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Pseudocomplemented semilattices are finite-to-finite relatively universal

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To Věra Trnková on the occasion of her 70^{th} *birthday.*

Abstract. It is shown that the category of directed graphs is isomorphic to a subcategory of the variety **S** of all pseudocomplemented semilattices which contains all homomorphisms whose images do not lie in the subvariety **B** of all Boolean pseudocomplemented semilattices. Moreover, the functor exhibiting the isomorphism may be chosen such that each finite directed graph is assigned a finite pseudocomplemented semilattice. That is to say, it is shown that the variety **S** of all pseudocomplemented semilattices is finite-to-finite **B**-relatively universal.

This illustrates the complexity of the endomorphism monoids of pseudocomplemented semilattices since it follows immediately that, for any monoid M, there exists a proper class of non-isomorphic pseudocomplemented semilattices such that, for each member S, the endomorphisms of S which do not have an image contained in the skeleton of S form a submonoid of the endomorphism monoid of S which is isomorphic to M.

1. Introduction

For a class **K** of algebras of similar type, let **H**(**K**), **S**(**K**), and **P**(**K**) respectively denote the classes of all homomorphic images, subalgebras, and products of algebras in **K**. A class **K** is a *variety* provided **K** = **HSP**(**K**), which, by a classical result of Birkhoff [5], is equivalent to being an equational class.

A *pseudocomplemented semilattice* $(S; \wedge, *, 0, 1)$ is an algebra where $(S; \wedge)$ is a semilattice with a least element 0, a greatest element 1, and a unary operation $*$ such that, for all $s, t \in S$, $s \wedge t = 0$ if and only if $t \leq s^*$. The class of pseudocomplemented semilattices is a variety, see Frink [9]. Further, as established by Jones [14] (see also Sankappanavar [26]), the lattice of all subvarieties of pseudocomplemented semilattices ordered by inclusion is a 3-element chain consisting of the trivial variety **T** of all 1-element algebras, the variety **B** (determined by the identity $x = x^{**}$) of Boolean pseudocomplemented semilattices (where $x \vee y = (x^* \wedge y^*)^*$), and the variety **S** of all pseudocomplemented semilattices.

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A variety $\mathbf{V}$ of algebras is *universal* if every category of algebras of finite type (or, equivalently, as shown by Pultr [23], Hedrlín and Pultr [13], and Vopěnka, Hedrlín, and Pultr [31], the category $\mathbf{G}$ of all connected directed graphs together with all compatible mappings) is isomorphic to a full subcategory of $\mathbf{V}$. If an embedding of $\mathbf{G}$ may be effected by a functor $\mathbf{\Phi}\colon \mathbf{G} \longrightarrow \mathbf{V}$ which assigns a finite algebra to each finite graph, then $\mathbf{V}$ is said to be *finite-to-finite* universal. A number of examples, as well as properties, of universal varieties are already known (see, for example, Pultr and Trnková [24].) In particular, if $\mathbf{V}$ is universal, then, for every monoid M, there exists a proper class of non-isomorphic algebras belonging to $\mathbf{V}$ each of which has an endomorphism monoid isomorphic to M. If $\mathbf{V}$ is finite-to-finite universal, then, in addition, for a finite monoid M, there exist infinitely many non-isomorphic finite algebras in $\mathbf{V}$ with the preceding property.

The variety $\mathbf{S}$ of all pseudocomplemented semilattices is not universal since, for any pseudocomplemented semilattice S, the mapping $\gamma_S\colon S \longrightarrow S$ given by $\gamma_S(x) = x^{**}$ is an endomorphism (referred to as the *Glivenko endomorphism*) onto the *skeleton* S^* of S, where $S^* = \{x^* : x \in S\}$ is Boolean and belongs to the subvariety $\mathbf{B}$. In particular, if S is not Boolean, then it has a non-trivial endomorphism onto its skeleton S^*. Furthermore, the endomorphism monoid of S then has at least as many endomorphisms as the skeleton (which is non-trivial whenever the Boolean skeleton is). It follows that every non-trivial pseudocomplemented semilattice has a non-trivial endomorphism. Since, by the above remarks, any universal variety contains a proper class of non-isomorphic algebras each of which has a trivial endomorphism monoid, it follows that the variety $\mathbf{S}$ of all pseudocomplemented semilattices is not universal.

On the other hand, as shown in [3], there does exist a proper class of non-isomorphic pseudocomplemented semilattices for each of which the identity is the only endomorphism which does not have an image contained in the respective skeleton (although, there is no bound on the cardinalities of the skeletons of the pseudocomplemented semilattices in this class). The situation is reminiscent of that for the variety of idempotent semigroups.

Universal varieties of semigroups have been completely characterised by Koubek and Sichler [18]. However, since any non-trivial idempotent semigroup has at least as many non-trivial endomorphism as there are elements in the semigroup, arguing as above, the variety of idempotent semigroups is not universal. This led Demlová and Koubek to introduce a notion of relatively universal. A variety $\mathbf{V}$ is *relatively universal* to a subvariety $\mathbf{W}$ (or, briefly, $\mathbf{W}$-*relatively universal*), providing $\mathbf{G}$ is isomorphic to a subcategory of $\mathbf{V}$ whose morphisms consist of all those homomorphisms whose images does not lie in the variety $\mathbf{W}$. In the course of a remarkable series of papers [6], [7], and [8], Demlová and Koubek completely determine which varieties of idempotent semigroups are relatively universal. (We remark that precursors to Demlová and Koubek's notion of **W**-relatively universal date back as far as Sichler [30].)

Our principal result is the following, which, since Boolean algebras with isomorphic endomorphism monoids are isomorphic (Magill [20], Maxson [21], and Schein [27]), is sharp.

Theorem 1.1. *The variety* **S** *of all pseudocomplemented semilattices is finite-to-finite* **B***-relatively universal, where* **B** *is the subvariety of all Boolean pseudocomplemented semilattices.*

An immediate consequence of Theorem 1.1 is a strengthening of the aforementioned result from [3], namely, for any monoid M, there exists a proper class of non-isomorphic pseudocomplemented semilattices such that, for each member S, the endomorphisms of S which do not have an image contained in S^* form a submonoid of the endomorphism monoid of S which is isomorphic to M. In passing, we mention that Theorem 1.1 was first conjectured to hold at the time of [3], but we were unable to prove it.

Although the primary objective here is a better understanding of pseudocomplemented semilattices and, in particular, of their endomorphisms, we mention a topical related notion.

A class $\mathbf{K}$ of algebras of similar type is a *quasivariety* provided $\mathbf{K} = \mathbf{ISPP_u}(\mathbf{K})$, where $\mathbf{I}(\mathbf{K})$ and $\mathbf{P_u}(\mathbf{K})$ respectively denote the classes of all isomorphic images and ultraproducts of algebras in $\mathbf{K}$. Every variety is a quasivariety, but not every quasivariety is a variety.

For a quasivariety $\mathbf{Q}$, let $L(\mathbf{Q})$ denote the lattice (ordered by set inclusion) of all quasivarieties contained in $\mathbf{Q}$. As defined by Sapir (see, for example, Gorbunov [10]), a variety $\mathbf{V}$ is *Q-universal* providing that, for any quasivariety $\mathbf{Q}$ of finite type, $L(\mathbf{Q})$ is a homomorphic image of a sublattice of $L(\mathbf{V})$. Amongst the most noteworthy properties of a Q-universal variety $\mathbf{V}$ are the facts that a free lattice on ω free generators is embeddable in $L(\mathbf{V})$ (hence, $L(\mathbf{V})$ fails to satisfy any non-trivial lattice identity) and $|L(\mathbf{V})| = 2^\omega$.

In [1], it was shown that every finite-to-finite universal variety is Q-universal. It is not known whether every finite-to-finite **W**-relatively universal variety **V** is Q-universal (see, for example, Koubek and Sichler [19]). It is in this context that we note Theorem 1.1 shows **S** is finite-to-finite **B**-relatively universal whilst, in [2], it was shown that **S** is Q-universal. That is, pseudocomplemented semilattices are an example in support of the existing conjecture that every finite-to-finite **W**-relatively universal variety **V** is Q-universal.

Returning to Theorem 1.1, to begin the proof we need to find a suitable pseudocomplemented semilattice that will act as a basic component in a *šíp*-type construction (see, for example, Mendelsohn [22] for a lucid general discussion of this technique). In [3], for each undirected connected graph, a pseudocomplemented semilattice was constructed. This appeared to be a promising source to find such algebras. Ultimately this proved inadequate for our purposes. Consequently, in §2.1 we give a new construction which associates a pseudocomplemented semilattice with every finite undirected graph. Then, in §2.2, we choose one such algebra in particular, to be denoted M.

To establish the universality of a variety, instead of all directed graphs it is sufficient to find a full embedding of the category $\mathbf{G_c}$ of all connected directed graphs which are (i) *strongly loopless* (that is, for vertices u and v, it is never the case that both (u, v) and (v, u) are edges), (ii) for every vertex v, there are edges (u, v) and (v, w), and (iii) considered as undirected graphs, are triangle-free (that is, they do not contain a subgraph isomorphic to K_3, the complete graph on 3 vertices). These properties are required for technical reasons in order to simplify the many constructions to follow. With this in mind, for each $G = (V; E)$ in $\mathbf{G_c}$, we will consider the $\mathbf{S}$-free product $\coprod_{\mathbf{S}}(M_e : e \in E)$ where, for $e \in E$, M_e denotes an isomorphic copy of M, a particular pseudocomplemented semilattice to be specified in §2.2. We then define a suitable congruence Θ over $\coprod_{\mathbf{S}}(M_e : e \in E)$ to obtain the *G-reduced free product* $S_G = (\coprod_{\mathbf{S}}(M_e : e \in E))/\Theta$. The desired functor $\mathbf{\Phi}$ will be determined by $\mathbf{\Phi}(G) = S_G$. The precise definition of $\mathbf{\Phi}$ will be given in §3.

We remark that the use of quotients of free products of algebras which, in some sense, forces them to act like graphs is not new. For example, in [11], Grätzer and Sichler consider a suitably defined quotient of the free $(0, 1)$-lattice generated by V for every triangle-connected undirected graph $G = (V; E)$, a particular instance of a so called *$\mathcal{C}$-reduced free product*.

Thus, having defined the functor $\mathbf{\Phi}$ in §3, two problems lie ahead. One is to unmask enough characteristics of G-reduced free products that it is possible to show that, with respect to their homomorphims, they mimic the compatible mappings of the corresponding graphs. The other is to do so without needing to solve the word problem in the process, and thereby avoid all that this would entail. As will be seen, we do so by the skin of our teeth.

It is to these ends that, in §4, we begin by first finding relevant properties of the skeletons of G-reduced free products. Since each of these is a quotient of a free products of Boolean algebras, our approach will be to use Stone's topological duality for Boolean algebras.

In §5, we return from the topological setting to essentially an algebraic one. Two constructions are presented which will be used in the proof of Theorem 1.1

as testing pseudocomplemented semilattices, thereby enabling us to side step the need to find a complete solution of the word problem for G-reduced free products.

In §6, with the information gleaned from §4 and §5, we proceed to show that, with respect to homomorphisms, G-reduced free products mimic their graphs, thereby completing the proof of Theorem 1.1.

Finally, we conclude with a tantalizing problem in §7.

2. Preliminaries

2.1. The basic construction. An *undirected graph* $G = (V; E)$ is a set V of *vertices* together with a set E of *edges* the members of which are 2-element subsets of V.

The immediate goal of this section is to associate a pseudocomplemented semilattice $S(G)$ to each finite undirected graph G with 4 or more elements (that G has 4 or more elements will not actually be needed until Lemma 2.6).

Recall that, for a pseudocomplemented semilattice S, the endomorphism $\gamma_S \colon S \longrightarrow S$ given by $\gamma(x) = x^{**}$ is referred to as the *Glivenko endomorphism.* The *Glivenko congruence* Γ_S is the congruence on S induced by γ_S, that is $\Gamma_S = \{(x, y) \in S \times S : x^* = y^*\}$. Accordingly, its congruence classes are called *Glivenko classes*. A Glivenko class is *trivial* iff it is a singleton. Note that S/Γ_S is isomorphic to the *skeleton* $S^* = \{x \in S : \ x = x^{**}\} = \{x \in S : \ x = y^* \text{ for some } y \in S\}$ of S.

For a finite undirected graph $G = (V; E)$, let $B(G)$ denote the Boolean lattice whose elements are all subsets of V ordered by inclusion. The pseudocomplemented semilattice $S(G)$ to be associated with G will have a copy of $B(G)$ as its skeleton, and prescribed Glivenko classes as follows: The classes of $\emptyset$ and V are trivial, if $|A| = 1$ or $A \in E$, its class is the two-element chain, and for all other A its class is a copy of the Boolean lattice of all subsets of $V \setminus A$.

More formally, let $S(G)$ be the set of all pairs $(A, B) \in B(G) \times B(G)$ satisfying one the four following mutually exclusive conditions:

(i) $A = B = \emptyset$. (ii) $|A| = 1$, and $B \in \{\emptyset, V\}$.

(iii) $|A| = 2$, $A \in E$, and $B \in \{\emptyset, V\}$. (iv) $|A| \geq 2$, $A \notin E$, and $A \subseteq B$.

and let $\leq$ denote the restriction of the order on $B(G) \times B(G)$ to $S(G)$.

Lemma 2.1. *$(S(G); \wedge)$ is a semilattice such that, for $(A, B), (C, D) \in S(G)$,*

$$(A,B) \wedge (C,D) = \begin{cases} (A \cap C, B \cap D) & \text{if } |A \cap C| \geq 2 \text{ and } A \cap C \notin E, \\ (A \cap C, V) & \text{if } A \cap C \in E \text{ and } B = D = V, \\ (A \cap C, \emptyset) & \text{if } A \cap C \in E \text{ and } B \cap D \subset V, \\ (A \cap C, V) & \text{if } A \cap C = \{x\} \text{ and } B = D = V, \\ (A \cap C, \emptyset) & \text{if } A \cap C = \{x\} \text{ and } B \cap D \subset V, \\ (\emptyset, \emptyset) & \text{if } A \cap C = \emptyset. \end{cases}$$

Proof. Since $(B(G); \subseteq)$ is a poset, so too is $(S(G); \leq)$.

Let $(A, B), (C, D) \in S(G)$. Since $(\emptyset, \emptyset) \in S(G)$, it is always the case that a lower bound for (A, B) and (C, D) exists. Suppose that $(R, S) \in S(G)$ is a lower bound of (A, B) and (C, D). In particular, $R \subseteq A \cap C$ and $S \subseteq B \cap D$.

If $|A \cap C| \geq 2$ and $A \cap C \notin E$, then, since $B \cap D \supseteq A \cap C$, $(A \cap C, B \cap D) \in S(G)$ and $(A, B) \wedge (C, D) = (A \cap C, B \cap D) \in S(G)$.

If $A \cap C = \{x, y\} \in E$, then either $B = D = V$ or $B \cap D \subset V$. If $B = D = V$, then $B \cap D = V$, $(A \cap C, B \cap D) \in S(G)$, and $(A, B) \wedge (C, D) = (A \cap C, V) \in S(G)$. If $B \cap D \subset V$, then either $R = \{x, y\}$, $\{x\}$, $\{y\}$, or $\emptyset$. In each case, since $(R, S) \in S(G)$, $S = \emptyset$ and, in particular, $(R, S) \leq (\{x, y\}, \emptyset)$. That is, $(A, B) \wedge (C, D) = (A \cap C, \emptyset)$.

Suppose $A \cap C = \{x\}$. If $B = D = V$, it follows that $(A, B) \wedge (C, D) = (A \cap C, V)$. Otherwise, since $B \cap D \subset V$ and $R = \{x\}$ or $\emptyset$, it again follows that $S = \emptyset$, giving $(A, B) \wedge (C, D) = (A \cap C, \emptyset)$.

Finally, if $A \cap C = \emptyset$, then $R = S = \emptyset$ and $(A, B) \wedge (C, D) = (\emptyset, \emptyset)$. □

Lemma 2.2. *$(S(G); \wedge, *, (\emptyset, \emptyset), (V, V))$ is a pseudocomplemented semilattice where $(V, V)^* = (\emptyset, \emptyset)$ and, for $(V, V) \neq (A, B) \in S(G)$, $(A, B)^* = (A^*, V)$. Hence, the skeleton $S(G)^*$ of $S(G)$ is isomorphic to $B(G)$.*

Proof. Then, $(V, V)^* = (\emptyset, \emptyset)$. If $(V, V) \neq (A, B) \in S(G)$, then it follows that $A \subset V$. Thus, $A^* \neq \emptyset$ and, in particular, $(A^*, V) \in S(G)$. By Lemma 2.1, $(A, B) \wedge (A^*, V) = (\emptyset, \emptyset)$. Further, by Lemma 2.1, if $(A, B) \wedge (C, D) = (\emptyset, \emptyset)$, then $A \cap C = \emptyset$. Thus, $C \subseteq A^*$ and, hence, $(C, D) \leq (A^*, V)$, as required. Finally, the isomorphism between $S(G)^*$ and $B(G)$ is given by sending $(A, V) \in S(G)$ to $A \in B(G)$. □

Having established that $S(G)$ is a pseudocomplemented semilattice for any undirected graph G, we now consider properties of $S(G)$ and, in particular, endomorphisms of $S(G)$ (see Lemma 2.6).

Lemma 2.3. *$S(G)$ is generated by $\{(V \setminus \{x\}, V \setminus \{x\}) : x \in V\}$.*

Proof. By Lemma 2.2, $(V \setminus \{x\}, V \setminus \{x\})^{**} = (V \setminus \{x\}, V)$, so $S(G)^*$ may be obtained from $\{(V \setminus \{x\}, V \setminus \{x\}) : x \in V\}$ by $\wedge$ and $*$. Since G is finite, for $(A, B) \in S(G)$ and $B \subset V$, $(A, B) = \bigwedge((V \setminus \{x\}, V\}) : x \in V \setminus A) \wedge \bigwedge((V \setminus \{x\}, V \setminus \{x\}) : x \in V \setminus B)$. □

As the following lemma shows, a non-skeletal element is only greater than the zero of the skeleton. This fact will prove crucial in establishing Lemma 2.6 (see Lemma 2.5).

Lemma 2.4. *For (A, B), $(C, D) \in S(G)$, if $(A, B) \geq (C, D)^* \neq (\emptyset, \emptyset)$, then $(A, B) \in S(G)^*$.*

Proof. By Lemma 2.2, $(C, D) \neq (V, V)$, $C \subset V$, and $(C, D)^* = (C^*, V)$. Thus, $\emptyset \neq C^* \subseteq A$ and $V \subseteq B$. Either $A = V$ and $(A, B) = (V, V)$ or else $A \subset V$ and $(A, B)^{**} = (A^*, V)^* = (A^{**}, V) = (A, B)$. Either way, $(A, B) \in S(G)^*$. □

Although Lemma 2.4 need not be true for all quotients of $S(G)$, as the following shows, when it holds we can sometimes conclude that the naturally induced congruence associated with the quotient contains the Glivenko congruence.

Lemma 2.5. *Let Θ be a congruence on $(S(G); \wedge, *, (\emptyset, \emptyset), (V, V))$. If, for some $x \in V$, $(V \setminus \{x\}, V \setminus \{x\}) \equiv (V \setminus \{x\}, V)(\Theta)$, then $\Theta \supseteq \Gamma_{S(G)}$ or, for some $(A, B), (C, D) \in S(G)$, $(A, B) \not\equiv (A, V)(\Theta)$ and $[(A, B)]_\Theta \geq [(C, D)^*]_\Theta \neq [(\emptyset, \emptyset)]_\Theta$.*

Proof. Suppose, for some $x \in V$, $(V \setminus \{x\}, V \setminus \{x\}) \equiv (V \setminus \{x\}, V)(\Theta)$ and that $\Theta \not\supseteq \Gamma_{S(G)}$, that is, for some $A, B \subseteq V$, $(A, B) \not\equiv (A, V)(\Theta)$.

First consider the case that, in addition, $(V \setminus \{x\}, V) \equiv (V, V)(\Theta)$. It follows that $(\{x\}, V) = (V \setminus \{x\}, V)^* \equiv (V, V)^* = (\emptyset, \emptyset)(\Theta)$ and that, for $y \in V \setminus \{x\}$, $(\{y\}, \emptyset) = (\{y\}, V) \wedge (V \setminus \{x\}, V \setminus \{x\}) \equiv (\{y\}, V) \wedge (V \setminus \{x\}, V) = (\{y\}, V)(\Theta)$. Since $A \subseteq V$ is finite, $A = \{x_0, \ldots, x_{n-1}\}$ for some $n < \omega$. If $(\{x_i\}, V) \equiv (\emptyset, \emptyset)(\Theta)$ for every $i < n$, then $(A, V) = \bigvee((\{x_i\}, V) : i < n) \equiv (\emptyset, \emptyset)(\Theta)$, which is absurd. Thus, for some $i < n$, $(\{x_i\}, V) \not\equiv (\emptyset, \emptyset)(\Theta)$. Since $x_i \neq x$ and $(A, B) \wedge (\{x_i\}, V) = (\{x_i\}, \emptyset)$, setting $(C, D)^* = (V \setminus \{x_i\}, V)^* = (\{x_i\}, V)$ will suffice.

Thus, it remains to consider the case that, for every $y \in V$, $(V \setminus \{y\}, V) \not\equiv (V, V)(\Theta)$. That is, we now need only consider the case when $\Theta \restriction S(G)^* = \Delta \restriction S(G)^*$ where $\Theta \restriction S(G)^*$ denotes the restriction of Θ to $S(G)^*$. If $(V \setminus \{x\}, V \setminus \{x\}) \equiv (V \setminus \{x\}, V)(\Theta)$, then, once more, for $y \in V \setminus \{x\}$, $(\{y\}, \emptyset) = (\{y\}, V) \wedge (V \setminus \{x\}, V \setminus \{x\}) \equiv (\{y\}, V) \wedge (V \setminus \{x\}, V) = (\{y\}, V)(\Theta)$. If there exists $x \neq y \in A$, then $(C, D)^* = (V \setminus \{y\}, V)^* = (\{y\}, V)$ will suffice since $(A, B) \wedge (\{y\}, V) = (\{y\}, \emptyset) \equiv (\{y\}, V)(\Theta)$. If there does not exist $x \neq y \in A$, then $A = \{x\}$ and $B = \emptyset$. Choose some $y \neq x$. Then $(\{x, y\}, R) \in S(G)$ for some $R \subset V$. Were it the case that $(\{x, y\}, R) \equiv (\{x, y\}, V)(\Theta)$, then it would follow that $(\{x\}, \emptyset) = (\{x\}, V) \wedge (\{x, y\}, R) \equiv (\{x\}, V) \wedge (\{x, y\}, V) = (\{x\}, V)(\Theta)$, contrary to the hypothesis that $(A, B) \not\equiv (A, V)(\Theta)$. Thus, $(\{x, y\}, R) \not\equiv (\{x, y\}, V)(\Theta)$. In particular, since $(\{y\}, \emptyset) \equiv (\{y\}, V)(\Theta)$, the proof is complete if the given A and B are replaced by $\{x, y\}$ and R, respectively, in the preceding argument. □

Let $\mathrm{End}(S(G))$ denote the monoid (semigroup with identity) of all endomorphisms of $S(G)$ with composition as multiplication. Let $\varphi \in \mathrm{End}(S(G))$ and let Θ denote the congruence on $S(G)$ induced by φ. Also, call a mapping $\psi \colon G \longrightarrow G$ *compatible* if $\{\psi(x), \psi(y)\} \in E$ whenever $\{x, y\} \in E$), and let $\mathrm{Aut}(G)$ be the automorphism group of G.

Lemma 2.6. *Let $\Theta \not\supseteq \Gamma_{S(G)}$.*

(i) *If* $(A,B) \in S(G)$ *and* $\varphi((A,B)) = (C,D)$*, then* $|C| = |A|$*, and*
(ii) *for* $x \in V$*,* $\varphi(\{x\},V) = (\{\psi(x)\},V)$ *defines a bijective compatible mapping* $\psi\colon G \longrightarrow G$*. Since* V *is finite,* $\psi \in \mathrm{Aut}(G)$*.*

Proof. If, for some $x \in V$, $\varphi((V\setminus\{x\},V)) = (V,V)$, then $(V\setminus\{x\},V\setminus\{x\}) \equiv (V\setminus\{x\},V)(\Theta)$. By Lemma 2.5, for some (A,B), $(C,D) \in S(G)$, $(A,B) \not\equiv (A,V)(\Theta)$ and $[(A,B)]_\Theta \geq [(C,D)^*]_\Theta \neq [(\emptyset,\emptyset)]_\Theta$. Since $\varphi \in \mathrm{End}(S(G))$ and $[(A,B)]_\Theta \notin (S(G)/\Theta)^*$, this contradicts Lemma 2.4. We conclude that, for $x \in V$, $\varphi((V\setminus\{x\},V)) \neq (V,V)$. In particular, $\varphi\restriction S(G)^*$ is one-to-one. Since $S(G)$ is finite, it follows that $\varphi\restriction S(G)^*$ is an automorphism. Hence, for any $(A,V) \in S(G)^*$, $\varphi(A,V) = (C,V)$ where $|C| = |A|$, completing the verification of (i).

By (i), for $x \in V$, $\varphi(\{x\},V) = (\{\psi(x)\},V)$ defines a bijection $\psi\colon V \longrightarrow V$. In particular, for $x,y \in V$, $\varphi(V\setminus\{x\},V) = (V\setminus\{\psi(x)\},V)$ and $\varphi((\{x,y\},V)) = \varphi((\{x\},V)\vee(\{y\},V)) = (\{\psi(x)\},V)\vee(\{\psi(y)\},V) = (\{\psi(x),\psi(y)\},V)$. Suppose $\{x,y\} \in E$. Since $|V| \geq 4$, it is possible to choose u, $v \in V\setminus\{x,y\}$. Then, by Lemma 2.1, $(\{x,y\},V)\wedge(V\setminus\{u\},V\setminus\{u\}) = (\{x,y\},\emptyset) = (\{x,y\},V)\wedge(V\setminus\{v\},V\setminus\{v\})$. Suppose $\{\psi(x),\psi(y)\} \notin E$. Then, by Lemma 2.1 again, $\varphi((\{x,y\},V)\wedge(V\setminus\{u\},V\setminus\{u\})) = (\{\psi(x),\psi(y)\},V)\wedge(V\setminus\{\psi(u)\},V\setminus\{\psi(u)\}) = (\{\psi(x),\psi(y)\},V\setminus\{\psi(u)\})$ and $\varphi((\{x,y\},V)\wedge(V\setminus\{v\},V\setminus\{v\})) = (\{\psi(x),\psi(y)\},V)\wedge(V\setminus\{\psi(v)\},V\setminus\{\psi(v)\}) = (\{\psi(x),\psi(y)\},V\setminus\{\psi(v)\})$. Since $V\setminus\{\psi(u)\} \neq V\setminus\{\psi(v)\}$, this is impossible. We conclude that, whenever $\{x,y\} \in E$, it follows that $\{\psi(x),\psi(y)\} \in E$, as required. □

2.2. A particular instance. We now choose a specific graph $\mathsf{G} = (\mathsf{V};\mathsf{E})$ where $\mathsf{V} = \{\mathsf{a},\mathsf{b}\} \cup \{\mathsf{c}_i : 0 \leq i < 4\}$ and $\mathsf{E} = \{\{\mathsf{a},\mathsf{c}_1\},\{\mathsf{b},\mathsf{c}_3\},\{\mathsf{c}_0,\mathsf{c}_2\}\} \cup \{\{\mathsf{c}_i,\mathsf{c}_{i+1}\} : 0 \leq i < 3\}$ (see Figure 1). In particular, $|\mathsf{V}| = 6$, which as required above is ≥ 4. We distinguish the pair of vertices a, $\mathsf{b} \in \mathsf{V}$ solely for future reference (see §3). Then, $|\mathrm{Aut}(\mathsf{G})| = 1$.

Henceforth, let M denote the pseudocomplemented semilattice $S(\mathsf{G})$ for this particular undirected graph G. It is M that will act as the basic component in the *šíp*-type construction presented here. Observe, again for future reference, that corresponding to a and b are $(\{\mathsf{a}\},\mathsf{V})$ and $(\{\mathsf{b}\},\mathsf{V})$, respectively, which are atoms of M^*, and $(\mathsf{V}\setminus\{\mathsf{a}\},\mathsf{V})$ and $(\mathsf{V}\setminus\{\mathsf{b}\},\mathsf{V})$, respectively, which are co-atoms of M^*.

By Lemma 2.6, for $\varphi \in \mathrm{End}(S(G))$, if $\Theta \not\supseteq \Gamma_{S(G)}$, then, for $x \in V$, $\varphi(\{x\},V) = (\{\psi(x)\},V)$ defines $\psi \in \mathrm{Aut}(G)$. For M, this implies

Lemma 2.7. *For* $\varphi \in \mathrm{End}(M)$*, if* $\Theta \not\supseteq \Gamma_M$ *where* Θ *denotes the congruence induced by* φ*, then* φ *is the identity.*

Proof. By Lemma 2.6 (i), for $\mathsf{x} \in \mathsf{V}$, $\varphi(\mathsf{V}\setminus\{\mathsf{x}\},\mathsf{V}) = (\mathsf{V}\setminus\{\mathsf{y}\},\mathsf{V})$ for some $\mathsf{y} \in \mathsf{V}$. By Lemma 2.5 and Lemma 2.4, $(\mathsf{V}\setminus\{\mathsf{x}\},\mathsf{V}\setminus\{\mathsf{x}\}) \not\equiv (\mathsf{V}\setminus\{\mathsf{x}\},\mathsf{V})(\Theta)$ for any $\mathsf{x} \in \mathsf{V}$.

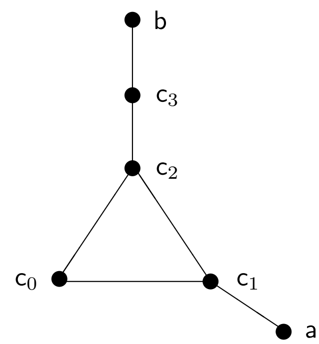

FIGURE 1. $\mathsf{G} = (\mathsf{V}; \mathsf{E})$

Thus, by Lemma 2.6, $\varphi((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})) = (\mathsf{V} \setminus \{\psi(\mathsf{x})\}, \mathsf{V} \setminus \{\psi(\mathsf{x})\})$ for every $\mathsf{x} \in \mathsf{V}$. Since M is generated by $\{(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\}) : \mathsf{x} \in \mathsf{V}\}$ by Lemma 2.3, it follows from the choice of $\mathsf{G} = (\mathsf{V}; \mathsf{E})$ that φ is the identity. □

3. The functor

In [12], Hedrlín and Pultr gave a full and faithful embedding from the category of all connected directed graphs to the category of all connected undirected graphs, based on a specific directed graph $G_0 = (V_0; E_0) \in \mathbf{G}$ with two distinguished vertices s_0 and t_0 as diagrammed in Figure 2, respectively on its *undirected* version $G'_0 = (V_0; E'_0)$. They specified a functor $\mathbf{\Psi}$ from $\mathbf{G}$ to the the category of all connected undirected graphs together with all of their compatible mappings as follows: For $G = (V; E) \in \mathbf{G}$, define an undirected graph $\mathbf{\Psi}(G)$ by taking a copy of G'_0 for every edge in E and, for $e, f \in E$, identifying the element t_0 of the copy representing e with the element s_0 of the copy representing f precisely when $e = (u, v)$ and $f = (v, w)$ for vertices $u, v, w \in V$.

If we proceed analogously, but with G_0 itself instead of G'_0, it is readily seen that we obtain a full and faithful functor $\mathbf{\Psi}\colon \mathbf{G} \longrightarrow \mathbf{G_c}$, where $\mathbf{G_c}$ denotes the category of all connected directed graphs which are (i) *strongly loopless* (that is, for vertices u and v, it is never the case that both (u, v) and (v, u) are edges), (ii) for every vertex v, there are edges (u, v) and (v, w), and (iii) considered as undirected graphs, they are triangle-free (that is, they do not contain a subgraph isomorphic to K_3, the complete graph on 3 vertices). Thus, in order for us to show that $\mathbf{S}$ is finite-to-finite $\mathbf{B}$-relatively universal, it is sufficient to define a suitable functor $\mathbf{\Phi}\colon \mathbf{G_c} \longrightarrow \mathbf{S}$.

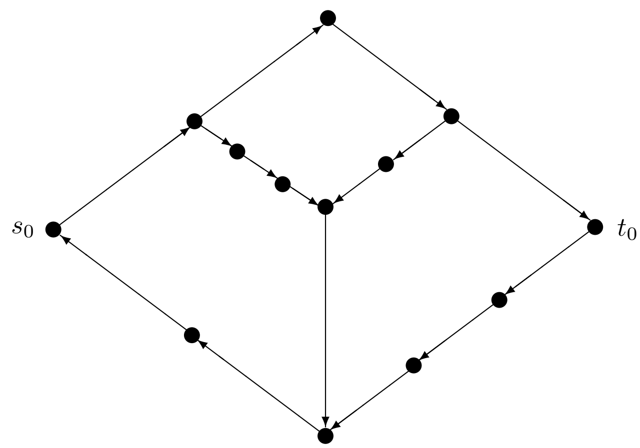

FIGURE 2. $G_0 = (V_0; E_0)$

For a variety $\mathbf{V}$ of pseudocomplemented semilattices, let $(S_i : i \in I)$ be a family of pseudocomplemented semilattices such that, for $i \in I$, $S_i \in \mathbf{V}$. A pseudocomplemented semilattice $S \in \mathbf{V}$ is a *free* $\mathbf{V}$*-product* of $(S_i : i \in I)$, denoted $\coprod_{\mathbf{V}}(S_i : i \in I)$, providing there is an embedding $\varepsilon_i\colon S_i \longrightarrow S$ for every $i \in I$ such that

(i) S is generated by $\bigcup(\varepsilon_i(S_i) : i \in I)$ in $\mathbf{V}$, and
(ii) if T is any pseudocomplemented semilattice in $\mathbf{V}$ and, for every $i \in I$, $\varphi_i\colon S_i \longrightarrow T$ is a homomorphism, then there exists a homomorphism $\varphi\colon S \longrightarrow T$ satisfying $\varphi_i = \varphi \circ \varepsilon_i$ for every $i \in I$.

In [16], Katriňák and Heleyová characterize $\mathbf{S}$-free products. In particular, they show that an $\mathbf{S}$-free product exists provided every component is a singleton or no component is a singleton. We remark that their characterization includes a characterization of free pseudocomplemented semilattices (that is, $\mathbf{S}$-free products of free pseudocomplemented semilattices with 1 free generator, each of which is order isomorphic to a 5-element non-modular lattice.) Free pseudocomplemented semilattices have also been considered in Balbes [4], Jones [15], and [29]. Although familiarity with [16] will not be required, we will make reference to it.

Let $G = (V; E)$ be a connected strongly loopless directed graph where, for $v \in V$, there exist (u, v) and $(v, w) \in E$ and, viewed as an undirected graph, no subgraph of it is isomorphic to K_3 (that is to say, $G \in \mathbf{G_c}$.) For each $e \in E$, let M_e denote a copy of the pseudocomplemented semilattice M, and $(A, B)_e$ denote the copy of $(A, B) \in M$ in M_e.

Notation: For the rest of this paper S will always denote the free product

$$S = \coprod_{\mathbf{S}}(M_e : e \in E).$$

Let Θ_G be the least congruence on S containing all pairs $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f)$ and $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_f)$ for $e = (u, v)$ and $f = (v, w)$ in E.

Let S_G be the G-*reduced free product*

$$S_G = (\coprod\nolimits_{\mathbf{S}}(M_e : e \in E))/\Theta_G$$

and set

$$\mathbf{\Phi}(G) = S_G.$$

Note that, since the $\mathbf{S}$-free product exists, $\mathbf{\Phi}$ is well-defined on objects. Further, since pseudocomplemented semilattices are locally finite (see Jones [14] and also Sankappanavar [26]), S_G is finite for finite G. In particular, the functor $\mathbf{\Phi}$ is finite-to-finite.

If $h\colon G \longrightarrow H$ is a compatible mapping between directed graphs $G = (V; E)$ and $H = (W; F) \in \mathbf{G_c}$, set

$$\mathbf{\Phi}(h) = \varphi,$$

where, for $e = (u, v) \in E$, $\varphi\colon S_G \longrightarrow S_H$ is determined by

$$\varphi([(A, B)_e]_{\Theta_G}) = [(A, B)_{(h(u),h(v))}]_{\Theta_H}$$

for $(A, B) \in M$.

Let $\psi\colon \coprod_{\mathbf{S}}(M_e : e \in E) \longrightarrow \coprod_{\mathbf{S}}(M_f : f \in F)$ be the homomorphism determined by $\psi((A, B)_e) = (A, B)_{(h(u),h(v))}$ for $e = (u, v) \in E$ and let $\theta_H\colon \coprod_{\mathbf{S}}(M_f : f \in F) \longrightarrow S_H$ be the natural homomorphism induced by Θ_H. To see that φ is a well-defined homomorphism, it is sufficient to show that the congruence Θ induced on $\coprod_{\mathbf{S}}(M_e : e \in E)$ by $\theta_H \circ \psi\colon \coprod_{\mathbf{S}}(M_e : e \in E) \longrightarrow S_H$ contains Θ_G. In particular, it is sufficient to show that, whenever $e = (u, v)$ and $f = (v, w) \in E$, each of the pairs $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f)$ and $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_f)$ are elements of Θ. Consider, for example, the pair $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f)$. By definition, $\psi((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e) = (\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_{(h(u),h(v))}$ and $\psi((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f) = (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_{(h(v),h(w))}$. Since $h\colon G \longrightarrow H$ is compatible, $(h(u), h(v))$ and $(h(v), h(w)) \in F$. Thus, $\theta_H(\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_{(h(u),h(v))}) = \theta_H(\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_{(h(v),h(w))})$, as required.

4. Free products of Boolean algebras

Let $G = (V; E)$ be a directed graph in $\mathbf{G_c}$. By Katriňák and Heleyová [16],

$$S^* = (\coprod\nolimits_{\mathbf{S}}(M_e : e \in E))^* = \coprod\nolimits_{\mathbf{B}}(M_e^* : e \in E) =: B,$$

where, by Lemma 2.2, $M^* = \{(\emptyset, \emptyset)\} \cup \{(A, \mathsf{V}) : \emptyset \neq A \subseteq \mathsf{V}\}$.

Let Θ_{G^*} be the least (Boolean) congruence relation on B containing all pairs $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f)$ for $e = (u, v)$ and $f = (v, w)$ in E and set

$$B_G = (\coprod_{\mathbf{B}}(M_e^* : e \in E))/\Theta_{G^*}.$$

As might be expected, the skeleton S_G^* of the G-reduced free product S_G is B_G, which will be confirmed in due course. Before proceeding to the proof of Theorem 1.1 in §6, we will establish some properties of B_G.

Since we are concerned only with Boolean algebras in this section, we can and will use Stone's topological representation for them. Although we will provide some basic facts and terminology, for more background on Boolean algebras and free **B**-products see, for example, Koppelberg [17].

Associated with each Boolean algebra $(B; \vee, \wedge, *, 0, 1)$ is a compact totally disconnected space $(X; \rho)$. The set X is the set of prime ideals of B. Each element of B is associated with the set of prime ideals to which it does not belong and, as such, the elements of B are recognizable as the clopen subsets of X (which form a basis for ρ). The Boolean operations join, meet, and complement are realized as set union, intersection, and complement, respectively.

For a family of Boolean algebras $(B_i : i \in I)$, the free **B**-product $B = \coprod_{\mathbf{B}}(B_i : i \in I)$ is associated with the cartesian product $P = \prod(X_i : \ i \in I)$ where the topology is the product topology. In particular, a subset $Y \subseteq P$ is associated with an element of B if and only if Y is a clopen subset of P if and only if Y is a finite union of sets of the type $H_{i_0} \times \cdots \times H_{i_{k-1}} \times \prod(X_i : i \neq i_0, \ldots, i_{k-1} \in I)$ where $i_0, \ldots, i_{k-1}$ is any finite selection of pairwise distinct indices from I and, for each $0 \leq j < k$, H_{i_j} is a clopen subset of X_{i_j}. For each $j \in I$, B contains a canonical copy B'_j of B_j as a subalgebra, given explicitly as the collection of all sets $\pi_j^{-1}(H_j)$ with $H_j \subseteq X_j$ clopen (where π_j is the canonical projection of $P = \prod(X_i : i \in I)$ onto X_j). If b_j is an element of B_j, then we will write b'_j for the copy of b_j in B'_j. Further, if b_j is associated with a clopen subset H_j of X_j, then, abusing notation, we will also write b'_j for $\pi_j^{-1}(H_j)$.

Actually, (i) B is generated by $\bigcup(B'_i : i \in I)$ (that is, the family $(\pi_i^{-1}(H_i) : b_i \in B_i$ for some $i \in I)$ is an open subbasis for the product topology on P) and (ii) whenever $b_{i_0}, \ldots, b_{i_{k-1}}$ is any finite collection of non-zero elements of pairwise distinct co-factors $B_{i_0}, \ldots, B_{i_{k-1}}$, then the meet of their copies $b'_{i_0}, \ldots, b'_{i_{k-1}}$ in B is non-zero (that is, for pairwise distinct spaces $X_{i_0}, \ldots, X_{i_{k-1}}$ and non-empty clopen subsets $H_{i_j} \subseteq X_{i_j}$, it is always the case that $H_{i_0} \times \cdots \times H_{i_{k-1}} \times \prod(X_i : i \neq i_1, \ldots, i_k \in I)$ is non-empty). In fact, properties (i) and (ii) characterize B up to isomorphism. The following special case of (ii) will be of interest. Whenever $i, j \in I$ and $i \neq j$, then $b'_i \leq b'_j$ implies $b'_i = 0$ or $b'_j = 1$ for any $b'_i, b'_j \in B$.

Let $G = (V; E)$ be a directed graph which, for the moment, we only assume to be strongly loopless. The topic at hand is the effect of the congruence Θ_{G^*} on $B := \coprod_{\mathbf{B}}(M_e^* : e \in E)$. Given an arrow $e = (u, v) \in E$, let σe and τe denote its *source* u and *target* v, respectively. Also, call vertices $u, v \in V$ *neighbours* provided either $(u, v) \in E$ or $(v, u) \in E$ (but not both, by strong looplessness) are in E.

Informally, we will assign labels a, b, or c to the arrows of G subject to the following admissibility rule. Whenever, at some vertex, *some* incoming arrow has label a or *some* outgoing arrow has label b, then at this vertex *all* incoming arrows must be labelled a and *all* outgoing arrows must be labelled b. Formally, a *labelling* for G is a map ℓ from E into the three-element set $\{a, b, c\}$. A labelling ℓ is called *G-admissible at a vertex* v if and only if the following holds: if $\tau e = v = \sigma f$ and $\ell(e) = a$ or $\ell(f) = b$, then $\ell(e') = a$ for all e' with $\tau e' = v$ and $\ell(f') = b$ for all f' with $\sigma f' = v$. The labelling ℓ is called *G-admissible* if and only if it is G-admissible at every vertex. Let ℓ_c be the labelling with constant value c and, for any $v \in V$, let ℓ_v be the labelling given by $\ell_v(e) = a$ if and only if $\tau e = v$, $\ell_v(e) = b$ if and only if $\sigma e = v$, and $\ell_v(e) = c$ in all other cases. Then, each of these labellings is G-admissible.

Recall that $M = S(\mathsf{G})$ for $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ where $\mathsf{V} = \{\mathsf{a}, \mathsf{b}\} \cup \{\mathsf{c}_k : 0 \leq k < 4\}$. In particular, by Lemma 2.2, $M^* = \{(\emptyset, \emptyset)\} \cup \{(A, \mathsf{V}) : \emptyset \subset A \subseteq \mathsf{V}\}$ ordered point-wise by inclusion. Thus, M^* is a finite Boolean algebra with 6 atoms. As such, it has an associated Stone space $(X; \rho)$ with $\mid X \mid = 6$ and discrete topology ρ. We will identify X with V and for each $e \in E$, V_e stands for a copy of the discrete space $(\mathsf{V}; \rho)$ with elements $\{\mathsf{a}_e, \mathsf{b}_e\} \cup \{\mathsf{c}_{k,e} : 0 \leq k < 4\}$. Let P be the product space $P := \prod(\mathsf{V}_e : e \in E)$.

An explicit description of the Stone space associated with B_{G^*} is now obtained as follows. An element $\mathbf{x} = (\ldots, \mathsf{x}_e, \ldots,) \in P$ is called *G-admissible* provided it satisfies $\mathsf{x}_e = \mathsf{a}_e$ if and only if $\mathsf{x}_f = \mathsf{b}_f$ whenever $\tau e = \sigma f$. It follows that if $\mathbf{x}$ is G-admissible, then the labelling ℓ of G given by $\ell(e) = a$ whenever $\mathsf{x}_e = \mathsf{a}_e$, $\ell(e) = b$ whenever $\mathsf{x}_e = \mathsf{b}_e$, and $\ell(e) = c$ whenever $\mathsf{x}_e \in \{\mathsf{c}_{k,e} : 0 \leq k < 4\}$ is G-admissible. Conversely, given a G-admissible labelling ℓ of G, any element $\mathbf{x} \in P$ satisfying $\mathsf{x}_e = \mathsf{a}_e$ whenever $\ell(e) = a$, $\mathsf{x}_e = \mathsf{b}_e$ whenever $\ell(e) = b$, and $\mathsf{x}_e \in \{\mathsf{c}_{k,e} : 0 \leq k < 4\}$ whenever $\ell(e) = c$ will be G-admissible. In which case, we will say that $\mathbf{x}$ is an *instance* of ℓ. Let AD denote the set of all G-admissible elements of P. Then B_{G^*} has as its Stone space the closed subset AD of P. The clopen subsets of AD are precisely the sets of the form $H \cap \mathsf{AD}$ for clopen $H \subseteq P$. The canonical projection $\pi_G \colon B \longrightarrow B_G$ is given explicitly as the map sending the element of B associated with the clopen set $H \subseteq P$ to the element of B_G associated with $H \cap \mathsf{AD}$.

We had agreed to write b'_e for $\pi_e^{-1}(H_e)$ when b_e is the element of M_e^* that is associated with $H_e \subseteq \mathsf{V}_e$. When b_e is an atom of M_e^*, $b'_e = \pi_e^{-1}(\{\mathsf{x}_e\})$ for some $\mathsf{x}_e \in \mathsf{V}_e$. In this case, abusing notation, we may write x'_e instead of b'_e. Likewise, when b_e is a co-atom of M_e^*, then $b'_e = \pi^{-1}(\mathsf{V}_e \setminus \{\mathsf{x}_e\})$ for some $\mathsf{x}_e \in \mathsf{V}_e$. Further abusing notation, we may write $(\mathsf{x}_e^*)'$ instead of b'_e in this case.

In order to make the presentation more readable, we will use the following abbreviations. For $\mathsf{x}_e \in \mathsf{V}_e$ and $\mathsf{y}_f \in \mathsf{V}_f$, write $\mathsf{x}_e \perp \mathsf{y}_f$ to mean $\pi_G(\mathsf{x}'_e) \wedge \pi_G(\mathsf{y}'_f) = 0$, and $\mathsf{x}_e \equiv \mathsf{y}_f$ to mean $\pi_G(\mathsf{x}'_e) = \pi_G(\mathsf{y}'_f)$. Throughout, k and k' will denote natural numbers between 0 and 3, if not stated otherwise explicitly.

As the following shows, there are enough admissible labellings with a prescribed value.

Lemma 4.1. *For any $e \in E$ and $\mathsf{k} \in \{a, b, c\}$ there is a G-admissible labelling ℓ of G such that $\ell(e) = \mathsf{k}$.*

Proof. If $\mathsf{k} = c$, let $\ell = \ell_c$. If $\mathsf{k} = a$, let $\ell = \ell_{\tau e}$, and if $\mathsf{k} = b$, let $\ell = \ell_{\sigma e}$. □

This already suffices to see that Θ_{G^*} does not collapse any of the subalgebras $(M^*_e)'$ of B.

Lemma 4.2. *The restriction of Θ_{G^*} to any of the subalgebras $(M^*_e)'$ of B is the identity; in other words, $\pi_G[(M^*_e)'] \cong (M^*_e)' \cong M^*_e$.*

Proof. It suffices to show that no atom x'_e of $(M^*_e)'$ is collapsed to 0 by Θ_{G^*}. Equivalently, it must be shown that, for $\mathsf{x}_e \in \mathsf{V}_e$, $\pi_e^{-1}(\{\mathsf{x}_e\}) \cap \mathsf{AD} \neq \emptyset$. Depending on whether $\mathsf{x}_e = \mathsf{a}_e$, b_e, or $\mathsf{c}_{k,e}$, let $\mathsf{k} = a$, b, or c, respectively. By Lemma 4.1, there is a G-admissible labelling ℓ of G such that $\ell(e) = \mathsf{k}$, so there is an instance $\mathbf{y}$ of ℓ such that $\mathsf{y}_e = \mathsf{x}_e$. In particular, $\mathbf{y} \in \pi_e^{-1}(\{\mathsf{x}_e\}) \cap \mathsf{AD}$. □

Recall that $b'_e \leq b'_f$ implies $b'_e = 0$ or $b'_f = 1$ for any $b'_e, b'_f \in B$ whenever $e \neq f$. This is no longer true in B_G. Indeed, if $\tau e = \sigma f$, there is certainly no G-admissible labelling ℓ of G such that $\ell(e) = \ell(f) = a$. Thus there is no G-admissible $\mathbf{y} \in P$ such that $\mathsf{y}_e = \mathsf{a}_e$ and $\mathsf{y}_f = \mathsf{a}_f$. In particular, it follows that $\pi_i^{-1}(\mathsf{a}_e) \cap \pi_j^{-1}(\mathsf{a}_f) \cap \mathsf{AD} = \emptyset$, in other words, $\pi_G(\mathsf{y}'_e) \wedge \pi_G(\mathsf{x}'_f) = 0$ in B_G, that is, $\mathsf{a}_e \perp \mathsf{a}_f$. But this is equivalent to $0 \neq \pi_G(\mathsf{a}'_e) \leq \pi_G((\mathsf{a}^*_f)') \neq 1$.

We will need to know exactly when $b'_e \leq b'_f$ in B_G. This happens iff $\mathsf{x}'_e \leq (\mathsf{x}^*_f)'$ for any atom $\mathsf{x}'_e \in (M^*_e)'$ with $\mathsf{x}'_e \leq b'_e$ and any coatom $(\mathsf{x}^*_f)' \in (M^*_f)'$ with $b'_f \leq (\mathsf{x}^*_f)$. But $\mathsf{x}'_e \leq (\mathsf{x}^*_f)'$ iff $\mathsf{x}'_e \wedge \mathsf{x}'_f = 0$, so the problem boils down to determining disjoint π_G-images of atoms coming from different co-factors. Following along the lines of the example in the preceding paragraph, we start by listing obstructions — imposed by our constraints on the edges of G — to the existence of admissible labellings with two prescribed values.

Lemma 4.3. *Let $e \neq f$ and $\mathsf{k}, \mathsf{l} \in \{a, b, c\}$. A G-admissible labelling ℓ satisfying $\ell(e) = \mathsf{k}$ and $\ell(f) = \mathsf{l}$ fails to exist exactly in the following 5 cases:*

(i) *$\mathsf{k} = \mathsf{l} = a$ and τe and τf are neighbours,*
(ii) *$\mathsf{k} = a$, $\mathsf{l} = b$ and τe and σf are neighbours,*
(iii) *$\mathsf{k} = \mathsf{l} = b$ and σe and σf are neighbours,*

(iv) $\mathsf{k} = c$, $\mathsf{l} = a$ *and* $\tau f \in \{\sigma e, \tau e\}$,
(v) $\mathsf{k} = c$, $\mathsf{l} = b$ *and* $\sigma f \in \{\sigma e, \tau e\}$.

Proof. Since the labelling with constant value c is admissible for any G, it is clear that at least one of k, l must be in $\{a, b\}$, which — taking advantage of symmetries — leads to the five cases listed.

(i) Let $\mathsf{k} = \mathsf{l} = a$. Suppose τe and τf are neighbours and assume, without loss of generality, that there is an arrow g from τe to τf. Since $\ell(f) = a$, we have $\ell(g) = a$ because ℓ is G-admissible at τf. But $\ell(e) = a$ and, since ℓ is G-admissible at τe, $\ell(g) = b$, a contradiction.

Conversely, assume τe and τf are not neighbours. If $\tau e = \tau f$, $\ell_{\tau e} = \ell_{\tau f}$ is a G-admissible labelling satisfying $\ell(e) = \ell(f) = a$. If $\tau e \neq \tau f$, define ℓ by $\ell(g) = a$ if and only if $\tau g = \tau e$ or $\tau g = \tau f$, $\ell(g) = b$ if and only if $\sigma g = \tau e$ or $\sigma g = \tau f$, and $\ell(g) = c$ in all other cases. This is G-admissible if τe and τf do not have a common neighbour v. If v is such a common neighbour, checking the 4 possible combinations (of arrow directions in and out of v) shows that no violation of admissibility at v occurs.

(ii) and (iii) are proved analogously.

(iv) Let $\mathsf{k} = c$ and $\mathsf{l} = a$. In particular, ℓ is G-admissible at τf. If $\tau f = \sigma e$, then $\ell(f) = a$ implies $\ell(e) = b$, a contradiction. If $\tau f = \tau e$, then $\ell(f) = a$ implies $\ell(e) = a$, a contradiction.

Conversely, suppose τf is different from σe and τe. Then the labelling $\ell_{\tau f}$ satisfies $\ell_{\tau f}(e) = c$ and $\ell_{\tau_f}(f) = a$.

(v) is proved analogously. □

In more algebraic terms, Lemma 4.3 reads:

Corollary 4.4. *Let* $e \neq f$ *and* $\mathsf{x}_e \in \mathsf{V}_e$, $\mathsf{y}_f \in \mathsf{V}_f$. *Then* $\mathsf{x}_e \perp \mathsf{y}_f$ *in* B_G *exactly if*

(i) $\mathsf{x}_e = \mathsf{a}_e$, $\mathsf{y}_f = \mathsf{a}_f$, *and* τe *and* τf *are neighbours, or*
(ii) $\mathsf{x}_e = \mathsf{a}_e$, $\mathsf{y}_f = \mathsf{b}_f$, *and* τe *and* σf *are neighbours, or*
(iii) $\mathsf{x}_e = \mathsf{b}_e$, $\mathsf{y}_f = \mathsf{b}_f$, *and* σe *and* σf *are neighbours, or*
(iv) $\mathsf{x}_e = \mathsf{c}_{k,e}$, $\mathsf{y}_f = \mathsf{a}_f$, *and* $\tau f \in \{\sigma e, \tau e\}$, *or*
(v) $\mathsf{x}_e = \mathsf{c}_{k,e}$, $\mathsf{y}_f = \mathsf{b}_f$, *and* $\sigma f \in \{\sigma e, \tau e\}$.

Moreover, if $\mathsf{c}_{k,e} \perp \mathsf{y}_f$ *for* some k *and* $\mathsf{y}_f = \mathsf{a}_f$ *or* $\mathsf{y}_f = \mathsf{b}_f$, *then* $\mathsf{c}_{k,e} \perp \mathsf{y}_f$ *for* all k.

Proof. The five numbered statements translate the corresponding ones in Lemma 4.3, while the last assertion follows from (iv) and (v), respectively, since the conditions given do not depend on the particular choice of k. □

The two following lemmata show that π_G-images of atoms coming from different co-factors are comparable only if they are equal, and that this only occurs for atoms of type $\pi_G(\mathsf{a}'_e), \pi_G(\mathsf{b}'_f)$ when forced directly by admissibility.

Lemma 4.5. *Let $e \neq f$, $\mathsf{x}_e \in \mathsf{V}_e$, $\mathsf{y}_f \in \mathsf{V}_f$. Then $\pi_G(\mathsf{x}'_e)$ is comparable with $\pi_G(\mathsf{y}'_f)$ in B_G iff $\mathsf{x}_e \equiv \mathsf{y}_f$.*

Proof. Assume $\pi_G(\mathsf{x}'_e) \leq \pi_G(\mathsf{y}'_f)$. Hence, $\mathsf{x}_e \perp \mathsf{z}_f$ for all $\mathsf{z}_f \in \mathsf{V}_f \setminus \{\mathsf{y}_f\}$. In particular,

$$\mathsf{V}_f \setminus \{\mathsf{y}_f\} \text{ must include } \mathsf{c}_{k,f} \text{ for some } k. \tag{†}$$

Since $\mathsf{c}_{k,e} \not\perp \mathsf{c}_{k',f}$ by Corollary 4.4, this implies $\mathsf{x}_e \in \{\mathsf{a}_e, \mathsf{b}_e\}$. On the other hand, by the last statement of Corollary 4.4, $\mathsf{V}_f \setminus \{\mathsf{y}_f\}$ must then contain $\mathsf{c}_{k,f}$ for all k and thus $\mathsf{y}_f \in \{\mathsf{a}_f, \mathsf{b}_f\}$.

Assume $\mathsf{x}_e = \mathsf{a}_e$. From (†) and Corollary 4.4 (iv) we obtain either $\tau e = \sigma f$ or $\tau e = \tau f$. In the first case, τf and τe are neighbours and we get $\mathsf{a}_e \perp \mathsf{a}_f$ by Corollary 4.4 (i), thus $\mathsf{y}_f = \mathsf{b}_f$ and $\mathsf{a}_e \equiv \mathsf{b}_f$ by admissibility at $\tau e = \sigma f$. In the second, τf and σf are neighbours and we arrive at $\mathsf{a}_e \equiv \mathsf{a}_f$ in the same way, using Corollary 4.4 (ii).

The same line of reasoning works also for $\mathsf{x}_e = \mathsf{b}_e$. □

In §6 we will also require the dual version of Lemma 4.5.

Corollary 4.6. *Let $e \neq f$, $\mathsf{x}_e \in \mathsf{V}_e$, $\mathsf{y}_f \in \mathsf{V}_f$. Then $\pi_G((\mathsf{V}_e \setminus \{\mathsf{x}_e\})')$ is comparable with $\pi_G((\mathsf{V}_f \setminus \{\mathsf{y}_f\})')$ in B_G iff $\pi_G((\mathsf{V}_e \setminus \{\mathsf{x}_e\})') = \pi_G((\mathsf{V}_f \setminus \{\mathsf{y}_f\})')$ iff $\mathsf{x}_e \equiv \mathsf{y}_f$.*

Lemma 4.7. *For $e \neq f$, $\mathsf{x}_e \in \mathsf{V}_e$, $\mathsf{y}_f \in \mathsf{V}_f$, we have $\mathsf{x}_e \equiv \mathsf{y}_f$ iff either (i) $\tau e = \tau f$, $\mathsf{x}_e = \mathsf{a}_e$, and $\mathsf{y}_f = \mathsf{a}_f$, or (ii) $\sigma e = \sigma f$, $\mathsf{x}_e = \mathsf{b}_e$, and $\mathsf{y}_f = \mathsf{b}_f$, or (iii) $\tau e = \sigma f$ and $\mathsf{x}_e = \mathsf{a}_e$, and $\mathsf{y}_f = \mathsf{b}_f$.*

Proof. One direction of each of the equivalences (i)–(iii) is immediate by the definition of admissibility. For the other direction where $\mathsf{x}_e \equiv \mathsf{y}_f$, it will suffice to show, using Lemma 4.5, that $\pi_G(\mathsf{x}'_e) \not\leq \pi_G(\mathsf{y}'_f)$ with the exception of the three cases listed.

Let $\mathsf{x}_e = \mathsf{c}_{k,e}$. Since $\mathsf{c}_{k,e} \not\perp \mathsf{c}_{k',f}$ for all k' by Corollary 4.4, we have $\pi_G(\mathsf{c}'_{k,e}) \not\leq \pi_G(\mathsf{y}'_f)$ for any y_f.

Let $\mathsf{x}_e = \mathsf{a}_e$. Assume e and f have no vertex in common and consider the admissible labelling $\ell_{\tau e}$ with values $\ell_{\tau e}(e) = a$ and $\ell_{\tau e}(f) = c$, and its instances. Again, it follows that $\mathsf{x}_e \not\perp \mathsf{c}_{k',f}$ for all k', and thus $\pi_G(\mathsf{a}'_e) \not\leq \pi_G(\mathsf{y}'_f)$ for any y_f. Assume e and f do have a vertex in common. If $\tau e = \tau f$, then $\mathsf{a}_e \equiv \mathsf{a}_f$; if $\tau e = \sigma f$, then $\mathsf{a}_e \equiv \mathsf{b}_f$; if $\sigma e \in \{\sigma f, \tau f\}$, consider the admissible labelling $\ell_{\tau e}$ and argue as above.

The case $\mathsf{x}_e = \mathsf{b}_e$ is handled dually. □

The preceding two lemmata combined with the next lemma, will show in Corollary 4.9 that π_G-images of any two elements from different co-factors are comparable only when forced directly by admissibility.

Lemma 4.8. *Assume $e \neq f$ and consider $\pi_G(h'_f) \in \pi_G[(M_f^*)'] \cong (M_f^*)'$ such that $\pi_G(h'_f)$ is neither an atom nor a coatom of $\pi_G[(M_f^*)']$. Then there is no atom $\pi_G(\mathsf{x}'_e)$ (where $\mathsf{x}_e \in \mathsf{V}_e$),* and *coatom $\pi_G((\mathsf{y}_e^*)')$ (where $\mathsf{y}_e \in \mathsf{V}_e$) of $\pi_G[(M_e^*)']$ such that* simultaneously $\pi_G(\mathsf{x}'_e) \leq \pi_G(h'_f)$ *and* $\pi_G(h'_f) \leq \pi_G((\mathsf{y}_e^*)')$.

Proof. Then, $\pi_G(h'_f)$ is neither an atom nor a coatom of $\pi_G[(M_f^*)']$ iff $h'_f = \pi_f^{-1}(H_f)$, $H_f \subseteq \mathsf{V}_f$, $1 <| H_f |< 5$. Put $H_f^* := \mathsf{V}_f \setminus H_f$. Now

$$\pi_G(\mathsf{x}'_e) \leq \pi_G(h'_f) \text{ iff } \mathsf{x}_e \perp \mathsf{z}_f \text{ for all } \mathsf{z}_f \in H_f^*, \tag{†}$$

and analogously

$$\pi_G(h'_f) \leq \pi_G((\mathsf{y}_e^*)') \text{ iff } \mathsf{y}_e \perp \mathsf{z}_f \text{ for all } \mathsf{z}_f \in H_f. \tag{††}$$

It is clear that $\mathsf{x}_e \neq \mathsf{y}_e$ whenever (†) and (††) are satisfied simultaneously.

Suppose, contrary to hypothesis, that there is an atom $\pi_G(\mathsf{x}'_e)$ and a coatom $\pi_G((\mathsf{y}_e^*)')$ of $\pi_G[(M_e^*)']$ such that simultaneously $\pi_G(\mathsf{x}'_e) \leq \pi_G(h'_f)$ and $\pi_G(h'_f) \leq \pi_G((\mathsf{y}_e^*)')$.

By Corollary 4.4 (iv) and (v), $\mathsf{z}_e \perp \mathsf{c}_{k,f}$ has a (unique) solution iff either $\tau e \in \{\sigma f, \tau f\}$ or $\sigma e \in \{\sigma f, \tau f\}$. Now at least one of H_f and H_f^* must contain some $\mathsf{c}_{k,f}$, so the solvability of (†) and (††) implies that exactly one of the four possible relative positions of e and f as indicated prevails. Assume $\mathsf{c}_{k,f} \in H_f$ for some k, and $\tau e = \sigma f$. Then, by Corollary 4.4 again, a_e is the unique candidate for y_e in (††). Turning to (†), suppose first that $\mathsf{c}_{k',f} \in H_f^*$ for some $k' \neq k$. The argument just used gives, again, a_e as the unique candidate for x_e, and thus $\mathsf{x}_e = \mathsf{y}_e$. If H_f^* does not contain any $\mathsf{c}_{k,f}$, we must have $H_f^* = \{\mathsf{a}_f, \mathsf{b}_f\}$. But then by Corollary 4.4 (i)–(iii), there is no common solution for $\mathsf{x}_e \perp \mathsf{a}_f$ and $\mathsf{x}_e \perp \mathsf{b}_f$. The cases $\tau e = \tau f$ and $\sigma e \in \{\sigma f, \tau f\}$ are handled analogously. □

Corollary 4.9. *Let $e \neq f$ and $D = (\pi_G[(M_e^*)'] \cap \pi_G[(M_f^*)']) \setminus \{0, 1\}$. Then $D \neq \emptyset$ iff either* (i) *$\tau e = \tau f$ and $D = \{\pi_G(\mathsf{a}'_e) = \pi_G(\mathsf{a}'_f),\ \pi_G((\mathsf{a}_e^*)') = \pi_G((\mathsf{a}_f^*)')\}$, or* (ii) *$\sigma e = \sigma f$ and $D = \{\pi_G(\mathsf{b}'_e) = \pi_G(\mathsf{b}'_f),\ \pi_G((\mathsf{b}_e^*)') = \pi_G((\mathsf{b}_f^*)')\}$, or* (iii) *$\tau e = \sigma f$ and $D = \{\pi_G(\mathsf{a}'_e) = \pi_G(\mathsf{b}'_f),\ \pi_G((\mathsf{a}_e^*)') = \pi_G((\mathsf{b}_f^*)')\}$.*

Proof. Combine Lemma 4.5, Lemma 4.7, and Lemma 4.8. □

The following lemma says that disjointness of π_G-images of atoms coming from different co-factors is rare, and that it is in some sense unique whenever it occurs. It depends heavily on the fact that G is triangle-free, so we will assume that G is strongly loopless *and* triangle-free from this point on.

Lemma 4.10. *Let $f \neq e$, $\mathsf{x}_f \in \mathsf{V}_f$ and suppose $\pi_G(\mathsf{x}'_f) \notin \pi_G[(M_e^*)']$. Then there exists at most one $\mathsf{y}_e \in \mathsf{V}_e$ such that $\mathsf{y}_e \perp \mathsf{x}_f$.*

Proof. There are five possibilities for the relative positions of e and f: (1) $\sigma e = \sigma f$, (2) $\tau e = \sigma f$, (3) $\sigma e = \tau f$, (4) $\tau e = \tau f$, and (5) e and f have no vertex in common. Then Lemma 4.7 implies for (1) that $\mathsf{b}_e \equiv \mathsf{b}_f$, for (2) that $\mathsf{a}_e \equiv \mathsf{b}_f$, for (3) that $\mathsf{b}_e \equiv \mathsf{a}_f$, and for (4) that $\mathsf{a}_e \equiv \mathsf{a}_f$.

Case 1: $\mathsf{x}_f = \mathsf{a}_f$.

- Assume (1).
  - $\mathsf{a}_f \perp \mathsf{a}_e$ iff τe and τf are neighbours by Corollary 4.4 (i) which violates triangle-free, so this case is not possible.
  - $\mathsf{a}_f \perp \mathsf{b}_e$ is true since $\mathsf{b}_e \equiv \mathsf{b}_f$.
  - $\mathsf{a}_f \perp \mathsf{c}_{k,e}$ iff $\tau f \in \{\sigma e, \tau e\}$ by Corollary 4.4 (iv), contradicting $\sigma e = \sigma f$, so this case is not possible.
- Assume (2).
  - $\mathsf{a}_f \perp \mathsf{a}_e$ is true since $\mathsf{a}_e \equiv \mathsf{b}_f$.
  - $\mathsf{a}_f \perp \mathsf{b}_e$ iff σe and τf are neighbours by Corollary 4.4 (ii) which violates triangle-free, so this case is not possible.
  - $\mathsf{a}_f \perp \mathsf{c}_{k,e}$ iff $\tau f \in \{\sigma e, \tau e\}$ by Corollary 4.4 (iv), contradicting $\sigma e = \sigma f$, so this case is not possible.
- Assume (3). Since $\mathsf{b}_e \equiv \mathsf{a}_f$, we infer $\pi_G(\mathsf{a}'_f) = \pi_G(\mathsf{b}'_e) \in \pi_G[(M^*_e)']$, violating our hypothesis.
- Assume (4). Since $\mathsf{a}_e \equiv \mathsf{a}_f$, we infer $\pi_G(\mathsf{a}'_f) = \pi_G(\mathsf{a}'_e) \in \pi_G[(M^*_e)']$, violating our hypothesis.
- Assume (5).
  - $\mathsf{a}_f \perp \mathsf{a}_e$ iff τe and τf are neighbours by Corollary 4.4 (ii). If g_1 is an arrow from τe to τf, then $\mathsf{a}_f \equiv \mathsf{a}_{g_1} \perp \mathsf{a}_e$ since $\mathsf{a}_e \equiv \mathsf{b}_{g_1}$ by Lemma 4.7. If g_2 is an arrow from τf to τe, then $\mathsf{a}_f \equiv \mathsf{b}_{g_2} \perp \mathsf{a}_e$ since $\mathsf{a}_e \equiv \mathsf{a}_{g_2}$ by Lemma 4.7. Note that exactly one of g_1 and g_2 exists since G is strongly loopless.
  - $\mathsf{a}_f \perp \mathsf{b}_e$ iff σe and τf are neighbours by Corollary 4.4 (ii). If g_1 is an arrow from σe to τf, then $\mathsf{a}_f \equiv \mathsf{a}_{g_1} \perp \mathsf{b}_e$ since $\mathsf{b}_e \equiv \mathsf{b}_{g_1}$ by Lemma 4.7. If g_2 is an arrow from τf to σe, then $\mathsf{a}_f \equiv \mathsf{b}_{g_2} \perp \mathsf{b}_e$ since $\mathsf{b}_e \equiv \mathsf{a}_{g_2}$ by Lemma 4.7.

    Note that the two preceding cases exclude one another: τf can't be a neighbour of σe and τe at the same time as G is triangle-free.
  - $\mathsf{a}_f \perp \mathsf{c}_{k,e}$ iff $\tau f \in \{\sigma e, \tau e\}$ by Corollary 4.4 (iv), contradicting the fact that e and f have no common vertex.

So $\mathsf{y}_e \perp \mathsf{a}_f$ has a (unique) solution just for (1): $\mathsf{y}_e = \mathsf{b}_e$, for (2): $\mathsf{y}_e = \mathsf{a}_e$ and for (5): $\mathsf{y}_e = \mathsf{a}_e$ iff τe and τf are neighbours or $\mathsf{y}_e = \mathsf{b}_e$ iff σe and τf are neighbours, respectively.

Case 2: $\mathsf{x}_f = \mathsf{b}_f$.

Arguing as in Case 1, we obtain that $\mathsf{y}_e \perp \mathsf{b}_f$ has a (unique) solution just for (3): $\mathsf{y}_e = \mathsf{b}_e$, for (4): $\mathsf{y}_e = \mathsf{a}_e$ and for (5): $\mathsf{y}_e = \mathsf{a}_e$ iff τe and σf are neighbours or $\mathsf{y}_e = \mathsf{b}_e$ iff σe and σf are neighbours, respectively.

Case 3: $\mathsf{x}_f = \mathsf{c}_{k,f}$.

- Assume (1).
  - $\mathsf{c}_{k,f} \perp \mathsf{a}_e$ iff $\tau e \in \{\sigma f, \tau f\}$ by Corollary 4.4 (iv), contradicting $\sigma e = \sigma f$, so this case is not possible.
  - $\mathsf{c}_{k,f} \perp \mathsf{b}_e$ is true by Corollary 4.4 (v) since $\sigma e \in \{\sigma f, \tau f\}$.
  - $\mathsf{c}_{k,f} \perp \mathsf{c}_{k',e}$ is not possible by Corollary 4.4.
- Assume (2).
  - $\mathsf{c}_{k,f} \perp \mathsf{a}_e$ is true by Corollary 4.4 (iv) since $\tau e \in \{\sigma f, \tau f\}$.
  - $\mathsf{c}_{k,f} \perp \mathsf{b}_e$ iff $\sigma e \in \{\sigma f, \tau f\}$ by Corollary 4.4 (v), contradicting $\tau e = \sigma f$, so this case is not possible.
  - $\mathsf{c}_{k,f} \perp \mathsf{c}_{k',e}$ is not possible by Corollary 4.4.
- Assume (3).
  - $\mathsf{c}_{k,f} \perp \mathsf{a}_e$ iff $\tau e \in \{\sigma f, \tau f\}$ by Corollary 4.4 (iv), contradicting $\sigma e = \tau f$, so this case is not possible.
  - $\mathsf{c}_{k,f} \perp \mathsf{b}_e$ is true by Corollary 4.4 (v) since $\sigma e \in \{\sigma f, \tau f\}$.
  - $\mathsf{c}_{k,f} \perp \mathsf{c}_{k',e}$ is not possible by Corollary 4.4.
- Assume (4).
  - $\mathsf{c}_{k,f} \perp \mathsf{a}_e$ is true by Corollary 4.4 (iv) since $\tau e \in \{\sigma f, \tau f\}$.
  - $\mathsf{c}_{k,f} \perp \mathsf{b}_e$ iff $\sigma e \in \{\sigma f, \tau f\}$ by Corollary 4.4 (v), contradicting $\tau e = \tau f$, so this case is not possible.
  - $\mathsf{c}_{k,f} \perp \mathsf{c}_{k',e}$ is not possible by Corollary 4.4.
- Assume (5).
  - $\mathsf{c}_{k,f} \perp \mathsf{y}_e$ is not possible for any $\mathsf{y}_e \in \mathsf{V}_e$ by Corollary 4.4 (iv) and (v).

So $\mathsf{y}_e \perp \mathsf{c}_{k,f}$ has a (unique) solution for (1): $\mathsf{y}_e = \mathsf{b}_e$, for (2): $\mathsf{y}_e = \mathsf{a}_e$, for (3): $\mathsf{y}_e = \mathsf{b}_e$, and for (4): $\mathsf{y}_e = \mathsf{a}_e$. □

Remark: Actually, Lemma 4.10 holds exactly if G is triangle-free. It is not hard to construct, given any three arrows e, f, g forming a triangle, an element $\mathsf{x}_f \in \mathsf{V}$ such that $\pi_G(\mathsf{x}'_f) \notin \pi_G[(M^*_e)']$, and two elements $\mathsf{y}_e \neq \mathsf{z}_e \in \mathsf{V}_e$ such that $\mathsf{x}_f \perp \mathsf{y}_e, \mathsf{z}_e$.

Our next concern is sets of atoms coming from different co-factors $\pi_G[(M^*_{e_i})']$ which are pairwise disjoint. It too will depend on the fact that G is triangle-free.

Lemma 4.11. *Let $e_1, \ldots, e_n \in E$ (for $n > 2$), and $\mathsf{x}_{e_i} \in \mathsf{V}_{e_i}$ for $1 \leq i \leq n$. Suppose $\mathsf{x}_{e_i} \perp \mathsf{x}_{e_j}$ for $1 \leq i < j \leq n$. Then, for some $1 \leq j \leq n$, there exist $\mathsf{y}_{i,e_j} \in \mathsf{V}_{e_j}$ (for $1 \leq i \leq n$) such that $\mathsf{x}_{e_i} \equiv \mathsf{y}_{i,e_j}$ and, in particular, it follows that $n \leq 6$.*

Proof. Let $2 < n$. Suppose (with no loss in generality) we find $\mathsf{x}_{e_i} \in \{\mathsf{a}_{e_i}, \mathsf{b}_{e_i}\}$ for $i = 1, 2, 3$. Select $\tau e_i \in V$ iff $\mathsf{x}_{e_i} = \mathsf{a}_{e_i}$, $\sigma e_i \in V$ iff $\mathsf{x}_{e_i} = \mathsf{b}_{e_i}$. The three selected vertices of G are pairwise neighbours by Corollary 4.4, violating triangle-free. So at most two of these x_{e_i} are in $\{\mathsf{a}_{e_i}, \mathsf{b}_{e_i}\}$, and thus (with no loss in generality) $\mathsf{x}_{e_3} = \mathsf{c}_{k,e_3}$ for some k. Suppose $\mathsf{x}_{e_1} = \mathsf{a}_{e_1}$. From $\mathsf{a}_{e_1} \perp \mathsf{c}_{k,e_3}$ we infer, using Corollary 4.4 (iv), that $\tau e_1 \in \{\sigma e_3, \tau e_3\}$. But then $\mathsf{a}_{e_1} \equiv \mathsf{b}_{e_3}$ or $\mathsf{a}_{e_1} \equiv \mathsf{a}_{e_3}$. The analogous argument based on Corollary 4.4 (v) shows that $\mathsf{x}_{e_1} = \mathsf{b}_{e_1}$ implies $\mathsf{b}_{e_1} \equiv \mathsf{b}_{e_3}$ or $\mathsf{b}_{e_1} \equiv \mathsf{a}_{e_3}$. Finally, $\mathsf{c}_{k,e_3} \not\perp \mathsf{c}_{k',e_i}$ for any k' and $i \neq 3$, so all the atoms $\pi_G(\mathsf{x}'_{e_i})$ are in $\pi_G[(M^*_{e_3})']$, and there are at most 6 of them. □

Let $1 \leq i < j \leq 6$ and assume that $y_1, \ldots, y_6 \in B_G$ satisfy $y_1 \vee \cdots \vee y_6 = 1$, $y_i \wedge y_j = 0$, and $y_i \geq \pi_G(\mathsf{x}'_{e_i})$ for $e_i \in E$ and $\mathsf{x}_{e_i} \in \mathsf{V}_{e_i}$. Since $\mathsf{x}_{e_i} \perp \mathsf{x}_{e_j}$ and $y_1 \vee \cdots \vee y_6 = 1$, it follows from Lemma 4.11 that, for some $e \in E$, $\mathsf{x}_{e_i} \equiv \mathsf{z}_{i,e}$ for $1 \leq i \leq 6$. However, $y_i \wedge \bigvee(\pi_G(\mathsf{z}'_{j,e}) : j \neq i) \leq y_i \wedge \bigvee(y_j : j \neq i) = 0$. In particular, $y_i \leq (\bigvee(\pi_G(\mathsf{z}'_{j,e}) : j \neq i))^* = \pi_G(\mathsf{z}'_{i,e})$. That is $y_i = \pi_G(\mathsf{z}'_{i,e})$. In §6, we will need a dual version of this.

Corollary 4.12. *Let $1 \leq q, q' \leq 6$ and assume $y_1, \ldots, y_6 \in B_G$ satisfy $y_1 \wedge \cdots \wedge y_6 = 0$, $y_q \vee y_{q'} = 1$ whenever $q \neq q'$, and $y_q \leq \pi_G((\mathsf{V}_{e_q} \setminus \{\mathsf{x}_{e_q}\})')$ for $e_q \in E$ and $\mathsf{x}_{e_q} \in \mathsf{V}_{e_q}$. Then there exists $e \in E$ such that $\{y_1, \ldots, y_6\} = \{\pi_G((\mathsf{V}_e \setminus \{\mathsf{x}_e\})') : \mathsf{x}_e \in \mathsf{V}_e\}$.*

5. Two constructions

5.1. Doubling elements of a Boolean lattice. For a subset X of a poset P, let $(X] = \{y : y \leq x$ for some $x \in X\}$ and $[X) = \{y : y \geq x$ for some $x \in X\}$. For brevity, whenever $X = \{x\}$, $(x]$ and $[x)$ will be used to denote $(X]$ and $[X)$, respectively. For $X \neq \emptyset$, X is an *order ideal* or an *order filter* providing $X = (X]$ or $[X)$, respectively.

We will now define a particular pseudocomplemented semilattice $B[\![F]\!]$ for each Boolean lattice B and non-trivial (that is, $\emptyset \subset F \subset B$) order filter F defined on it. The skeleton of $B[\![F]\!]$ will be B, and the non-trivial Glivenko classes in $B[\![F]\!]$ will be 2-element chains containing an element from $(B \setminus F) \setminus \{0\}$.

Thus let

$$B[\![F]\!] = (B \times \mathbf{2}) \setminus ((F \times \{0\}) \cup \{(0, 1)\}),$$

where $\mathbf{2}$ denotes the 2-element chain $\{0, 1\}$ with $0 < 1$. Let $\leq$ denote the restriction of the order on $B \times \mathbf{2}$ to $B[\![F]\!]$.

Lemma 5.1. $(B[\![F]\!];\wedge,*,(0,0),(1,1))$ *is a pseudocomplemented semilattice such that, for* (a,i), $(b,j)\in B[\![F]\!]$,

$$(a,i)\wedge(b,j)=\begin{cases}(a\wedge b,i\wedge j) & \text{if } a\wedge b\neq 0,\\ (0,0) & \text{if } a\wedge b=0,\end{cases}$$

where $(1,1)^*=(0,0)$ *and, for* $(a,i)\neq(1,1)$, $(a,i)^*=(a^*,1)$.

Proof. Since $B[\![F]\!]$ is a subset of $B\times\mathbf{2}$, $(B[\![F]\!];\leq)$ is a poset.

Since $(0,0)\in B[\![F]\!]$, (a,i) and (b,j) always have a lower bound. Say (c,k) is one such. Then, as $c\leq a\wedge b$ and $k\leq i\wedge j$, $(a\wedge b,i\wedge j)$ will be the greatest lower bound providing it is an element of $B[\![F]\!]$. If $(a\wedge b,i\wedge j)\notin B[\![F]\!]$, then either $a\wedge b=0$ and $i\wedge j=1$ or else $a\wedge b\in F\setminus\{0\}$ and $i\wedge j=0$. In the former case, $a\wedge b=0$ and $i\wedge j=1$. Then, since $((0,1)]=\{(0,0),(0,1)\}$ in $B\times\mathbf{2}$, $(a,i)\wedge(b,j)=(0,0)$ in $B[\![F]\!]$. In the latter case, $a\wedge b\in F\setminus\{0\}$. Then $a,b\in F$ and, in particular, $i=j=1$. Thus, contrary to hypothesis, $i\wedge j=1$, and we conclude that this case does not arise.

Certainly, $(1,1)^*=(0,0)$. If $(a,i)\neq(1,1)$, $a<1$. If $(a,i)\wedge(b,j)=(0,0)$, then $a\wedge b=0$ and, in particular, $b\leq a^*$. Since $(a,i)\wedge(a^*,1)=(0,0)$, $(a,i)^*=(a^*,1)$, as required. □

Lemma 5.2. *For* (a,i), $(b,j)\in B[\![F]\!]$, *if* $(a,i)\geq(b,j)^*\neq(0,0)$, *then* $(a,i)\in B[\![F]\!]^*$.

Proof. By Lemma 5.1, $(b,j)\neq(1,1)$, $b<1$, and $(b,j)^*=(b^*,1)$. Thus, $0<b^*\leq a$ and $1\leq i$. Either $a=1$ and $(a,i)=(1,1)$ or else $a<1$ and $(a,i)^{**}=(a^*,1)^*=(a^{**},1)=(a,1)=(a,i)$. □

We are interested in the special case where B equals M^* and F is the nontrivial order filter $\{(\mathsf{V},\mathsf{V})\}$ on M^*.

Lemma 5.3. (i) $M^*[\![\{(\mathsf{V},\mathsf{V})\}]\!]$ *is isomorphic to* $M_{\mathsf{V}}:=\{(\emptyset,\emptyset),(\mathsf{V},\mathsf{V})\}\cup\{(A,B);\emptyset\subset A\subset\mathsf{V}$ *and* $B=\emptyset$ *or* $B=\mathsf{V}\}$, *ordered by* $\subseteq$ *component-wise.*

(ii) *The function* $\varphi_M\colon M\longrightarrow M_{\mathsf{V}}$, *given by*

$$\varphi_M((A,B))=\begin{cases}(A,\mathsf{V}) & \text{if } B=\mathsf{V},\\ (A,\emptyset) & \text{if } B\subset\mathsf{V},\end{cases}$$

for $(A,B)\in M$, *is a homomorphism.*

Proof. (i) By Lemma 2.2, $M^*=S(\mathsf{G})^*$ is isomorphic to $B(\mathsf{G})=\mathcal{P}(\mathsf{V})$, the Boolean lattice of all subsets V, and the order filter $\{(\mathsf{V},\mathsf{V})\}$ on M^* corresponds to the order filter $\{\mathsf{V}\}$ on $\mathcal{P}(\mathsf{V})$ under this isomorphism. Realize $\mathcal{P}(\mathsf{V})[\![\{\mathsf{V}\}]\!]$ as defined above but with $\mathbf{2}$ as $\{\emptyset,\mathsf{V}\}$ ordered by $\emptyset\subset\mathsf{V}$. We obtain an isomorphic copy M_{V} of $M^*[\![\{(\mathsf{V},\mathsf{V})\}]\!]$ as described.

(ii) Consider (A,B), $(C,D) \in M$. Case 1: $A \cap C = \emptyset$. Then $(A,B) \wedge (C,D) = (\emptyset,\emptyset)$ by Lemma 2.1 and thus $\varphi_M((A,B) \wedge (C,D)) = (\emptyset,\emptyset) = \varphi_M((A,B)) \wedge \varphi_M((C,D))$ by Lemma 5.1. Case 2: $A \cap C \neq \emptyset$. Subcase 2.1: $B \cap D = \mathsf{V}$. Then $(A,\mathsf{V}) \wedge (C,\mathsf{V}) = (A \cap C,\mathsf{V})$ and thus $\varphi_M((A,B) \wedge (C,D)) = (A \cap C,\mathsf{V}) = \varphi_M((A,\mathsf{V})) \wedge \varphi_M((C,\mathsf{V}))$. Subcase 2.2: $B \cap D \neq \mathsf{V}$. Then, with no loss in generality, $B \neq \mathsf{V}$ and $(A,B) \wedge (C,D) = (A \cap C, Z)$ for some $\emptyset \subseteq Z \subset \mathsf{V}$, which implies $\varphi_M((A,B) \wedge (C,D)) = (A \cap C,\emptyset)$. But then also $\varphi_M((A,B)) = (A,\emptyset)$ and thus $\varphi_M((A,B)) \wedge \varphi_M((C,D)) = (A \cap C,\emptyset)$. This shows that φ_M preserves meets. Preservation of pseudocomplements is immediate from the definitions. □

A (closely related) case of interest is the following. Define a nontrivial order ideal I_G of B_G by putting an element $h \in B_G$ into I_G iff there are an edge $e \in E$ and a coatom q of $\pi_G[(M_e^*)']$ such that $h \leq q$. Consequently, $F_G := B_G \setminus I_G$ is a nontrivial order filter of B_G.

Imitating the description of $M^*[\![\{(\mathsf{V},\mathsf{V})\}]\!]$ given in Lemma 5.3 (i), we can realize $\mathbf{2}$ as $\{\emptyset, X\}$ this time, ordered by $\emptyset \subset X$ where X is any nonempty set (the choice of X will be one purely of notational convenience, but in most cases it will be chosen to be V or V_e for some $e \in E$). It is straightforward to see that $B_G[\![F_G]\!]$ is isomorphic to $B_G[\![F_G]\!]_X := \{(\emptyset,\emptyset)\} \cup \{(\pi_G(H), B) : \emptyset \neq \pi_G(H) \in B_G$ and $B = \emptyset$ or $B = X\} \setminus \{(\pi_G(H),\emptyset) : \pi_G(H) \in F_G\}$, ordered by $\subseteq$ component-wise (where, since it will be only a matter of notational convenience, $B_G[\![F_G]\!]_X \cong B_G[\![F_G]\!]_{\mathsf{V}} \cong B_G[\![F_G]\!]_{\mathsf{V}_e} \cong B_G[\![F_G]\!]$.

Recall that B_G contains a canonical copy of M_e^* as a subalgebra for any $e \in E$ by Lemma 4.2, realized as $\{\pi_G(A') : \emptyset \subseteq A \subseteq \mathsf{V}_e\}$ ordered by $\subseteq$. Appealing to Lemma 5.3 (i) again, with V_e in place of V, we see that $M_e^*[\![\{(\mathsf{V}_e,\mathsf{V}_e)\}]\!]$ is isomorphic to $M_{\mathsf{V}_e} := \{(\emptyset,\emptyset),(\mathsf{V}_e,\mathsf{V}_e)\} \cup \{(\pi_G(A'), B) : \emptyset \subset A \subset \mathsf{V}_e$ and $B = \emptyset$ or $B = \mathsf{V}_e\}$, ordered by $\subseteq$ component-wise. Observing that $\pi_G(A') \in I_G$ whenever $A \subset \mathsf{V}_e$, we conclude that M_{V_e} is, in fact, a subalgebra of $B_G[\![F_G]\!]_{\mathsf{V}_e}$.

Let $\varphi_e \colon M_e \longrightarrow B_G[\![F_G]\!]_{\mathsf{V}}$ be the map given by

$$\varphi_e((A,B)) = \begin{cases} (\pi_G(A'),\mathsf{V}) & \text{if } B = \mathsf{V}_e, \\ (\pi_G(A'),\emptyset) & \text{if } B \subset \mathsf{V}_e, \end{cases}$$

By Lemma 5.3 (ii), φ_e is a homomorphism from M_e into $B_G[\![F_G]\!]_{\mathsf{V}} \cong B_G[\![F_G]\!]$.

Corollary 5.4. *There exists a homomorphism* $\overline{\varphi} \colon S = \coprod_{\mathbf{S}}(M_e : e \in E) \longrightarrow B_G[\![F_G]\!]$ *extending* φ_e *for every* $e \in E$.

5.2. A test algebra for G-reduced free products. Let F_G and I_G be as above and choose a fixed vertex $f \in E$ for the rest of this paragraph. A subset $Z_{G,f}$ of I_G

is now defined as the set of all $h \in I_G$ such that there are an atom p of $\pi_G[(M_f^*)']$ and a coatom q of $\pi_G[(M_f^*)']$ satisfying $p \leq h \leq q$.

Since $\pi_G[(M_f^*)']$ is a *finite* subalgebra of B_G, there exists, for any $h \in B_G$, a uniquely determined largest element $a \in \pi_G[(M_f^*)']$ such that $a \leq h$; we call it the *lower cover of* h *in* $\pi_G[(M_f^*)']$ and denote it by $\lambda(h)$. Note that $\lambda\colon B_G \longrightarrow \pi_G[(M_f^*)']$ is a $\wedge$-retraction of B_G onto $\pi_G[(M_f^*)']$.

Our goal is to construct a pseudocomplemented semilattice S_f with skeleton B_G and prescribed Glivenko classes, as follows: Let $h \in B_G$. Then its Glivenko class in S_f shall be $\{h\}$ if $h = 0$ or $h \in F_G$, a copy of the Glivenko class of $\lambda(h)$ in $M_f \supseteq M_f^* \cong \pi_G[(M_f^*)'] \ni \lambda(h)$ if $h \in Z_{G,f}$, and a two-element set if $h \neq 0$ and $h \in I_G \setminus Z_{G,f}$. The construction of S_f generalizes that of $S(G)$ given in §2, and is based on the description of B_G given in §4.

Recall that elements $h \in B_G$ are represented as sets $\pi_G(H) = H \cap \mathsf{AD}$ where H is any clopen subset of $P = \prod(\mathsf{V}_e : e \in E)$. Given such $\pi_G(H)$, there is a unique largest subset $A \subseteq \mathsf{V}_f$ such that $\pi_G(A') \subseteq \pi_G(H)$, to be denoted by $A = \Lambda(\pi_G(H))$. It is easy to see that whenever $\pi_G(H)$ represents $h \in B_G$ and $A = \Lambda(\pi_G(H))$, then the lower cover $\lambda(h)$ of h is represented by $\pi_G(A')$; further, we have $\Lambda(\pi_G(H_1 \cap H_2)) = \Lambda(\pi_G(H_1)) \cap \Lambda(\pi_G(H_2))$. Finally, the subset I_G of B_G defined above is realized as the collection of all $\pi_G(H)$ such that $\pi_G(H) \subseteq \pi_G((\mathsf{V}_e \setminus \{\mathsf{y}\})')$ for some $e \in E$ and some $\mathsf{y} \in \mathsf{V}_e$. Further, the subset $Z_{G,f}$ of B_G defined above is realized as the collection of all $\pi_G(H)$ such that $\pi_G(\{\mathsf{x}\}') \subseteq \pi_G(H) \subseteq \pi_G((\mathsf{V}_f \setminus \{\mathsf{y}\})')$ for some $\mathsf{x}, \mathsf{y} \in \mathsf{V}_f$. In particular, $\Lambda(\pi_G(H)) \neq \emptyset$ whenever $\pi_G(H) \in Z_{G,f}$.

The pseudocomplemented semilattice S_f announced above is now defined as a subset of $B_G \times \mathcal{P}(\mathsf{V}_f)$, ordered by set inclusion component-wise (where $\mathcal{P}(\mathsf{V}_f)$ is the power set of V_f), as follows.

Definition 5.5. A pair $(\pi_G(H), B) \in B_G \times \mathcal{P}(\mathsf{V}_f)$ belongs to S_f iff one of the following holds:

(i) $\pi_G(H) = \emptyset$ and $B = \emptyset$,
(ii) $\pi_G(H) \in F_G$ and $B = \mathsf{V}_f$,
(iii) $\pi_G(H) \in Z_{G,f}$, and $(\Lambda(\pi_G(H)), B) \in M_f$,
(iv) $\emptyset \neq \pi_G(H) \in I_G \setminus Z_{G,f}$, and $B = \emptyset$ or $B = \mathsf{V}_f$.

The cases listed are mutually exclusive. Writing $\leq$ for the order relation defined on S_f, we will show that $(S_f, \leq)$ is a pseudocomplemented semilattice. The following fact, a direct consequence of Lemma 4.10, is crucial.

Lemma 5.6. *Let $\pi_G(H_1) \in Z_{G,f}$, $\pi_G(H_2) \in I_G \setminus Z_{G,f}$ and $\pi_G(H_1) \subseteq \pi_G(H_2)$. Then $\Lambda\pi_G(H_1) = \{\mathsf{x}_f\}$ for some $\pi_G(\mathsf{x}_f') \in Z_{G,f}$.*

Proof. Let $A := \Lambda\pi_G(H_1)$. Then $A \neq \emptyset$ since $\pi_G(H_1) \in Z_{G,f}$. Pick $\mathsf{x}_f \in A$. It follows that $\pi_G(\mathsf{x}_f') \subseteq \pi_G(A') \subseteq \pi_G(H_1)$. Now if $\pi_G(H_1) \subseteq \pi_G(H_2)$, then $\pi_G(\mathsf{x}_f') \subseteq$

$\pi_G(H_2) \in I_G \setminus Z_{G,f}$ implies that $\pi_G(H_2) \subseteq \pi_G((\mathsf{V}_e \setminus \{\mathsf{y}_e\})')$ for some $\mathsf{y}_e \in \mathsf{V}_e$ such that $e \neq f$ and $\mathsf{y}_e \not\equiv \mathsf{z}_f$ for any $\mathsf{z}_f \in \mathsf{V}_f$. It follows that $\pi_G(\mathsf{x}'_f) \subseteq \pi_G((\mathsf{V}_e \setminus \{\mathsf{y}_e\})')$, in other words, $\mathsf{x}_f \perp \mathsf{y}_e$. But by Lemma 4.10, this equation has at most one solution for x_f, given y_e as specified. □

Lemma 5.7. $(S_f, \leq)$ *is a pseudocomplemented semilattice.*

Proof. Fix any pair $(\pi_G(H_1), B_1)$, $(\pi_G(H_2), B_2)$ of elements of S_f and consider an arbitrary element $(\pi_G(H), B) \in S_f$ such that $(\pi_G(H), B) \leq (\pi_G(H_i), B_i)$ for $i = 1, 2$. We have to show that there is a unique $\leq$-maximal element in S_f with this property, to be denoted it by $(\pi_G(H_m), B_m)$.

Define $\pi_G(H_0) := \pi_G(H_1) \cap \pi_G(H_2)$ $(= \pi_G(H_1 \cap H_2))$ and $B_0 := B_1 \cap B_2$. It is clear that $\pi_G(H) \subseteq \pi_G(H_0)$ and $B \subseteq B_0$. Note that $(\pi_G(H_0), B_0)$ is not necessarily a member of S_f. However, we will show that in all cases, we have either $(\pi_G(H_m), B_m) = (\pi_G(H_0), B_0)$ (with $B_0 \neq \emptyset$) or $(\pi_G(H_m), B_m) = (\pi_G(H_0), \emptyset)$.

We may assume that $\pi_G(H_0) \neq \emptyset$ for otherwise $(\pi_G(H_m), B_m)$ equals $(\emptyset, \emptyset)$. Observe further that $(\pi_G(H), \mathsf{V}_f) \in S_f$ for any $\emptyset \neq \pi_G(H) \in B_G$. This readily implies that $(\pi_G(H_m), B_m) = (\pi_G(H_0), \mathsf{V}_f)$ whenever $B_1 = B_2 = \mathsf{V}_f$, and takes care, in particular, of all cases where $\pi_G(H_1)$, $\pi_G(H_2)$ both are in F_G (by Definition 5.5 (ii)) and where $\emptyset \neq \pi_G(H_0) \in F_G$ (since then both of $\pi_G(H_1)$, $\pi_G(H_2)$ must be in $F_G = B_G \setminus I_G$). So we may assume $B_0 \neq \mathsf{V}_f$ and $\emptyset \neq \pi_G(H_0) \in I_G$ in the sequel. We will determine, given such $\pi_G(H_0)$ and B_0, all elements $(\pi_G(H), B) \in S_f$ satisfying $\pi_G(H) \subseteq \pi_G(H_0)$ and $B \subseteq B_0$.

For the purpose of this proof, call a subset of V_f *small* if it is a singleton or an edge of $\mathsf{G}_f = (\mathsf{V}_f, \mathsf{E}_f)$. We distinguish three cases:

(1) $\pi_G(H_0) \in I_G \setminus Z_{G,f}$.
(2) $\pi_G(H_0) \in Z_{G,f}$ and $\Lambda\pi_G(H_0)$ is small.
(3) $\pi_G(H_0) \in Z_{G,f}$ and $\Lambda\pi_G(H_0)$ is not small.

Assume (1). If $\pi_G(H) \subseteq \pi_G(H_0)$ then either (1.1) $\pi_G(H) \in I_G \setminus Z_{G,f}$ or (1.2) $\pi_G(H) \in Z_{G,f}$. If (1.1) holds, then $(\pi_G(H), B) \in S_f$ implies $B = \emptyset$ by Definition 5.5 (iv) since $B = \mathsf{V}_f$ is ruled out by $B \subseteq B_0 \neq \mathsf{V}_f$. If (1.2) holds, then $\Lambda(\pi_G(H))$ is small by Lemma 5.6, which also implies $B = \emptyset$ by Definition 5.5 (iii) and the properties of $M_f \cong M$. Now $(\pi_G(H_0), \emptyset)$ itself is in S_f by Definition 5.5 (iv), and we conclude that indeed $(\pi_G(H_m), B_m) = (\pi_G(H_0), \emptyset)$.

Assume (2). If $\pi_G(H) \subseteq \pi_G(H_0)$ then either (2.1) $\pi_G(H) \in I_G \setminus Z_{G,f}$ or (2.2) $\pi_G(H) \in Z_{G,f}$. If (2.1) holds, then $B = \emptyset$ as in subcase (1.1). If (2.2) holds, then $\Lambda(\pi_G(H))$ is small since $\pi_G(H) \subseteq \pi_G(H_0)$ implies $\Lambda(\pi_G(H)) \subseteq \Lambda(\pi_G(H_0))$, and so $B = \emptyset$ as in subcase (1.2). Moreover, $(\pi_G(H_0), \emptyset) \in S_f$ by Definition 5.5 (iii) and the properties of M_f, and thus again $(\pi_G(H_m), B_m) = (\pi_G(H_0), \emptyset)$.

Assume (3). Observe first that $\pi_G(H_0) \in Z_{G,f}$ and $\Lambda\pi_G(H_0)$ not small excludes $\pi_G(H_i) \in I_G \setminus Z_{G,f}$ by Lemma 5.6 for $i = 1, 2$. So either $\pi_G(H_i) \notin I_G$ and $B_i = \mathsf{V}_f$, or $\pi_G(H_i) \in Z_{G,f}$ with $\Lambda(\pi_G(H_i))$ not small and thus $\Lambda(\pi_G(H_i)) \subseteq B_i \subseteq \mathsf{V}_f$. We conclude that $\Lambda(\pi_G(H_0)) = \Lambda(\pi_G(H_1 \cap H_2)) = \Lambda(\pi_G(H_1)) \cap \Lambda(\pi_G(H_2)) \subseteq B_1 \cap B_2 = B_0$ (and thus $B_0 \neq \emptyset$). But this means that $(\pi_G(H_0), B_0) \in S_f$ by Definition 5.5 (iii) and the properties of M_f. Hence $(\pi_G(H_m), B_m) = (\pi_G(H_0), B_0)$ (with $B_0 \neq \emptyset$).

With that, we have established that $(S_f, \leq)$ is a meet-semilattice with zero $(\emptyset, \emptyset)$. Pseudocomplements are easy: consider $(\pi_G(H_i), B_i) \in S_f$ for $i = 1, 2$, and suppose $(\pi_G(H_1), B_1) \wedge (\pi_G(H_2), B_2) = (\emptyset, \emptyset)$. This implies $\pi_G(H_1) \cap \pi_G(H_2) = \emptyset$. Let $\pi_G(H)$ be the complement of $\pi_G(H_1)$ in B_G; then certainly $\pi_G(H_2) \subseteq \pi_G(H)$, and $(\pi_G(H), \mathsf{V}_f)$ is the largest element of S_f disjoint from $(\pi_G(H_1), B_1)$. □

Consider M_f. Then B_G contains a canonical copy of M_f^* as a subalgebra by Lemma 4.2, realized as $\{\pi_G(A') : \emptyset \subseteq A \subseteq \mathsf{V}_f\}$ (ordered by $\subseteq$). Observe that if $A \neq \emptyset, \mathsf{V}_f$, then $\pi_G(A') \in Z_{G,f}$ and $\Lambda(\pi_G(A')) = A$ by the definition of Λ. It follows that $\psi_f \colon M_f \longrightarrow S_f$, defined by

$$\psi_f((A, B)) = \begin{cases} (\emptyset, \emptyset) & \text{if } A = \emptyset, \\ (\pi_G(A'), B) & \text{otherwise} \end{cases}$$

is an embedding of M_f into S_f.

Consider M_e, $e \neq f$ and the description of $M^*[\![\{(\mathsf{V}, \mathsf{V})\}]\!]$ given in the proof of Lemma 5.3. Using $\mathcal{P}(\mathsf{V}_e)$ as the Boolean lattice and realizing $\mathbf{2}$ as $\{\emptyset, \mathsf{V}_f\}$ (ordered by $\emptyset \subset \mathsf{V}_f$), we obtain an isomorphic copy of $M_e^*[\![\{(\mathsf{V}_e, \mathsf{V}_e)\}]\!]$ given as $M_{\mathsf{V}_e} := \{(\emptyset, \emptyset), (\mathsf{V}_e, \mathsf{V}_f)\} \cup \{(A, B) : \emptyset \subset A \subset \mathsf{V}_e \text{ and } B = \emptyset \text{ or} B = \mathsf{V}_f\}$, ordered by $\subseteq$ component-wise.

Again, B_G contains a canonical copy of M_e^* as a subalgebra, realized as $\{\pi_G(A') : \emptyset \subseteq A \subseteq \mathsf{V}_e\}$ (ordered by $\subseteq$). Define $\varepsilon_e \colon M_{\mathsf{V}_e} \longrightarrow S_f \subseteq B_G \times \mathcal{P}(\mathsf{V}_f)$ by

$$\varepsilon_e((A, B)) = \begin{cases} (\pi_G(A'), \mathsf{V}_f) & \text{if } B = \mathsf{V}_e, \\ (\pi_G(A'), \emptyset) & \text{if } B = \emptyset. \end{cases}$$

We show that ε_e is an embedding of M_{V_e} into S_f, provided $\pi_G[(M_e^*)'] \cap \pi_G[(M_f^*)'] = \{0, 1\}$. Assume not. Then there is $\emptyset \subset A_e \subset \mathsf{V}_e$ such that $\pi_G(A_e') \in \pi_G[(M_e^*)'] \cap \pi_G[(M_f^*)']$. It follows by Corollary 4.9 that $\pi_G(A_e') = \pi_G(A_f')$ (where $\emptyset \neq A_f \neq \mathsf{V}_f$ is of the same cardinality as A_e) is a uniquely determined common atom or coatom of $\pi_G[(M_e^*)']$ and $\pi_G[(M_f^*)']$, and thus $\pi_G(A_f') \in Z_{G,f}$. We need to redefine $\varepsilon_e((A, B))$ in case $B = \emptyset$ to account for clause (iii) of Definition 5.5. Recall that $\Lambda(\pi_G(A_f')) = A_f$. If $\pi_G(A_f')$ is an atom, $(A_f, C) \in M_f$ iff $C = \emptyset$ and no modification of ε_e is necessary. So assume $\pi_G(A_f')$ is a coatom. To obtain $(A_f, C) \in M_f$ with $C \neq \mathsf{V}_f$, we must set $\varepsilon_e((A_e, \emptyset)) := (\pi_G(A_f'), A_f) = (\pi_G(A_e'), A_f)$ in this case.

It remains to show that ε_e so modified preserves meets. Consider $(A_1, B_1) \in M_{\mathsf{V}_e}$ such that $(A_1, B_1) \neq (A_e, \emptyset)$; the meet of $(A_e, \emptyset)$ and (A_1, B_1) in M_{V_e} is $(A_e \cap A_1, \emptyset)$. Turning to ε_e-images, observe that neither $\pi_G(A_1')$ nor $\pi_G((A_e \cap A_1)')$ are in $Z_{G,f}$ by Corollary 4.9. So $\varepsilon_e((A_1, B_1))$ is either $(\pi_G(A_1'), \mathsf{V}_f)$ or $(\pi_G(A_1'), \emptyset)$, and $\varepsilon_e((A_e \cap A_1, \emptyset))$ is $(\pi_G((A_e \cap A_1)'), \emptyset)$, while still $\varepsilon_e((A_e, \emptyset)) = (\pi_G(A_e'), A_f)$. By the proof of Lemma 5.7, case (1), the meet – taken in S_f – of $(\pi_G(A_e'), A_f) = (\pi_G(A_f'), A_f)$ with either $(\pi_G(A_1'), \mathsf{V}_f)$ or $(\pi_G(A_1'), \emptyset)$ is indeed $(\pi_G((A_f \cap A_1)'), \emptyset) = (\pi_G((A_e \cap A_1)'), \emptyset)$, as required.

On the other hand, we have the homomorphism $\varphi_{M_e} \colon M_e \longrightarrow M_{\mathsf{V}_e}$ provided by Lemma 5.3. The composition $\varepsilon_e \circ \varphi_{M_e}$ thus defines a homomorphism $\psi_e \colon M_e \longrightarrow S_f$ given explicitly by

$$\psi_e((A, B)) = \begin{cases} (\pi_G(A'), \mathsf{V}_f) & \text{if } B = \mathsf{V}_e, \\ (\pi_G(A'), A_f) & \text{if } B \subset \mathsf{V}_e,\ |A| = 5, \text{ and} \\ & \qquad \pi_G(A') = \pi_G(A_f') \in Z_{G,f} \text{ for } A_f \subset \mathsf{V}_f, \\ (\pi_G(A'), \emptyset) & \text{if } B \subset \mathsf{V}_e, \text{ otherwise.} \end{cases}$$

Corollary 5.8. *There exists a homomorphism* $\overline{\psi} \colon S = \coprod_{\mathbf{S}}(M_e : e \in E) \longrightarrow S_f$ *extending* ψ_f *and* ψ_e *for every* $e \in E$ *with* $e \neq f$.

6. Proof of Theorem 1.1

To establish Theorem 1.1, it is sufficient to show that (i) for any directed graphs G and $H \in \mathbf{G_c}$, $\Phi(h)(S_G) \not\subseteq S_H^*$ whenever $h \colon G \longrightarrow H$ is a compatible mapping, and that (ii) there exists a compatible mapping $h \colon G \longrightarrow H$ such that $\varphi = \Phi(h)$ whenever $\varphi \colon S_G \longrightarrow S_H$ is a homomorphism for which $\varphi(S_G) \not\subseteq S_H^*$.

Let $\overline{\pi}_G \colon S \longrightarrow S_G$ be the canonical projection with kernel Θ_G. There is an intimate connection between $\overline{\pi}_G$ and π_G. The canonical copy of M_e^* within $S_G = S/\Theta_G$ is $\{\overline{\pi}_G((A, \mathsf{V})_e)\}$ for $e \in E$ and $A \subseteq \mathsf{V}$ and the canonical copy of M_e^* within B_G, as constructed in §4, is $\{\pi_G(A')\}$ for $A \subseteq \mathsf{V}_e$. We will freely switch between these representations without explicitly specifying the associated isomorphism.

We start by observing that $\Theta_G \subseteq \ker \overline{\varphi}$ and $\Theta_G \subseteq \ker \overline{\psi}$ where $\overline{\varphi} \colon S \longrightarrow B_G[\![F_G]\!]$ and $\overline{\psi} \colon S \longrightarrow S_f$ (for any $f \in E$) are the homomorphisms given by Corollaries 5.4 and 5.8, respectively. Indeed, recall that Θ_G is the least congruence on S containing all pairs $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f)$ and $((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_e, (\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_f)$ for $e = (u, v)$ and $f = (v, w)$ in E. But then $\tau e = \sigma f$ and $\pi_G((\mathsf{V} \setminus \{\mathsf{a}\})_e') = \pi_G((\mathsf{V}_e \setminus \{\mathsf{a}_e\})') = \pi_G((\mathsf{V}_f \setminus \{\mathsf{b}_f\})') = \pi_G((\mathsf{V} \setminus \{\mathsf{b}\})_f')$. It follows that the partial homomorphisms φ_e and ψ_e used to define $\overline{\varphi}$ and $\overline{\psi}$ take the same values on both components of the pairs generating Θ_G (check the definitions of φ_e and ψ_e), and so these pairs are in $\ker \overline{\varphi}$ and $\ker \overline{\psi}$, respectively.

It follows that there exists homomorphisms

$$\varphi_G\colon S_G \longrightarrow B_G[\![F_G]\!] \text{ and } \psi_G\colon S_G \longrightarrow S_f$$

such that $\overline{\varphi} = \varphi_G \circ \overline{\pi}_G$ and $\overline{\psi} = \psi_G \circ \overline{\pi}_G$.

There are a number of immediate conclusions that we wish to make from these observations, namely Lemmas 6.1–6.5.

Lemma 6.1. *$\overline{\pi}_G$ is one-to-one on M_e for each $e \in E$.*

Proof. Consider some fixed $f \in E$. Then $\overline{\psi}$ extends ψ_f (see §5.2) which embeds M_f into S_f. Hence $\psi_G \circ \overline{\pi}_G$, and with that, $\overline{\pi}_G$ must be one-to-one on M_f. □

Lemma 6.2. *Let $x \in M_e$. Then $\overline{\pi}_G(x) \in S_G^*$ iff $x \in M_e^*$.*

Proof. Consider $x \in M_e \setminus M_e^*$. If $\overline{\pi}_G(x) \in S_G^*$, then $\overline{\pi}_G(x^{**}) = \overline{\pi}_G(x)$. But $x \neq x^{**} \in M_e^*$, violating Lemma 6.1. □

Lemma 6.3. *For $G = (V; E)$ and $H = (W; F) \in \mathbf{G_c}$, if $h\colon G \longrightarrow H$ is a compatible mapping, then $\mathbf{\Phi}(h)(S_G) \nsubseteq S_H^*$.*

Proof. Let $(A, B)_e \in M_e$. By Lemma 6.2, $\overline{\pi}_G((A, B)_e) \notin S_G^*$ iff $(A, B)_e \notin M_e^*$ which is the case iff $A \neq \emptyset$ and $B \subset \mathsf{V}$, by the properties of M_e. Since $\mathbf{\Phi}(h)(\overline{\pi}_G((A, B)_e)) = \overline{\pi}_H((A, B)_{(h(u),h(v))})$ for $e = (u, v) \in E$, $\mathbf{\Phi}(h)(S_G) \nsubseteq S_H^*$. □

Thus Lemma 6.3 provides part (i) of our proof of Theorem 1.1.

The following applies to any pseudocomplemented semilattice S. Let Z be any generating set for S, and $x \in S$. Then x may be written as $x = x^{**} \wedge z_0 \wedge \cdots \wedge z_{n-1}$ for some $n \in \mathbb{N}$ and $z_i \in Z \setminus S^*$ (see, for example, [29]). In any such representation, $x \notin S^*$ iff $n \neq 0$; thus, $x \notin S^*$ implies $x \leq z$ for some $z \in Z \setminus S^*$.

Lemma 6.4. *For $y, z \in S_G$, if $0 \neq y^* \leq z$, then $z \in S_G^*$.*

Proof. By Lemma 2.3, S_G is generated by $\{\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e) : \mathsf{x} \in \mathsf{V}$ and $e \in E\}$. Assume $y, z \in S_G$ and $y^* \leq z \notin S_G^*$. Thus $y^* \leq \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e)$ for some $\mathsf{x} \in \mathsf{V}$ and $e \in E$ by the preceding remark. Now $\varphi_G(y^*) \in B_G[\![F_G]\!]^*$ while $\varphi_G(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e)) = \overline{\varphi}((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e) = \varphi_e((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e) = (\pi_G((\mathsf{V}_e \setminus \{\mathsf{x}_e\})'), \emptyset) \notin B_G[\![F_G]\!]^*$. As φ_G is order-preserving, this implies $\varphi_G(y^*) = (\emptyset, \emptyset) = 0$ in $B_G[\![F_G]\!]^*$ and thus $y^* = 0$ since φ_G is one-to-one on $S_G^* \cong B_G[\![F_G]\!]^* \cong B_G$. □

Lemma 6.5. *For $\mathsf{x} \in \mathsf{V}$ and $e \in E$, the Glivenko class of $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)$ in S_G has precisely two elements, namely $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)$ and $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e)$.*

Proof. By definition, the Glivenko class of $(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e$ in M_e has precisely two elements, namely $(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e$ and $(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e$. By Lemma 6.1, the Glivenko class of $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)$ in S_G thus contains *at least* the two elements specified by the Lemma. Assume $y \in S_G$ is another member of this class, that is, $y^{**} = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)$. By the remark preceding Lemma 6.4, y may be written as $y = y^{**} \wedge z_0 \wedge \cdots \wedge z_{n-1}$ where the z_i belong to $Z \setminus S_G^*$, Z a set generating S_G; thus $y^{**} = y^{**} \wedge z_0^{**} \wedge \cdots \wedge z_{n-1}^{**}$ and $y^{**} \leq z_i^{**}$ for all $0 \leq i \leq n-1$. By Lemma 2.3, Z may be chosen as $\{\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e) : \mathsf{x} \in \mathsf{V} \text{ and } e \in E\}$, whence $Z \cap S_G^* = \emptyset$. Consider a fixed z_i. Then z_i^{**} has the form $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{w}\}, \mathsf{V})_f) = \overline{\pi}_G((\mathsf{V}_f \setminus \{\mathsf{w}_f\}, \mathsf{V}_f)$ for some $f \in E$ and $\mathsf{w}_f \in \mathsf{V}_f$, and we obtain $y^{**} = \overline{\pi}_G((\mathsf{V}_e \setminus \{\mathsf{x}_e\}, \mathsf{V}_e)) \leq \overline{\pi}_G((\mathsf{V}_f \setminus \{\mathsf{w}_f\}, \mathsf{V}_f)$. This comparability is within the Boolean algebra B_G, so we may take complements and obtain $\overline{\pi}_G((\{\mathsf{x}_e\}, \mathsf{V}_e)) \geq \overline{\pi}_G((\{\mathsf{w}_f\}, \mathsf{V}_f)$, in terms of B_G as constructed in §4 this reads as $\pi_G(\mathsf{x}'_e) \geq \pi_G(\mathsf{w}'_f)$. By Lemma 4.5 we obtain $\pi_G(\mathsf{x}'_e) = \pi_G(\mathsf{w}'_f)$, thus back in terms of S_G we have $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{w}\}, \mathsf{V})_f)$. By the definition of Θ_G then also $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{w}\}, \mathsf{V} \setminus \{\mathsf{w}\})_f)$ and we are done since this applies for every z_i $(0 \leq i \leq n-1)$. □

To obtain part (ii) of the proof of Theorem 1.1, assume that, for $G = (V, E)$ and $H = (W, F) \in \mathbf{G_c}$, $\varphi\colon S_G \longrightarrow S_H$ is any homomorphism for which $\varphi(S_G) \not\subseteq S_H^*$. It must be shown that $\varphi = \Phi(h)$ for some compatible map $h\colon G \longrightarrow H$.

Suppose that, for *some* $\mathsf{x} \in \mathsf{V}$ and $e \in E$, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)) = \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e))$. Denoting the restriction of $\varphi \circ \overline{\pi}_G$ to $M_e \subseteq S$ by α, we obtain by Lemma 2.5 that either $\ker \alpha \supseteq \Gamma_{M_e}$ or else that there exist $(A, B)_e, (C, D)_e \in M_e$ such that $\alpha((A, B)_e) \neq \alpha((A, \mathsf{V})_e)$ and $\alpha((A, B)_e) \geq \alpha((C, D)_e^*) \neq \alpha((\emptyset, \emptyset)_e)$. The latter option is ruled out by Lemma 6.4, thus $\ker \alpha \supseteq \Gamma_{M_e}$. In particular, we have $\alpha(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e) = \alpha((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e)$ for *all* $\mathsf{x} \in \mathsf{V}$. Consider $f \in E$ such that either $\sigma f = \tau e$ or $\tau f = \sigma e$. Then, by the definition of Θ_G, we have $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_e) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_f)$ or $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_f)) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_e)$, hence $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_f) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V} \setminus \{\mathsf{b}\})_f)$ or $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_f) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V} \setminus \{\mathsf{a}\})_f)$. So there is *some* $\mathsf{y} \in \mathsf{V}$ and such that $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{y}\}, \mathsf{V})_f)) = \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{y}\}, \mathsf{V} \setminus \{\mathsf{y}\})_f))$. Repeating the procedure and using connectivity of G, this implies by Lemma 6.5 that φ collapses the full Glivenko classes of all generators of S_G, implying $\varphi(S_G) \subseteq S_H^*$, contrary to hypothesis.

We conclude that $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)) \neq \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e))$ for every $\mathsf{x} \in \mathsf{V}$ and every $e \in E$.

Observe that S_H is generated by $\{\overline{\pi}_H((\mathsf{V} \setminus \{\mathsf{y}\}, \mathsf{V} \setminus \{\mathsf{y}\})_f) : \mathsf{y} \in \mathsf{V} \text{ and } f \in F\}$. Since, for every $\mathsf{x} \in \mathsf{V}$ and every $e \in E$, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)) \neq \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e))$, we conclude that, for each $\mathsf{x} \in \mathsf{V}$ and $e \in E$, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)) \leq \overline{\pi}_H((\mathsf{V} \setminus \{\mathsf{y}\}, \mathsf{V})_f)$ for some $\mathsf{y} \in \mathsf{V}$ and $f \in F$. Separated as it is by §5, Corollary 4.12 feels a long way away. However, this is precisely where it is needed. For each $e \in E$, by Corollary 4.12 with $\{y_i : 1 \leq i \leq 6\} = \phi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e) : \mathsf{x} \in \mathsf{V}\}$, there exists $\eta(e) \in F$ such that $\phi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e) = \overline{\pi}_H((\mathsf{V} \setminus \{\zeta(\mathsf{x})\}, \mathsf{V})_{\eta(e)})$ where $\zeta : \mathsf{V} \longrightarrow \mathsf{V}$ is one-to-one (and thus bijective). Furthermore, by Lemma 6.5, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus$

$\{\mathsf{x}\})_e)) = \overline{\pi}_H((\mathsf{V} \setminus \{\zeta(\mathsf{x})\}, \mathsf{V} \setminus \{\zeta(\mathsf{x})\})_{\eta(e)})$. By Lemma 2.7, ζ is the identity, that is, for every $\mathsf{x} \in \mathsf{V}$ and $e \in E$, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_e)) = \overline{\pi}_H((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V})_{\eta(e)})$ and $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e)) = \overline{\pi}_H((\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_{\eta(e)})$.

Consider $v \in V$. Then, since $G \in \mathbf{G_c}$, there exist $e_1, e_2 \in E$ such that $\tau e_1 = v = \sigma e_2$. In particular, $\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_{e_1}) = \overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_{e_2})$. Thus, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_{e_1})) = \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_{e_2}))$. Whence, $\varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{a}\}, \mathsf{V})_{\eta(e_1)})) = \varphi(\overline{\pi}_G((\mathsf{V} \setminus \{\mathsf{b}\}, \mathsf{V})_{\eta(e_2)}))$. Even further away are Corollary 4.6 and Lemma 4.7. However, this is the point where they are needed. By Corollary 4.6 and Lemma 4.7, this is only possible if $\eta(e_1) = f_1$ and $\eta(e_2) = f_2$, for $f_1, f_2 \in F$ such that $\tau f_1 = w = \sigma f_2$. Set $h(v) = w$. By Corollary 4.6 and Lemma 4.7, $h\colon V \longrightarrow W$ is a well-defined compatible mapping. Since $\mathbf{\Phi}(h)(\overline{\pi}_G((V \setminus \{\mathsf{x}\}, V \setminus \{\mathsf{x}\})_e)) = \varphi(\overline{\pi}_G((V \setminus \{\mathsf{x}\}, V \setminus \{\mathsf{x}\})_e))$ for every $\mathsf{x} \in \mathsf{V}$ and $e \in E$ and S_G is generated by $\{(\mathsf{V} \setminus \{\mathsf{x}\}, \mathsf{V} \setminus \{\mathsf{x}\})_e : \mathsf{x} \in \mathsf{V}$ and $e \in E\}$, we have $\varphi = \mathbf{\Phi}(h)$, as required.

7. Concluding remark

A pseudocomplemented semilattice is *relatively rigid* providing the only endomorphism $\varphi\colon S \longrightarrow S$ for which $\varphi(S) \nsubseteq S^*$ is the identity. Since there exists a proper class of non-isomorphic rigid connected strongly loopless graphs, it follows from Theorem 1.1 that there exists a proper class of non-isomorphic relatively rigid pseudocomplemented semilattices. Inspection of the functor $\mathbf{\Phi}$ shows that, for a graph $G = (V; E) \in \mathbf{G_c}$, $|\Phi(S_G)^*| \geq |E|$, which increases with $|V|$. This leads us to the following problem. Does there exist some cardinal κ for which there is a proper class of non-isomorphic relatively rigid pseudocomplemented semilattices such that, for each member S, $|S^*| \leq \kappa$?

Since the time of submission, Václav Koubek and Jiří Sichler have shown that every finite-to-finite almost universal variety $\mathbf{V}$ is Q-universal (*Almost ff-universality imples Q-universality*, to appear).

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References

[1] M. E. Adams and W. Dziobiak, *Finite-to-finite universal quasivarieties are Q-universal*, Algebra Universalis **46** (2001), 253–283.

[2] M. E. Adams, W. Dziobiak, M. Gould, and J. Schmid, *Quasivarieties of pseudocomplemented semilattices*, Fund. Math. **146** (1995), 295–312.

[3] M. E. Adams and M. Gould, *A construction for pseudocomplemented semilattices and two applications*, Proc. Amer. Math. Soc. **106** (1989), 899–905.

[4] R. Balbes, *On free pseudo-complemented and relatively pseudo-complemented semilattices*, Fund. Math. **78** (1973), 119–131.

[5] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

[6] M. Demlová and V. Koubek, *Endomorphism monoids of bands*, Semigroup Forum **38** (1989), 305–329.

[7] M. Demlová and V. Koubek, *Endomorphism monoids in small varieties of bands*, Acta Sci. Math. (Szeged) **55** (1991), 9–20.

[8] M. Demlová and V. Koubek, *Endomorphism monoids in varieties of bands*, Acta Sci. Math. (Szeged) **66** (2000), 477–516.

[9] O. Frink, *Pseudo-complements in semilattices*, Duke Math. J. **29** (1962), 505–514.

[10] V. A. Gorbunov, *Algebraic Theory of Quasivarieties*, Plenum Publishing Co., New York, 1998.

[11] G. Grätzer and J. Sichler, *On the endomorphism semigroup (and category) of bounded lattices*, Pacific J. Math. **35** (1970), 639–647.

[12] Z. Hedrlín and A. Pultr, *Symmetric relations (undirected graphs) with given semigroup*, Monatsh. Math. **69** (1965), 318–322.

[13] Z. Hedrlín and A. Pultr, *On full embeddings of categories of algebras*, Illinois J. Math. **10** (1966), 392–406.

[14] G. T. Jones, *Pseudocomplemented semilattices*, Ph.D. dissertation, U.C.L.A., 1972.

[15] G. T. Jones, *Projective pseudo-complemented semilattices*, Pacific J. Math. **52** (1974), 443–456.

[16] T. Katriňák and Z. Heleyová, *Free products of pseudocomplemented semilattices*, Semigroup Forum **60** (2000), 450–469.

[17] S. Koppelberg, *Handbook of Boolean Algebras I*, North-Holland, Amsterdam, 1989.

[18] V. Koubek and J. Sichler, *Universal varieties of semigroups*, J. Austral. Math. Soc. (Series A) **36** (1984), 143–152.

[19] V. Koubek and J. Sichler, *On relative universality and Q-universality*, Studia Logica **78** (2004), 279–291.

[20] K. D. Magill, *The semigroup of endomorphisms of a Boolean ring*, Semigroup Forum **4** (1972), 411–416.

[21] C. J. Maxson, *On semigroups of Boolean ring endomorphisms*, Semigroup Forum **4** (1972), 78–82.

[22] E. Mendelsohn, *On a technique for representing semigroups as endomorphism semigroups of graphs with given properties*, Semigroup Forum **4** (1972), 283–294.

[23] A. Pultr, *Concerning universal categories*, Comment. Math. Univ. Carolinae **6** (1964), 227–239.

[24] A. Pultr and V. Trnková, *Combinatorial, Algebraic, and Topological Representations of Groups, Semigroups, and Categories*, North-Holland, Amsterdam, 1980.

[25] H. P. Sankappanavar, *Principal congruences of pseudocomplemented semilattices and congruence extension property*, Proc. Amer. Math. Soc. **73** (1979), 308–312.

[26] H. P. Sankappanavar, *Remarks on subdirectly irreducible pseudocomplemented semi-lattices and distributive pseudocomplemented lattices*, Math. Japon. **25** (1980), 519–521.

[27] B. M. Schein, *Ordered sets, semilattices, distributive lattices, and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. **68** (1970), 31–50.

[28] J. Schmid, *Lee classes and sentences for pseudocomplemented semilattices*, Algebra Universalis **25** (1988), 223–232.

[29] J. Schmid, *On the structure of free pseudocomplemented semilattices*, Houston J. Math. **16** (1990), 71–85.

[30] J. Sichler, *Nonconstant endomorphisms of lattices*, Proc. Amer. Math. Soc. **34** (1972), 67–70.

[31] P. Vopěnka, Z. Hedrlín, and A. Pultr, *A rigid relation exists on any set*, Comment. Math. Univ. Carolinae **6** (1965), 149–155.

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