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SPECTRAL THEORY OF THE KLEIN–GORDON EQUATION IN KREIN SPACES

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Dedicated to the memory of Peter Jonas.

Abstract In this paper the spectral properties of the abstract Klein–Gordon equation are studied. The main tool is an indefinite inner product known as the charge inner product. Under certain assumptions on the potential V, two operators are associated with the Klein–Gordon equation and studied in Krein spaces generated by the charge inner product. It is shown that the operators are self-adjoint and definitizable in these Krein spaces. As a consequence, they possess spectral functions with singularities, their essential spectra are real with a gap around 0 and their non-real spectra consist of finitely many eigenvalues of finite algebraic multiplicity which are symmetric to the real axis. One of these operators generates a strongly continuous group of unitary operators in the Krein space; the other one gives rise to two bounded semi-groups. Finally, the results are applied to the Klein–Gordon equation in $\mathbb{R}^n$.

Keywords: Klein–Gordon equation; Krein space; charge inner product; self-adjoint operators in Krein spaces; definitizable operators

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1. Introduction

The Klein–Gordon equation

$$\left(\left(\frac{\partial}{\partial t} - \mathrm{i}eq\right)^2 - \Delta + m^2\right)\psi = 0 \tag{1.1}$$

describes the motion of a relativistic spinless particle of mass m and charge e in an electrostatic field with potential q (the velocity of light being normalized to 1); here ψ is a complex-valued function of $x \in \mathbb{R}^n$ and of $t \in \mathbb{R}$.

If in (1.1) we replace the uniformly positive self-adjoint operator generated by the differential expression $-\Delta + m^2$ in the Hilbert space $L_2(\mathbb{R}^n)$ by a uniformly positive

* Sadly, Professor Branko Najman died in August 1996.

self-adjoint operator H_0 in a Hilbert space $\mathcal{H}$ and the operator of multiplication with the function eq by a symmetric operator V in $\mathcal{H}$, we obtain the abstract Klein–Gordon equation

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}V\right)^2 + H_0\right)u = 0, \tag{1.2}$$

where u is a function of $t \in \mathbb{R}$ with values in $\mathcal{H}$. Equation (1.2) can be transformed into a first-order differential equation for a vector function $\boldsymbol{x}$ in an appropriate space, formally given by $\mathcal{H} \oplus \mathcal{H}$, and a linear operator A therein:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}A\boldsymbol{x}. \tag{1.3}$$

However, this is in general not possible with a self-adjoint operator A in a Hilbert space.

In the literature two inner products have been associated with the Klein–Gordon equation (1.1), one of them representing the charge and the other one representing the energy of the particle. The energy inner product is used in numerous papers to study the spectral and scattering properties of (1.1), see, for example, [**5**, **11**, **12**, **16**, **20**, **28**, **29**, **35**–**37**, **41**, **42**, **49**, **50**]. The charge inner product has been suggested by the physics literature (see [**13**]); it was first used in the pioneering work of Veselić [**45**–**47**] and subsequently in the papers [**34**, **38**, **48**], the unpublished manuscript* [**26**], as well as in [**9**, **17**–**19**]. It is also called the number norm since it is related to the number operator in the theory of second quantization (see [**3**]).

The reason for the preference of the energy inner product $\langle\cdot\,,\cdot\rangle$ may be that it is positive definite and generates a Hilbert space if the potential V is small with respect to $H_0^{1/2}$; this can be seen from its formal definition

$$\langle\boldsymbol{x}, \boldsymbol{x}'\rangle = \left(\begin{pmatrix} H_0 - V^2 & 0 \\ 0 & I \end{pmatrix}\boldsymbol{x}, \boldsymbol{x}'\right) = ((H_0 - V^2)x, x') + (y, y') \tag{1.4}$$

for suitable elements $\boldsymbol{x} = (x\ y)^{\mathrm{T}}$, $\boldsymbol{x}' = (x'\ y')^{\mathrm{T}}$ of $\mathcal{H} \oplus \mathcal{H}$, where $(\cdot\,,\cdot)$ denotes the scalar product in $\mathcal{H}$. The charge inner product $[\cdot\,,\cdot]$, however, is always indefinite: it is defined on elements $\boldsymbol{x} = (x\ y)^{\mathrm{T}}$, $\boldsymbol{x}' = (x'\ y')^{\mathrm{T}}$ of $\mathcal{H} \oplus \mathcal{H}$ by a relation of the form

$$[\boldsymbol{x}, \boldsymbol{x}'] = \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\boldsymbol{x}, \boldsymbol{x}'\right) = (x, y') + (y, x'). \tag{1.5}$$

Therefore, it is negative on an infinite-dimensional subspace (if $\mathcal{H}$ is infinite dimensional) and hence leads to a so-called Krein space. At first glance, these indefinite structures seem to be less convenient from the mathematical point of view. However, they allow deeper insight into the spectral properties of the Klein–Gordon equation, e.g. by providing a classification of the points of the spectrum into points of positive, negative or neutral type and sufficient conditions for the existence of corresponding strongly continuous groups of operators which are unitary with respect to the indefinite inner product (1.5).

* This manuscript, dating back to the late 1980s, was the starting point for the present paper and also for the papers [**18**, **27**].

In this paper we associate two operators, A_1 and A_2, with the abstract Klein–Gordon equation (1.2). Formally, both operators arise from the second-order differential equation (1.2) by means of the substitution

$$x = u, \qquad y = \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V\right)u, \tag{1.6}$$

which leads to a first-order differential equation (1.3) of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}\begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}\boldsymbol{x} \tag{1.7}$$

for the vector function $\boldsymbol{x} = (x\ y)^{\mathrm{T}}$. However, we consider the block operator matrix in (1.7) in two different Krein spaces induced by the charge inner product (1.5), and we prove the self-adjointness of the corresponding operators A_1 and A_2 in these Krein spaces.

The main results of this paper concern the structure and classification of the spectrum for A_1 and A_2, the existence of a spectral function with singularities, the generation of a strongly continuous group of unitary operators, the existence of solutions of the Cauchy problem for the abstract Klein–Gordon equation (1.2) and an application of these results to the Klein–Gordon equation (1.1) in $\mathbb{R}^n$. The main tools for this are techniques from the theory of block operator matrices and results from the theory of self-adjoint operators in Krein spaces.

The paper is organized as follows: in § 2 we present some basic notation and definitions from spectral theory and we give a brief review of results from the theory of self-adjoint and definitizable operators in Krein spaces. In § 3 we introduce the first operator, A_1, associated with (1.7), which acts in the Hilbert space $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$. Equipped with the charge inner product (1.5), the space $\mathcal{G}_1$ becomes a Krein space, which we denote by $\mathcal{K}_1$. The block operator matrix in (1.7) is formally symmetric with respect to the charge inner product $[\cdot\,,\cdot]$ since, for $\boldsymbol{x} \in (\mathcal{D}(V) \cap \mathcal{D}(H_0)) \oplus \mathcal{D}(V)$, $\boldsymbol{x}' \in \mathcal{H} \oplus \mathcal{H}$,

$$\left[\begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}\boldsymbol{x}, \boldsymbol{x}'\right] = \left(\begin{pmatrix} H_0 & V \\ V & I \end{pmatrix}\boldsymbol{x}, \boldsymbol{x}'\right). \tag{1.8}$$

We show that if V is relatively bounded with respect to $H_0^{1/2}$, then the block operator matrix in (1.7) is essentially self-adjoint in $\mathcal{K}_1$, and we denote its self-adjoint closure by A_1. Using a certain factorization of $A_1 - \lambda$, $\lambda \in \mathbb{C}$, we relate the spectral properties of A_1 to those of the quadratic operator polynomial L_1 in $\mathcal{H}$ given by

$$L_1(\lambda) = I - (S - \lambda H_0^{-1/2})(S^* - \lambda H_0^{-1/2}), \quad \lambda \in \mathbb{C},$$

where S is the bounded operator $S = VH_0^{-1/2}$.

In § 4 we define the second operator, A_2, associated with (1.7) in the more complicated Hilbert space $\mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$; here $\mathcal{H}_\alpha$, $-1 \leqslant \alpha \leqslant 1$, is a scale of Hilbert spaces induced by the fractional powers H_0^α of the uniformly positive operator H_0. We equip $\mathcal{G}_2$

with the charge inner product (1.5), where now the brackets on the right-hand side of (1.5) denote the duality

$$(x,y)=(H_0^{\alpha}x,H_0^{-\alpha}y),\quad x\in\mathcal{H}_\alpha,\ y\in\mathcal{H}_{-\alpha},$$

between the spaces $\mathcal{H}_\alpha$ and $\mathcal{H}_{-\alpha}$ for $\alpha=\frac{1}{4}$ and $\alpha=-\frac{1}{4}$; the corresponding space $\mathcal{K}_2$ is a Krein space. An analogue of formula (1.8) in $\mathcal{H}_{1/4}\oplus\mathcal{H}_{-1/4}$ shows that the block operator matrix in (1.7), now considered as an operator in $\mathcal{G}_2$, is formally symmetric with respect to the charge inner product $[\cdot\,,\cdot]$. If the resolvent set $\rho(L_1)$ is non-empty, we associate a self-adjoint operator A_2 in the Krein space $\mathcal{K}_2$ with the block operator matrix in (1.7).

The operator A_2 was first considered by Veselić [**45**] (see also [**34**, **46**, **47**]). He showed that, under his assumptions, A_2 is similar to a self-adjoint operator in a Hilbert space. The operators A_1 and A_2 were also introduced by Jonas [**18**] using so-called range restriction (see [**17**]). In [**18**] and in [**45**] it was supposed that $VH_0^{-1/2}$ is compact. The operator A_2 was also studied in [**9**,**34**] for the cases when either $VH_0^{-1/2}$ is compact or $\|VH_0^{-1/2}\|<1$. The main results of the present paper are proved under the more general condition

$$S:=VH_0^{-1/2}=S_0+S_1,\quad \|S_0\|<1,\ S_1\text{ compact.}\tag{1.9}$$

Under this assumption, in § 5, we study and compare the spectral properties of the operators A_1 and A_2. We show that A_1 and A_2 are definitizable (for the definition of definitizability see § 2), that their spectra, essential spectra and point spectra coincide and are symmetric to the real axis, that their essential spectra are real and have a gap around 0, and that the non-real spectrum consists of a finite number of complex conjugate pairs of eigenvalues of finite algebraic multiplicity; this number is bounded by the number κ of negative eigenvalues of the operator $I-S^*S$ in $\mathcal{H}$. As a consequence of the definitizability, the operators A_1 and A_2 possess spectral functions with at most finitely many singularities.

At the end of § 5 we compare the results for A_1 with results for another operator associated with the Klein–Gordon equation in [**27**]. This operator, A, arises from the second-order differential equation (1.2) by means of the substitution

$$x=u,\qquad y=-\mathrm{i}\frac{\mathrm{d}u}{\mathrm{d}t},\tag{1.10}$$

which leads to a first-order differential equation of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\mathrm{i}\begin{pmatrix}0 & I\\ H_0-V^2 & 2V\end{pmatrix}\boldsymbol{x}.\tag{1.11}$$

The operator A, formally given by the block operator matrix in (1.11), acts in the space $\mathcal{G}=\mathcal{H}_{1/2}\oplus\mathcal{H}$; it is defined if V is $H_0^{1/2}$-bounded and $1\in\rho(S^*S)$. These two assumptions guarantee that a self-adjoint operator $H=H_0^{1/2}(I-S^*S)H_0^{1/2}$ can be associated with the entry H_0-V^2 in (1.11). Under the additional assumption 1.9, the space $\mathcal{K}$ is a Pontryagin space and A is a self-adjoint (and hence definitizable) operator in $\mathcal{K}$. We prove that the spectra, essential spectra and point spectra of A and of A_1 (and hence also of A_2) coincide.

In § 6 we show that A_2 generates a strongly continuous group $(\mathrm{e}^{\mathrm{i}tA_2})_{t\in\mathbb{R}}$ of unitary operators in the Krein space $\mathcal{K}_2$. Hence, the Cauchy problem

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}A_2\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1.12}$$

has a unique classical solution given by

$$\boldsymbol{x}(t) = \mathrm{e}^{\mathrm{i}tA_2}\boldsymbol{x}_0, \quad t \in \mathbb{R}, \tag{1.13}$$

if $\boldsymbol{x}_0 \in \mathcal{D}(A_2)$; if only $\boldsymbol{x}_0 \in \mathcal{K}_2$, then (1.13) is a mild solution of (1.12). To this end, we prove that ∞ is a regular critical point of A_2, thus generalizing a result in [**9**] for the special cases $S_0 = 0$ and $S_1 = 0$. The regularity of ∞ in fact implies that A_2 is the sum of a bounded operator and of an operator which is similar to a self-adjoint operator in a Hilbert space (see § 2). For the operator A_1, however, ∞ is in general a singular critical point if H_0 is unbounded; thus, A_1 does not generate a group of unitary operators in $\mathcal{K}_1$. In fact, the construction in [**17**] of the operator A_2 from A_1 is designed to make ∞ a regular critical point.

In § 7, for bounded V, we show the existence of maximal non-negative and non-positive invariant subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively, for the definitizable self-adjoint operator A_1. These subspaces admit so-called angular operator representations, e.g.

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{D}(K) \right\}$$

with a closed linear operator K in $\mathcal{H}$. This yields solutions on the negative and positive half-axes of the second-order initial-value problem

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + V\right)^2 - H_0\right)v = 0, \quad v(0) = v_0, \tag{1.14}$$

which arises from (1.2) by means of the substitution $\tau = -\mathrm{i}t$, $v(\tau) = u(t)$. The solutions on the positive half-axis are given by

$$v(\tau) = \mathrm{e}^{-\tau(K+V)}v_0, \quad \tau \geqslant 0,$$

and the admissible set of initial values v_0 is the domain $\mathcal{D}((K + V)^2)$, which in the case when $V = 0$ amounts to $v_0 \in \mathcal{D}(H_0)$. The solution on the positive half-axis corresponds, roughly speaking, to the spectrum of positive type and the spectrum in the upper (or lower) half-plane, whereas the solution on the negative half-axis corresponds to the spectrum of negative type and the spectrum in the upper (or lower) half-plane of A_1.

Finally, in § 8, we apply the results of the previous sections to the Klein–Gordon equation (1.1) in $\mathbb{R}^n$. We prove that in the space $W_2^{-1/2}(\mathbb{R}^n)$ it has a unique classical solution if the initial values $\psi_0 = \psi(\cdot\,,0)$ and $\psi_1 = \partial\psi(\cdot\,,0)/\partial t$ belong to $W_2^1(\mathbb{R}^n)$ and $W_2^{1/2}(\mathbb{R}^n)$, respectively, and $(-\Delta - V^2)\psi_0 \in W_2^{-1/2}(\mathbb{R}^n)$.

2. Preliminaries

2.1. Notation and definitions from spectral theory

For Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $L(\mathcal{H},\mathcal{H}')$ denotes the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$, and we write $L(\mathcal{H})$ if $\mathcal{H}=\mathcal{H}'$.

For a closed linear operator A in a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$, we denote by $\rho(A)$, $\sigma(A)$ and $\sigma_{\mathrm{p}}(A)$ its *resolvent set*, *spectrum* and *point spectrum* or *set of eigenvalues*, respectively. For $\lambda\in\sigma_{\mathrm{p}}(A)$ the algebraic eigenspace of A at λ is denoted by $\mathcal{L}_\lambda(A)$. The operator A is called *Fredholm* if its kernel is finite dimensional and its range is finite codimensional (and hence closed; see, for example, [**15**, Chapter IV, § 5.1]). The *essential spectrum* of A is defined by

$$\sigma_{\mathrm{ess}}(A):=\{\lambda\in\mathbb{C}: A-\lambda \text{ is not Fredholm}\}.$$

An eigenvalue $\lambda_0\in\sigma_{\mathrm{p}}(A)$ is called *of finite type* if λ_0 is isolated (i.e. a punctured neighbourhood of λ_0 belongs to $\rho(A)$) and $A-\lambda_0$ is Fredholm or, equivalently, the corresponding Riesz projection is finite dimensional. The set of all eigenvalues of finite type is called the *discrete spectrum* of A and denoted by $\sigma_{\mathrm{d}}(A)$.

For an analytic operator function $F:\mathbb{C}\to L(\mathcal{H})$, the *resolvent set* and the *spectrum* of F are defined by

$$\rho(F):=\{\lambda\in\mathbb{C}: 0\in\rho(F(\lambda))\},\qquad \sigma(F):=\mathbb{C}\setminus\rho(F),$$

and the *point spectrum* or *set of eigenvalues* of F is the set

$$\sigma_{\mathrm{p}}(F):=\{\lambda\in\mathbb{C}: 0\in\sigma_{\mathrm{p}}(F(\lambda))\}$$

(see [**14**, **30**]). The *essential spectrum* of F is given by

$$\sigma_{\mathrm{ess}}(F):=\{\lambda\in\mathbb{C}: F(\lambda) \text{ is not Fredholm}\}.$$

An eigenvalue $\lambda_0\in\sigma_{\mathrm{p}}(F)$ is called *of finite type* if λ_0 is isolated (i.e. a punctured neighbourhood of λ_0 belongs to $\rho(F)$) and $F(\lambda_0)$ is Fredholm. If $\rho_0\subset\mathbb{C}$ is an open connected subset of $\mathbb{C}\setminus\sigma_{\mathrm{ess}}(F)$ such that $\rho_0\cap\rho(F)\neq\emptyset$, then $\rho_0\cap\sigma(F)$ is at most countable with no accumulation point in ρ_0 and consists of eigenvalues of finite type of F, and $F(\cdot)^{-1}$ is a *finitely meromorphic operator function* on ρ_0. This means that $F(\cdot)^{-1}$ is a meromorphic operator function from ρ_0 to $L(\mathcal{H})$ for which the coefficients of the principal parts of the Laurent expansions at the poles of $F(\cdot)^{-1}$ are all operators of finite rank (cf. [**15**, Corollary XI.8.4, Theorem XVII.2.1]). Conversely, if for some open set $\rho_0\subset\mathbb{C}$ the operator function $F(\cdot)^{-1}$ is finitely meromorphic, then $\rho_0\cap\sigma_{\mathrm{ess}}(F)=\emptyset$. The operator function F is called *self-adjoint* if

$$F(\overline{\lambda})=F(\lambda)^*,\quad \lambda\in\mathbb{C};$$

in particular, for $\lambda\in\mathbb{R}$ the values $F(\lambda)$ are self-adjoint operators. The spectrum of a self-adjoint operator function is symmetric with respect to the real axis.

2.2. Self-adjoint operators in Krein spaces

For the definition and simple properties of Krein spaces and the linear operators therein we refer the reader to [**4**, **7**, **25**]. In the following we recall some of the basic notions. A *Krein space* $(\mathcal{K}, [\cdot\,,\cdot])$ is a linear space $\mathcal{K}$ which is equipped with an (indefinite) inner product $[\cdot\,,\cdot]$ such that $\mathcal{K}$ can be written as

$$\mathcal{K} = \mathcal{G}_+ [\dot{+}] \mathcal{G}_-, \tag{2.1}$$

where $(\mathcal{G}_\pm, \pm[\cdot\,,\cdot])$ are Hilbert spaces and $[\dot{+}]$ means that the sum of $\mathcal{G}_+$ and $\mathcal{G}_-$ is direct and $[\mathcal{G}_+, \mathcal{G}_-] = \{0\}$. The norm topology on a Krein space $\mathcal{K}$ is the norm topology of the orthogonal sum of the Hilbert spaces $\mathcal{G}_\pm$ in (2.1). It can be shown that this norm topology is independent of the particular decomposition (2.1); all the topological notions in $\mathcal{K}$ refer to this norm topology.

Krein spaces often arise as follows: in a given Hilbert space $(\mathcal{G}, (\cdot\,,\cdot))$ any bounded self-adjoint operator G in $\mathcal{G}$ with $0 \in \rho(G)$ induces an inner product

$$[x, y] := (Gx, y), \quad x, y \in \mathcal{G},$$

such that $(\mathcal{G}, [\cdot\,,\cdot])$ becomes a Krein space. In the particular case where G has the additional property $G^2 = I$, that is, G is the difference of two complementary orthogonal projections P and Q, $G = P - Q$ with $P + Q = I$, one often writes $G = J$ and in the decomposition (2.1) one can choose $\mathcal{G}_+ = P\mathcal{G}$, $\mathcal{G}_- = Q\mathcal{G}$.

For a closed linear operator A in a Krein space $\mathcal{K}$ with dense domain $\mathcal{D}(A)$, the (Krein space) *adjoint* A^+ of A is the densely defined operator in $\mathcal{K}$ with

$$\mathcal{D}(A^+) = \{y \in \mathcal{K} : [A\cdot\,, y] \text{ is a continuous linear functional on } \mathcal{D}(A)\}$$

and

$$[Ax, y] = [x, A^+ y], \quad x \in \mathcal{D}(A),\ y \in \mathcal{D}(A^+).$$

The operator A is called *symmetric* in $\mathcal{K}$ if $A \subset A^+$, and *self-adjoint* if $A = A^+$. A self-adjoint operator A in a Krein space $\mathcal{K}$ may have a non-real spectrum, which is always symmetric with respect to the real axis, and both the spectrum $\sigma(A)$ and the resolvent set $\rho(A)$ may be empty.

An element $x \in \mathcal{K}$ is called *positive* (respectively, *non-positive, neutral*, etc.) if $[x, x] > 0$ (respectively, $[x, x] \leqslant 0$, $[x, x] = 0$, etc.), a subspace of $\mathcal{K}$ is called *positive* (respectively, *non-positive, neutral*, etc.) if all its non-zero elements are positive (respectively, non-positive, neutral, etc.). If for a self-adjoint operator A in a Krein space $\mathcal{K}$ with $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ all the eigenvectors at λ_0 are positive (respectively, negative), then λ_0 is called an *eigenvalue of positive type* (respectively, *negative type*). An eigenvector x_0 of A at λ_0 that is positive or negative does not have any associated vectors.

2.3. Definitizable operators in Krein spaces

A self-adjoint operator A in a Krein space $\mathcal{K}$ is called *definitizable* if $\rho(A) \neq \emptyset$ and there exists a polynomial p such that

$$[p(A)x, x] \geqslant 0, \quad x \in \mathcal{D}(p(A)).$$

The spectrum of a definitizable operator A is real with the possible exception of finitely many pairs of eigenvalues $\lambda, \overline{\lambda}$, which are necessarily zeros of each definitizing polynomial p; at such an eigenvalue λ or $\overline{\lambda}$ the resolvent of A has a pole of order not greater than the order of λ as a zero of the polynomial p. The closed linear span of all the algebraic eigenspaces $\mathcal{L}_\lambda(A)$ corresponding to the eigenvalues λ of A in the open upper (or lower) half-plane is a neutral subspace of $\mathcal{K}$. The algebraic eigenspaces $\mathcal{L}_\lambda(A), \mathcal{L}_{\overline{\lambda}}(A)$ corresponding to non-real $\lambda, \overline{\lambda} \in \sigma_{\mathrm{p}}(A)$ are skewly linked, that is, to each non-zero $x \in \mathcal{L}_\lambda(A)$ there exists a $y \in \mathcal{L}_{\overline{\lambda}}(A)$ such that $[x, y] \neq 0$ and to each non-zero $y \in \mathcal{L}_{\overline{\lambda}}(A)$ there exists an $x \in \mathcal{L}_\lambda(A)$ such that $[x, y] \neq 0$.

If $\lambda \in \sigma_{\mathrm{p}}(A)$, the maximal dimension (up to ∞) of a non-negative (non-positive, respectively) subspace of $\mathcal{L}_\lambda(A)$ is denoted by $\kappa_\lambda^+(A)$ ($\kappa_\lambda^-(A)$, respectively), and the dimension of the so-called *isotropic subspace* $\mathcal{L}_\lambda(A) \cap \mathcal{L}_\lambda(A)^{[\perp]}$ of $\mathcal{L}_\lambda(A)$ is denoted by $\kappa_\lambda^\circ(A)$. Observe that if, for example, $\mathcal{L}_\lambda(A)$ is one-dimensional and neutral, then $\kappa_\lambda^+(A) = \kappa_\lambda^-(A) = \kappa_\lambda^\circ(A) = 1$; for a non-real eigenvalue λ of A the number $\kappa_\lambda^\circ(A)$ coincides with the dimension of $\mathcal{L}_\lambda(A)$.

A definitizable operator A with definitizing polynomial p has a *spectral function with critical points*. To introduce it, we call a bounded real interval Γ *admissible* for the operator A if some definitizing polynomial p of A does not vanish at the end points of Γ. Then, for each admissible bounded interval Γ, there exists an orthogonal projection $E(\Gamma)$ in $\mathcal{K}$ such that the range $E(\Gamma)\mathcal{K}$ is invariant under A, the spectrum of the restriction $A|_{E(\Gamma)\mathcal{K}}$ is contained in $\overline{\Gamma}$, and the real spectrum of the restriction $A|_{(I-E(\Gamma))\mathcal{K}}$ is contained in $\overline{\mathbb{R} \setminus \Gamma}$.

A real spectral point $\lambda \in \sigma(A)$ is called of *positive type* if there exists an admissible open interval Γ such that $\lambda \in \Gamma$ and $(E(\Gamma)\mathcal{K}, [\cdot\,,\cdot])$ is a Hilbert space; this is equivalent to the fact that $[x, x] \geqslant 0$, $x \in E(\Gamma)\mathcal{K}$ (which implies that $[x, x] > 0$ if $x \neq 0$). The set of all spectral points of positive type of A is denoted by $\sigma_+(A)$. Clearly, if $(E(\Gamma)\mathcal{K}, [\cdot\,,\cdot])$ is a Hilbert space, the restriction $A|_{E(\Gamma)\mathcal{K}}$ has the same spectral properties as a self-adjoint operator in a Hilbert space. If a definitizing polynomial p is positive on an admissible interval Γ, then $\Gamma \cap \sigma(A)$ consists only of spectral points of positive type. Similarly, a real point $\lambda \in \sigma(A)$ for which there exists an open admissible interval Γ such that $\lambda \in \Gamma$ and $(E(\Gamma)\mathcal{K}, -[\cdot\,,\cdot])$ is a Hilbert space is called a spectral point of *negative type of* A; the set of all spectral points of negative type of A is denoted by $\sigma_-(A)$. Finally, $\lambda_0 \in \mathbb{R}$ is called a *critical point* of A if, for each admissible open interval Γ with $\lambda_0 \in \Gamma$, the range $E(\Gamma)\mathcal{K}$ contains both positive and negative elements. The set of all critical points of A is denoted by $\sigma_{\mathrm{crit}}(A)$; it is always finite. In fact, each critical point is a zero of every definitizing polynomial p. From these definitions it follows that

$$\sigma(A) \cap \mathbb{R} = \sigma_+(A) \cup \sigma_-(A) \cup \sigma_{\mathrm{crit}}(A).$$

A real eigenvalue of A with a neutral eigenvector is always a critical point of A. An eigenvalue λ of positive type is a critical point of A if in each neighbourhood of λ there are spectral points of negative type of A. If, however, the eigenvalue λ of positive type is an isolated spectral point, then it is a spectral point of positive type.

Similarly, ∞ is called a *critical point* of A if, outside of each compact real interval, there are spectral points of positive and of negative type of A. If ∞ is a critical point of the self-adjoint operator A, it is called a *regular critical point of* A if there exists a constant $\gamma > 0$ such that $\|E(\Gamma)\| \leqslant \gamma$ for all sufficiently large intervals Γ centred at 0; otherwise, ∞ is called a *singular critical point*; here $\|\cdot\|$ denotes the operator norm corresponding to the norm in $\mathcal{K}$ induced by any of the equivalent decompositions (2.1). Note that the norm of a self-adjoint projection in a Krein space can be arbitrarily large. If ∞ is a regular critical point of A, then outside of a sufficiently large compact interval the operator A has the same spectral properties as a self-adjoint operator in a Hilbert space; in particular, the bounded operators $\mathrm{e}^{\mathrm{i}tA}$, $t \in \mathbb{R}$, can be defined and form a group of unitary operators in $\mathcal{K}$. Recall that a *unitary operator* U in a Krein space $\mathcal{K}$ is a bounded operator such that

$$UU^{+} = U^{+}U = I.$$

The operators occurring in this paper belong to a special class of definitizable operators: they are self-adjoint operators A in a Krein space $\mathcal{K}$ for which $\rho(A) \neq 0$ and the sesquilinear form

$$[Ax, y], \quad x, y \in \mathcal{D}(A), \tag{2.2}$$

has a finite number κ of negative squares; recall that the latter means that each subspace $\mathcal{L}$ of $\mathcal{K}$ for which

$$[Ax, x] < 0, \quad x \in \mathcal{L},\ x \neq 0, \tag{2.3}$$

is of dimension less than or equal to κ and for at least one κ-dimensional subspace $\mathcal{L}$ the relation (2.3) holds. In this case the definitizing polynomial is of the form $p(\lambda) = \lambda q(\lambda)\overline{q(\overline{\lambda})}$ with some polynomial q of degree less than or equal to κ. Then all the algebraic eigenspaces corresponding to non-real eigenvalues are finite dimensional, the positive spectrum of A consists of spectral points of positive type with the possible exception of a finite number of eigenvalues with a negative or neutral eigenvector, and the negative spectrum of A consists of spectral points of negative type with the possible exception of a finite number of eigenvalues with a positive or neutral eigenvector. If, additionally, A is boundedly invertible, the following equality holds:

$$\kappa = \sum_{\lambda \in \sigma_{\mathrm{p}}(A) \cap (0,+\infty)} \kappa_{\lambda}^{-}(A) \quad + \sum_{\lambda \in \sigma_{\mathrm{p}}(A) \cap (-\infty,0)} \kappa_{\lambda}^{+}(A) \quad + \sum_{\lambda \in \sigma_{\mathrm{p}}(A) \cap \mathbb{C}^{+}} \kappa_{\lambda}^{\mathrm{o}}(A). \tag{2.4}$$

The Krein space $\mathcal{K}$ is called a *Pontryagin space with negative index* κ if in one (and hence in all) decompositions of the form (2.1) the space $\mathcal{G}_{-}$ has finite dimension κ. Any self-adjoint operator A in a Pontryagin space is definitizable and hence has a spectral function with critical points. With the exception of finitely many points, the real spectral points of A are of positive type; the exceptional points are eigenvalues with a negative or neutral eigenvector and

$$\kappa = \sum_{\lambda \in \sigma_{p}(A) \cap \mathbb{R}} \kappa_{\lambda}^{-}(A) + \sum_{\lambda \in \sigma_{p}(A) \cap \mathbb{C}^{+}} \kappa_{\lambda}^{\mathrm{o}}(A).$$

3. Operators associated with the abstract Klein–Gordon equation: A_1 in $\mathcal{H} \oplus \mathcal{H}$

Let $(\mathcal{H}, (\cdot\,,\cdot))$ be a Hilbert space with corresponding norm $\|\cdot\|$, let H_0 be a uniformly positive self-adjoint operator in $\mathcal{H}$, $H_0 \geqslant m^2 > 0$, and let V be a densely defined symmetric operator in $\mathcal{H}$.

By $\mathcal{G}_1$ we denote the orthogonal sum $\mathcal{G}_1 := \mathcal{H} \oplus \mathcal{H}$ with its inner product

$$(\boldsymbol{x}, \boldsymbol{x}')_{\mathcal{G}_1} := (x, x') + (y, y'), \quad \boldsymbol{x} = (x\ y)^{\mathrm{T}},\ \boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{G}_1.$$

Equipped with the indefinite inner product

$$[\boldsymbol{x}, \boldsymbol{x}'] := (x, y') + (y, x'), \quad \boldsymbol{x} = (x\ y)^{\mathrm{T}},\ \boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{G}_1, \tag{3.1}$$

the space $\mathcal{K}_1 := (\mathcal{G}_1, [\cdot\,,\cdot])$ becomes a Krein space. This is clear since the inner product $[\cdot\,,\cdot]$ can be defined by $[\boldsymbol{x}, \boldsymbol{x}'] = (G\boldsymbol{x}, \boldsymbol{x}')_{\mathcal{G}_1}$ with the Gram operator

$$G = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

In $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$, with the abstract first-order differential equation (1.7), we associate the block operator matrix $\hat{A}_1$ given by

$$\hat{A}_1 := \begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}. \tag{3.2}$$

With its natural domain $\mathcal{D}(\hat{A}_1) := (\mathcal{D}(V) \cap \mathcal{D}(H_0)) \oplus \mathcal{D}(V)$, the operator $\hat{A}_1$ need not be densely defined nor closable. The main assumption here and below is as follows.

Assumption 3.1. $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$.

Since V is closable (with closure denoted by $\overline{V}$), Assumption 3.1 is satisfied if and only if V is $H_0^{1/2}$-bounded (see [**22**, §§ IV.1.1, IV.1.3]). Since, in addition, H_0 is assumed to be boundedly invertible, Assumption 3.1 implies that the operator

$$S := V H_0^{-1/2} \tag{3.3}$$

is defined on all of $\mathcal{H}$ and bounded in $\mathcal{H}$. Together with

$$H_0^{-1/2} V \subset H_0^{-1/2} V^* \subset (V H_0^{-1/2})^* = S^*, \tag{3.4}$$

this shows that $\overline{H_0^{-1/2} V} = S^*$.

Under Assumption 3.1, the domain of $\hat{A}_1$ takes the form

$$\mathcal{D}(\hat{A}_1) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}(H_0),\ y \in \mathcal{D}(V) \right\}. \tag{3.5}$$

Theorem 3.2. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, then $\hat{A}_1$ is essentially self-adjoint in the Krein space $\mathcal{K}_1$. Its closure is the self-adjoint operator $A_1 = (\hat{A}_1)^+$ given by*

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}(H_0^{1/2}),\ H_0^{1/2}x + S^* y \in \mathcal{D}(H_0^{1/2}) \right\},$$
$$A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Vx + y \\ H_0^{1/2}(H_0^{1/2}x + S^* y) \end{pmatrix}.$$

Proof. First we show that

$$(\hat{A}_1)^+ = A_1. \tag{3.6}$$

In order to prove the inclusion $(\hat{A}_1)^+ \subset A_1$, let $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}((\hat{A}_1)^+)$. Then, for $(\hat{A}_1)^+ \boldsymbol{x} =: \boldsymbol{u} = (u\ v)^{\mathrm{T}} \in \mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$, we have

$$[\hat{A}_1 \boldsymbol{x}', \boldsymbol{x}] = [\boldsymbol{x}', \boldsymbol{u}], \quad \boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{D}(\hat{A}_1),$$

that is,

$$(Vx' + y', y) + (H_0 x' + V y', x) = (x', v) + (y', u), \quad x' \in \mathcal{D}(H_0),\ y' \in \mathcal{D}(V). \tag{3.7}$$

Choosing $y' = 0$, we conclude that for all $x' \in \mathcal{D}(H_0)$ we have the equality

$$\begin{aligned}
&(Vx', y) + (H_0 x', x) = (x', v) \\
&\qquad \iff \quad (VH_0^{-1/2}(H_0^{1/2}x'), y) + (H_0^{1/2}(H_0^{1/2}x'), x) = (H_0^{-1/2}(H_0^{1/2}x'), v) \\
&\qquad \iff \quad (H_0^{1/2}x', S^* y) + (H_0^{1/2}(H_0^{1/2}x'), x) = (H_0^{1/2}x', H_0^{-1/2}v) \\
&\qquad \iff \quad (H_0^{1/2}(H_0^{1/2}x'), x) = (H_0^{1/2}x', H_0^{-1/2}v - S^* y) \\
&\qquad \iff \quad (H_0^{1/2}w', x) = (w', H_0^{-1/2}v - S^* y)
\end{aligned}$$

with $w' := H_0^{1/2}x'$ being an arbitrary element of $\mathcal{D}(H_0^{1/2})$. Hence, $x \in \mathcal{D}(H_0^{1/2})$ and

$$H_0^{1/2}x = H_0^{-1/2}v - S^* y,$$

that is,

$$H_0^{1/2}x + S^* y = H_0^{-1/2}v \in \mathcal{D}(H_0^{1/2})$$

and

$$v = H_0^{1/2}(H_0^{1/2}x + S^* y).$$

Choosing $x' = 0$ in (3.7) and using the symmetry of V, we obtain that, for all $y' \in \mathcal{D}(V)$,

$$(y', y) + (Vy', x) = (y', u).$$

Since $x \in \mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and $\mathcal{D}(V)$ is dense in $\mathcal{H}$, we see that $u = Vx + y$.

The inclusion $A_1 \subset (\hat{A}_1)^+$ follows if we observe that, for arbitrary $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}(A_1)$ and $\boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{D}(\hat{A}_1)$,

$$\begin{aligned}[\hat{A}_1\boldsymbol{x}', \boldsymbol{x}] &= (Vx', y) + (y', y) + (H_0x', x) + (Vy', x)\\ &= (H_0^{1/2}x', S^*y) + (y', y) + (H_0^{1/2}x', H_0^{1/2}x) + (Vy', x)\\ &= (H_0^{1/2}x', H_0^{1/2}x + S^*y) + (y', Vx + y)\\ &= [\boldsymbol{x}', A_1\boldsymbol{x}].\end{aligned}$$

By the definition of $\hat{A}_1$ and A_1 and by (3.6), we have $\hat{A}_1 \subset A_1 = (\hat{A}_1)^+$. Thus, $\hat{A}_1$ is symmetric in $\mathcal{K}_1$ and hence closable. Since $(\hat{A}_1)^+$ is closed, it follows that

$$\overline{\hat{A}_1} \subset (\hat{A}_1)^+ = A_1,$$

and the theorem is proved if we show that $A_1 \subset \overline{\hat{A}_1}$.

To this end, let $(x\ y)^{\mathrm{T}} \in \mathcal{D}(A_1)$. Then $x \in \mathcal{D}(H_0^{1/2})$ and $y \in \mathcal{H}$ are such that $z := H_0^{1/2}x + S^*y \in \mathcal{D}(H_0^{1/2})$. Since $\mathcal{D}(V)$ is dense in $\mathcal{H}$, there exists a sequence $(y_n) \subset \mathcal{D}(V)$ with $y_n \to y$ and, since S is bounded, $H_0^{-1/2}Vy_n = S^*y_n \to S^*y$. If we define

$$\tilde{x}_n := z - H_0^{-1/2}Vy_n \in \mathcal{D}(H_0^{1/2}) \quad \text{and} \quad x_n := H_0^{-1/2}\tilde{x}_n \in \mathcal{D}(H_0),$$

then we have, for $n \to \infty$,

$$x_n = H_0^{-1/2}z - H_0^{-1/2}H_0^{-1/2}Vy_n \to H_0^{-1/2}(H_0^{1/2}x + S^*y) - H_0^{-1/2}S^*y = x,$$
$$H_0^{1/2}x_n = \tilde{x}_n \to z - S^*y = H_0^{1/2}x.$$

Since V is $H_0^{1/2}$-bounded by Assumption 3.1, this implies that also (Vx_n) and hence $(Vx_n + y_n)$ converges. Finally,

$$H_0x_n + Vy_n = H_0^{1/2}(H_0^{1/2}x_n + H_0^{-1/2}Vy_n) = H_0^{1/2}(\tilde{x}_n + H_0^{-1/2}Vy_n) = H_0^{1/2}z,$$

and hence $(H_0x_n + Vy_n)$ converges. This proves that $(x\ y)^{\mathrm{T}} \in \mathcal{D}(\overline{\hat{A}_1})$. □

In order to ensure that the resolvent set of A_1 is non-empty, an additional condition is required in § 5. In this respect, the self-adjoint quadratic operator polynomial

$$L_1(\lambda) := I - (S - \lambda H_0^{-1/2})(S^* - \lambda H_0^{-1/2}), \quad \lambda \in \mathbb{C}, \tag{3.8}$$

in the Hilbert space $\mathcal{H}$ is useful as it reflects the spectral properties of A_1; note that, according to Assumption 3.1, the values $L_1(\lambda)$ are bounded operators in $\mathcal{H}$.

Proposition 3.3. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, then, for $\lambda \in \mathbb{C}$,*

$$A_1 - \lambda = \begin{pmatrix} I & (S - \lambda H_0^{-1/2})H_0^{1/2} \\ 0 & H_0 \end{pmatrix} \begin{pmatrix} 0 & L_1(\lambda) \\ I & H_0^{-1/2}(S^* - \lambda H_0^{-1/2}) \end{pmatrix}. \tag{3.9}$$

Proof. The formal equality in (3.9) follows immediately from formula (3.8). To prove the equality of the domains, we observe that $(S - \lambda H_0^{-1/2})H_0^{1/2} = V - \lambda$ on $\mathcal{D}(H_0)$. Then the domain $\mathcal{D}_1$ of the product on the right-hand side of (3.9) is given by

$$\begin{aligned}
\mathcal{D}_1 &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x + H_0^{-1/2}(S^* - \lambda H_0^{-1/2})y \in \mathcal{D}(H_0) \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x + H_0^{-1/2} S^* y \in \mathcal{D}(H_0) \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}(H_0^{1/2}),\ H_0^{1/2}x + S^* y \in \mathcal{D}(H_0^{1/2}) \right\},
\end{aligned}$$

which coincides with the domain of A_1 by Theorem 3.2. □

Proposition 3.4. *Let $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$. Then $\rho(L_1) \subset \rho(A_1)$ and, for $\lambda \in \rho(L_1)$,*

$$\begin{aligned}
&(A_1 - \lambda)^{-1} \\
&\quad = \begin{pmatrix} -H_0^{-1/2}(S^* - \lambda H_0^{-1/2})L_1(\lambda)^{-1} & I \\ L_1(\lambda)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -(S - \lambda H_0^{-1/2})H_0^{-1/2} \\ 0 & H_0^{-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & H_0^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -H_0^{-1/2}(S^* - \lambda H_0^{-1/2}) \\ I \end{pmatrix} L_1(\lambda)^{-1} \begin{pmatrix} I & -(S - \lambda H_0^{-1/2})H_0^{-1/2} \end{pmatrix}.
\end{aligned}$$

Moreover,

$$\sigma_{\mathrm{ess}}(A_1) \subset \sigma_{\mathrm{ess}}(L_1). \tag{3.10}$$

Proof. The first and the second claim are immediate consequences of the factorization (3.9). If $\lambda_0 \notin \sigma_{\mathrm{ess}}(L_1)$, then $L_1(\cdot)^{-1}$ is a finitely meromorphic operator function in a neighbourhood of λ_0 and hence so is $(A_1 - \cdot\,)^{-1}$ by the second formula for $(A_1 - \lambda)^{-1}$ above. This shows that $\lambda_0 \notin \sigma_{\mathrm{ess}}(A_1)$. □

4. Operators associated with the abstract Klein–Gordon equation: A_2 in $\mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$

In order to associate a second operator A_2 with the abstract Klein–Gordon equation (1.2), we introduce a scale of Hilbert spaces $(\mathcal{H}_\alpha, \|\cdot\|_\alpha)$, $-1 \leqslant \alpha \leqslant 1$, induced by the operator H_0 as follows. If $0 \leqslant \alpha \leqslant 1$, we set

$$\mathcal{H}_\alpha := \mathcal{D}(H_0^\alpha), \quad \|x\|_\alpha := \|H_0^\alpha x\|, \quad x \in \mathcal{H}_\alpha,\ 0 \leqslant \alpha \leqslant 1. \tag{4.1}$$

Obviously, $\mathcal{H}_0 = \mathcal{H}$. If $-1 \leqslant \alpha < 0$, then $\mathcal{H}_\alpha$ is defined to be the corresponding space with negative norm (see [**6**]), which can also be considered as the completion of $\mathcal{D}(H_0^\alpha) = \mathcal{H}$ with respect to the norm

$$\|x\|_\alpha := \|H_0^\alpha x\|, \quad x \in \mathcal{H},\ -1 \leqslant \alpha < 0.$$

All powers of H_0 extend in a natural way to operators between the spaces of the scale $\mathcal{H}_\alpha$, $-1 \leqslant \alpha \leqslant 1$; in particular, $H_0^\beta \in L(\mathcal{H}_\alpha, \mathcal{H}_{\alpha-\beta})$ for $\alpha, \beta, \alpha - \beta \in [-1, 1]$. The duality between $\mathcal{H}_\alpha$ and $\mathcal{H}_{-\alpha}$ is again denoted by $(\cdot\,,\cdot)$:

$$(x, y) := (H_0^\alpha x, H_0^{-\alpha} y), \quad x \in \mathcal{H}_\alpha,\ y \in \mathcal{H}_{-\alpha}. \tag{4.2}$$

Let $\mathcal{G}_2$ be the orthogonal sum $\mathcal{G}_2 := \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ with its inner product

$$(\boldsymbol{x}, \boldsymbol{x}')_{\mathcal{G}_2} = (H_0^{1/4} x, H_0^{1/4} x') + (H_0^{-1/4} y, H_0^{-1/4} y')$$

for $\boldsymbol{x} = (x\ y)^{\mathrm{T}}$, $\boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{G}_2$. In addition, we equip the space $\mathcal{G}_2$ with the indefinite inner product

$$[\boldsymbol{x}, \boldsymbol{x}'] := (H_0^{1/4} x, H_0^{-1/4} y') + (H_0^{-1/4} y, H_0^{1/4} x')$$

for $\boldsymbol{x} = (x\ y)^{\mathrm{T}}$, $\boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{G}_2$, that is,

$$[\boldsymbol{x}, \boldsymbol{x}'] = (x, y') + (y, x'); \tag{4.3}$$

the brackets on the right-hand side of (4.3) denote the duality (4.2) between $\mathcal{H}_{1/4}$ and $\mathcal{H}_{-1/4}$ and vice versa.

The space $\mathcal{K}_2 := (\mathcal{G}_2, [\cdot\,,\cdot])$ is a Krein space. To see this, we observe that $H_0^{-1/2}$ is a unitary mapping from $\mathcal{G}_{-1/4}$ onto $\mathcal{G}_{1/4}$, $H_0^{1/2}$ is a unitary mapping from $\mathcal{G}_{1/4}$ onto $\mathcal{G}_{-1/4}$, and that

$$[\boldsymbol{x}, \boldsymbol{x}'] = (y, x') + (x, y') = \left(\begin{pmatrix} H_0^{-1/2} y \\ H_0^{1/2} x \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right)_{\mathcal{G}_2} = (J\boldsymbol{x}, \boldsymbol{x}')_{\mathcal{G}_2};$$

here the operator J in $\mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ is given by

$$J := \begin{pmatrix} 0 & H_0^{-1/2} \\ H_0^{1/2} & 0 \end{pmatrix} \tag{4.4}$$

and satisfies $J = J^*$ as well as $J^2 = I$ (see § 2.2).

Imposing Assumption 3.1, that is, $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, we may consider the symmetric operator V also as an operator in the scale of spaces $\mathcal{H}_\alpha$, $-1 \leqslant \alpha \leqslant 1$.

Remark 4.1. Assumption 3.1 is equivalent to the boundedness of V regarded as an operator from $\mathcal{H}_{1/2}$ to $\mathcal{H}$, that is, $V \in L(\mathcal{H}_{1/2}, \mathcal{H})$. Due to its symmetry, V then admits an extension as a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{-1/2}$; by interpolation, it can also be defined as a bounded operator from $\mathcal{H}_\alpha$ into $\mathcal{H}_{\alpha-1/2}$ for all $\alpha \in [0, \frac{1}{2}]$, see [**39**, Chapter IX.4, Appendix, Example 3]. All these extensions and restrictions of the operator V, originally given in $\mathcal{H}$, which act between the spaces of the scale $\mathcal{H}_\alpha$, are also denoted by V:

$$V \in L(\mathcal{H}_\alpha, \mathcal{H}_{\alpha-1/2}), \quad \alpha \in [0, \tfrac{1}{2}]. \tag{4.5}$$

As a consequence of (4.5), for the corresponding extensions and restrictions of the operators $S = VH_0^{-1/2}$ and the adjoint $S^* = (VH_0^{-1/2})^*$ of S in $\mathcal{H}$, we have

$$S = VH_0^{-1/2} \in L(\mathcal{H}_\alpha), \quad \alpha \in [-\tfrac{1}{2}, 0],$$

and

$$S^* = H_0^{-1/2} V \in L(\mathcal{H}_\alpha), \quad \alpha \in [0, \tfrac{1}{2}].$$

Note that in § 3 the operator S^* had to be written as $S^* = \overline{H_0^{1/2} V}$ since there V acts only within $\mathcal{H}$ and thus requires the domain $\mathcal{D}(V) \subset \mathcal{H}$; observe also that in $\mathcal{H}_\alpha$, $\alpha \neq 0$, the operator S^* is not the adjoint of S.

Under Assumption 3.1, we now consider the operator A_2 in $\mathcal{G}_2$ given by

$$\left.\begin{aligned} \mathcal{D}(A_2) &:= \biggl\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} : x \in \mathcal{D}(H_0^{1/2}),\ y \in \mathcal{H}, \\ &\qquad\qquad\qquad\qquad Vx + y \in \mathcal{H}_{1/4},\ H_0 x + Vy \in \mathcal{H}_{-1/4} \biggr\}, \\ A_2 \begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} Vx + y \\ H_0 x + Vy \end{pmatrix}. \end{aligned}\right\} \tag{4.6}$$

Remark 4.2. The domain of A_2 can also be written as

$$\mathcal{D}(A_2) = \biggl\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} : y \in \mathcal{H},\ Vx + y \in \mathcal{H}_{1/4},\ H_0 x + Vy \in \mathcal{H}_{-1/4} \biggr\};$$

in fact, if $y \in \mathcal{H}$, then $H_0 x + Vy \in \mathcal{H}_{-1/4}$ automatically implies $x \in \mathcal{D}(H_0^{1/2})$.

In the following, we first show that A_2 is a symmetric operator in the Krein space $\mathcal{K}_2$. In the theorem below we give a sufficient condition for the self-adjointness of A_2 in $\mathcal{K}_2$.

Proposition 4.3. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, then A_2 is symmetric in $\mathcal{K}_2$.*

Proof. Let $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}(A_2) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$. Then, according to the formula (4.3) for the inner product $[\cdot\,,\cdot]$,

$$[A_2 \boldsymbol{x}, \boldsymbol{x}] = (Vx + y, y) + (H_0 x + Vy, x) = (Vx, y) + (y, y) + (H_0 x, x) + (Vy, x).$$

Note that the first two terms are the inner products in $\mathcal{H}$, whereas the last two terms denote the duality between $\mathcal{G}_{-1/2}$ and $\mathcal{G}_{1/2}$. Since

$$(Vy, x) = (H_0^{-1/2} Vy, H_0^{1/2} x) = (S^* y, H_0^{1/2} x) = (y, S H_0^{1/2} x) = (y, Vx) = \overline{(Vx, y)},$$

it follows that $[A_2 \boldsymbol{x}, \boldsymbol{x}] \in \mathbb{R}$. □

Theorem 4.4. *Suppose that $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$. Then*

$$\rho(L_1) \subset \rho(A_2),$$

the operator A_2 is self-adjoint in $\mathcal{K}_2$ if $\rho(L_1) \neq \emptyset$ and, for $\lambda \in \rho(L_1)$,

$$\begin{aligned} &(A_2 - \lambda)^{-1} \\ &\quad = \begin{pmatrix} 0 & H_0^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -H_0^{-1/2}(S^* - \lambda H_0^{-1/2}) \\ I \end{pmatrix} L_1(\lambda)^{-1} \left(I - (S - \lambda H_0^{-1/2}) H_0^{-1/2} \right). \end{aligned} \tag{4.7}$$

For the proof of Theorem 4.4, we use the following simple lemma.

Lemma 4.5. *If A is a symmetric operator in a Krein space $\mathcal{K}$ such that there exists a point $\lambda \in \mathbb{C}$ with $\lambda, \overline{\lambda} \in \rho(A)$, then A is self-adjoint in $\mathcal{K}$.*

Proof. Let $\lambda \in \mathbb{C}$ with $\lambda, \overline{\lambda} \in \rho(A)$. Then $\lambda \in \rho(A) \cap \rho(A^+)$ and the symmetry of A implies that

$$(A^+ - \overline{\lambda})^{-1} \supset (A - \overline{\lambda})^{-1}.$$

Since $\overline{\lambda} \in \rho(A)$, the right-hand side is defined on all of $\mathcal{K}$. Hence, the last inclusion is in fact an equality and so $A = A^+$. □

Proof of Theorem 4.4. First we prove that $\rho(L_1) \subset \rho(A_2)$. Let $\lambda_0 \in \rho(L_1)$ and let $\boldsymbol{u} = (u\ v)^{\mathrm{T}} \in \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ be arbitrary. We have to show that there exists a unique element $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}(A_2) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$ such that $(A_2 - \lambda_0)\boldsymbol{x} = \boldsymbol{u}$ or, equivalently,

$$(V - \lambda_0)x + y = u, \tag{4.8}$$

$$H_0 x + (V - \lambda_0)y = v. \tag{4.9}$$

Since $V \in L(\mathcal{H}_{1/2}, \mathcal{H})$, we have $u - (V - \lambda_0)H_0^{-1}v \in \mathcal{H}$ and thus

$$y := L_1(\lambda_0)^{-1}(u - (V - \lambda_0)H_0^{-1}v) \in \mathcal{H}, \tag{4.10}$$

$$x := H_0^{-1}(v - (V - \lambda_0)y) \in \mathcal{D}(H_0^{1/2}). \tag{4.11}$$

Then, by the definition of x, (4.9) holds. Relation (4.8) follows since

$$\begin{aligned}(V - \lambda_0)x + y &= (V - \lambda_0)H_0^{-1}(v - (V - \lambda_0)y) + y \\ &= (V - \lambda_0)H_0^{-1}v + (I - (V - \lambda_0)H_0^{-1}(V - \lambda_0))y \\ &= (V - \lambda_0)H_0^{-1}v + (I - (VH_0^{-1/2} - \lambda_0 H_0^{-1/2})(H_0^{-1/2}V - \lambda_0 H_0^{-1/2}))y \\ &= (V - \lambda_0)H_0^{-1}v + L_1(\lambda_0)y \\ &= u;\end{aligned}$$

here we have used the fact that $H_0^{-1/2}V = S^*$ according to Remark 4.1. This proves that $A_2 - \lambda_0$ is surjective. In order to show that $A_2 - \lambda_0$ is injective, let $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}(A_2) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$ be such that $(A_2 - \lambda_0)\boldsymbol{x} = \boldsymbol{0}$ or, equivalently,

$$(V - \lambda_0)x + y = 0,$$

$$H_0 x + (V - \lambda_0)y = 0.$$

The second relation yields $x = -H_0^{-1}(V - \lambda_0)y$. Inserting this into the first relation, we obtain $L_1(\lambda_0)y = 0$. Now $y \in \mathcal{H}$ implies that $y = 0$ and hence also that $x = 0$.

Since L_1 is a self-adjoint operator function in $\mathcal{H}$, its resolvent set $\rho(L_1)$ is symmetric to $\mathbb{R}$. Hence, $\rho(L_1) \neq \emptyset$ implies that A_2 is self-adjoint in $\mathcal{K}_2$ by Lemma 4.5.

It remains to prove the representation for $(A_2-\lambda)^{-1}$. From (4.10), (4.11) it follows that, for $\lambda \in \rho(L_1)$,

$$(A_2-\lambda)^{-1} = \begin{pmatrix} -H_0^{-1}(V-\lambda)L_1(\lambda)^{-1} & H_0^{-1} + H_0^{-1}(V-\lambda)L_1(\lambda)^{-1}(V-\lambda)H_0^{-1} \\ L_1(\lambda)^{-1} & -L_1(\lambda)^{-1}(V-\lambda)H_0^{-1} \end{pmatrix}.$$

Using the identities

$$(V-\lambda)H_0^{-1} = (S-\lambda H_0^{-1/2})H_0^{-1/2}, \qquad H_0^{-1}(V-\lambda) = H_0^{-1/2}(S^*-\lambda H_0^{-1/2}),$$

we arrive at (4.7). Finally, we observe that the operators

$$\begin{gathered} \begin{pmatrix} I & -(S-\lambda H_0^{-1/2})H_0^{-1/2} \end{pmatrix} : \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} \to \mathcal{H}, \\ L_1(\lambda)^{-1} : \mathcal{H} \to \mathcal{H}, \\ \begin{pmatrix} -H_0^{-1/2}(S^*-\lambda H_0^{-1/2}) \\ I \end{pmatrix} : \mathcal{H} \to \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} \end{gathered}$$

are bounded and so the right-hand side of (4.7) is a bounded operator in the space $\mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$. $\square$

Corollary 4.6. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, we have $\rho(L_1) \subset \rho(A_1) \cap \rho(A_2)$ and the resolvents of A_1 and A_2 coincide on the dense subset $\mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}$ of $\mathcal{K}_1$ and $\mathcal{K}_2$:*

$$(A_1-\lambda)^{-1}\boldsymbol{x} = (A_2-\lambda)^{-1}\boldsymbol{x}, \quad \boldsymbol{x} \in \mathcal{K}_1 \cap \mathcal{K}_2, \quad \lambda \in \rho(L_1) \subset \rho(A_1) \cap \rho(A_2). \tag{4.12}$$

Proof. The claim follows easily from the formulae for the resolvents of A_1 and A_2 in Proposition 3.4 and Theorem 4.4. $\square$

5. Spectral properties of the operators A_1 and A_2

In this section we investigate the spectral properties of the self-adjoint operators A_1 and A_2 in the respective Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$.

We recall that under Assumption 3.1, that is, $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, the operator A_1 in $\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{H}$ is given by (see Theorem 3.2)

$$\begin{aligned} \mathcal{D}(A_1) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : x \in \mathcal{D}(H_0^{1/2}),\ H_0^{1/2}x + S^*y \in \mathcal{D}(H_0^{1/2}) \right\}, \\ A_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} Vx + y \\ H_0^{1/2}(H_0^{1/2}x + S^*y) \end{pmatrix}. \end{aligned}$$

Under the assumption that $\rho(L_1) \neq \emptyset$, the operator A_2 in $\mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ is given by (see (4.6) and Remark 4.2)

$$\mathcal{D}(A_2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} : x \in \mathcal{D}(H_0^{1/2}),\ y \in \mathcal{H}, \right. \\ \left. Vx + y \in \mathcal{H}_{1/4},\ H_0 x + Vy \in \mathcal{H}_{-1/4} \right\}$$
$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4} : y \in \mathcal{H},\ Vx + y \in \mathcal{H}_{1/4},\ H_0 x + Vy \in \mathcal{H}_{-1/4} \right\},$$
$$A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Vx + y \\ H_0 x + Vy \end{pmatrix}.$$

The definitions of A_1 and A_2 imply that, for $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{D}(A_j) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$,

$$[A_j \boldsymbol{x}, \boldsymbol{x}] = \|y + S\hat{x}\|^2 + ((I - S^*S)\hat{x}, \hat{x}), \quad j = 1, 2, \tag{5.1}$$

with $\hat{x} := H_0^{1/2} x$ in $\mathcal{H}$. Indeed, using $S\hat{x} = Vx$, we obtain

$$\begin{aligned} [A_2 \boldsymbol{x}, \boldsymbol{x}] &= (Vx + y, y) + (H_0 x + Vy, x) \\ &= \|H_0^{1/2} x\|^2 + \|y\|^2 + (Vx, y) + (y, Vx) \\ &= \|H_0^{1/2} x\|^2 + \|y\|^2 + (S\hat{x}, y) + (y, S\hat{x}) \\ &= \|y + S\hat{x}\|^2 + ((I - S^*S)\hat{x}, \hat{x}); \end{aligned}$$

the proof for A_1 is similar.

Relation (5.1) shows that the number of negative squares of the Hermitian forms $[A_j \boldsymbol{x}, \boldsymbol{y}], \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}(A_j)$, $j = 1, 2$, coincides with the dimension of the spectral subspace of the self-adjoint operator $I - S^*S$ in $\mathcal{H}$ corresponding to the negative half-axis. The following assumption is crucial for the remaining part of this paper; it guarantees that this number is finite.

Assumption 5.1. *$S = VH_0^{-1/2} = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact in $\mathcal{H}$.*

Obviously, such a decomposition of S is not unique, and the operator S_1 can be chosen to have some more particular properties.

Lemma 5.2. *In Assumption 5.1, without loss of generality, we can suppose that $S_1 = \sum_{i=1}^n (\cdot\,, w_i) v_i$ with $v_i \in \mathcal{H}$ and $w_i \in \mathcal{D}(H_0^{1/2})$, $i = 1, \dots, n$.*

Proof. Since a compact operator is the sum of an operator with arbitrarily small norm and an operator of finite rank, S_1 can be chosen to be of finite rank, say $S_1 = \sum_{i=1}^n (\cdot\,, w_i) v_i$ with $v_i, w_i \in \mathcal{H}$, $i = 1, \dots, n$. Moreover, by means of an additive perturbation of arbitrarily small norm, the elements w_i can be chosen in the dense subset $\mathcal{D}(H_0^{1/2})$ of $\mathcal{H}$. □

Lemma 5.3. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact, then $\rho(L_1) \neq \emptyset$.*

Proof. According to the assumption on S, for $\mu \in \mathbb{R}$ the operator $L_1(\mathrm{i}\mu)$ can be written as

$$L_1(\mathrm{i}\mu) = (I + \mu^2 H_0^{-1})^{1/2}(F(\mu) + K(\mu) + \mathrm{i}G(\mu))(I + \mu^2 H_0^{-1})^{1/2} \tag{5.2}$$

with the bounded self-adjoint operators

$$\begin{aligned} F(\mu) &:= I - (I + \mu^2 H_0^{-1})^{-1/2} S_0 S_0^* (I + \mu^2 H_0^{-1})^{-1/2}, \\ K(\mu) &:= -(I + \mu^2 H_0^{-1})^{-1/2}(S_0 S_1^* + S_1 S_0^* + S_1 S_1^*)(I + \mu^2 H_0^{-1})^{-1/2}, \\ G(\mu) &:= \mu(I + \mu^2 H_0^{-1})^{-1/2}(S H_0^{-1/2} + H_0^{-1/2} S^*)(I + \mu^2 H_0^{-1})^{-1/2}. \end{aligned} \tag{5.3}$$

By (5.2), $\mathrm{i}\mu \in \rho(L_1)$ if and only if $0 \in \rho(F(\mu) + K(\mu) + \mathrm{i}G(\mu))$. Since $\|S_0\| < 1$ and $0 \leqslant (I + \mu^2 H_0^{-1})^{-1} \leqslant I$, we have $F(\mu) \geqslant I - S_0 S_0^* \geqslant \gamma$ with some $\gamma > 0$ for all $\mu \in \mathbb{R}$. The self-adjointness of $G(\mu)$ implies that $0 \in \rho(F(\mu) + \mathrm{i}G(\mu))$ for all $\mu \in \mathbb{R}$, and the claim now follows if we show that $\|K(\mu)\| < \gamma$ for $\mu \in \mathbb{R}$ sufficiently large.

To this end we observe that, for $x \in \mathcal{H}$,

$$\|(I + \mu^2 H_0^{-1})^{-1/2} x\|^2 = \int_0^{\|H_0^{-1}\|} \frac{1}{1 + \mu^2 t}\,\mathrm{d}(E(t)x, x) \to 0, \quad \mu \to \infty,$$

where E is the spectral function of H_0^{-1}. Hence, the rightmost factor in (5.3) tends to 0 strongly for $\mu \to \infty$. The middle factor in (5.3) is compact since S_1 is compact and the leftmost factor is uniformly bounded for all μ. Therefore, $\|K(\mu)\| \to 0$ for $\mu \to \infty$ (see, for example, [**51**, § 6.1]). □

Lemma 5.4. *Suppose that $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and $S = V H_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Then the number of negative squares of the Hermitian form $[A_1 \boldsymbol{x}, \boldsymbol{y}]$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}(A_1)$, in $\mathcal{H}_1$ and of the Hermitian form $[A_2 \boldsymbol{x}, \boldsymbol{y}]$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}(A_2)$, in $\mathcal{H}_2$ is finite; it is equal to the number κ of negative eigenvalues of $I - S^* S$ counted with multiplicities.*

Proof. Both claims follow from relation (5.1) and from the fact that, due to the assumption on S,

$$I - S^* S = I - S_0^* S_0 + K$$

with a compact operator K. Observe that Lemma 5.3 implies that $\rho(L_1) \neq \emptyset$ so that A_2 is self-adjoint by Theorem 4.4. □

In the following, we first consider the particular case $\|S\| < 1$, which means that the operator $I - S^* S$ is uniformly positive. Recall that $m > 0$ is such that $H_0 \geqslant m^2$.

Lemma 5.5. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and $\|S\| < 1$, then*

$$\sigma(L_1) \subset \mathbb{R} \setminus (-\alpha, \alpha),$$

where $\alpha := (1 - \|S\|)m$.

Proof. In the proof we use the numerical range $W(L_1)$ of the operator polynomial L_1 in (3.8). By definition, $W(L_1)$ consists of all points $\lambda \in \mathbb{C}$ for which there exists an element $x \in \mathcal{H}$, $x \neq 0$, such that $(L_1(\lambda)x, x) = 0$. Since $0 \in \rho(L_1)$, we have $\sigma(L_1) \subset \overline{W(L_1)}$ (see [**30**, Theorem 26.6]). Hence, it is sufficient to show that $W(L_1)$ is real and does not contain points of the interval $(-\alpha, \alpha)$. The first claim follows from the fact that, for arbitrary $x \in \mathcal{H}$ with $\|x\| = 1$, the quadratic polynomial

$$(L_1(\lambda)x, x) = \|x\|^2 - ((S^* - \lambda H_0^{-1/2})x, (S^* - \overline{\lambda} H_0^{-1/2})x)$$

is positive at $\lambda = 0$ since $\|S\| < 1$ and tends to $-\infty$ if $\lambda \to \pm\infty$. For the second claim, using $\|H_0^{-1/2}\| \leqslant 1/m$, we obtain that, for $|\lambda| < \alpha$,

$$\begin{aligned} |(L_1(\lambda)x, x)| &\geqslant \|x\|^2 - \|(S^* - \lambda H_0^{-1/2})x\|^2 \\ &\geqslant 1 - \left(\|S\| + \frac{|\lambda|}{m}\right)^2 \\ &> 1 - \left(\|S\| + \frac{\alpha}{m}\right)^2 \\ &\geqslant 0, \end{aligned}$$

and hence $\lambda \notin W(L_1)$. □

The above lemma can be used to obtain information about the essential spectrum of L_1 in the more general situation of Assumption 5.1.

Lemma 5.6. *If $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and $S = V H_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact, then*

$$\sigma_{\mathrm{ess}}(L_1) \subset \mathbb{R} \setminus (-\alpha, \alpha)$$

where $\alpha := (1 - \|S_0\|)m$.

Proof. By the assumption on S, the operator function L_1 can be written as

$$L_1(\lambda) = L_0(\lambda) - K(\lambda), \quad \lambda \in \mathbb{C}; \tag{5.4}$$

here the operator function L_0 is given by

$$L_0(\lambda) := I - (S_0 - \lambda H_0^{-1/2})(S_0^* - \lambda H_0^{-1/2}), \quad \lambda \in \mathbb{C}, \tag{5.5}$$

and

$$K(\lambda) := S_1(S_0^* - \lambda H_0^{-1/2}) + (S_0 - \lambda H_0^{-1/2})S_1^*$$

is compact for all $\lambda \in \mathbb{C}$ since S_1 is compact. By Lemma 5.5 applied to L_0, we have $\sigma(L_0) \subset \mathbb{R} \setminus (-\alpha, \alpha)$. Hence, $\sigma(L_0)$ has empty interior as a subset of $\mathbb{C}$ and $\mathbb{C} \setminus \sigma(L_0)$ consists of only one component. By the proof of Lemma 5.3, this component contains points $\mathrm{i}\mu \in \rho(L_1)$ for $\mu \in \mathbb{R}$ sufficiently large. Now [**40**, Lemma XIII.4] shows that

$$\sigma_{\mathrm{ess}}(L_1) = \sigma_{\mathrm{ess}}(L_0) \subset \mathbb{R} \setminus (-\alpha, \alpha). \tag{5.6}$$

□

Theorem 5.7. *Suppose that $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and that the operator $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Let $m > 0$ be such that $H_0 \geqslant m^2$ and let κ be the number of negative eigenvalues of $I - S^*S$ counted with multiplicities. Then the following statements hold.*

(i) *The self-adjoint operator A_1 is definitizable in the Krein space $\mathcal{K}_1$.*

(ii) *The non-real spectrum of A_1 is symmetric to the real axis and consists of at most κ pairs of eigenvalues $\lambda, \overline{\lambda}$ of finite type; the algebraic eigenspaces corresponding to λ and $\overline{\lambda}$ are isomorphic.*

(iii) *For the essential spectrum of A_1 we have, with $\alpha := (1 - \|S_0\|)m$,*

$$\sigma_{\mathrm{ess}}(A_1) \subset \mathbb{R} \setminus (-\alpha, \alpha).$$

(iv) *If V is $H_0^{1/2}$-compact, then*

$$\sigma_{\mathrm{ess}}(A_1) = \{\lambda \in \mathbb{R} : \lambda^2 \in \sigma_{\mathrm{ess}}(H_0)\} \subset \mathbb{R} \setminus (-m, m).$$

(v) *If $\|VH_0^{-1/2}\| < 1$, then the operator A_1 is uniformly positive in the Krein space $\mathcal{K}_1$ and, with $\alpha := (1 - \|VH_0^{-1/2}\|)m$,*

$$\sigma(A_1) \subset \mathbb{R} \setminus (-\alpha, \alpha).$$

Corollary 5.8. *If, in addition to the assumptions of Theorem 5.7, $1 \notin \sigma_{\mathrm{p}}(S^*S)$ or, equivalently, $0 \notin \sigma_{\mathrm{p}}(A_1)$, then*

$$\kappa = \sum_{\lambda \in \sigma_{\mathrm{p}}(A_1) \cap (0, +\infty)} \kappa_\lambda^-(A_1) + \sum_{\lambda \in \sigma_{\mathrm{p}}(A_1) \cap (-\infty, 0)} \kappa_\lambda^+(A_1) + \sum_{\lambda \in \sigma_{\mathrm{p}}(A_1) \cap \mathbb{C}^+} \kappa_\lambda^{\mathrm{o}}(A_1) \tag{5.7}$$

(see (2.4)). As a consequence, the following are true.

(i) *The number of positive eigenvalues of A_1 that have a non-positive eigenvector plus the number of negative eigenvalues of A_1 that have a non-negative eigenvector plus the number of all eigenvalues of A_1 in the open upper half-plane $\mathbb{C}^+$ is at most κ.*

(ii) *With the exception of the real eigenvalues in (i), the spectrum of A_1 on the positive half-axis is of positive type and the spectrum of A_1 on the negative half-axis is of negative type.*

(iii) *If $\|VH_0^{-1/2}\| < 1$, that is, $\kappa = 0$, then all the spectrum of A_1 on the positive half-axis is of positive type and all the spectrum of A_1 on the negative half-axis is of negative type.*

(iv) *If $\kappa \geqslant 1$, there exists at least one eigenvalue with the properties mentioned in (i) and, in particular, A_1 has at least one eigenvalue.*

Proof of Theorem 5.7. (i) We have $\rho(L_1) \neq \emptyset$ by Lemma 5.3, and $\rho(L_1) \subset \rho(A_1)$ by Proposition 3.4. Thus, $\rho(A_1) \neq \emptyset$ and hence the claim follows from Lemma 5.4 and [**24**, Chapter I.3] (see § 2.3).

(ii) All claims follow from the definitizability of A_1 (see (i) and § 2.3).

For the proof of the remaining statements, we observe that, by (ii), $\sigma(A_1)$ has empty interior as a subset of $\mathbb{C}$. From Proposition 3.4 it follows that

$$(L_1(\lambda)^{-1}x, y) = \left[(A_1 - \lambda)^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right], \quad x, y \in \mathcal{H},\ \lambda \in \rho(L_1). \tag{5.8}$$

This shows that the analytic operator function $L_1(\cdot)^{-1}$ on $\rho(L_1)$ can be continued analytically to $\rho(A_1)$ and hence $\rho(A_1) \subset \rho(L_1)$. Since $\rho(L_1) \subset \rho(A_1)$ by Proposition 3.4, we arrive at

$$\rho(A_1) = \rho(L_1). \tag{5.9}$$

The relation (5.8) also implies that if $\lambda_0 \notin \sigma_{\text{ess}}(A_1)$, and hence $(A_1 - \cdot)^{-1}$ is a finitely meromorphic operator function in a neighbourhood of λ_0, then so is $L_1(\cdot)^{-1}$, that is, $\lambda_0 \notin \sigma_{\text{ess}}(L_1)$. Since $\sigma_{\text{ess}}(A_1) \subset \sigma_{\text{ess}}(L_1)$ by (3.10), we obtain

$$\sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(L_1). \tag{5.10}$$

(iii) By (5.10) and Lemma 5.6 we have

$$\sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(L_1) \subset \mathbb{R} \setminus (-\alpha, \alpha).$$

(iv) If V is $H_0^{1/2}$-compact, we can choose $S_0 = 0$, and so (5.4) and (5.5) yield that $L_1(\lambda) = L_0(\lambda) + K(\lambda)$, where $K(\lambda)$ is compact and L_0 is now given by

$$L_0(\lambda) := I - \lambda^2 H_0^{-1}, \quad \lambda \in \mathbb{C}.$$

Together with (5.10) and (5.6), we obtain that

$$\sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(L_1) = \sigma_{\text{ess}}(L_0) = \{\lambda \in \mathbb{R} : \lambda^2 \in \sigma_{\text{ess}}(H_0)\}.$$

(v) If $\|S\| = \|VH_0^{-1/2}\| < 1$, then equality (5.9) and Lemma 5.5 show that

$$\sigma(A_1) = \sigma(L_1) \subset \mathbb{R} \setminus (-\alpha, \alpha).$$

□

Theorem 5.9. *Suppose that $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and that the operator $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Let $m > 0$ be such that $H_0 \geqslant m^2$ and let κ be the number of negative eigenvalues of $I - S^*S$ counted with multiplicities. Then the following statements hold.*

(i) *The self-adjoint operator A_2 is definitizable in the Krein space $\mathcal{K}_2$.*

(ii) *The spectrum, essential spectrum and point spectrum of A_1 and A_2 coincide:*

$$\sigma(A_1) = \sigma(A_2), \qquad \sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(A_2), \qquad \sigma_{\text{p}}(A_1) = \sigma_{\text{p}}(A_2); \tag{5.11}$$

moreover, A_1 and A_2 have the same Jordan chains and their spectral points are of the same (positive or negative) type.

(iii) *The statements (ii)–(v) of Theorem 5.7 continue to hold for A_2.*

Remark 5.10. Although A_1 and A_2 have the same spectra, there is an essential difference in the behaviour of their spectral functions at ∞ (see § 6).

Proof of Theorem 5.9. (i) We have $\rho(L_1) \neq \emptyset$ by Lemma 5.3, and $\rho(L_1) \subset \rho(A_2)$ by Theorem 4.4. Thus, $\rho(A_2) \neq \emptyset$ and hence the claim follows from Lemma 5.4 and [**24**, Chapter I.3] (see § 2.3).

(ii) First we observe that, by (5.9) and Theorem 4.4, we have

$$\rho(A_1) = \rho(L_1) \subset \rho(A_2) \tag{5.12}$$

and that for $\lambda \in \rho(A_1)$, by (4.12),

$$[(A_1 - \lambda)^{-1}\boldsymbol{x}, \boldsymbol{y}] = [(A_2 - \lambda)^{-1}\boldsymbol{x}, \boldsymbol{y}], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}. \tag{5.13}$$

If λ_0 is an eigenvalue of finite type of A_1, that is, $\lambda_0 \in \sigma_{\rm d}(A_1)$, then it is either an isolated eigenvalue of A_2 or it belongs to $\rho(A_2)$ by (5.12). Then, for $j = 1, 2$, the corresponding Riesz projection is

$$P_{\lambda_0,j} := -\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}_{\lambda_0}} (A_j - z)^{-1}\,\mathrm{d}z,$$

where $\mathcal{C}_{\lambda_0}$ is a closed Jordan curve in $\rho(A_1)$ surrounding λ_0 and no other point of the spectra of A_1 and A_2. Its range is the algebraic eigenspace of A_j at λ_0 and the Jordan structure of A_j in λ_0 is determined by the coefficients of the principal part of the Laurent series of $(A_j - \lambda)^{-1}$, given by

$$\frac{1}{\lambda - \lambda_0} P_{\lambda_0,j} + \sum_{k=1}^{p_j - 1} \frac{1}{(\lambda - \lambda_0)^{k+1}} B_j^k,$$

where $p_j \in \mathbb{N} \cup \{\infty\}$ and $B_j := (A_j - \lambda_0)P_{\lambda_0,j}$ (see [**15**, Chapter XV.2]). Since λ_0 was assumed to be an eigenvalue of finite type of A_1, the projection $P_{\lambda_0,1}$ is finite dimensional, p_1 is finite and B_1 is a finite rank operator. By (5.13), we have

$$[A_1^k P_{\lambda_0,1}\boldsymbol{x}, \boldsymbol{y}] = [A_2^k P_{\lambda_0,2}\boldsymbol{x}, \boldsymbol{y}], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}_1 \cap \mathcal{K}_2. \tag{5.14}$$

Since $\mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}$ is dense in $\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{H}$ and in $\mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$, the coefficients of the principal parts in the Laurent expansions of $(A_1 - \lambda)^{-1}$ and $(A_2 - \lambda)^{-1}$ at λ_0 coincide. Hence, we have shown that

$$\sigma_{\rm d}(A_1) \subset \sigma_{\rm d}(A_2). \tag{5.15}$$

Since A_1 and A_2 are definitizable, we have

$$\rho(A_j) \cup \sigma_{\rm d}(A_j) \cup \sigma_{\rm ess}(A_j) = \mathbb{C}, \quad j = 1, 2.$$

This, together with (5.12) and (5.15), implies that $\sigma_{\rm ess}(A_2) \subset \sigma_{\rm ess}(A_1) \subset \mathbb{R}$. It remains to be shown that

$$\sigma_{\rm ess}(A_1) \subset \sigma_{\rm ess}(A_2). \tag{5.16}$$

Since the essential spectra of A_1 and A_2 are both real, the spectral projections E_j of A_j can be represented by means of contour integrals over the resolvent as follows (see [**24**, Chapter I.3]): for $j=1,2$ and a bounded interval $\Gamma \subset \mathbb{R}$ which is admissible for A_j and has end points a, b, $a \leqslant b$, consider the operator

$$\hat{E}_j(\Gamma) := -\frac{1}{2\pi \mathrm{i}} \int'_{\mathcal{C}_\Gamma} (A_j - z)^{-1}\, \mathrm{d}z.$$

Here $\mathcal{C}_\Gamma$ is the positively oriented rectangle with corners $a+\mathrm{i}\varepsilon$, $b+\mathrm{i}\varepsilon$, $b-\mathrm{i}\varepsilon$ and $a-\mathrm{i}\varepsilon$ for $\varepsilon>0$ so small that $\mathcal{C}_\Gamma$ does not surround any non-real spectral points of A_1 and A_2, and the prime denotes the Cauchy principal value of the integral at a and b. If the end points a, b of Γ are not eigenvalues of A_j, then $\hat{E}_j(\Gamma) = E_j(\Gamma)$; if a is an eigenvalue of A_j and b is not an eigenvalue of A_j, then $\hat{E}_j(\Gamma) = E_j(\Gamma) + \frac{1}{2}E_j(\{a\})$, and similarly in all other cases for a and b. In any case, the operators $\hat{E}_j(\Gamma)$ determine the spectral projections of A_j, whence

$$[E_1(\Gamma)\boldsymbol{x}, \boldsymbol{y}] = [E_2(\Gamma)\boldsymbol{x}, \boldsymbol{y}], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}_1 \cap \mathcal{K}_2.$$

In particular, $\dim E_1(\Gamma)(\mathcal{K}_1 \cap \mathcal{K}_2) = \dim E_2(\Gamma)(\mathcal{K}_1 \cap \mathcal{K}_2)$ and, consequently, $E_1(\Gamma)$ and $E_2(\Gamma)$ have the same rank, which proves (5.16).

It remains to be shown that the Jordan chains of A_1 and A_2 coincide. If $\lambda_0 \in \sigma_{\mathrm{p}}(A_1)$ with Jordan chain $(\boldsymbol{x}_k)_{k=0}^r \subset \mathcal{D}(A_1) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$, $r \in \mathbb{N} \cup \{\infty\}$, then, with $\boldsymbol{x}_{-1} := \boldsymbol{0}$,

$$(A_1 - \lambda_0)\boldsymbol{x}_k = \boldsymbol{x}_{k-1}, \quad k = 0, 1, \dots, r.$$

Hence, for $\boldsymbol{x}_k =: (x_k y_k)^{\mathrm{T}}$, we have $y_k \in \mathcal{H}$ and

$$\left.\begin{aligned} V x_k + y_k &= x_{k-1} + \lambda_0 x_k \in \mathcal{D}(H_0^{1/2}) \subset \mathcal{H}_{1/4}, \\ H_0^{1/2} x_k + S^* y_k &= H_0^{-1/2}(y_{k-1} + \lambda_0 y_k) \in \mathcal{D}(H_0^{1/2}) \subset \mathcal{H}_{1/4}, \end{aligned}\right\} \tag{5.17}$$

and thus $H_0 x_k + V y_k \in \mathcal{H}_{-1/4}$. This shows that $(\boldsymbol{x}_k)_{k=0}^r \subset \mathcal{D}(A_2)$ and so $(\boldsymbol{x}_k)_{k=0}^r$ is a Jordan chain of A_2 at λ_0. Conversely, if $(\boldsymbol{x}_k)_{k=0}^r \subset \mathcal{D}(A_2) \subset \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$ is a Jordan chain of A_2 at λ_0, then, in (5.17), the element $V x_k + y_k$ belongs to $\mathcal{D}(H_0^{1/2}) \subset \mathcal{H}$ and the element $H_0^{1/2} x_k + S^* y_k$ belongs to $\mathcal{D}(H_0^{1/2})$. Thus, $(\boldsymbol{x}_k)_{k=0}^r \subset \mathcal{D}(A_1)$ and so $(\boldsymbol{x}_k)_{k=0}^r$ is a Jordan chain of A_1 at λ_0. $\square$

To conclude this section, we compare the spectral properties of A_1 with those of the operator A associated with the abstract Klein–Gordon equation in [**27**]. This operator acts in the space $\mathcal{G} := \mathcal{H}_{1/2} \oplus \mathcal{H}$; if Assumption 3.1 holds and if $1 \in \rho(S^*S)$, it is defined as

$$A = \begin{pmatrix} 0 & I \\ H & 2V \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{D}(H_0^{1/2}), \tag{5.18}$$

with $H := H_0^{1/2}(I - S^*S)H_0^{1/2}$. The operator A is related to the operator A_1 introduced in § 3 by the formula

$$A = W A_1 W^{-1}, \tag{5.19}$$

where W acts from $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{G} = \mathcal{H}_{1/2} \oplus \mathcal{H}$,

$$W := \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}, \qquad \mathcal{D}(W) := \mathcal{H}_{1/2} \oplus \mathcal{H}.$$

It was shown in [**27**] that the operator A is self-adjoint in $\mathcal{K} := (\mathcal{G}, \langle \cdot\,, \cdot \rangle)$ with inner product

$$\langle \boldsymbol{x}, \boldsymbol{x}' \rangle = ((I - S^*S)H_0^{1/2}x, H_0^{1/2}x') + (y, y'), \quad \boldsymbol{x} = (x\ y)^{\mathrm{T}},\ \boldsymbol{x}' = (x'\ y')^{\mathrm{T}} \in \mathcal{G},$$

which is a Pontryagin space due to Assumption 5.1. Note that the negative index of this Pontryagin space equals the number κ of negative eigenvalues of $I - S^*S$ counted with multiplicities (cf. Lemma 5.4). Between the indefinite inner products $\langle \cdot\,, \cdot \rangle$ of $\mathcal{K}$ and $[\cdot\,, \cdot]$ of $\mathcal{K}_1$ we have the relation (see [**27**, Proposition 4.4 (i)])

$$\langle \boldsymbol{x}, \boldsymbol{x}' \rangle = [A_1 W^{-1}\boldsymbol{x}, W^{-1}\boldsymbol{x}'], \quad \boldsymbol{x} \in W\mathcal{D}(A_1),\ \boldsymbol{x}' \in \mathcal{G}. \tag{5.20}$$

Theorem 5.11. *Suppose that $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, that the operator $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact, and that $1 \in \rho(S^*S)$. Then*

$$\sigma(A) = \sigma(A_1) = \sigma(A_2), \quad \sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_2), \quad \sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{p}}(A_1) = \sigma_{\mathrm{p}}(A_2).$$

Proof. By [**27**, Theorem 5.2 (iii)] and Theorem 5.7 (iii), the non-real spectra of A and A_1 consist of finitely many eigenvalues of finite type. If $\lambda \in \rho(A) \cap \rho(A_1)$, then for $\boldsymbol{w}, \boldsymbol{x}' \in \mathcal{G}$ we obtain, from (5.20) with $\boldsymbol{x} = (A - \lambda)^{-1}\boldsymbol{w} \in \mathcal{D}(A) \subset W\mathcal{D}(A_1)$ and (5.19),

$$\begin{aligned} \langle (A-\lambda)^{-1}\boldsymbol{w}, \boldsymbol{x}' \rangle &= [A_1 W^{-1}(A-\lambda)^{-1}\boldsymbol{w}, W^{-1}\boldsymbol{x}'] \\ &= [A_1(A_1 - \lambda)^{-1}W^{-1}\boldsymbol{w}, W^{-1}\boldsymbol{x}'] \\ &= [W^{-1}\boldsymbol{w}, W^{-1}\boldsymbol{x}'] + \lambda[(A_1-\lambda)^{-1}W^{-1}\boldsymbol{w}, W^{-1}\boldsymbol{x}']. \end{aligned}$$

In a similar way as in the proof of Theorem 5.9, observing that the range of W^{-1}, which is given by $\mathcal{D}(W) = \mathcal{H}_{1/2} \oplus \mathcal{H}$, is dense in $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$, one can show that all eigenvalues of finite type of A_1 and A and also their essential spectra coincide.

The equality of the point spectra of A and A_1 follows from (5.19): if λ is an eigenvalue of A with eigenvector $x \in \mathcal{D}(A)$, then $W^{-1}x \in \mathcal{D}(A_1)$ and $W^{-1}x$ is an eigenvector of A_1 corresponding to the eigenvalue λ. Conversely, if λ is an eigenvalue of A_1 with eigenvector $x \in \mathcal{D}(A_1)$, then $Wx \in \mathcal{D}(A)$ and Wx is an eigenvector of A corresponding to the eigenvalue λ.

The equalities with the various parts of $\sigma(A_2)$ follow from Theorem 5.9. □

6. The critical point ∞: A_2 as a generator of a strongly continuous unitary group

Although the spectra of A_1 and A_2 coincide, their spectral functions E_1 and E_2 behave differently at ∞ if H_0 is unbounded. This will be proved first for the case $V = 0$; for

$V \neq 0$ satisfying Assumption 5.1, a perturbation theorem due to Ćurgus (see [**8**]) applies and shows that ∞ is a regular critical point for A_2.

The unperturbed operators $A_{1,0}$ in $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ and $A_{2,0}$ in $\mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ are given by

$$A_{1,0} := \begin{pmatrix} 0 & I \\ H_0 & 0 \end{pmatrix}, \qquad \mathcal{D}(A_{1,0}) := \mathcal{D}(H_0) \oplus \mathcal{H},$$
$$A_{2,0} := \begin{pmatrix} 0 & I \\ H_0 & 0 \end{pmatrix}, \qquad \mathcal{D}(A_{2,0}) := \mathcal{D}(H_0^{3/4}) \oplus \mathcal{D}(H_0^{1/4}).$$

In fact, if $V = 0$, then the operators A_1 in $\mathcal{G}_1$ and A_2 in $\mathcal{G}_2$ coincide with the operators $A_{1,0}$ and $A_{2,0}$.

Lemma 6.1. *If H_0 is unbounded, then ∞ is a regular critical point of $A_{2,0}$, whereas it is a singular critical point of $A_{1,0}$.*

Proof. The resolvents of $A_{1,0}$ and $A_{2,0}$ are defined for $\lambda \in \mathbb{C}$ such that $\lambda^2 \notin \sigma(H_0)$; they are of the form

$$(A_{j,0} - \lambda)^{-1} = \begin{pmatrix} \lambda(H_0 - \lambda^2)^{-1} & (H_0 - \lambda^2)^{-1} \\ I + \lambda^2(H_0 - \lambda^2)^{-1} & \lambda(H_0 - \lambda^2)^{-1} \end{pmatrix}, \quad j = 1, 2.$$

Denote the spectral function of $H_0^{1/2}$ in $\mathcal{H}$ by E_0 and let $\Gamma \subset \mathbb{R}$ be a bounded interval with $\Gamma > 0$ or $\Gamma < 0$. Using the equality

$$(H_0 - \lambda^2)^{-1} = (H_0^{1/2} - \lambda)^{-1}(H_0^{1/2} + \lambda)^{-1} = \frac{1}{2\lambda}((H_0^{1/2} - \lambda)^{-1} - (H_0^{1/2} + \lambda)^{-1})$$

and observing that $(H_0^{1/2} - \cdot\,)^{-1}$ is holomorphic on Γ if $\Gamma < 0$ and $(H_0^{1/2} + \cdot\,)^{-1}$ is holomorphic on Γ if $\Gamma > 0$, we find that the spectral projection $E_j(\Gamma)$ of $A_{j,0}$ in $\mathcal{K}_j$ for $j = 1, 2$ is given by

$$E_j(\Gamma) = \pm\frac{1}{2}\begin{pmatrix} E_0(\Gamma) & H_0^{-1/2}E_0(\Gamma) \\ H_0^{1/2}E_0(\Gamma) & E_0(\Gamma) \end{pmatrix}.$$

Hence, for elements $\boldsymbol{x} = (x0)^{\mathrm{T}} \in \mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$, we have

$$\|E_1(\Gamma)\boldsymbol{x}\|^2_{\mathcal{G}_1} = \tfrac{1}{4}(\|E_0(\Gamma)x\|^2 + \|H_0^{1/2}E_0(\Gamma)x\|^2).$$

If H_0 is unbounded, the last term does not remain bounded if $\Gamma > 0$ extends to ∞ and $x \notin \mathcal{D}(H_0^{1/2})$.

On the other hand, for every $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$,

$$\begin{aligned} \|E_2(\Gamma)\boldsymbol{x}\|^2_{\mathcal{G}_2} &= \tfrac{1}{4}\|H_0^{1/4}(E_0(\Gamma)x + H_0^{-1/2}E_0(\Gamma)y)\|^2 \\ &\qquad + \tfrac{1}{4}\|H_0^{-1/4}(H_0^{1/2}E_0(\Gamma)x + E_0(\Gamma)y)\|^2 \\ &\leqslant \|H_0^{1/4}x\|^2 + \|H_0^{-1/4}y\|^2 \\ &= \|\boldsymbol{x}\|^2_{\mathcal{G}_2}, \end{aligned}$$

and therefore $\|E_2(\Gamma)\| \leqslant 1$. $\square$

Since for the operator A_1 already in the unperturbed situation (that is, $V = 0$ and $A_1 = A_{1,0}$) the point ∞ is a singular critical point if H_0 is unbounded, it will remain a singular critical point for large classes of perturbations (e.g. for bounded perturbations).

For A_2, however, in the unperturbed situation (that is, $V = 0$ and $A_2 = A_{2,0}$) the point ∞ is a regular critical point. Due to Assumptions 3.1 and 5.1, we may apply a perturbation result of Ćurgus (see [**8**, Corollary 3.6]), which guarantees that under certain additive perturbations the critical point ∞ remains regular.

In order to see this, we introduce sesquilinear forms $\mathfrak{h}_0$ and $\mathfrak{v}$ in the Hilbert space $\mathcal{G}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ on $\mathcal{D}(\mathfrak{h}_0) = \mathcal{D}(\mathfrak{v}) = \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$ by

$$\begin{aligned}\mathfrak{h}_0(\boldsymbol{x}, \boldsymbol{y}) &:= (H_0^{1/2} x_1, H_0^{1/2} y_1) + (x_2, y_2),\\ \mathfrak{v}(\boldsymbol{x}, \boldsymbol{y}) &:= (V x_1, y_2) + (x_2, V y_1)\end{aligned}$$

for $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$, $\boldsymbol{y} = (y_1, y_2)^{\mathrm{T}} \in \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$. It is not difficult to see that the form $\mathfrak{h}_0$ is closed, symmetric and uniformly positive. Moreover, for $\boldsymbol{x} \in \mathcal{D}(A_{2,0})$, $\boldsymbol{y} \in \mathcal{D}(\mathfrak{h}_0)$, we have

$$\mathfrak{h}_0(\boldsymbol{x}, \boldsymbol{y}) = [A_{2,0}\boldsymbol{x}, \boldsymbol{y}] = (J A_{2,0} \boldsymbol{x}, \boldsymbol{y})_{\mathcal{G}_2}, \tag{6.1}$$

where J is defined as in (4.4), $[\cdot\,,\cdot]$ is the indefinite inner product of $\mathcal{K}_2 = \mathcal{H}_{1/4} \oplus \mathcal{H}_{-1/4}$ (see (4.3)) and $(\cdot\,,\cdot)_{\mathcal{G}_2}$ is the corresponding Hilbert space inner product.

Lemma 6.2. *Let $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and suppose that $S = V H_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Then*

(i) *the form $\mathfrak{v}$ is $\mathfrak{h}_0$-bounded with $\mathfrak{h}_0$-bound less than 1,*

(ii) *the form $\mathfrak{h} := \mathfrak{h}_0 + \mathfrak{v}$ defined on $\mathcal{D}(\mathfrak{h}_0) = \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$ is closed, symmetric and bounded from below.*

Proof. (i) By the assumption on S and Lemma 5.2, we may choose the operator S_1 to be of the form $S_1 = \sum_{i=1}^{n} (\cdot\,, w_i) v_i$ with $v_i \in \mathcal{H}$ and $w_i \in \mathcal{D}(H_0^{1/2})$, $i = 1, \dots, n$. Then the operator $S_1 H_0^{1/2}$ in $\mathcal{H}$ is bounded on $\mathcal{D}(H_0^{1/2})$:

$$\|S_1 H_0^{1/2} x\| \leqslant \sum_{i=1}^{n} |(x, H_0^{1/2} w_i)|\, \|v_i\| \leqslant \biggl(\sum_{i=1}^{n} \|H_0^{1/2} w_i\|\, \|v_i\|\biggr) \|x\| =: a\|x\|$$

for $x \in \mathcal{D}(H_0^{1/2})$. If we set $b := \|S_0\| < 1$, we obtain that

$$\|Vx\| = \|(S_0 + S_1) H_0^{1/2} x\| \leqslant a\|x\| + b\|H_0^{1/2} x\|, \quad x \in \mathcal{D}(H_0^{1/2}), \tag{6.2}$$

that is, V is $H_0^{1/2}$-bounded with relative bound less than 1. It is not difficult to check (see [**21**, § V4.1]) that (6.2) implies that there exist constants $a', b' \geqslant 0$, $b' < 1$, such that

$$\|Vx\|^2 \leqslant a'^2 \|x\|^2 + b'^2 \|H_0^{1/2} x\|^2, \quad x \in \mathcal{D}(H_0^{1/2}). \tag{6.3}$$

Now let $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathcal{D}(\mathfrak{v}) = \mathcal{D}(H_0^{1/2}) \oplus \mathcal{H}$. Using (6.3) and the inequality

$$\|H_0^{1/4} x_1\|^2 = |(H_0^{1/2} x_1, x_1)| \geqslant m\|x_1\|^2,$$

we obtain the estimate

$$\begin{aligned}
\mathfrak{v}(\boldsymbol{x},\boldsymbol{x}) &= 2\,\mathrm{Re}(Vx_1,x_2)\\
&\leqslant 2\|Vx_1\|\|x_2\|\\
&\leqslant \frac{1}{b'}\|Vx_1\|^2 + b'\|x_2\|^2\\
&\leqslant \frac{1}{b'}({a'}^2\|x_1\|^2 + {b'}^2\|H_0^{1/2}x_1\|^2) + b'\|x_2\|^2\\
&\leqslant \frac{{a'}^2}{b'}\|x_1\|^2 + b'(\|H_0^{1/2}x_1\|^2 + \|x_2\|^2)\\
&\leqslant \frac{{a'}^2}{b'm}(\|H_0^{1/4}x_1\|^2 + \|H_0^{-1/4}x_2\|^2) + b'(\|H_0^{1/2}x_1\|^2 + \|x_2\|^2)\\
&= \frac{{a'}^2}{b'm}(\boldsymbol{x},\boldsymbol{x})_{\mathcal{G}_2} + b'\mathfrak{h}_0(\boldsymbol{x},\boldsymbol{x}).
\end{aligned}$$

(ii) It is easy to see that $\mathfrak{h}$ is symmetric since $\mathfrak{h}_0$ and $\mathfrak{v}$ are also symmetric. All other claims follow from the perturbation result [**21**, Theorem VI.1.33]. □

Theorem 6.3. *Let $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and suppose that $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Then ∞ is a regular critical point of A_2.*

Proof. According to Lemma 6.2, Assumptions (A) and (B) of [**9**, § 2] are satisfied. By (6.1) and the first representation theorem (see [**21**, Theorem VI.2.1]), the operator $JA_{2,0}$ is the self-adjoint operator associated with the closed form $\mathfrak{h}_0$ in $\mathcal{H}_2$. Analogously, it follows that JA_2 is the self-adjoint operator associated with the perturbed form $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{v}$ in $\mathcal{H}_2$. Since A_2 is definitizable by Theorem 5.9 and ∞ is a regular critical point for $A_{2,0}$ by Lemma 6.1, it is also a regular critical point for A_2 by [**9**, Proposition 2.1]. □

Remark 6.4. Theorem 6.3 was proved in [**9**, Theorem 3.5] for the particular cases when either $S_0 = 0$ or $S_1 = 0$.

An important consequence of the regularity of the critical point ∞ is that A_2 is the generator of a unitary group in $\mathcal{K}_2$ and thus we obtain information on the solvability of the Cauchy problem for the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}A_2\boldsymbol{x}.$$

A function $\boldsymbol{x} : \mathbb{R} \to \mathcal{K}_2$ is called a *classical solution* of this differential equation if

$$\boldsymbol{x} \in C^1(\mathbb{R},\mathcal{K}_2), \quad \boldsymbol{x}(t) \in \mathcal{D}(A_2), \quad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t) = \mathrm{i}A_2\boldsymbol{x}(t), \quad t \in \mathbb{R}.$$

Theorem 6.5. *Let $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$ and suppose that $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. Then the operator A_2 is the* *infinitesimal generator of a strongly continuous group* $(\mathrm{e}^{\mathrm{i}tA_2})_{t\in\mathbb{R}}$ *of unitary operators in* $\mathcal{K}_2$*. If* $\boldsymbol{x}_0 \in \mathcal{D}(A_2)$*, the Cauchy problem*

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}A_2\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{6.4}$$

has a unique classical solution given by

$$\boldsymbol{x}(t) = \mathrm{e}^{\mathrm{i}tA_2}\boldsymbol{x}_0, \quad t \in \mathbb{R}. \tag{6.5}$$

Proof. The regularity of the critical point ∞ by Theorem 6.3 implies that A_2 is the sum of a bounded operator and an operator similar to a self-adjoint operator. As a consequence, the operators $\mathrm{e}^{\mathrm{i}tA_2}$, $t \in \mathbb{R}$, are defined and form a strongly continuous group of unitary operators in $\mathcal{K}_2$. The last claim follows in a similar way as corresponding results for semi-groups (see, for example, [**23**, Theorem 1.1]). □

Remark 6.6. If only $\boldsymbol{x}_0 \in \mathcal{K}_2$, then (6.5) is the unique mild solution of (6.4), that is,

$$\boldsymbol{x} \in C(\mathbb{R}, \mathcal{K}_2), \quad \int_s^t \boldsymbol{x}(\tau)\,\mathrm{d}\tau \in \mathcal{D}(A_2), \quad \boldsymbol{x}(t) = \boldsymbol{x}(s) + \mathrm{i}A_2\int_s^t \boldsymbol{x}(\tau)\,\mathrm{d}\tau, \quad s,t \in \mathbb{R}$$

(see the analogous definitions and results for semi-groups, e.g. in [**1**, § 1.2], [**2**, 3.1.12]).

7. Semi-groups associated with A_1

In this section we restrict ourselves to the case that the symmetric operator V in $\mathcal{H}$ is everywhere defined and hence bounded. Then Assumption 3.1 used in § 3 is automatically satisfied and the operator $\hat{A}_1$ in $\mathcal{G}_1 = \mathcal{H} \oplus \mathcal{H}$ defined in (3.2) is already closed:

$$A_1 = \hat{A}_1 = \begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}, \qquad \mathcal{D}(A_1) = \mathcal{D}(H_0) \oplus \mathcal{H}.$$

Moreover, Assumption 5.1 used in § 5 holds if V can be decomposed as $V = V_0 + V_1$ such that $\|V_0\| < m$ and $V_1H_0^{-1/2}$ is compact.

Then, according to Theorem 5.7, the operator A_1 is definitizable in $\mathcal{K}_1$ and the interval $(-(m-\|V_0\|), m-\|V_0\|)$ contains only eigenvalues of finite type; in particular, there exists a real point $\mu \in \rho(A_1)$.

Lemma 7.1. *Assume that* V *is bounded and can be decomposed as* $V = V_0 + V_1$ *such that* $\|V_0\| < m$ *and* $V_1H_0^{-1/2}$ *is compact. Let* κ *be the number of negative eigenvalues of* $I - S^*S$ *counted with multiplicities and let* $\mu \in \rho(A_1) \cap \mathbb{R}$*. There then exists a maximal non-negative subspace* $\mathcal{L}_+ \subset \mathcal{K}_1$ *that is invariant under* $(A_1 - \mu)^{-1}$ *and such that*

$$\operatorname{Im}\sigma((A_1-\mu)^{-1}|_{\mathcal{L}_+}) \leqslant 0.$$

The subspace $\mathcal{L}_+$ *can be chosen so that it contains all the algebraic eigenspaces of* A_1 *corresponding to the eigenvalues in the open upper half-plane, all the positive spectral subspaces and a non-negative eigenvector of* A_1 *at all real eigenvalues of* A_1 *that are not of negative type; the negative spectral subspaces of* A_1 *are orthogonal to* $\mathcal{L}_+$*. Furthermore,*

$$\dim \mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]} \leqslant \kappa. \tag{7.1}$$

Remark 7.2. For z in a complex neighbourhood of μ, the relation

$$(A_1 - z)^{-1} = \sum_{j=0}^{\infty} (z-\mu)^j (A_1 - \mu)^{-j-1}$$

holds and implies that the subspace $\mathcal{L}_+$ from Lemma 7.1 is also invariant under $(A_1 - z)^{-1}$. Since $\rho(A_1)$ is connected, an analytic continuation argument shows that this continues to hold for all $z \in \rho(A_1)$.

Proof of Lemma 7.1. Since $(A_1 - \mu)^{-1}$ is a bounded definitizable operator, the existence of a maximal non-negative invariant subspace $\mathcal{L}_+^0$ for $(A_1 - \mu)^{-1}$ follows from [**24**, Satz 3.2]. If $\lambda_0 \in \mathbb{C}^+$ is an eigenvalue of A_1 with algebraic eigenspace $\mathcal{L}_{\lambda_0}(A_1)$, then, together with $\mathcal{L}_+^0$, the subspace

$$\mathcal{L}_+^1 := \mathcal{L}_+^0 \cap (\mathcal{L}_{\lambda_0}(A_1) + \mathcal{L}_{\overline{\lambda_0}}(A_1))^{[\perp]} \dotplus \mathcal{L}_{\lambda_0}(A_1)$$

is also a maximal non-negative invariant subspace for $(A_1 - \mu)^{-1}$ and it contains $\mathcal{L}_{\lambda_0}(A_1)$. Since A_1 has only a finite number of non-real eigenvalues, after repeating this argument a finite number of times, the new subspace $\mathcal{L}_+ := \mathcal{L}_+^n$ contains all the algebraic eigenspaces of A_1 corresponding to eigenvalues in the open upper half-plane. If $\mathcal{L}_+$ does not contain all positive subspaces of the form $E_1(\varGamma)\mathcal{K}_1$, adding such a subspace to $\mathcal{L}_+$ would still give a non-negative invariant subspace, a contradiction to the maximality of $\mathcal{L}_+$. In a similar way, it can be shown that it contains non-negative eigenelements of A_1 at all real eigenvalues of A_1 that are not of negative type. Finally, the subspace on the left-hand side of (7.1) is neutral with respect to the inner product $[A_1 \cdot, \cdot]$ and hence of dimension less than or equal to κ. $\square$

Lemma 7.3. *If $\mathcal{L}_+$ is a maximal non-negative subspace as in Lemma 7.1, then*

$$\begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathcal{L}_+ \quad \Longrightarrow \quad y = 0. \tag{7.2}$$

Proof. It is easy to see that the resolvent of A_1 admits the representation

$$(A_1 - z)^{-1} = \begin{pmatrix} -D(z)^{-1}(V-z) & D(z)^{-1} \\ I + (V-z)D(z)^{-1}(V-z) & -(V-z)D(z)^{-1} \end{pmatrix}, \quad z \in \rho(A_1),$$

where $D(z) := H_0 - (V-z)^2$, $z \in \mathbb{C}$. For any $z \in \rho(A_1)$, we have $(A_1 - z)^{-1}\mathcal{L}_+ \subset \mathcal{L}_+$ which implies that $(A_1 - \overline{z})^{-1}\mathcal{L}_+^{[\perp]} \subset \mathcal{L}_+^{[\perp]}$. Hence,

$$(A_1 - z)^{-1}(\mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}) \subset \mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}, \quad z \in \rho(A_1),$$

that is, $(A_1 - z)^{-1}$ maps the isotropic subspace of $\mathcal{L}_+$ into itself. Since an element $\boldsymbol{x} = (0 y)^{\mathrm{T}} \in \mathcal{L}_+$ as in (7.2) is neutral in $\mathcal{K}_1$, we have $\boldsymbol{x} \in \mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}$ and so

$$0 = [(A_1 - z)^{-1}\boldsymbol{x}, (A_1 - z)^{-1}\boldsymbol{x}] = -2((V - \operatorname{Re} z)D(z)^{-1}y, D(z)^{-1}y).$$

Choosing $z \in \mathbb{C}$ such that $V - \operatorname{Re} z > 0$, we obtain $y = 0$. $\square$

Lemma 7.3 shows that $\mathcal{L}_+$ is the graph of a closed linear operator K in $\mathcal{H}$,

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{D}(K) \right\}. \tag{7.3}$$

Since for $\boldsymbol{x} = (x\ y)^{\mathrm{T}} \in \mathcal{K}_1$ we have $[\boldsymbol{x}, \boldsymbol{x}] = 2\,\mathrm{Re}(x, y)$, the non-negativity of $\mathcal{L}_+$ yields that the operator K is *accretive* in $\mathcal{H}$, that is, $\mathrm{Re}(Kx, x) \geqslant 0$, $x \in \mathcal{D}(K)$ (see [**21**, Chapter V.3.10]). The fact that the subspace $\mathcal{L}_+$ is maximal non-negative implies that the operator K in (7.3) is *maximal accretive* in $\mathcal{H}$, that is, it admits no proper accretive extension.

Remark 7.4. For a general definitizable operator in a Krein space, the maximal non-negative invariant subspace $\mathcal{L}_+$ is not uniquely determined, and so we do not have a uniqueness result for $\mathcal{L}_+$ in Lemma 7.1. However, if $V = 0$, uniqueness can be proved and the maximal non-negative invariant subspace is of the form

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ H_0^{1/2} x \end{pmatrix} : x \in \mathcal{H} \right\},$$

which shows that in this case $K = H_0^{1/2}$.

In the following we consider the operator $K + V$ which is, in general, only quasi-accretive (see [**21**, Chapter V.3.10]). Therefore, we define

$$\nu := \min\left\{ 0, \inf_{x \neq 0} \frac{(Vx, x)}{(x, x)} \right\} \leqslant 0. \tag{7.4}$$

Then the operator $K + V - \nu$ is maximal accretive in $\mathcal{H}$ and thus generates the contractive strongly continuous semi-group

$$S(\tau) := \mathrm{e}^{-\tau(K+V-\nu)}, \quad \tau \geqslant 0.$$

As a consequence, the operator $K + V$ generates the quasi-bounded strongly continuous semi-group

$$T(\tau) := \mathrm{e}^{-\tau(K+V)}, \quad \tau \geqslant 0, \tag{7.5}$$

in $\mathcal{H}$ such that $\|T(\tau)\| \leqslant \mathrm{e}^{\tau\nu}$ (cf. [**10**, Theorem 3.1]).

The next theorem establishes the existence of solutions of the abstract differential equation (1.14).

Theorem 7.5. *Suppose that V is bounded and that the operator $S = VH_0^{-1/2}$ can be decomposed as $S = S_0 + S_1$ with $\|S_0\| < 1$ and S_1 compact. There then exists a maximal accretive operator K in $\mathcal{H}$ such that, with the semi-group $(T(\tau))_{\tau \geqslant 0}$ given by $T(\tau) := \mathrm{e}^{-\tau(K+V)}$, $\tau \geqslant 0$, for any initial value $v_0 \in \mathcal{D}((K + V)^2)$, the function*

$$v(\tau) := T(\tau)v_0, \quad \tau \geqslant 0, \tag{7.6}$$

is a classical solution of the Cauchy problem

$$\ddot{v}(\tau) + 2V\dot{v}(\tau) + V^2 v(\tau) - H_0 v(\tau) = 0, \quad \tau \geqslant 0, \ v(0) = v_0. \tag{7.7}$$

The spectrum of the infinitesimal generator $K+V$ of the semi-group $(T(\tau))_{\tau\geqslant 0}$ contains the eigenvalues of A_1 in the open upper half-plane, the spectral points of positive type of A_1 and the real eigenvalues of A_1 that are not of negative type.

Proof. By Theorem 5.7 (ii), the non-real spectrum of A_1 consists only of finitely many points. Thus, we can choose $\beta \in \rho(A_1)$ such that $\operatorname{Re}\beta < \nu$, where ν is given by (7.4). Then, according to Lemma 7.1 and Remark 7.2, there exists at least one maximal non-negative subspace $\mathcal{L}_+$ in $\mathcal{K}_1$ such that $(A_1-\beta)^{-1}\mathcal{L}_+ \subset \mathcal{L}_+$, and hence

$$\mathcal{L}_+ \subset (A_1-\beta)(\mathcal{L}_+ \cap \mathcal{D}(A_1)). \tag{7.8}$$

According to Lemma 7.3 and the remarks following its proof, there exists a maximal accretive operator K in $\mathcal{H}$ so that $\mathcal{L}_+$ is the graph of K, that is, (7.3) holds. For the function v defined in (7.6) we have $v(\tau) \in \mathcal{D}((K+V)^2)$, since $v_0 \in \mathcal{D}((K+V)^2)$ and thus the derivatives $\dot v(\tau)$, $\ddot v(\tau)$ exist in the norm topology of $\mathcal{H}$:

$$\dot v(\tau) = -(K+V)v(\tau), \quad \ddot v(\tau) = (K+V)^2 v(\tau), \quad \tau \geqslant 0. \tag{7.9}$$

We introduce the function

$$w(\tau) := (K+V-\beta)v(\tau), \quad \tau \geqslant 0. \tag{7.10}$$

Then $w(\tau) \in \mathcal{D}(K+V) = \mathcal{D}(K)$ and $(w(\tau) K w(\tau))^{\mathrm{T}} \in \mathcal{L}_+$ for all $\tau \geqslant 0$. According to (7.8), there exists an element $(\hat v(\tau) K \hat v(\tau))^{\mathrm{T}} \in \mathcal{L}_+ \cap \mathcal{D}(A_1)$ such that

$$\begin{pmatrix} w(\tau) \\ Kw(\tau) \end{pmatrix} = (A_1-\beta)\begin{pmatrix} \hat v(\tau) \\ \hat v(\tau) \end{pmatrix}, \quad \tau \geqslant 0.$$

This equation is equivalent to the system

$$(K+V-\beta)\hat v(\tau) = w(\tau), \tag{7.11}$$
$$(H_0 + (V-\beta)K)\hat v(\tau) = Kw(\tau) \tag{7.12}$$

for all $\tau \geqslant 0$. Since $\operatorname{Re}\beta < \nu$ and K is maximal accretive, we have $\beta \in \rho(K+V)$. Thus, (7.11) and (7.10) yield $\hat v(\tau) = v(\tau)$, $\tau \geqslant 0$. Now (7.11) and (7.12) imply that

$$(H_0 + (V-\beta)K)v(\tau) = K(K+V-\beta)v(\tau), \quad \tau \geqslant 0,$$

or, equivalently,

$$H_0 v(\tau) + 2V(K+V)v(\tau) - V^2 v(\tau) = (K+V)^2 v(\tau), \quad \tau \geqslant 0.$$

From this we obtain (7.7) using (7.9).

To prove the last claim, we first consider a point λ_0 that is either an eigenvalue of A_1 in the open upper half-plane or a real eigenvalue of A_1 that is not of negative type. In both cases, Lemma 7.1 shows that there exists an eigenvector $\boldsymbol{x}_0$ of A_1 at λ_0 that belongs to $\mathcal{L}_+$. This means that there exists an $x_0 \in \mathcal{D}(K) = \mathcal{D}(K+V)$ so that $\boldsymbol{x}_0 = (x_0\ Kx_0)^{\mathrm{T}}$ and

$$\begin{pmatrix} V & I \\ H_0 & V \end{pmatrix} \begin{pmatrix} x_0 \\ Kx_0 \end{pmatrix} = \lambda_0 \begin{pmatrix} x_0 \\ Kx_0 \end{pmatrix}.$$

Comparing the first components, we find $(K+V)x_0 = \lambda_0 x_0$, i.e. $\lambda_0 \in \sigma_{\mathrm{p}}(K+V)$. Finally, we consider a spectral point λ_0 of positive type of A_1. In this case, there exists a bounded admissible open interval $\varGamma \subset \mathbb{R}$ such that $\lambda_0 \in \varGamma$ and $E(\varGamma)\mathcal{K}_1$ is a positive subspace, which is contained in $\mathcal{L}_+$ by Lemma 7.1. Then λ_0 is an eigenvalue or approximate eigenvalue of the restriction of A_1 to $E(\varGamma)\mathcal{K}_1$; hence, there exists a sequence $(\boldsymbol{x}_n) \subset E(\varGamma)\mathcal{K}_1 \subset \mathcal{L}_+$ such that $\|\boldsymbol{x}_n\| = 1$ and $(A_1 - \lambda_0)\boldsymbol{x}_n \to \mathbf{0}$, $n \to \infty$. As above, it is easy to see that, with $\boldsymbol{x}_n = (x_n\ Kx_n)^{\mathrm{T}}$, we have $\liminf_{n\to\infty} \|x_n\| > 0$ and $(K+V)x_n - \lambda_0 x_n \to 0$, $n \to \infty$, and hence $\lambda_0 \in \sigma(K+V)$. □

Remark 7.6. Using the spectral mapping theorem, it can be shown that the spectrum of $K+V$ consists exactly of the points described in the last sentence of the theorem.

Remark 7.7. The function $v(\tau) = T(\tau)v_0$, $\tau \geqslant 0$, is defined for arbitrary elements $v_0 \in \mathcal{H}$. However, if H_0 is unbounded, it does not necessarily have a second derivative, and hence v is only a solution in some weaker sense.

Similar considerations as in this section apply to a maximal non-positive invariant subspace of A_1, which leads to a semi-group and also to a solution of the differential equation (1.14) for $\tau \leqslant 0$.

8. Application to the Klein–Gordon equation in $\mathbb{R}^n$

In this section we consider the Klein–Gordon equation (1.1) in $\mathbb{R}^n$. Here $\mathcal{H} = L_2(\mathbb{R}^n)$ with corresponding inner product $(\cdot\,,\cdot)_2$ and norm $\|\cdot\|_2$, $H_0 = -\Delta + m^2$, and $V = eq$ is the maximal multiplication operator by the real-valued measurable function $eq : \mathbb{R}^n \to \mathbb{R}$. In the following we formulate necessary and sufficient conditions for Assumptions 3.1 and 5.1 and we apply the results of the previous sections to the Klein–Gordon equation in $\mathbb{R}^n$.

Many different sufficient conditions for the relative boundedness and for the relative compactness of a multiplication operator with respect to $H_0^{1/2} = (-\Delta + m^2)^{1/2}$ have been established (see, for example, [**22**, **39**, **43**] and the more specialized references therein). Examples considered in [**27**, § 6] include Rollnik potentials, which are $(-\Delta + m^2)^{1/2}$-bounded, and potentials belonging to $L_p(\mathbb{R}^n)$ with $n \leqslant p < \infty$, which are $(-\Delta + m^2)^{1/2}$-compact. The most general description in terms of necessary and sufficient conditions, which we present here, has been given by Maz′ya and Shaposhnikova in [**31**, **33**].

8.1. Assumptions 3.1 and 5.1

It is well known (see [**44**, §§ 1.3.1, 1.3.2]) that, for $H_0 = -\Delta + m^2$, the spaces $\mathcal{H}_\alpha = \mathcal{D}(H_0^\alpha)$, $\alpha \in [0,1]$, introduced in (4.1) are the Sobolev spaces of order 2α associated with

$L_2(\mathbb{R}^n)$ (see the definition given below):

$$\mathcal{H}_\alpha = W_2^{2\alpha}(\mathbb{R}^n), \quad \alpha \in [0,1].$$

Hence, Assumption 3.1, which reads $\mathcal{H}_{1/2} = \mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$, now becomes

Assumption 8.1. $W_2^1(\mathbb{R}^n) \subset \mathcal{D}(V)$.

This is equivalent to $V \in L(W_2^1(\mathbb{R}^n), L_2(\mathbb{R}^n))$ or to the fact that V is $(-\Delta + m^2)^{1/2}$-bounded; the latter means that there exist constants $a, b \geqslant 0$ such that

$$\|Vu\|_2 \leqslant a\|u\|_2 + b\|(-\Delta + m^2)^{1/2}u\|_2, \quad u \in W_2^1(\mathbb{R}^n). \tag{8.1}$$

Therefore, Assumption 5.1 (and hence Assumption 3.1) is satisfied if the restriction of V to $W_2^1(\mathbb{R}^n)$ can be decomposed as follows.

Assumption 8.2. *$V = V_0 + V_1$ such that V_0 is $(-\Delta + m^2)^{1/2}$-bounded and satisfies (8.1) with $a/m + b < 1$ and V_1 is $(-\Delta + m^2)^{1/2}$-compact.*

Before we formulate abstract necessary and sufficient conditions for Assumptions 8.1 and 8.2, we consider an example for which both assumptions may be checked directly using Hardy's inequality and Sobolev's embedding theorems (see [**27**, Theorem 6.1, Proposition 6.4 and Example 6.5]).

Example 8.3. Let $n \geqslant 3$. Assumption 8.2 is satisfied for

$$V(x) = \frac{\gamma}{|x|} + V_1(x), \quad x \in \mathbb{R}^n \setminus \{0\},$$

with $\gamma \in \mathbb{R}$, $|\gamma| < (n-2)/2$, and $V_1 \in L_p(\mathbb{R}^n)$, $n \leqslant p < \infty$.

Instead of the Coulomb part $V_0(x) = \gamma/|x|$, we could also assume that V_0 is bounded, $V_0 \in L_\infty(\mathbb{R}^n)$ with $\|V_0\|_\infty < m$, or that V_0^2 is a so-called Rollnik potential (see [**39**, **43**]) with Rollnik norm $\|V_0^2\|_R < 4\pi$ (see [**27**, Theorem 6.2]).

Note that the admission of the relatively compact part V_1 of V, which is not subject to any relative norm bound, may give rise to complex eigenvalues, even if V is a bounded potential. This was observed in [**42**] for potentials represented by a sufficiently deep well.

In general, the property that V_0 is $(-\Delta+m^2)^{1/2}$-bounded means that V_0 belongs to the space $M(W_2^1(\mathbb{R}^n), L_2(\mathbb{R}^n))$ of bounded multipliers from the Sobolev space $W_2^1(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$, the property that V_1 is $(-\Delta+m^2)^{1/2}$-compact means that V_1 belongs to the space $M^\circ(W_2^1(\mathbb{R}^n), L_2(\mathbb{R}^n))$ of compact multipliers from $W_2^1(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$ (see [**33**]). Necessary and sufficient conditions for functions V to belong to a space $M(W_2^m(\mathbb{R}^n), W_2^l(\mathbb{R}^n))$ of bounded multipliers or the corresponding space of compact multipliers have been established by Maz′ya and Shaposhnikova for integer m and l in [**31**] and for fractional m and l in [**32**] (see [**33**]). In view of Assumption 3.1, we restrict ourselves to the case $l = 0$. In the following we introduce all necessary definitions to formulate these criteria in Theorem 8.4, below.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are Banach spaces of functions on $\mathbb{R}^n$, then a function $\gamma : \mathcal{S}_1 \to \mathcal{S}_2$ is called a *bounded multiplier* from $\mathcal{S}_1$ into $\mathcal{S}_2$, $\gamma \in M(\mathcal{S}_1, \mathcal{S}_2)$, if

$$\|\gamma\|_{M(\mathcal{S}_1,\mathcal{S}_2)} := \sup\{\|\gamma u\|_{\mathcal{S}_2} : u \in \mathcal{S}_1,\ \|u\|_{\mathcal{S}_1} = 1\} < \infty.$$

With any Banach space $\mathcal{S}$ of functions on $\mathbb{R}^n$, we associate the space

$$\mathcal{S}_{\mathrm{loc}} := \{u : \mathbb{R}^n \to \mathbb{C} : \text{ for all } \eta \in C_0^\infty(\mathbb{R}^n) \text{ and } \eta u \in \mathcal{S}\}.$$

For $s > 0$, we denote by $[s]$ and $\{s\}$ the integer and fractional part, respectively, of s, i.e. $s = [s] + \{s\}$ with $[s] \in \mathbb{N}$ and $0 \leqslant \{s\} < 1$. For $k \in \mathbb{N}$, we denote by ∇_k the *gradient of order* k, that is,

$$\nabla_k = \left(\frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right)_{\alpha_1+\cdots+\alpha_n=k}.$$

For $s > 0$ and $1 \leqslant p < \infty$, the *fractional derivative* $D_{p,s}$ of order s in $L_p(\mathbb{R}^n)$ of a function u on $\mathbb{R}^n$ is given by

$$(D_{p,s}u)(x) := \left(\int_{\mathbb{R}^n} |(\nabla_{[s]}u)(x+h) - (\nabla_{[s]}u)(x)||h|^{n-p\{s\}}\,\mathrm{d}h\right)^{1/p}.$$

The fractional Sobolev space $W_p^s(\mathbb{R}^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{p,s} := \|D_{s,p}u\|_p + \|u\|_p, \quad u \in W_p^s(\mathbb{R}^n); \tag{8.2}$$

henceforth $\|\cdot\|_p$ denotes the standard norm in $L_p(\mathbb{R}^n)$. For integer $s = k \in \mathbb{N}$, the space $W_p^s(\mathbb{R}^n)$ coincides with the classical Sobolev space

$$W_p^k(\mathbb{R}^n) := \{u \in L_p(\mathbb{R}^n) : \nabla_j u \in L_p(\mathbb{R}^n),\ j = 1, 2, \ldots, k\}$$

and the norm (8.2) is equivalent to

$$\|u\|_{p,k} := \left(\sum_{j=0}^{k} \|\nabla_j u\|_p^2\right)^{1/2}, \quad u \in W_p^k(\mathbb{R}^n).$$

Finally, for a compact subset $e \subset \mathbb{R}^n$, we define the (p,s)-*capacity* of e by

$$\operatorname{cap}(e, W_p^s(\mathbb{R}^n)) := \inf\{\|u\|_{p,s} : u \in C_0^\infty(\mathbb{R}^n),\ u|_e \geqslant 1\}.$$

In the following, if ω varies in some set Ω and a, b depend on ω, we write $a(\omega) \sim b(\omega)$ if there exist constants $c_1, c_2 > 0$ such that $c_1 b(\omega) \leqslant a(\omega) \leqslant c_2 b(\omega)$ for all $\omega \in \Omega$.

Theorem 8.4. *Let $V : \mathbb{R}^n \to \mathbb{C}$ be a measurable function, $m \in \mathbb{N}$, and $p \in (1, \infty)$. Then $V \in M(W_p^m(\mathbb{R}^n), L_p(\mathbb{R}^n))$ (the space of bounded multipliers) if and only if there exists a constant $c > 0$ such that, for any compact subset $e \subset \mathbb{R}^n$,*

$$\|V; e\|_p^p := \int_e |V(x)|^p\,\mathrm{d}x \leqslant c\operatorname{cap}(e, W_p^m(\mathbb{R}^n)),$$

and

$$\|V\|_{M(W_p^m(\mathbb{R}^n),L_p(\mathbb{R}^n))} \sim \sup_{\substack{e\subset\mathbb{R}^n \text{ compact},\\ \operatorname{diam} e\leqslant 1}} \frac{\|V;e\|_p}{\operatorname{cap}(e,W_p^m(\mathbb{R}^n))^{1/p}}.$$

Furthermore, $V\in M^\circ(W_p^m(\mathbb{R}^n),L_p(\mathbb{R}^n))$ *(the space of compact multipliers) if and only if*

$$\lim_{\delta\to 0}\sup_{\substack{e\subset\mathbb{R}^n \text{ compact},\\ \operatorname{diam} e\leqslant \delta}} \frac{\|V;e\|_p}{\operatorname{cap}(e,W_p^m(\mathbb{R}^n))^{1/p}}=0.$$

The proof of this theorem may be found in [**33**, § 2.1.4 and Lemma 2.2.2/1].

8.2. Application of Theorems 5.7, 5.9 and 6.5

If the potential V satisfies Assumption 8.2 (and hence Assumptions 3.1 and 5.1), then all results of the previous sections apply to the Klein–Gordon equation in $\mathbb{R}^n$ and we obtain the following statements.

Recall that, by Assumption 8.1, the operator V maps the space $W_2^k(\mathbb{R}^n)$ boundedly onto $W_2^{k-1}(\mathbb{R}^n)$ for all $k\in[0,1]$ (see Remark 4.1, (4.5)).

Theorem 8.5. *Suppose that* $W_2^1(\mathbb{R}^n)\subset\mathcal{D}(V)$ *and that* $V=V_0+V_1$*, where* V_0 *is* $(-\Delta+m^2)^{1/2}$*-bounded satisfying (8.1) with* $a/m+b<1$ *and* V_1 *is* $(-\Delta+m^2)^{1/2}$*-compact.*

(i) *The operator* A_2 *given by*

$$\mathcal{D}(A_2):=\left\{\begin{pmatrix}x\\y\end{pmatrix}\in W_2^{1/2}(\mathbb{R}^n)\oplus W_2^{-1/2}(\mathbb{R}^n): x\in W_2^1(\mathbb{R}^n),\ y\in L_2(\mathbb{R}^n),\right.$$
$$\left. Vx+y\in W_2^{1/2}(\mathbb{R}^n), (-\Delta+m^2)x+Vy\in W_2^{-1/2}(\mathbb{R}^n)\right\},$$
$$A_2\begin{pmatrix}x\\y\end{pmatrix}:=\begin{pmatrix}Vx+y\\(-\Delta+m^2)x+Vy\end{pmatrix}$$

is a self-adjoint and definitizable operator in the Krein space given by $\mathcal{K}_2=W_2^{1/2}(\mathbb{R}^n)\oplus W_2^{-1/2}(\mathbb{R}^n)$ *with indefinite inner product*

$$[\boldsymbol{x},\boldsymbol{x}']:=((-\Delta+m^2)^{1/4}x,(-\Delta+m^2)^{-1/4}y')_2+((-\Delta+m^2)^{-1/4}y,(-\Delta+m^2)^{1/4}x')_2,$$

for $\boldsymbol{x}=(x\ y)^{\mathrm{T}}$, $\boldsymbol{x}'=(x'\ y')^{\mathrm{T}}\in W_2^{1/2}(\mathbb{R}^n)\oplus W_2^{-1/2}(\mathbb{R}^n)$.

(ii) *The non-real spectrum of* A_2 *is symmetric to the real axis and consists of at most finitely many pairs of eigenvalues* $\lambda,\overline{\lambda}$ *of finite type; the algebraic eigenspaces corresponding to* λ *and* $\overline{\lambda}$ *are isomorphic. There are no complex eigenvalues if*

$$\|V(-\Delta+m^2)^{-1/2}\|<1.$$

(iii) *The essential spectrum of* A_2 *is real and has a gap around* 0*; more precisely,*

$$\sigma_{\mathrm{ess}}(A_2)\subset\mathbb{R}\setminus(-\alpha,\alpha),\quad \alpha:=\left(1-\left(\frac{a}{m}+b\right)\right)m.$$

(iv) *The operator A_2 is generates a strongly continuous group $(\mathrm{e}^{\mathrm{i}tA_2})_{t\in\mathbb{R}}$ of unitary operators in the Krein space $\mathcal{K}_2 = W_2^{1/2}(\mathbb{R}^n) \oplus W_2^{-1/2}(\mathbb{R}^n)$.*

(v) *For every initial value $\boldsymbol{x}_0 \in \mathcal{D}(A_2)$, the Cauchy problem*

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathrm{i}A_2\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

has a unique classical solution $\boldsymbol{x} \in C^1(\mathbb{R}, W_2^{1/2}(\mathbb{R}^n) \oplus W_2^{-1/2}(\mathbb{R}^n))$ given by $\boldsymbol{x}(t) = \mathrm{e}^{\mathrm{i}tA_2}\boldsymbol{x}_0$, $t \in \mathbb{R}$.

The Cauchy problem for A_2 is equivalent to an initial-value problem for the Klein–Gordon equation (1.1). This leads to the following consequence of Theorem 8.5 (v).

Theorem 8.6. *Suppose that the potential V satisfies the assumptions of Theorem 8.5. Then the initial-value problem*

$$\left(\left(\frac{\partial}{\partial t} - \mathrm{i}eq\right)^2 - \Delta + m^2\right)\psi = 0, \quad \psi(\cdot\,,0) = \psi_0, \quad \frac{\partial\psi}{\partial t}(\cdot\,,0) = \psi_1, \tag{8.3}$$

in the space $W_2^{-1/2}(\mathbb{R}^n)$ has a unique classical solution ψ_s if $\psi_0 \in W_2^1(\mathbb{R}^n)$, $\psi_1 \in W_2^{1/2}(\mathbb{R}^n)$ and $(-\Delta - V^2)\psi_0 \in W_2^{-1/2}(\mathbb{R}^n)$, and the function $t \mapsto \psi_\mathrm{s}(\cdot\,,t)$ belongs to $C^1(\mathbb{R}, W_2^{1/2}(\mathbb{R}^n)) \cap C^2(\mathbb{R}, W_2^{-1/2}(\mathbb{R}^n))$.

Proof. The Cauchy problem for A_2 arises from (8.3) by means of the substitution

$$x(t) = \psi(\cdot\,,t), \quad y(t) = \left(-\mathrm{i}\frac{\partial}{\partial t} - V\right)\psi(\cdot\,,t), \quad t \in \mathbb{R};$$

hence, the initial value for the Cauchy problem for A_2 is given by

$$\boldsymbol{x}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \psi(\cdot\,,0) \\ -\mathrm{i}\dfrac{\partial\psi}{\partial t}(\cdot\,,0) - V\psi(\cdot\,,0) \end{pmatrix} = \begin{pmatrix} \psi_0 \\ -\mathrm{i}\psi_1 - V\psi_0 \end{pmatrix}.$$

Since V maps $W_2^1(\mathbb{R}^n)$ boundedly in $L_2(\mathbb{R}^n)$ by Assumption 8.1, the assumptions on ψ_0 and ψ_1 guarantee that $\boldsymbol{x}_0 \in \mathcal{D}(A_2)$. Hence, Theorem 8.5 (v) yields a unique solution $\boldsymbol{x} = (x\ y)^\mathrm{T} \in C^1(\mathbb{R}, W_2^{1/2}(\mathbb{R}^n) \oplus W_2^{-1/2}(\mathbb{R}^n))$ satisfying the equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \mathrm{i}(Vx + y) \quad \text{in } W_2^{1/2}(\mathbb{R}^n), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mathrm{i}((-\Delta + m^2)x + Vy) \quad \text{in } W_2^{-1/2}(\mathbb{R}^n). \end{aligned}$$

Differentiating the first equation and using the second one, we obtain that $x \in C^1(\mathbb{R}, W_2^{1/2}(\mathbb{R}^n)) \cap C^2(\mathbb{R}, W_2^{-1/2}(\mathbb{R}^n))$ and that x satisfies (8.3) in $W_2^{-1/2}(\mathbb{R}^n)$. $\square$

Remark 8.7. If only $\psi_0 \in W_2^{1/2}(\mathbb{R}^n)$ and $\psi_1 \in W_2^{-1/2}(\mathbb{R}^n)$, then $\boldsymbol{x}_0 \in W_2^{1/2}(\mathbb{R}^n) \oplus W_2^{-1/2}(\mathbb{R}^n)$. In this case we only obtain mild solutions of the Cauchy problem for A_2 (see Remark 6.6) and thus solutions of (8.3) in some weaker sense.

If the potential V is $(-\Delta + m^2)^{1/2}$-compact, a similar result was proved in [**18**, § 5], also using the regularity of the critical point ∞ of A_2.

The above theorem should also be compared with a corresponding result which can be obtained using the operator A introduced in [**27**] (see § 5). Since A is a self-adjoint operator in the Pontryagin space $\mathcal{K} = \mathcal{H}_{1/2} \oplus \mathcal{H} = W_2^1(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)$, it generates a group $(\mathrm{e}^{\mathrm{i}tA})_{t\in\mathbb{R}}$ of unitary operators in this space. By means of the substitution

$$x(t) = \psi(\cdot\,, t), \quad y(t) = -\mathrm{i}\frac{\partial\psi}{\partial t}(\cdot\,, t), \quad t \in \mathbb{R},$$

one can prove an existence and uniqueness result for classical solutions of (8.3) in $L_2(\mathbb{R}^n)$. Here, stronger assumptions on the initial values have to be imposed which ensure that

$$\boldsymbol{x}(0) = (\psi_0 - \mathrm{i}\psi_1)^{\mathrm{T}} \in \mathcal{D}(A) = \mathcal{D}(H) \oplus \mathcal{D}(H_0^{1/2}) \subset W_2^2(\mathbb{R}^n) \oplus W_2^1(\mathbb{R}^n)$$

($H = H_0^{1/2}(I - S^*S)H_0^{1/2}$ being the operator associated with the differential expression $-\Delta + V^2$).

Remark 8.8. Suppose that the potential V satisfies the assumptions of Theorem 8.5, and that $1 \in \rho(S^*S)$. Then the initial problem (8.3) in $L_2(\mathbb{R}^n)$ has a unique classical solution ψ_{s} if $\psi_0 \in \mathcal{D}(H) \subset W_2^2(\mathbb{R}^n)$, $\psi_1 \in W_2^1(\mathbb{R}^n)$, and the function $t \mapsto \psi_{\mathrm{s}}(\cdot\,, t)$ belongs to $C^1(\mathbb{R}, W_2^1(\mathbb{R}^n)) \cap C^2(\mathbb{R}, L_2(\mathbb{R}^n))$.

This shows that, for smooth initial values as in Remark 8.8, the operator A studied in [**27**] gives classical solutions of the Klein–Gordon equation (8.3) in $L_2(\mathbb{R}^n)$; for less smooth initial values as in Theorem 8.6, the operator A_2 studied in the present paper still gives classical solutions of the Klein–Gordon equation, but only in $W_2^{-1/2}(\mathbb{R}^n)$.

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