb{C}^2$ such that,if $\beta_i(t) = \alpha(\gamma_i(t))$, $\Gamma' = \alpha(\Gamma')$ and $E' = \alpha(E)$, then the following holds:

(1) $\lim_{|t|\to\infty} |\pi_1(\beta_i(t))| = \infty \quad i = 1, 2, \ldots, m$
(2) There is an $M \in \mathbb{R}$ such that for all $R \geq M$ $\mathbb{C} \setminus (\pi_1(\Gamma') \cup \overline{\Delta}_R)$ does not contain any relatively compact connected components.
(3) The restriction of the projection π_1 to $\Gamma' \cup E'$ is a proper map into $\mathbb{C}$.

Lemma 2.2 *Given a polynomially convex compact set $K \subset \mathbb{C}^2$, a finite set of points $c_1, c_2, \ldots, c_l \in K$ and a ball B containing K and a positive number $\varepsilon > 0$. Moreover given a finite number of real curves $\Gamma = \bigcup_{i=1}^m \Gamma_i$ and a discrete subset $E \subset \mathbb{C}^2 \setminus \Gamma$ as in the definition above having the nice projection property , such that $(\Gamma \cup E) \cap K = \varnothing$. Then there is an automorphism $\psi \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$ such that*

(a) $\sup_{z\in K} |\psi(z) - z| < \varepsilon$
(b) $\psi(c_i) = c_i \quad i = 1, 2, \ldots, l$
(c) $\psi(\Gamma \cup E) \subset \mathbb{C}^2 \setminus B$

Proof To simplify notation, we will assume we already have applied α, i.e. that (1),(2) and (3) hold with β_i,Γ', E' replaced by γ_i, Γ, E. It is clear that the result will follow by conjugating with α, if we choose a slightly larger polynomially convex set and a sufficiently big ball.

In order to construct ψ assume that $B = B_R$ where R is so big that $R > M$ (remember M is from the nice projection property) and $K \subset \overline{\Delta}_R \times \mathbb{C}$.

Set $\tilde{\Gamma} = \Gamma \cap (\overline{\Delta}_R \times \mathbb{C})$ and $\tilde{E} = E \cap (\overline{\Delta}_R \times \mathbb{C})$. By the nice projection property (3) $\tilde{\Gamma} \cup \tilde{E}$ is compact. Take an isotopy of diffeomorphisms removing $\tilde{\Gamma} \cup \tilde{E}$ from B not intersecting K at any time (first do it for the curves Γ, this will automatically remove all points from E except finitely many, then remove the finite number of remaining points) and apply the Andersen-Lempert theorem to $K \cup \tilde{\Gamma} \cup \tilde{E}$. By a theorem of Stolzenberg [16] $K \cup \tilde{\Gamma}$ is polynomially convex. By lemma 2.3 $K \cup \tilde{\Gamma} \cup \tilde{E}$ is also polynomially convex and the same is true for all isotopies of that set. We get a holomorphic automorphism $\varphi \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$ with

(a') $\sup_{z\in K} |\varphi(z) - z| < \frac{\varepsilon}{2}$
(b') $\varphi(c_i) = c_i \quad i = 1, 2, \ldots, l$
(c') $\varphi(\tilde{\Gamma} \cup \tilde{E}) \subset \mathbb{C}^2 \setminus \overline{B}_R$

To achieve (b') either correct the approximating automorphism by suitable shears which are small on a certain ball containing the whole situation or working in the proof of the Andersen- Lempert theorem with the geometric structure of vector fields vanishing on a finite number of points in $\mathbb{C}^2$.

We will correct this automorphism φ, which could move points from $\Gamma \cup E$ into B_R which have not been there before, by precomposing it with a shear. For that set $\Gamma E_R = \{z \in \Gamma \cup E : \varphi(z) \in B = B_R\}$. By assumption the complement of $\overline{\Delta}_R \cup \pi_1(\Gamma \cup E)$ does not contain any bounded component and by construction $\pi_1(\Gamma E_R) \subset \pi_1(\Gamma \cup E) \setminus \Delta_R$.

Now define a Mergelyan setting like in the proof of lemma 1 in [6] on $\overline{\Delta}_R \cup \pi_1(\Gamma \cup E)$ to construct a shear automorphism s of $\mathbb{C}^2$ of the form $s(z, w) = (z, w + f(z))$ which removes ΓE_R from the compact $\varphi^{-1}(B_R)$ not bringing new points from the set E into $\varphi^{-1}(B_R)$. To achieve this last property the crucial facts are first that the limit set of the sequence E is contained in Γ and second that the projection π_1 restricted to $\Gamma \cup E$ is proper. Finally observe that in the approximating function f from Mergelyan's theorem can be chosen to be zero at the finite number of points $\pi_1(c_i) \quad i = 1, 2, \ldots, l$ contained in Δ_R (where f approximates zero). Finally, let $\psi = \varphi \circ s$. □

We shall refer to K, $F = \{c_1, \cdots, c_l\}$, B, Γ, E as data for the lemma.

Lemma 2.3 *Given a polynomially convex compact set $M \subset \mathbb{C}^N$ and a finite or countably infinite set $E \subset \mathbb{C}^N \setminus M$ such that $M \cup E$ is compact. Then $M \cup E$ is polynomially convex.*

Proof Let $z \in \mathbb{C}^N$ be an arbitrary point in the complement of $M \cup E$. Choose a polynomially convex compact neighborhood $\tilde{M}$ of M which contains M in its interior but does not contain the point z. Observe that $E \setminus \tilde{M}$ consists of finitely many points.

Let $f, g \in \mathcal{O}(\mathbb{C}^N)$ be a holomorphic functions with

$$f(z) = 1 \quad \sup_{w \in \tilde{M}} |f(w)| < \frac{1}{2}$$

and

$$g(z) = 1 \quad g(w) = 0 \forall w \in E \setminus \tilde{M}.$$

Then $h = f^n g$ satisfies $1 = h(z) > \sup_{w \in M \cup E} |h(w)|$ for n sufficiently big. Thus z does not belong to the polynomial convex hull of $M \cup E$. □

Definition 2.4 An open Riemann surface $X \subset \mathbb{C}^2$ together with a discrete sequence without repetition $A = \{a_j\} \subset X$ are called *suitable* if X is a bordered submanifold of $\mathbb{C}^2$ such that ∂X is a collection $\partial_1, \cdots, \partial_m$ of unbounded smooth curves, and (Γ, A) satisfies the nice projection property, where $\Gamma = \bigcup_{i=1}^m \partial_i$.

Here is a lemma on polynomial convexity that is needed for our main lemma:

Lemma 2.5 *Let $X \subset \mathbb{C}^2$ be a bordered Riemann surface with unbounded boundary components $\partial_1, \cdots, \partial_m$. Then there is an exhaustion M_j of X by polynomially convex compact sets such that if $K \subset \mathbb{C}^2 \setminus \partial X$ is polynomially convex and $K \cap X \subset M_i$ for some i, then $K \cup M_i$ is polynomially convex.*

Proof This follows from (the proof of) Proposition 3.1 of [7]. □

Lemma 2.6 *Let $X \subset \mathbb{C}^2$ be an open Riemann surface and $A = \{a_j\} \subset X$ a discrete sequence without repetition which are suitable. Let $\{b_j\} \subset \mathbb{C}^2$ be a discrete sequence without repetition. Let $B \subset B' \subset \mathbb{C}^2$ be closed balls such that $\Gamma \cap B' = \varnothing$ and let $L = X \cap B'$. Assume that if $b_j \in B \cup L$ then $b_j = a_j$ and if $b_j \notin B \cup L$ then $a_j \notin B'$, i.e. $a_j \in X \setminus L$. Given $\varepsilon > 0$ and a compact set $K \subset X$, there exist a ball $B'' \subset \mathbb{C}^2$ containing B' (B'' may be chosen as large as desired), a compact polynomially convex set $M \subset X$ with $L \cup K \subset M$, and a holomorphic automorphism θ of $\mathbb{C}^2$ satisfying the following properties*:

(i) $|\theta(z) - z| < \varepsilon$ *for all* $z \in B \cup L$,
(ii) *if* $a_j \in M$ *for some index* j *then* $\theta(a_j) = b_j \in B''$,
(iii) *If* $b_j \in B' \cup (\theta(X) \cap B'')$, *then* $a_j \in M$
(iv) $\theta(M) \subset \mathrm{Int} B''$
(v) *if* $a_j \in X \setminus M$ *for some* j *then* $\theta(a_j) \notin B''$.
(vi) $\theta(\Gamma) \cap B'' = \varnothing$

Remark 2.7 This is the fundamental inductive step of the construction. Notice that the lemma states that the geometric situation is preserved after applying θ, i.e. that if X, A, B, B', Γ are replaced by $\theta(X)$, $\theta(A)$, B', B'', $\theta(\Gamma)$, then the hypotheses of the lemma still hold.

Proof An automorphism θ with the required properties will be constructed in two steps, $\theta = \psi \circ \varphi$.

By lemma 2.5 there is a polynomially convex compact set $M \subset X$ such that $L \cup K \cup \{a_j; b_j \in B'\} \subset M$ and $B \cup M$ is polynomially convex.

Since $B \cup L$ is also polynomially convex, by (repeated application of) Proposition 2.1 of [3] there is an automorphism φ such that

(a) $|\varphi(z) - z| < \frac{\varepsilon}{2}$ for all $z \in B \cup L$
(b) $\varphi(a_j) = b_j$ for all $a_j \in M$

Now, $\varphi(B \cup M)$ is polynomially convex and if $E' = \{a_j; a_j \notin M\}$, then $(\varphi(\Gamma), \varphi(E'))$ has the nice projection property. Let B'' be a large ball containing $\varphi(B \cup M) \cup B'$. By lemma 2.2, applied to the data $\varphi(B \cup M)$, $F = \{b_j; a_j \in M\}$, B'', $\varphi(\Gamma)$, $\varphi(E')$, there is an automorphism ψ satisfying the following:

(a') $|\psi(w) - w| < \frac{\varepsilon}{2}$ when $w \in \varphi(B \cup M)$,
(b') $\psi(b_j) = b_j$ for all $b_j \in F$
(c') $\psi(\varphi(\Gamma) \cup \varphi(E')) \subset \mathbb{C}^2 \setminus B''$

Let $\theta = \psi \circ \varphi$. (i) follows from (a) and (a'). (ii) follows from (b) and (b'). (iv) follows from (a') and the definition of B''. (v) follows from (c') and the definition of E'. (vi) follows from (c').

To prove (iii), notice that if $b_j \in B'$, then $a_j \in M$ by the definition of M. Let $F' = \{b_j \in B''\} \supset F$. If $b_j \in \theta(X) \setminus F$, then there is a shear s such that s is close to the identity on B'', $s(b_i) = b_i$ for all $b_i \in F \setminus \{b_j\}$ and such that $s \circ \theta(X)$ avoids b_j. Replacing θ by $s \circ \theta$ does not destroy the other properties. Hence we may assume that $\theta(X)$ avoids b_j and therefore all points in $F' \setminus F$. This implies (iii). □

Theorem 2 *Let X be an open Riemann surface and $A = \{a_j\} \subset X$ a discrete sequence without repetition which are suitable. Let $\{b_j\} \subset \mathbb{C}^2$ be a discrete sequence without repetition. Then there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^2$ satisfying $f(a_j) = b_j$ for $j = 1, 2, \ldots$.*

Proof Choose an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ by compact sets. Fix a number ε with $0 < \varepsilon < 1$. We shall inductively construct the following:

(a) a sequence of holomorphic automorphisms Φ_k of $\mathbb{C}^2$,
(b) an exhaustion $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} L_j = X$ by compact, polynomially convex sets
(c) a sequence of balls $B_1 \subset B_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} B_j = \mathbb{C}^2$ centered at $0 \in \mathbb{C}^2$ whose radii satisfy $r_{k+1} \geq r_k + 1$ for $k = 1, 2, \ldots$,

such that the following hold for all $k = 1, 2, \ldots$ (conditions (iv) and (v) are vacuous for $k = 1$):

(i) $\Phi_k(L_k) = \Phi_k(X) \cap B_{k+1}$,
(ii) if $a_j \in L_k$ for some j then $\Phi_k(a_j) = b_j$,
(iii) if $b_j \in \Phi_k(L_k) \cup B_k$ for some j then $a_j \in L_k$ and $\Phi_k(a_j) = b_j$,
(iv) $L_{k-1} \cup K_{k-1} \subset \mathrm{Int} L_k$,
(v) $|\Phi_k(z) - \Phi_{k-1}(z)| < \varepsilon\, 2^{-k}$ for all $z \in B_{k-1} \cup L_{k-1}$.
(vi) $\Phi_k(\Gamma) \subset \mathbb{C}^2 \setminus B_{k+1}$

To begin we set $B_0 = \varnothing$, $L_0 = \varnothing$ and $K_0 = \varnothing$ and choose a pair of balls $B_1 \subset B_2 \subset \mathbb{C}^2$ of radii 1 and 2. Let $A_0 = \{a_j; b_j \in B_2\}$. There is an automorphism φ such that $\varphi(a_j) = b_j$

for all $a_j \in A_0$ (for instance by Proposition 2.1 of [3]). Taking $c_i = b_i$ for $b_i \in B_2$, $E = \varphi(A \setminus A_0)$ and replacing Γ by $\varphi(\Gamma)$ in lemma 2.2, there is an automorphism ψ such that, if we let $\Phi_1 = \psi \circ \varphi$, we have $\Phi_1(a_j) = b_j$ for all those (finitely many) indices j for which $b_j \in B_2$, $\Phi_1(a_j) \in \mathbb{C}^2 \setminus B_2$ for the remaining indices j, and $\Phi_1(\Gamma) \subset \mathbb{C}^2 \setminus B_2$. Setting $L_1 = \{z \in X : \Phi_1(z) \in B_2\}$, the properties (i), (ii), (iii) and (vi) are satisfied for $k = 1$ and the remaining two properties (iv), (v) are void.

Assume inductively that we have already found sets $L_1, \ldots, L_k \subset X$, balls $B_1, \ldots, B_{k+1} \subset \mathbb{C}^2$ and automorphisms $\Phi_1, \ldots, \Phi_k$ such that (i)–(vi) hold up to index k. We now apply lemma 2.6 with $B = B_k$, $B' = B_{k+1}$, X replaced by $X_k = \Phi_k(X) \subset \mathbb{C}^2$, Γ replaced by $\Phi_k(\Gamma)$, A by $\Phi_k(A)$, K by $\Phi_k(K)$ and $L = \Phi_k(L_k) \subset X_k$. This gives us a compact polynomially convex set $M = M_k \subset X_k$ containing $\Phi_k(K_k \cup L_k)$, an automorphism $\theta = \theta_k$ of $\mathbb{C}^2$, and a ball $B'' = B_{k+2} \subset \mathbb{C}^2$ of radius $r_{k+2} \geq r_{k+1} + 1$ such that the conclusion of lemma 2.6 holds. In particular, $\theta_k(M_k) \subset B_{k+2}$, the interpolation condition is satisfied for all points $b_j \in \theta_k(M_k) \cup B_{k+1}$, and the remaining points in the sequence $\{\Phi_k(a_j)\}_{j \in \mathbb{N}}$ together with the curves $\Phi_k(\Gamma)$ are sent by θ_k out of the ball B_{k+2}. Setting

$$\Phi_{k+1} = \theta_k \circ \Phi_k, \quad L_{k+1} = \{z \in X : \Phi_{k+1}(z) \in B_{k+2}\}$$

one easily checks that the properties (i)–(vi) hold for the index $k+1$ as well. (Note that L_{k+1} corresponds to the set L' in remark 2.7). The induction may now continue.

Let $\Omega \subset \mathbb{C}^2$ denote the set of points $z \in \mathbb{C}^2$ for which the sequence $\{\Phi_k(z) : k \in \mathbb{N}\}$ remains bounded. Proposition 5.2 in [3] (p. 108) implies that $\lim_{k \to \infty} \Phi_k = \Phi$ exists on Ω, the convergence is uniform on compacts in Ω, and $\Phi : \Omega \to \mathbb{C}^2$ is a biholomorphic map of Ω onto $\mathbb{C}^2$ (a Fatou-Bieberbach map). In fact, $\Omega = \bigcup_{k=1}^{\infty} \Phi_k^{-1}(B_k)$ (Proposition 5.1 in [3]). >From (v) we see that $X \subset \Omega$, from (vi) it follows that $\Gamma \cup \Omega = \varnothing$ i.e. X is a closed subset of Ω implying that Φ restricted to X gives a proper holomorphic embedding into $\mathbb{C}^2$. Properties (ii), (iii) imply the interpolation condition $\Phi(a_j) = b_j$ for all $j = 1, 2, \ldots$. This completes the proof of the theorem. □

We may now prove the following generalization of Theorem 1 in [7]:

Theorem 3 *Let $X \subset \mathbb{C}^2$ be a Riemann surface whose boundary components are smooth Jordan curves $\partial_1, \ldots, \partial_m$. Assume that there are points $p_i \in \partial_i$ such that*

$$\pi_1^{-1}(\pi_1(p_i)) \cap \overline{X} = p_i.$$

Assume that $\overline{X}$ is a smoothly embedded surface, and that all p_i are regular points of the projection π_1. If in addition $\pi_1(\overline{X}) \subset \mathbb{C}$ is bounded, then X embeds into $\mathbb{C}^2$ with interpolation.

Proof Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ be constants and define the following rational map $F : \mathbb{C}^2 \to \mathbb{C}^2$:

$$F(x, y) = \left(x, y + \sum_{j=1}^{m} \frac{\alpha_j}{x - p_j} \right).$$

Let Γ denote $\partial F(X)$. It is not hard to see that the constants α_j can be chosen such that the Γ satisfies the conditions on the curves in the definition of the nice projection property. Let $A = \{a_j\} \subset X$ be a discrete sequence without repetition. Since $\pi_1(X)$ is bounded it follows the π_2 is proper when restricted to $\Gamma \cup A$ so the pair (Γ, A) has the nice projection property. The result follows from Theorem 2. □

Proof of Theorem 1 We start by proving (1): Let X be a finitely connected planar domain, and let $x_1, \ldots, x_k$ be the complementary components of X consisting of isolated points (if such components exist). Let $g\colon X \hookrightarrow \mathbb{C}^2$ be the embedding

$$g(x) := \left(x, \sum_{j=1}^{k} \frac{1}{x - x_i}\right).$$

If there are no other complementary components than the points x_i then $(\partial g(X), A)$ has the nice projection property for any discrete sequence $A = \{a_j\} \subset X$ (project onto the plane $x = y$). If there are more complementary components we may assume that X is a circled subset of the unit disk, and it is clear that $g(X)$ satisfies the condition in Theorem 3.

To prove (2) let $S_1, \ldots, S_m \subset \mathbb{C}$ be smooth compact slits with an endpoint q_i for each curve, let L denote the closed negative real axis, and let $\{x_j\} \subset \mathbb{C}$ be a discrete sequence without repetition. Let these sets be pairwise disjoint. We may assume that X is of one of the following two types

(a) $X = \mathbb{C}\setminus(\cup_{i=1}^m S_m \cup \{x_j\})$,

(b) $X = \mathbb{C}\setminus(\cup_{i=1}^m S_m \cup \{x_j\} \cup L)$,

and that all S_i are contained in $\triangle$. Let $A = \{a_j\} \subset X$ be a discrete sequence without repetition. We want to construct an embedding $f\colon X \hookrightarrow \mathbb{C}^2$ such that the boundary $\Gamma = \partial f(X)$ satisfies the conditions on the curves in the definition of the nice projection property with projection on the plane $x = y$, and such that $f(A)$ is contained in the set

$$D := \left\{(x, y) \subset \mathbb{C}^2; |x| \le 1 \text{ or } |y| \le 1 \text{ or } |y| \ge 2|x|\right\}.$$

In that case we see that the projection onto the plane $x = y$ is proper when restricted to $\Gamma \cup f(A)$.

We will define f as a mapping on the form $f(\zeta) = (\zeta, h(\zeta) + g(\zeta))$ with $h(\zeta) = \sum_{j=1}^m \frac{\alpha_j}{\zeta - q_j}$, $g(\zeta) = \sum_{j=1}^\infty (\frac{\beta_j}{\zeta - x_j})^{N_j}$ for some choice of constants $\alpha_j, \beta_j \in \mathbb{C}$, $N_j \in \mathbb{N}$.

If $\varepsilon > 0$ is small enough we have that generic choices of $\alpha_j \in \triangle_\varepsilon$ gives that the map $\zeta \mapsto (\zeta, h(\zeta))$ maps X onto a surface with a nice projection of the boundary curves onto the plane $x = y$. If we choose g such that $\|g\|_{\mathcal{C}^1(\partial X \setminus \{x_j\})}$ is small then f will have the same property. If ε is small we also have $|h(a_j)| < \frac{1}{2}$ for all $a_j \in \mathbb{C}\setminus\overline{\triangle}$.

For each $j \in \mathbb{N}$ choose $\beta_j > 0$ such that the disks $\overline{\triangle}_j := \overline{\triangle}_{\beta_j}(x_j)$ are pairwise disjoint and such that $\overline{\triangle}_{\beta_j}(x_j) \cap \partial X \setminus x_j = \varnothing$ for all $j \in \mathbb{N}$. Make sure that $\partial \triangle_{\beta_j}(x_j) \cap A = \varnothing$ for each j. Since $(\frac{\beta_j}{\zeta - x_j})^N \to \infty$ as $N \to \infty$ on $\triangle_j$ and $(\frac{\beta_j}{\zeta - x_j})^N \to 0$ as $N \to \infty$ on $\mathbb{C}\setminus\overline{\triangle}_j$ it is clear that we may choose the sequence N_j such that $|g(a_i)| \ge 3|a_i|$ if $a_i \in \triangle_j$ for some j and $g(a_i) < \frac{1}{2}$ otherwise. For any choice of $\delta > 0$ we may also choose the N_j's such that $\|g\|_{\mathcal{C}^1(\partial X \setminus \{x_j\})} < \delta$. If δ is small enough then $\partial f(X)$ has a nice projection onto the plane $x = y$ and $f(A) \subset D$.

To prove (3) let $\lambda \in \mathbb{C}$ be contained in the upper half plane and let ϱ_λ be the Weierstrass $p-$function:

$$\varrho_\lambda(\zeta) := \frac{1}{\zeta^2} + \sum_{m,n \in \mathbb{N}^2 \setminus \{0\}} \frac{1}{(\zeta - (m + n\lambda))^2} - \frac{1}{(m + n\lambda)^2}.$$

If $2p \in \mathbb{C}$ is not contained in the lattice $L_\lambda := \{\zeta \in \mathbb{C}; \zeta = m + \lambda n\}$ we have that the map

$$\varphi_p(\zeta) := (\varrho(\zeta), \varrho(\zeta - p)),$$

determines is a proper holomorphic embedding of $\mathbb{T}_\lambda \setminus \{[0], [p]\}$ into $\mathbb{C}^2$, where $\mathbb{T}_\lambda$ is the torus obtained by dividing out $\mathbb{C}$ by the lattice group L_λ, and $[0]$ and $[p]$ are the equivalence classes of the points 0 and p (see [7] for details). We treat three different cases: Assume first that the complementary components of X are two distinct points, i.e. X is some quotient $\mathbb{T}_\lambda$ with two points $[x_1]$ and $[x_2]$ removed. By a linear change of coordinates on $\mathbb{C}$ we may assume that $x_2 = -x_1$ and $2x_1 \notin L_\lambda$. Then φ_{x_1} is a proper embedding of $\mathbb{T}_\lambda \setminus \{[0], [x_1]\}$ into $\mathbb{C}^2$. Now $\varrho_\lambda(x_1) = \varrho_\lambda(x_2) = q \in \mathbb{C}$ so we may chose a Möbius transformation $m : \widehat{\mathbb{C}} \hookrightarrow \widehat{\mathbb{C}}$ such that $m(q) = \infty$. We get that the map

$$f(\zeta) = (f_1(\zeta), f_2(\zeta)) = (m \circ \varrho_\lambda(\zeta), \varrho_\lambda(\zeta - x_1))$$

determines a proper embedding of $X = \mathbb{T} \setminus \{[x_1], [x_2]\}$ into $\mathbb{C}^2$. Moreover since $\lim_{\zeta \to x_j} f_1(\zeta) = \infty$ for $j = 1, 2$ we have that the pair $(\partial f(X), f(A))$ has the nice projection property for all discrete sequences $A = \{a_j\} \subset X$. Next assume that the complementary components K_1 and K_2 of X are not both points. If neither of them are points the result follows from (4) so we may assume that K_1 is the point [0] and that K_2 is not a point. We may then assume that K_2 is a smoothly bounded disk in $\mathbb{T}_\lambda$ and by choosing $[p] \in K_2$ appropriately one sees that the map φ_p embeds X onto a surface in $\mathbb{C}^2$ satisfying the conditions in Theorem 3 above after the coordinate change $(x, y) \mapsto (y, x)$ (see [7] for more details).

To prove (4) we recall from Theorem 1' in [8] that a subset of a torus without isolated points in the boundary embeds onto a surface in $\mathbb{C}^2$ satisfying the conditions in Theorem 3 above.

Next let X be as in (5). Then X can be obtained by removing a finite set $D_1, \ldots, D_m$ of smoothly bounded (topological) disks from a closed Riemann surface $\mathcal{R}$, so $X = \mathcal{R} \setminus \cup_{i=1}^m \overline{D}_i$. There exists a separating pair of inner functions $f, g \in \mathcal{A}(X)$, i.e. f and g separate points on $\overline{X}$ and $|f(x)| = |g(x)| = 1$ for all $x \in \partial X$ [11,15]. Then the map $h := (f, g)$ embeds X properly into the unit polydisk in $\mathbb{C}^2$, and by perturbing the boundary $h(\partial X)$ slightly one obtains a surface satisfying the conditions in Theorem 3 above.

If X is a surface as in (6) we have by the maximum principle that either X is a planar domain or the projection π_1 takes its maximum at a finite set of points $q_1, \ldots, q_s \in \partial X$. By a linear change of coordinates X satisfies the conditions in Theorem 3. □

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