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Density property for hypersurfaces $UV = P(\bar{X})$

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Abstract We study hypersurfaces of $\mathbb{C}^{n+2}_{\bar{x},u,v}$ given by equations of form $uv = p(\bar{x})$ where the zero locus of a polynomial p is smooth reduced. The main result says that the Lie algebra generated by algebraic completely integrable vector fields on such a hypersurface coincides with the Lie algebra of all algebraic vector fields. Consequences of this result for some conjectures of affine algebraic geometry and for the Oka-Grauert-Gromov principle are discussed.

Keywords Density property · Oka-Grauert-Gromov principle · Andersén-Lempert-theory · Affine space

Mathematics Subject Classification (2000) Primary 32M05 · 14R20; Secondary 14R10 · 32M25

1 Introduction

The ground-breaking papers of Andersén and Lempert [1,3] were a starting point for intensive study of the holomorphic automorphism group of $\mathbb{C}^n$ $(n \geq 2)$. Their central observation was that

Each polynomial vector field on $\mathbb{C}^n$ $(n \geq 2)$ *is a finite sum of completely integrable polynomial vector fields*

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where a holomorphic vector field on a complex manifold is completely integrable if its flow generates a holomorphic $\mathbb{C}_+$-action on this manifold. This observation lead to their main result that implies, in particular, that any local holomorphic flow on a Runge domain Ω in $\mathbb{C}^n$ can be approximated by global holomorphic automorphisms of $\mathbb{C}^n$ (for an exact statement see Theorem 2.1 in [14]). This theorem has deep applications among which there are examples of non-rectifiable proper holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ [1] [11] (see also [5,8,9]) in sharp contrast with the algebraic situation where the famous Abhyankar–Moh–Suzuki theorem states that any algebraic embedding of a line in a plane is always equivalent to a linear one [4,28]. This was crucial for Derksen and the second author who constructed counterexamples to the Holomorphic Linearization Problem (they showed existence of non-linearizable holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^m$, $m \geq 4$ and, moreover, existence of such non-linearizable homomorphic actions for any compact Lie group K on $\mathbb{C}^n$ with n sufficiently large see [7,8]).

We would like to mention in this context a well-known conjecture in the algebraic setting:

Conjecture 1.1 (Abhyankar–Sathaye) Every polynomial embedding $\varphi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ is equivalent to the standard embedding, i.e., there is an algebraic automorphism of the ambient space $\alpha \in \mathrm{Aut}_{alg}(\mathbb{C}^n)$ such that $\alpha \circ \varphi(z) = (z, 0) \quad \forall z \in \mathbb{C}^{n-1}$.

The next step in the development of the Andersén–Lempert theory was made by Varolin who extended it from Euclidean spaces to a wider class of algebraic complex manifolds. He realized also that instead of presenting algebraic vector fields as a finite sum of integrable algebraic fields one can use Lie combinations of those fields (see Remark 5.6). This leads to the following.

Definition 1.2 A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly the algebraic density property implies the density property).

In this terminology Varolin's version of the Andersén–Lempert observation says that $\mathbb{C}^n$ ($n \geq 2$) has the algebraic density property. Varolin and Tóth [29,30,32] established the density property for some manifolds including semisimple complex Lie groups and some homogenous spaces of semisimple Lie groups.

In fact, in our next paper, using new criteria, we shall prove the algebraic density property for all linear algebraic groups different from tori or $\mathbb{C}_+$ [22]. However, in some cases this new approach does not work (at least in our hands) while computations that are closer to the original ideas of Andersén and Lempert give the desired result.

One of the aims of the paper is to present this computation which implies, in particular, the algebraic density property for the following important class of affine algebraic varieties (see Theorem 1).

Theorem (Main Theorem) *Let $p \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a polynomial with a smooth reduced zero fiber. Then the hypersurface*

[1] Some non-rectifiable proper holomorphic embeddings of $\mathbb{C}^k$ into $\mathbb{C}^n$ were constructed earlier via the Rosay–Rudin theory. This includes non-rectifiable embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^n$ with $n \geq 3$ [19]. However, the case of $n = 2$ remained resistant until the Andersén–Lempert theory.

$$X_p := \{(\bar{x}, u, v) \in \mathbb{C}^{n+2} : uv = p(\bar{x}), \quad \bar{x} = (x_1, x_2, \ldots, x_n)\}$$

has the algebraic density property.

At this point we like to mention the ambitious problem to find a characterization of affine space $\mathbb{C}^n$ among affine algebraic manifolds ($n \geq 3$) in the algebraic category and among Stein manifolds ($n \geq 2$) in the holomorphic category. One attempt is to use the group of automorphisms. Mentioning that all known examples of manifolds with the density property except $\mathbb{C}^n$ itself have nontrivial topology, Varolin and Tóth formulated the following conjecture:

Conjecture 1.3 (Tóth–Varolin) If X is a Stein manifold with the density property and X is diffeomorphic to $\mathbb{C}^n$, then X is biholomorphic to $\mathbb{C}^n$.

There is also the following question relating the algebraic and holomorphic settings.

Question 1.4 (Zaidenberg) *Is there a complex affine algebraic variety biholomorphic to $\mathbb{C}^n$ but not isomorphic to $\mathbb{C}^n$?*

Among the hypersurfaces in our main theorem there is a class of smooth contractible affine algebraic varieties (which are automatically diffeomorphic to Euclidean spaces as real manifolds starting with complex dimension 3, see [6]) that do not admit obvious (algebraic or holomorphic) isomorphisms with Euclidean spaces (as an example one can consider the hypersurface of $\mathbb{C}^6$ given by $uv = x + x^2y + z^2 + t^3$).

These examples allow us to relate the above mentioned conjectures of Tóth–Varolin and of Abhyankar–Sathaye to Zaidenberg's question. They give a counterexample to one of the two conjectures or a positive answer to Zaidenberg's question (see Proposition 4.6). Unfortunately we can not decide at the moment which of the cases holds.

As an application of the holomorphic version of our main theorem (see Theorem 2) we prove in the present paper that the Tóth–Varolin-Conjecture implies the existence of a holomorphic action of the rotation group $SO_2(\mathbb{R}) = S^1$ on $\mathbb{C}^n$ ($n \geq 4$) with fixed point set biholomorphic to the unit disc (see the Remarks after lemma 4.7).

Some implications of the density property which may not be so well-known to the specialists in affine algebraic geometry are considered at the end of the paper:

We prove that any complex manifold X with the density property satisfies the Oka–Grauert–Gromov principle since X admits a spray. Therefore, by the work of Gromov (see [18] and for complete proofs see [12,13]) we extend the classical results of Grauert, Forster and Ramspott about the validity of this principle to submersions with fibers that are smooth hypersurfaces of the form given in the Main Theorem.

Another implication of the density property is that every point of a contractible smooth hypersurface $uv = p(\bar{x})$ in $\mathbb{C}^{n+2}$, ($n \geq 2$) possesses a (Fatou–Bieberbach) neighborhood biholomorphic to $\mathbb{C}^{n+1}$.

The paper is organized as follows. In Sect. 2 we give criteria ensuring the algebraic density property for a hypersurface $uv = p(\bar{x})$ (this is the main technical part of the paper). In Sect. 3 we give a holomorphic version of our results. In Sect. 4 we describe relations between the density property for these hypersurfaces and the conjectures and to group actions on $\mathbb{C}^n$. Finally, in Sect. 5 we list some implications of the density property. Also we establish the existence of a Gromov's spray for manifolds with the density property and as a consequence we prove the Oka–Grauert–Gromov principle for morphisms with fiber isomorphic to hypersurfaces from Theorem 1.

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2 Algebraic density property for a hypersurface $uv = p(\bar{x})$

Recall that a holomorphic vector field $V \in \mathrm{VF}_{hol}(\mathbb{C}^n)$ is completely integrable if for any initial value $z \in \mathbb{C}^n$ there is an entire holomorphic function $\gamma : \mathbb{C} \to \mathbb{C}^n$ solving the ordinary differential equation

$$\dot{\gamma}(t) = V(\gamma(t)), \quad \gamma(0) = z. \tag{1}$$

In this case the flow (i.e. the map $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(t, z) \mapsto \gamma_z(t)$) is a holomorphic action of the additive group $\mathbb{C}_+$ on $\mathbb{C}^n$, where index z in γ_z denotes the dependence on the initial value. It is worth mentioning that this action is not necessarily algebraic in the case of an algebraic vector field $V \in \mathrm{VF}_{\mathrm{alg}}(\mathbb{C}^n)$.

In addition to the density property given in Definition 1.2, Varolin introduced the following notion of volume density property.

Definition 2.1 Let ω be a holomorphic nowhere vanishing n-form on a complex manifold X of dimension n (we call such ω a volume form). We say that X has the volume density property with respect to ω if the Lie algebra $\mathrm{Lie}^{\omega}_{\mathrm{hol}}$ generated by completely integrable holomorphic vector fields ν such that $L_\nu(\omega) = 0$, is dense in the Lie algebra $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ of all holomorphic vector fields that annihilate ω.

If X is affine algebraic we say that X has the algebraic volume density property with respect to ω if the Lie algebra $\mathrm{Lie}^{\omega}_{\mathrm{alg}}$ generated by completely integrable algebraic vector fields ν such that $L_\nu(\omega) = 0$, coincides with the Lie algebra $\mathrm{VF}^{\omega}_{\mathrm{alg}}(X)$ of all algebraic vector fields that annihilate ω.

Let us discuss some simple properties of the divergence $\mathrm{div}_\omega(\nu)$ of a vector field ν on X with respect to this volume form ω. The divergence is defined by the equation

$$\mathrm{div}_\omega(\nu)\omega = L_\nu(\omega) \tag{2}$$

where L_ν is the Lie derivative. Furthermore, for any vector fields ν_1, ν_2 on X we have the following relation between divergence and Lie bracket

$$\mathrm{div}_\omega([\nu_1, \nu_2]) = L_{\nu_1}(\mathrm{div}_\omega(\nu_2)) - L_{\nu_2}(\mathrm{div}_\omega(\nu_1)). \tag{3}$$

In particular, when $\mathrm{div}_\omega(\nu_1) = 0$ we have

$$\mathrm{div}_\omega([\nu_1, \nu_2]) = L_{\nu_1}(\mathrm{div}_\omega(\nu_2)). \tag{4}$$

Another useful formula is

$$\mathrm{div}_\omega(f\nu) = f\,\mathrm{div}_\omega(\nu) + \nu(f) \tag{5}$$

for any holomorphic function f on X.

Lemma 2.2 *Let Y be a Stein complex manifold with a volume form Ω on it, and X be a submanifold of Y which is a strict complete intersection (that is, the defining ideal of X is generated by holomorphic functions $P_1, \ldots, P_k$ on Y, where k is the codimension of X in Y). Suppose that ν is a vector field on X and μ is its extension to Y such that $\mu(P_i) = 0$ for every $i = 1, \ldots, k$. Then*

(i) *there exists a volume form ω on X such that $\Omega|_X = dP_1 \wedge \ldots \wedge dP_k \wedge \omega$; and*
(ii) $\mathrm{div}_\omega(\nu) = \mathrm{div}_\Omega(\mu)|_X$.

Proof Let $x_1, \ldots, x_n$ be a local holomorphic coordinate system in a neighborhood of a point in X. Then $P_1, \ldots, P_k, x_1, \ldots, x_n$ is a local holomorphic coordinate system in a neighborhood of this point in Y. Hence in the last neighborhood $\Omega = h dP_1 \wedge \ldots \wedge dP_k \wedge dx_1 \wedge \ldots \wedge dx_n$ where h is a holomorphic function. Set $\omega = h|_X dx_1 \wedge \ldots \wedge dx_n$. This is the desired volume form in (i).

Recall that $L_\nu = d \circ \iota_\nu + \iota_\nu \circ d$ where ι_ν is the interior product with respect to ν ([24], Chapt. 1, Proposition 3.10). Since $\mu(P_i) = 0$ we have $L_\mu(dP_i) = 0$. Hence by formula (2) we have $\mathrm{div}_\Omega(\mu)\Omega|_X = L_\mu \Omega|_X = L_\mu(dP_1 \wedge \ldots \wedge dP_k \wedge \omega)|_X = dP_1 \wedge \ldots \wedge dP_k|_X \wedge L_\nu \omega + L_\mu(dP_1 \wedge \ldots \wedge dP_k)|_X \wedge \omega = \mathrm{div}_\omega(\nu)(dP_1 \wedge \ldots \wedge dP_k)|_X \wedge \omega = \mathrm{div}_\omega(\nu)\Omega|_X$ which is (ii). ☐

Remark 2.3 1. Lemma 2.2 enables us conveniently to compute the divergence of a vector field on X via the divergence of a vector field extension on an ambient space. It is worth mentioning that there is another simple way to compute divergence on X which leads to the same formulas in Lemma 2.6 below. Namely, X that we are going to consider will be an affine modification $\sigma : X \to Z$ of another affine algebraic manifold Z with a volume form ω_0 (for definitions of affine and pseudo-affine modifications see [21] and Appendix 5.9). In particular, for some divisors $D \subset Z$ and $E \subset X$ the restriction of σ produces an isomorphism $X \setminus E \to Z \setminus D$. One can suppose that D coincides with the zero locus of a regular (or holomorphic) function α on Z. In the situation we are going to study, the function $\tilde{\alpha} = \alpha \circ \sigma$ has simple zeros on E. Consider the form $\sigma^* \omega_0$ on X. It may vanish on E only. Dividing this form by some power $\tilde{\alpha}^k$ we get a volume form on X. In order to compute divergence of a vector field on X it suffices to find this divergence on the Zariski open subset $X \setminus E \simeq Y \setminus D$, i.e. we need to compute the divergence of a vector field ν on $Y \setminus D$ with respect to a volume form $\beta\omega_0$ where $\beta = \alpha^{-k}$. The following formula relates it with the divergence with respect to ω_0:

$$\mathrm{div}_{\beta\omega_0}(\nu) = \mathrm{div}_{\omega_0}(\nu) + L_\nu(\beta)/\beta.$$

In the cases, we need to consider, β will be often in the kernel of ν, i.e. $\mathrm{div}_{\beta\omega_0}(\nu) = \mathrm{div}_{\omega_0}(\nu)$ in these cases.

2. If the normal bundle of $X \subset \mathbb{C}^n$ is trivial we may choose ω as the restriction of the standard volume form on $\mathbb{C}^n$ by Lemma 2.2. Indeed, taking n sufficiently large we can always assume that X is a complete intersection in $\mathbb{C}^n$ (see for example [27]).

The condition in Lemma 2.2 that an algebraic field ν on X has an extension μ on Y with $\mu(P_i) = 0$ is also very mild. We consider it in the case of hypersurfaces only.

Lemma 2.4 *Let X be a smooth hypersurface in a complex Stein* (*resp. affine algebraic*) *manifold Y given by zero of a reduced holomorphic* (*resp. algebraic*) *function P on Y. Then every holomorphic* (*resp. algebraic*) *vector field ν on X has a similar extension μ to Y such that $\mu(P) = 0$.*

Proof Consider, for instance, the algebraic case, i.e. P belongs to the ring $\mathbb{C}[Y]$ of regular functions on Y. Since μ must be tangent to X we see that $\mu(P)$ vanishes on X, i.e. $\mu(P) = PQ$ where $Q \in \mathbb{C}[Y]$. Any other algebraic extension of ν is of form $\tau = \mu - P\theta$ where $\theta \in \mathrm{VF}_{\mathrm{alg}}(Y)$. Thus if $\theta(P) = Q$ then we are done.

In order to show that such θ can be found consider the set $M = \{\theta(P) | \theta \in \mathrm{VF}_{\mathrm{alg}}(Y)\}$. One can see that M is an ideal of $\mathbb{C}[Y]$. Therefore, it generates a coherent sheaf $\mathcal{F}$ over Y. The restriction $Q|_{Y \setminus X}$ is a section of $\mathcal{F}|_{Y \setminus X}$ because $Q = \mu(P)/P$. Since X is smooth for every point $x \in X$ there are a Zariski open neighborhood of X and an algebraic vector field ∂ such that $\partial(P)$ does not vanish on U. Hence $Q|_U$ is a section of $\mathcal{F}|_U$. Since $\mathcal{F}$ is coherent this implies that Q is a global section of $\mathcal{F}$ and, therefore, $Q \in M$ which is the desired conclusion. □

2.5 Terminology and notation

In the rest of this section X is a closed affine algebraic submanifold of $\mathbb{C}^n$, ω is a volume form on X, p is a regular function on X such that the divisor $p^*(0)$ is smooth reduced, X' is the hypersurface in $Y = \mathbb{C}^2_{u,v} \times X$ given by the equation $P := uv - p = 0$.[2] Note that X' is smooth and, therefore, Lemma 2.4 is applicable. We shall often use the fact that every regular function f on X' can be presented uniquely as the restriction of a regular function on Y of the form

$$f = \sum_{i=1}^{m} (a_i u^i + b_i v^i) + a_0 \tag{6}$$

where $a_i = \pi^*(a_i^0)$, $b_i = \pi^*(b_i^0)$ are lift-ups of regular functions a_i^0, b_i^0 on X via the natural projection $\pi : Y \to X$ (as we mentioned by abusing terminology we shall say that a_i and b_i themselves are regular functions on X).

Let $\Omega = du \wedge dv \wedge \omega$, i.e. it is a volume form on Y. By Lemma 2.2 there is a volume form ω' on X' such that $\Omega|_{X'} = dP \wedge \omega'$. Furthermore, for any vector field μ such that $\mu(P) = 0$ and $\nu' = \mu|_{X'}$ we have $\mathrm{div}_{\omega'}(\nu') = \mathrm{div}_{\Omega}(\mu)|_X$. Note also that any vector field ν on X generates a vector field κ on Y that annihilates u and v. We shall always denote $\kappa|_{X'}$ by $\tilde{\nu}$. It is useful to note for further computations that $u^i \pi^*(\mathrm{div}_\omega(\nu)) = \mathrm{div}_\Omega(u^i \kappa)$ for every $i \geq 0$. Note also that every algebraic vector field λ on X' can be written uniquely in the form

$$\lambda = \tilde{\mu}_0 + \sum_{i=1}^{m} (u^i \tilde{\mu}_i^1 + v^i \tilde{\mu}_i^2) + f_0 \partial/\partial u + g_0 \partial/\partial v \tag{7}$$

where μ_0, μ_i^j are algebraic vector fields on X, and f_0, g_0 are regular functions on X'.

For any algebraic manifold Z with a volume form ω we denote by $\mathrm{Lie}_{\mathrm{alg}}(Z)$ (resp. $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(Z)$) the Lie algebra generated by algebraic completely integrable vector fields on Z (resp. that annihilate ω) and by $\mathrm{VF}_{\mathrm{alg}}(Z)$ we denote the Lie algebra of all algebraic vector fields on Z. We have a linear map

$$\widetilde{\mathrm{Pr}} : \mathrm{VF}_{\mathrm{alg}}(X') \to \mathrm{VF}_{\mathrm{alg}}(X)$$

defined by $\widetilde{\mathrm{Pr}}(\lambda) = \mu_0$ where λ and μ_0 are from formula (7). The following facts are straightforward calculations that follow easily from Lemma 2.2.

Lemma 2.6 *Let ν_1, ν_2 be vector fields on X, and f be a regular function on X. For $i \geq 0$ consider the algebraic vector fields*

$$\nu_1' = u^{i+1} \tilde{\nu}_1 + u^i \nu_1(p) \partial/\partial v, \quad \nu_2' = v^{i+1} \tilde{\nu}_2 + v^i \nu_2(p) \partial/\partial u$$

and $\mu_f = f(u\partial/\partial u - v\partial/\partial v)$ on Y. Then

(i) *ν_i' and μ_f are tangent to X' (actually they are tangent to fibers of $P = uv - p(x)$), i.e., they can be viewed as vector fields on X';*

[2] By abusing notation we treat p in this formula as a function on Y, and, if necessary, we treat it as a function on X'. Furthermore, by abusing notation, for any regular function on X we denote its lift-up to Y or X' by the same symbol.

(ii) μ_f *is always completely integrable on* X', *and* ν_i' *is completely integrable on* X' *if* ν_i *is completely integrable on* X;

(iii) $\operatorname{div}_{\omega'}(\mu_f) = 0$, $\operatorname{div}_{\omega'}(\nu_1') = u^{i+1}\operatorname{div}_\omega(\nu_1)$, $\operatorname{div}_{\omega'}(\nu_2') = v^{i+1}\operatorname{div}_\omega(\nu_2)$, *and*

$$\operatorname{div}_{\omega'}([\mu_f, \nu_1'] = (i+1)u^{i+1} f \operatorname{div}_\omega(\nu_1),\ \operatorname{div}_{\omega'}([\nu_2', \mu_f] = (i+1)v^{i+1} f \operatorname{div}_\omega(\nu_2);$$

(iv) *we have the following Lie brackets*

$$[\mu_f, \nu_1'] = (i+1)u^{i+1} f\tilde{\nu}_1 + \alpha_1 \partial/\partial u + \beta_1 \partial/\partial v,$$
$$[\nu_2', \mu_f] = (i+1)v^{i+1} f\tilde{\nu}_2 + \alpha_2 \partial/\partial u + \beta_2 \partial/\partial v,$$

where α_i *and* β_i *are some regular functions on* X';

(v) *more precisely, if* $i = 0$ *in formulas for* ν_1' *and* ν_2' *then*

$$[\mu_f, \nu_1'] = fu\tilde{\nu}_1 - u^2\nu_1(f)\partial/\partial u + \nu_1(fp)\partial/\partial v,$$
$$[\nu_2', \mu_f] = fv\tilde{\nu}_2 - v^2\nu_2(f)\partial/\partial v + \nu_2(fp)\partial/\partial u;$$

and

$$\widetilde{\Pr}([[\mu_f, \nu_1'], \nu_2']) = \nu_1(fp)\nu_2 - \nu_2(fp)\nu_1 + fp[\nu_1, \nu_2]. \tag{8}$$

2.7 Additional notation

For every affine algebraic manifold Z let $\mathbb{C}[Z]$ be the algebra of its regular functions, $\operatorname{IVF}_{\mathrm{alg}}(Z)$ be the set of completely integrable algebraic vector fields on Z. If there is a volume form ω on Z then we denote by $\operatorname{Div}_Z : \operatorname{VF}_{\mathrm{alg}}(Z) \to \mathbb{C}[Z]$ the map that assigns to each vector field its divergence with respect to ω, and set $\operatorname{IVF}^\omega_{\mathrm{alg}}(Z) = \operatorname{Ker} \operatorname{Div}_Z \cap \operatorname{IVF}_{\mathrm{alg}}(Z)$, $\operatorname{VF}^\omega_{\mathrm{alg}}(Z) = \operatorname{Ker} \operatorname{Div}_Z \cap \operatorname{VF}_{\mathrm{alg}}(Z)$. For a closed submanifold C of Z denote by $\operatorname{VF}_{\mathrm{alg}}(Z, C)$ the Lie algebra of vector fields on Z that are tangent to C. Formula (6) yields a monomorphism of vector spaces $\iota : \mathbb{C}[X'] \hookrightarrow \mathbb{C}[Y]$ and the natural embedding $X \hookrightarrow X \times (0,0) \subset Y$ generates a projection $\Pr : \mathbb{C}[Y] \to \mathbb{C}[X]$. Note that $\Pr(\iota(f)) = a_0$ in the notation of formula (6).

Proposition 2.8 *Let* C *be the smooth zero locus of* p *in* X. *Suppose also that the following conditions hold:*

(A1) *the linear space* $\operatorname{VF}_{\mathrm{alg}}(X, C)$ *is generated by vector fields that are of the form* $\widetilde{\Pr}([[\mu_f, \nu_1'], \nu_2'])$ *where* μ_f *and* ν_i' *are as in formula* (8) *from Lemma* 2.6 *with* $\nu_i \in \operatorname{IVF}_{\mathrm{alg}}(X)$;

(A2) $\operatorname{VF}_{\mathrm{alg}}(X)$ *is generated by* $\operatorname{IVF}^\omega_{\mathrm{alg}}(X)$ *as a module over* $\mathbb{C}[X]$;

(A3) $\operatorname{Div}_X(\operatorname{VF}_{\mathrm{alg}}(X))$ *is generated by* $\operatorname{Div}_X(\operatorname{IVF}_{\mathrm{alg}}(X))$ *over* $\mathbb{C}[X]$ (*for instance, the ideal generated by* $\operatorname{Div}_X(\operatorname{IVF}_{\mathrm{alg}}(X))$ *in* $\mathbb{C}[X]$ *coincides with* $\mathbb{C}[X]$).

Then $\operatorname{Lie}_{\mathrm{alg}}(X')$ *coincides with* $\operatorname{VF}_{\mathrm{alg}}(X')$, *i.e.,* X' *has the algebraic density property.*

Proof Let λ, f_0, and g_0 be as in formula (7) and $\Lambda = \iota(\lambda)$ be the extension of λ to Y also given by formula (7). By formula (6) f_0 and g_0 can be written uniquely in the form

$$f_0 = \sum_{i=1}^m (a_i u^i + b_i v^i) + a_0 \text{ and } g_0 = \sum_{i=1}^m (\hat{a}_i u^i + \hat{b}_i v^i) + \hat{a}_0$$

where $a_i, \hat{a}_i, b_i, \hat{b}_i \in \mathbb{C}[X]$.

Since Λ is a vector field tangent to $X' = P^{-1}(0)$ we have $\Lambda(P)|_{X'} = 0$. Thus $0 = \Pr(\iota(\Lambda(P)|_{X'})) = p(a_1 + \hat{b}_1) - \mu_0(p)$ (recall that $P = uv - p(x)$). Hence $\mu_0(p)$ vanishes on C, i.e. $\mu_0 \in \operatorname{VF}_{\mathrm{alg}}(X, C)$. Let $\mu_f, \nu_i' \in \operatorname{IVF}_{\mathrm{alg}}(X')$ be as in Lemma 2.6. Condition (A1) implies that adding elements of the form $[[\mu_f, \nu_1'], \nu_2']$ to λ we can suppose that $\mu_0 = 0$.

Now we have $p(a_1 + \hat{b}_1) = 0$, i.e. $a_1 = -\hat{b}_1$. This implies that $\Pr(\iota(\mathrm{div}_\Omega(\Lambda)|_{X'})) = 0$. By Lemma 2.4 there exists a vector field of the form $\tau = \Lambda - P\theta$ on Y such that $\tau(P) = 0$. Hence

$$(uv - p)\theta(uv - p) = \Lambda(uv - p).$$

Formula (7) and the form of f_0 and g_0 imply that $\Lambda(P)$ does not contain nonzero monomials $u^k v^l$ with both $k \geq 2$ and $l \geq 2$ (as a polynomial in u and v). Thus $\theta(P)$ is of form given by formula (6). We have also $\Pr(\theta(P)) = 0$ because $\Pr(\Lambda(P)) = 0$, and, therefore, $\Pr(\iota(\theta(P)|_{X'}) = 0$. By Lemma 2.2 and formula (5) $\mathrm{div}_{\omega'}(\lambda) = \mathrm{div}_\Omega(\tau)|_{X'} = (\mathrm{div}_\Omega(\Lambda) - P\,\mathrm{div}_\Omega(\theta) - \theta(P))|_{X'}$. Since $\iota(P\,\mathrm{div}_\Omega(\theta)|_{X'}) = 0$ we have $\Pr(\iota(\mathrm{div}_{\omega'}(\lambda))) = 0$ now. By condition (A3) $\mathrm{Div}_X(\mathrm{VF}_{\mathrm{alg}}(X))$ is generated by $\mathrm{Div}_X(\mathrm{IVF}_{\mathrm{alg}}(X))$ over $\mathbb{C}[X]$ and, therefore, adding vector fields of the form $[\mu_f, \nu_1']$ and $[\mu_f, \nu_2']$, we can suppose by Lemma 2.6 (iii) that $\mathrm{div}_{\omega'}(\lambda) = 0$. Note that this addition leaves μ_0 equal to 0 since $\widetilde{\Pr}([\mu_f, \nu_i']) = 0$. Taking into consideration condition (A2) and Lemma 2.6 (iv) we can make $\mu_i^j = 0$ by adding fields of the form $[\mu_f, \nu_i']$ with $\nu_i \in \mathrm{IVF}^\omega_{\mathrm{alg}}(X)$. Note that this addition leaves not only μ_0 equal to 0 but also $\mathrm{div}_{\omega'}(\lambda)$ equal to 0, since $\mathrm{div}_{\omega'}([\mu_f, \nu_i']) = 0$ as soon as $\mathrm{div}_\omega(\nu_i) = 0$. Hence $\lambda = f\partial/\partial u + g\partial/\partial v$ and $\Lambda(P)|_{X'} = fv + gu = 0$.

Using formula (6) one can see that f must be divisible by u, and g by v. That is, there exists a regular function h on X' for which $f = uh$ and $g = -vh$. Hence $\lambda = h(u\partial/\partial u - v\partial/\partial v)$. Note that $\Lambda(P) = 0$ now. Thus $0 = \mathrm{div}_{\omega'}(\lambda) = \mathrm{div}_\Omega(\Lambda)|_{X'} = (u\partial h/\partial u - v\partial h/\partial v)|_{X'}$. Taking h as in formula (6) we see that h is independent of u and v. Thus λ is integrable by Lemma 2.6 (ii). □

Lemma 2.9 *Condition* (A1) *in Proposition* 2.8 *is a consequence of the following two conditions:*

(B1) $\mathrm{VF}_{\mathrm{alg}}(X)$ *is generated as a* $\mathbb{C}[X]$*-module by vector fields of the form* $[\nu_1, \nu_2]$ *where the vector fields* $\nu_1, \nu_2 \in \mathrm{IVF}_{\mathrm{alg}}(X)$ *are proportional* (*here proportional is meant in the* $\mathbb{C}[X]$*-module structure, i.e., there are functions* f_1 *and* f_2 *such that* $f_1\nu_1 + f_2\nu_2 = 0$)*;*

(B2) *If C is the zero fiber of p then the* $\mathbb{C}[C]$*-module* $\mathrm{VF}_{\mathrm{alg}}(C)$ *is generated by vector fields of the form* $\gamma|_C$ *where* $\gamma = \nu_1(p)\nu_2 - \nu_2(p)\nu_1$ *with* $\nu_1, \nu_2 \in \mathrm{IVF}_{\mathrm{alg}}(X)$.

Proof Note that when ν_1 and ν_2 are proportional $\widetilde{\Pr}([[\mu, \nu_i'], \nu_2'] = fp[\nu_1, \nu_2]$ in formula (8). Condition (B1) implies that linear combinations of such vector fields produce any vector field of the form $p\nu$ with $\nu \in \mathrm{VF}_{\mathrm{alg}}(X)$. Hence condition (B1) has the consequence that for every $\kappa \in \mathrm{VF}_{\mathrm{alg}}(X)$ there exists $\lambda \in \mathrm{Lie}(X')$ [more specifically λ is a linear combination of vector fields of the form $\widetilde{\Pr}([[\mu_f, \nu_1'], \nu_2'])$ as required in Condition (A1)] with $\widetilde{\Pr}(\lambda) = p\kappa$.

Note further that γ in condition (B2) differs from a vector field in formula (8) by a vector field divisible by p, i.e., of the form $p\kappa$.

Now take an arbitrary $\theta \in \mathrm{VF}_{\mathrm{alg}}(X, C)$. By condition (B2) its restriction to C is a sum of fields of the form $f(\nu_1(p)\nu_2 - \nu_2(p)\nu_1)$. Thus subtracting from θ the corresponding sum of fields of the form $\widetilde{\Pr}([[\mu_f, \nu_1'], \nu_2'])$ we get a field which is divisible by p, i.e., vanishes on C. The mentioned consequence of condition (B1) concludes now the proof. □

As a first application we set $p(x)$ in the definition of X' equal to a nonzero constant, in that case C is empty and X' is isomorphic to $X \times \mathbb{C}^*$. This special case seems worth to be stated separately.

Corollary 2.10 *Let X be an affine algebraic manifold with a volume form ω such that condition (B1) from Lemma* 2.9 *and conditions* (A2) *and* (A3) *from Proposition* 2.8 *hold. Then* $X \times \mathbb{C}^*$ *has the density property.*

Here comes the next application of Proposition 2.8 and Lemma 2.9:

Lemma 2.11 *Let $X = \mathbb{C}^n$ with a coordinate system $(x_1, \ldots, x_n)$ and $p_1, \ldots, p_n$ be the partial derivatives of p. Then:*

(i) *Condition (B1) holds.*
(ii) *Condition (B2) follows from the smoothness of the zero fiber C of p (i.e. from the fact that the partial derivatives $p_i = \partial p/\partial x_i$ of p have no common zeros on C).*
(iii) *In particular, condition (A1) from Lemma 2.8 holds, when the zero fiber C of p is smooth.*

Proof Taking $\nu_1 = \partial/\partial x_i$ and $\nu_2 = x_i\partial/\partial x_i$ we get the vector field $[\nu_1, \nu_2] = \partial/\partial x_i$ which implies condition (i).

There is nothing to prove when $n = 1$. For $n > 1$ take $\nu_1 = \partial/\partial x_i$ and $\nu_2 = \partial/\partial x_j$. We get γ from condition (B2) equal to $p_j\partial/\partial x_i - p_i\partial/\partial x_j$. Hence $\mathbb{C}[X]$-combinations of such fields include any field of the form

$$\lambda = \sum_{i,j} q_{i,j}(p_j\partial/\partial x_i - p_i\partial/\partial x_j) \tag{9}$$

where $q_{i,j}$ are arbitrary polynomials on X.

Since the partial derivatives of p have no common zeros on C, such vector fields $\lambda|_C$ generate the tangent bundle TC of C at each point. By a standard application of Theorem B of Serre they generate the global sections of TC as $\mathbb{C}[C]$-module. This is (ii). □

Combining Lemmas 2.11, 2.9 and Proposition 2.8 we conclude the main result of this section:

Theorem 1 *Let $p \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a polynomial with smooth reduced zero fibre, i.e., the partial derivatives $p_i = \partial p/\partial x_i$ of p have no common zeros on the zero fiber of p. Then the hypersurface*

$$X_p := \{(\bar{x}, u, v) \in \mathbb{C}^{n+2} : uv = p(\bar{x}), \quad \bar{x} = (x_1, x_2, \ldots, x_n)\}$$

has the algebraic density property.

Proof We check that the conditions in Proposition 2.8 are fulfilled. Lemma 2.11 (iii) ensures condition (A1). $\mathrm{IVF}^{\omega}_{\mathrm{alg}}(\mathbb{C}^n)$ with respect to the standard volume form ω contains the fields $\partial/\partial x_i$ and those generates $\mathrm{VF}_{\mathrm{alg}}(\mathbb{C}^n)$ as $\mathbb{C}[x_1, x_2, \ldots, x_n]$ module. Finally $\mathrm{Div}_X(\mathrm{IVF}_{\mathrm{alg}}(\mathbb{C}^n))$ contains $\mathrm{div}_\omega(x_1\partial/\partial x_1) \equiv 1$ and thus the ideal generated by it is equal to $\mathbb{C}[x_1, x_2, \ldots, x_n]$. □

Example 2.12 (1) For $n = 1$ we get the density property for Danilievsky surfaces $uv = p(x)$ where the polynomial p has no multiple roots.
(2) For $n = 2$ (i.e. $X = \mathbb{C}^2_{x,y}$) we have the density property for the hypersurface X' in $\mathbb{C}^4_{x,y,u,v}$ given by $xy - uv = 1$ which is, of course, $SL(2, \mathbb{C})$. Thus we got a new proof of the fact that $SL(2, \mathbb{C})$ has the density property, established first in [31].

3 Generalizations to analytic hypersurfaces

3.1 The analytic case

We need to change terminology and notation in order to give a holomorphic analog of the results in Sect. 2.

In this section X is a Stein manifold, ω is a holomorphic volume form on X, p is a holomorphic function on X such that the divisor $p^*(0)$ is smooth reduced, X' is the hypersurface in $Y = X \times \mathbb{C}^2_{u,v}$ given by the equation $P := uv - p = 0$. Again by abusing notation, for any holomorphic function on X (say p) we denote its lift-up to Y or X' by the same symbol. Instead of working with holomorphic functions on $Y = \mathbb{C}^2_{u,v} \times X$ we use the polynomial algebra $\mathcal{O}(X)[u, v]$ in two variables u and v over the algebra $\mathcal{O}(X)$ of holomorphic functions on X. We put $\mathcal{A}(X') := \mathcal{O}(X)[u, v]|_{X'}$ (note that every function $f \in \mathcal{A}(X')$ can be presented uniquely as the restriction of a function from $\mathcal{O}(X)[u, v]$ of form given by Eq. (6) but with $a_i = \pi^*(a_i^0)$, $b_i = \pi^*(b_i^0)$ being now the lift-ups of holomorphic functions a_i^0, b_i^0 on X via the natural projection $\pi : Y \to X$).

Consider the Lie algebra $\mathrm{VF}_{\mathrm{mixed}}(Y)$ of vector fields on Y each of which maps the algebra $\mathcal{O}(X)[u, v]$ into itself (if, say, Y is an affine algebraic variety then we have $\mathrm{VF}_{\mathrm{mixed}}(Y) := \mathcal{O}(X)[u, v] \otimes_{\mathbb{C}[Y]} \mathrm{VF}_{\mathrm{alg}}(Y)$). By $\mathrm{VF}_{\mathrm{mixed}}(X')$ we denote the restrictions to X' of fields from $\mathrm{VF}_{\mathrm{mixed}}(Y)$ that are tangent to X'.

Every such vector field $\lambda \in \mathrm{VF}_{\mathrm{mixed}}(X')$ can be written again uniquely in the form given by formula (7) but with μ_0, μ_i^j being holomorphic vector fields on X, and $f_0, g_0 \in \mathcal{A}(X')$. In particular, we have the linear map

$$\widetilde{\mathrm{Pr}} : \mathrm{VF}_{\mathrm{mixed}}(X') \to \mathrm{VF}_{\mathrm{hol}}(X)$$

defined by $\widetilde{\mathrm{Pr}}(\lambda) = \mu_0$.

Let $\mathrm{Lie}_{\mathrm{mixed}}(X')$ denote the Lie algebra generated by the set $\mathrm{IVF}_{\mathrm{mixed}}(X')$ of completely integrable vector fields in $\mathrm{VF}_{\mathrm{mixed}}(X')$. Then instead of Proposition 2.8 one has.

Proposition 3.2 *Let C be the smooth zero locus of p in X and* $\mathrm{VF}_{\mathrm{hol}}(X, C) \subset \mathrm{VF}_{\mathrm{hol}}(X)$ *consist of vector fields on X tangent to C. Suppose also that the following conditions hold:*

(A1′) *the linear space* $\mathrm{VF}_{\mathrm{hol}}(X, C)$ *is generated by vector fields that are of the form* $\widetilde{\mathrm{Pr}}([[\mu_f, \nu_1'], \nu_2'])$ *where μ_f and ν_i' are as in the holomorphic version of formula (8) from Lemma 2.6 with* $\nu_i \in \mathrm{IVF}_{\mathrm{hol}}(X)$*;*

(A2′) $\mathrm{VF}_{\mathrm{hol}}(X)$ *is generated by* $\mathrm{IVF}^{\omega}_{\mathrm{hol}}(X)$ *as a module over* $\mathcal{O}(X)$*;*

(A3′) $\mathrm{Div}_X(\mathrm{VF}_{\mathrm{hol}}(X))$ *is generated by* $\mathrm{Div}_X(\mathrm{IVF}_{\mathrm{hol}}(X))$ *over* $\mathcal{O}(X)$ (*for instance, the ideal generated by* $\mathrm{Div}_X(\mathrm{IVF}_{\mathrm{hol}}(X))$ *in* $\mathcal{O}(X)$ *coincides with* $\mathcal{O}(X)$).

Then $\mathrm{Lie}_{\mathrm{mixed}}(X')$ *coincides with* $\mathrm{VF}_{\mathrm{mixed}}(X')$*, i.e., X' has the density property.*

The proof of this Proposition goes *Mutatis Mutandis* and the only place that requires additional comments is the following. The holomorphic vector field μ from Lemma 2.4 such that $\mu(P) = \mu(uv - p) = 0$ on Y may not be a priori from $\mathrm{VF}_{\mathrm{mixed}}(Y)$. However this μ is of form $\Lambda - P\theta$ where $\Lambda \in \mathrm{VF}_{\mathrm{mixed}}(Y)$ and then the argument as in the proof of Proposition 2.8 shows that $\theta(P) \in \mathcal{O}(X)[u, v]$. This fact suffices to continue the argument practically without change.

Lemmas 2.9 and 2.11 have obvious holomorphic reformulations that lead to the following.

Theorem 2 *Let $p \in \mathcal{O}(\mathbb{C}^n)$ be a holomorphic function with a smooth reduced zero fibre, i.e., the partial derivatives $p_i = \partial p/\partial x_i$ of p have no common zeros on the zero fiber of p. Then the Stein manifold*

$$X_p := \{(\bar{x}, u, v) \in \mathbb{C}^{n+2} : uv = p(\bar{x}), \quad \bar{x} = (x_1, x_2, \dots, x_n)\}$$

has the density property.

4 Connections to the conjectures

4.1 Contractible hypersurfaces and related conjectures

Note that X_p from Sect. 5.1 can be viewed as an affine modification of $\mathbb{C}^{n+1}_{\bar{x},v}$ over the divisor $D = \{v = 0\}$ with center $C = \{v = p(\bar{x}) = 0\}$. Then by Theorem 3.5 in [20] (see also [21], Corollary 3.1), we have the following.

Proposition 4.2 *Let C be contractible. Then X_p is contractible. Furthermore, X_p is diffeomorphic as a real manifold to $\mathbb{R}^{2n+2}$ in the case of complex dimension* $\dim X_p = n + 1 \geq 3$.

It is an easy exercise to check that when C can be sent onto a coordinate hyperplane of $\mathbb{C}^n_{\bar{x}}$ by a polynomial automorphism of $\mathbb{C}^n_{\bar{x}}$ (we call such C rectifiable) then X_p is isomorphic to $\mathbb{C}^{n+1}$. It is not clear whether this isomorphism is preserved when $C \simeq \mathbb{C}^{n-1}$ is not rectifiable. The existence of non-rectifiable embeddings $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ is an open question for $n \geq 3$ and we have the following.

Conjecture 4.3 (Abhyankar–Sathaye) Every polynomial embedding $\varphi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ is rectifiable.

However, one can use a smooth contractible C non-isomorphic to $\mathbb{C}^{n-1}$. In this case it is also unknown whether a hypersurface from Proposition 4.2 is isomorphic (or even biholomorphic) to $\mathbb{C}^{n+1}$. As an example one can consider as C a zero locus in $\mathbb{C}^3$ of the polynomial $p(x, y, z) = [(xz + 1)^3 - (yz + 1)^2 - z]/z$ which is a contractible (Ramanujam) surface of Kodaira logarithmic dimension 1 [10] or the Russell cubic which is the hypersurface in $\mathbb{C}^4_{x,y,z,t}$ given by $x + x^2y + z^2 + t^3 = 0$ [25]. That is, we have hypersurfaces

$$H_1 = \{P_1(x, y, z, u, v) = uv - [(xz + 1)^3 - (yz + 1)^2 - z]/z = 0\} \subset \mathbb{C}^5$$

and

$$H_2 = \{P_2(x, y, z, t, u, v) = uv - (x + x^2y + z^2 + t^3) = 0\} \subset \mathbb{C}^6.$$

If any of them were isomorphic to a Euclidean space we would have a counterexample to the Abhyankar–Sathaye Conjecture since there is a singular fiber of P_1 and nonzero fibers of P_2 are not homeomorphic to the zero one. By Theorem 1 the hypersurfaces X_p from Proposition 4.2 have the algebraic density property. Hence H_1 and H_2 would be biholomorphic to complex Euclidean spaces if the following were true.

Conjecture 4.4 (Tóth–Varolin) If X is a Stein manifold with the density property and X is diffeomorphic to $\mathbb{R}^{2n}$, then X is biholomorphic to $\mathbb{C}^n$.

Though we cannot say whether hypersurfaces like H_1 and H_2 are biholomorphic to Euclidean spaces it is worth mentioning that by property (1) from Sect. 5.1 every point of an ($(n + 1)$-dimensional) hypersurface X_p from Theorem 1 has a Fatou–Bieberbach neighborhood biholomorphic to $\mathbb{C}^{n+1}$. These arguments give hope that a hypersurface like H_1 or H_2 can produce a positive answer to the following question of Zaidenberg.

Question 4.5 (Zaidenberg) *Is there a complex affine algebraic variety biholomorphic to $\mathbb{C}^n$ but not isomorphic to $\mathbb{C}^n$?*

Summarizing we get

Proposition 4.6 *The hypersurfaces H_1 and H_2 are either counterexamples to one of the conjectures of Abhyankar–Sathaye resp. Tóth–Varolin or they give a positive answer to Zaidenbergs question.*

Surfaces like H_1 and H_2 may be also viewed also as "potential counterexamples" to the problem of linearizing of an algebraic $\mathbb{C}^*$-action on a Euclidean space. Consider say, the $\mathbb{C}^*$-action on H_2 given by $(u, v, x, y, z, t) \to (\lambda u, \lambda^{-1} v, x, y, z, t)$. Its fixed point set is $\{u = v = 0\}$ is isomorphic to the Russell cubic and, therefore, if H_2 were isomorphic to $\mathbb{C}^5$ the action would not be linearizable.

Furthermore, if the Tóth–Varolin conjecture were valid one would be able to construct in a similar manner a holomorphic $\mathbb{C}^*$-action on a Euclidean spaces whose fixed point set is a disc. This would yield examples of a new type for non-linearizable holomorphic $\mathbb{C}^*$-actions on Euclidean spaces (the examples constructed by Derksen and the second author were based on a different fact; namely on existence of actions with a non-rectifiable embedding of the fixed point set into the quotient).

More precisely, consider a proper holomorphic embedding of the unit disc $\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ into $\mathbb{C}^2$ which exists by the result of Kasahara, Nishino [23] (see also [2], [16]). By the solution of the second Cousin problem the image of the disc coincides with the zeros of a reduced holomorphic function $F \in \mathcal{O}(\mathbb{C}^2_{x,y})$. Note that the partial derivatives of F have no common zero on the zero fiber $F^{-1}(0)$.

As usual X_F is the hypersurface in $\mathbb{C}^4_{x,y,u,v}$ given by

$$X_F = \{(x, y, u, v) : F(x, y) = uv\}. \tag{10}$$

Lemma 4.7 *For all $n \geq 0$ there is a holomorphic action of $\mathbb{C}^*$ on $X_F \times \mathbb{C}^n$ with fixed point set biholomorphic to the unit disc Δ.*

Proof Consider the linear action $\mathbb{C}^* \times \mathbb{C}^4_{x,y,u,v} \times \mathbb{C}^n_{\bar{w}} \to \mathbb{C}^4_{x,y,u,v} \times \mathbb{C}^n_{\bar{w}}$ given by

$$(\theta, (x, y, u, v, \bar{w})) \mapsto (x, y, \theta u, \theta^{-1} v, \theta \bar{w}).$$

It leaves invariant the hypersurface $X_F \times \mathbb{C}^n$ and restricts therefore to a $\mathbb{C}^*$-action on $X_F \times \mathbb{C}^n$. The fixed point set of this action is given by $u = v = 0$, $\bar{w} = 0$, $F(x, y) = 0$ which concludes the proof. □

By Theorem 2, X_F and, therefore, $X_F \times \mathbb{C}^n$ have the density property. In combination with Proposition 5.10 this would lead to promised examples provided the Tóth–Varolin conjecture were true. By the same reasoning one can prove that the Tóth–Varolin conjecture implies the existence of holomorphic $\mathbb{C}^*$ actions on affine spaces with fixed point set biholomorphic to any contractible domain Ω in $\mathbb{C}^n$ which can be properly holomorphically embedded into $\mathbb{C}^{n+1}$.

5 Implications of the density property

5.1 Properties inherited from the density property

We suppose below that $X_p \subset \mathbb{C}^{n+2}$ is an algebraic hypersurface in $\mathbb{C}^{n+2}$ given by $uv = p(\bar{x})$ where $\bar{x} = (x_1, \ldots, x_n)$ and $(\bar{x}, u, v)$ is a coordinate system in $\mathbb{C}^{n+2}_{\bar{x},u,v}$. Again we assume the zero fiber $C \subset \mathbb{C}^n_{\bar{x}}$ of p to be smooth and reduced which implies that X_p is also smooth.

We list below some properties of X_p that hold for any complex manifold with the density property (e.g., see [33] Corollaries 4.1, 4.3).

1. (Fatou–Bieberbach maps of the first kind) For each point $x \in X_p$ there is an injective but not surjective holomorphic map $f : \mathbb{C}^{n+1} \to X_p$ with $f(0) = x$. In particular all Eisenman measures on X_p vanish identically.
2. (Fatou–Bieberbach maps of the second kind) For each point $x \in X_p$ there is an injective but not surjective holomorphic map $f : X_p \to X_p$ with $f(x) = x$.
3. The holomorphic automorphism group of X acts k-transitively on X for all $k \in \mathbb{N}$.

For the affine modifications from Theorem 1 property (3) was originally proved by Zaidenberg and the first author in [21] even for the algebraic automorphism group. In the holomorphic case it is a consequence of Theorem 0.2 in [33] whose weak version is as follows.

Proposition 5.2 *Let X be a Stein manifold with the density property, K be a compact in X, $x, y \in X$ be two points outside the holomorphic hull of K. Suppose also that $x_1, \dots, x_n \in K$. Then there exists a holomorphic automorphism Ψ of X such that $\Psi(x_i) = x_i$, $\Psi|_K$ is as close to the identical map as we wish, and $\Psi(y) = x$.*

5.3 Density implies Gromov's spray

The Oka principle is a fundamental principle in complex analysis stating that on Stein manifolds analytic problems which can be formulated in cohomological terms have only topological obstructions.

Definition 5.4 Let $h : Z \to W$ be a holomorphic submersion of complex manifolds, and let $\mathrm{Cont}(W, Z)$ (resp. $\mathrm{Holo}(W, Z)$) be the set of continuous (resp. holomorphic) sections of h with compact-open topology. We say that this submersion h satisfies the Oka–Grauert–Gromov principle if the natural embedding $\mathrm{Holo}(W, Z) \hookrightarrow \mathrm{Cont}(W, Z)$ is a weak homotopy equivalence (see [18]) (this implies, in particular, that each continuous section $f^0 : W \to Z$ of h can be deformed to a holomorphic section $f^1 : W \to Z$ through a homotopy of continuous sections $f^t : W \to Z$ $(0 \le t \le 1)$, and any two holomorphic sections which are homotopic through continuous sections are also homotopic through holomorphic sections).

The most powerful version of this principle was introduced by Gromov [18] who extended the classical results of Oka [26], Grauert [17], and Forster and Ramspott [15] to manifolds admitting sprays (for exact statements and proofs see [12,13]).

Definition 5.5 1. A (dominating) *spray* on a complex manifold F is a holomorphic vector bundle $\rho : E \to F$, together with a holomorphic map $s : E \to F$, such that s is identical on the zero section $F \hookrightarrow E$, and for each $x \in F$ the induced differential map sends the fibre $E_x = \rho^{-1}(x)$ (which is viewed as a linear subspace of $T_x E$) surjectively onto $T_x F$.
2. A fiber-dominating spray for a surjective submersion $h : Z \to W$ of complex manifolds is a vector bundle $\rho : E \to Z$ together with a map $s : E \to Z$ identical on the zero section $Z \hookrightarrow E$ and such that $h \circ s = h \circ \rho$ and for every $z \in Z$ the induced differential map sends $E_z = \rho^{-1}(z)$ (which is viewed as a linear subspace of $T_z E$) surjectively onto the subspace of $T_z Z$ tangent to the fiber $h^{-1}(h(z))$.

Let us recall Theorem 4.5 from [18] (for a proof see [13])

Theorem 3 *Suppose that $h : Z \to W$ is a holomorphic submersion of a complex manifold Z onto a Stein manifold W for which every $x \in W$ has a neighborhood $U \subset W$ such that $h^{-1}(U) \to U$ admits a fiber-dominating spray. Then $h : Z \to W$ satisfies the Oka–Grauert–Gromov principle.*

Before we come to the main point of this subsection let us make a useful remark.

Remark 5.6 For a complex manifold Z the closure (in the compact-open topology) of $\mathrm{Lie}_{\mathrm{hol}}(Z) \subset \mathrm{VF}_{\mathrm{hol}}(Z)$ (resp. $\mathrm{Lie}_{\mathrm{alg}}(Z) \subset \mathrm{VF}_{\mathrm{alg}}(Z)$) coincides with the closure of the linear span $\mathrm{Span}(Z)$ generated by completely integrable holomorphic (resp. algebraic) vector fields. Indeed, it suffices to prove that the closure of $\mathrm{Span}(Z)$ is a Lie algebra. This is so since the bracket of two completely integrable vector fields V and W, $[V, W] = \lim_{t\to 0} \frac{(\varphi^t)^*(V)-V}{t}$ can be approximated by $\frac{(\varphi^t)^*(V)-V}{t}$ for small t where φ_t is the flow of W. The desired conclusion follows now from the fact that $\frac{(\varphi^t)^*(V)}{t}$, $\frac{V}{t}$ are both completely integrable because global change of variables and multiplication by a constant factor preserve integrability.

Theorem 4 *Any Stein manifold X with the density property admits a spray.*

Proof For each point x in X there are finitely many completely integrable holomorphic vector fields which span the tangent space T_xX. Indeed, assume the contrary, i.e. there is a point x and a nonzero linear functional $l : T_xX \to \mathbb{C}$ such that $l(V(x)) = 0$ for all completely integrable holomorphic vector fields V on X. By the Remark before $l(V(x)) = 0$ would hold for all vector fields in the closure of $\mathrm{Span}(X)$, i.e., for all holomorphic vector fields on X, a contradiction to the fact that X is Stein. If a finite number of holomorphic vector fields span the tangent space at a single point, they span the tangent space at all points outside a proper analytic subset A. In the algebraic case by standard induction over the dimension of A we find finally many completely integrable holomorphic vector fields $V_1, V_2, \ldots, V_n$ which span the tangent space T_xX at each point. Let φ_i^t be the flow of V_i on X and let $s : X \times \mathbb{C}^n \to X$ be given by

$$s(x; t_1, t_2, \ldots, t_n) = \varphi_1^{t_1} \circ \varphi_2^{t_2} \circ \cdots \circ \varphi_n^{t_n}(x).$$

Then $s(x; 0, 0, \ldots, 0) = x$ and $\partial s/\partial t_i = V_i(x)$. Since these vectors span T_xX for each x, holomorphic submersion s is a spray on X. This construction of a spray is due to Gromov [18].

In the holomorphic case the argument must be a bit more accurate since A may have an infinite number of irreducible components $A_1, A_2, A_3, \ldots$ By transitivity property (3) from Sect. 5.1 we can choose integrable vector fields $\theta_1, \ldots, \theta_k$ that span the tangent space at a given point $x_1 \in A_1$. Our aim is to construct a holomorphic automorphism $\Phi \in \mathrm{Aut}_{\mathrm{hol}} X$ such that $\Phi_*(\theta_1), \ldots, \Phi_*(\theta_k)$ span the tangent space at a general point of each A_i, $i = 1, 2, \ldots$ Then the induction by dimension will work as in the algebraic case which yields the desired conclusion.

For this aim note that the topological space $\mathrm{Aut}_{\mathrm{hol}}(X)$ of holomorphic automorphisms of X (with compact-open topology) is metrizable by the Urysohn theorem. More precisely, since X is Stein it can be treated as a closed analytic subset of $\mathbb{C}^m$ and, therefore, each holomorphic map $g : X \to X$ is given by coordinate functions $g_1, \ldots, g_m$. Choose a sequence of compacts $\{K_i\}$ such that $\bigcup_i K_i = X$. For $g, h \in \mathrm{Aut}_{\mathrm{hol}}(X)$ set $\kappa_i(g, h) = \min(\max(||g_1 - h_1||_i, \ldots, ||g_m - h_m||_i), 1)$ where for any continuous function λ on K_i its maximal absolute value is denoted by $||\lambda||_i$. Define the distance between h and g by $\sum_i \kappa_i(g, h)/2^i$. This metric generates the compact-open topology and, what is important, makes $\mathrm{Aut}_{\mathrm{hol}}(X)$ an open subset in a complete metric space. Denote by B_i a subset of $\mathrm{Aut}_{\mathrm{hol}}(X)$ such that for each $\Psi \in B_i$ the integrable vector fields $\Psi_*(\theta_1), \ldots, \Psi_*(\theta_k)$ span the tangent space at a general point of A_i. Clearly, B_i is an open subset of $\mathrm{Aut}_{\mathrm{hol}}(X)$. By the argument before there is a completely integrable vector field ν_i not tangent to A_i. Then for its flow ψ_i^t and any automorphism $\Theta \notin B_i$ the composition $\psi_i^t \circ \Theta \in B_i$ for small values of time t. Therefore, B_i is dense in $\mathrm{Aut}_{\mathrm{hol}}(X)$. By the Baire category theorem $\bigcap_i B_i$ is not empty and we choose the desired $\Phi \in \bigcap_i B_i$. □

Remark 5.7 In fact, instead of the density a weaker assumption ensures the existence of a spray for X. That is, it follows from the proof that it suffices to require the existence of a single completely integrable vector field on X, the transitivity on X of the group of holomorphic automorphisms and the fact that for some point the isotropy subgroup of this group acts irreducible on the tangent space.

One can include *holomorphic dependence of parameters* in the formulation of Theorems 1 and 2 with obvious adjustment of the arguments. This implies now the following.

Corollary 5.8 *Let $h : Z \to W$ be a surjective submersion of complex manifolds such that W is Stein and for every $w_0 \in W$ there is a neighborhood U for which $h^{-1}(U)$ is naturally isomorphic to a hypersurface in $\mathbb{C}^{n+2}_{\bar{x},u,v} \times U$ given by $uv = p(\bar{x}, w)$ where p is a holomorphic function on $\mathbb{C}^n_{\bar{x}} \times U$ (independent of u and v). Suppose, furthermore, that $p^*(0) \cap (\mathbb{C}^n_{\bar{x}} \times w)$ is a smooth reduced proper (may be empty) submanifold of $\mathbb{C}^n_{\bar{x}} \times w$ for every point $w \in U$. Then h satisfies the Oka–Grauert–Gromov principle.*

5.9 Appendix: topology of pseudo-affine modifications

Let X be a Stein manifold, D be its smooth reduced analytic divisor, and C be a proper closed complex submanifold of D. Suppose that Z is the result of blowing up X along C and deleting the proper transform of D. Then Z is smooth Stein and the natural projection $\sigma : Z \to X$ is called a basic pseudo-affine modification [21]. Geometrically it means deleting D and replacing it with a divisor $E = \sigma^{-1}(C)$ which is biholomorphic to the projectivised normal bundle of C in X from which a section is deleted. In the case that C is contractible this divisor is simply $\simeq C \times \mathbb{C}^k$ in Z where $k = \mathrm{codim}_D C$.

Proposition 5.10 *Let $\sigma : Z \to X$ be a basic pseudo-affine modification of Stein manifolds with D and C as before. Suppose that the pair (X, D) is diffeomorphic to $(\mathbb{R}^{2n}, \mathbb{R}^{2n-2})$ where the embedding in the latter pair is linear and $n \geq 3$. Suppose also that C is contractible and admits a proper surjective smooth function $\varphi_1 : C \to \mathbb{R}_{\geq 0}$ with all critical values less then some $t_0 > 0$. Then Z is diffeomorphic to $\mathbb{R}^{2n}$.*

Proof (Sketch.) By Corollary 3.1 in [21] Z is contractible. Suppose that one can construct a compact manifold $K \subset Z$ with boundary ∂K such that K is a deformation retract of Z and ∂K is simply connected. Then we are done by the h-cobordism theorem which implies that K is diffeomorphic to a ball. Thus our aim is to show the existence of such K.

Let φ_3 be a distance on $X \simeq \mathbb{R}^{2n}$ and $\varphi_2 = \varphi_3|_D$ on D. That is, $B_i(R) = \varphi_i^{-1}([0, R])$ is diffeomorphic to a closed ball for each $R > 0$ and $i = 2, 3$ while $B_1(R)$ is contractible for $R \geq t_0$. Since X is Stein and contractible there is a holomorphic function $p \in \mathcal{O}(X)$ generating the ideal of the divisor D.

Consider small closed tubular neighborhoods U_1 of C in D (resp. U_2 of D in X) with projection $\varrho_1 : U_1 \to C$ (resp. $\varrho_2 : U_2 \to D$). Choose smooth positive functions $R(t)$ and $r(t)$ on $\{t \in \mathbb{R} | t \geq t_0\}$ such that

(i) $R(t)$ is a strictly increasing function for which the interior of $B_2(R(t))$ contains $\varrho_1^{-1}(B_1(t))$;

(ii) $r = r(t)$ is a strictly decreasing positive function going to zero such that $B_2'(t) := p^{-1}(\bar{\Delta}_{r(t)}) \cap \varrho_2^{-1}(B_2(R(t)))$ is contained in the interior of U_2 and $B_2'(t)$ is naturally diffeomorphic to $\bar{\Delta}_r \times B_2(R(t))$ where $\bar{\Delta}_r = \{\zeta \in \mathbb{C} \,|\, |z| \leq r\}$.

Since C is Stein contractible its normal bundle in X (resp. D) is trivial and U_2 (resp. U_1) can be viewed as neighborhood of its zero section. Choose a trivialization $C \times \mathbb{C}^k_w \times \mathbb{C}_z$ of this bundle (where w is a coordinate system on the second factor) such that ρ_2 is the natural projection $C \times \mathbb{C}^k_w \times \mathbb{C}_z \to C \times \mathbb{C}^k_w$ and ρ_1 is the natural projection $C \times \mathbb{C}^k_w \to C$. Set $k = 1$ for simplicity. Then U_1 is biholomorphic to $\{|w| \le f(c)\}$, where $c \in C$ and f is a strictly positive function. Also $U_2' = \rho_2^{-1}(U_1)$ is biholomorphic to $\{|z| < g(c, w)\}$ where the function g is also positive. Decreasing g if necessary one can suppose that $z = p|_{U_2'}$. Note that the preimage $\tilde{U}_2'$ of U_2' under the pseudo-affine modification is biholomorphic to the hypersurface in $U_1' \times \mathbb{C}_\xi$ given by $p\xi = w$.

Set $B_1'(t) := \varrho_2^{-1}(\varrho_1^{-1}(B_1(t))) \cap p^{-1}(\bar{\Delta}_{r(t)}) \subset B_2'(t)$ and $S_t^1 = (B_2'(t) \setminus B_1'(t)) \cap p^{-1}(\partial\bar{\Delta}_{r(t)})$. Then the part $\Gamma(t)$ of the boundary of S_t^1 that meets $B_1'(t)$ consists of two pieces given by the equations

$$\varphi(c) = t, \quad |p| = r(t), \quad |w| \le f(c) \tag{A1}$$

and

$$\varphi(c) \le t, \quad |p| = r(t), \quad |w| = f(c) \tag{A2}$$

Note that $\Gamma(t)$ can be also viewed as the boundary of the real surface S_2^t in $\tilde{U}_2'$ consisting of the two pieces

$$\varphi(c) = t, \quad |p| \le r(t), \quad |\xi| \le \frac{f(c)}{r(t)} \tag{B1}$$

and

$$\varphi(c) \le t, \quad |p| \le r(t), \quad |\xi| = \frac{f(c)}{r(t)} \tag{B2}$$

Except for a simple case when C is a point (which we omit) S_t^2 is simply connected. Also S_t^2 and $S_{t'}^2$ are disjoint for $t \ne t'$. Furthermore, except for a compact piece the divisor $E = \sigma^{-1}(C)$ is contained in $\bigcup_{t \ge t_0} S_t^2$. Set $S_t^3 = \partial B_3(R(t)) \setminus p^{-1}(\bar{\Delta}_{r(t)})$. One can easily check that S_t^i is diffeomorphic to $S_{t'}^i$ for $i = 1, 2, 3$ and $t, t' > t_0$.

Consider the compact piece-wise differential $(2n-1)$-dimensional manifold S_t in Z that is the union of S_t^1, S_t^2, and S_t^3. Using partition of unity and local flows one can make diffeomorphisms between pieces of S_t and $S_{t'}$ agreeable on the boundaries so that for $W = \bigcup_{t \ge t_0} S_t$ the natural projection $\varphi : W \to [t_0, \infty)$ becomes a proper locally trivial fibration with fibers S_t. Thus the closure of $Z \setminus W$ is a compact manifold which is a deformation retract of Z and whose boundary is S_{t_0}. Application of the Seifert-Van Kampen theorem implies that S_{t_0} is simply connected. At non-smooth points S_{t_0} is locally a union of at most three smooth pieces meeting transversally. Thus rounding these corners we obtain a desired compact manifold K. □

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