Math. Ann. (2008) 340:775–780
DOI 10.1007/s00208-007-0168-1

Mathematische Annalen

A Fatou–Bieberbach domain in $\mathbb{C}^2$ which is not Runge

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Received: 7 February 2007 / Accepted: 8 May 2007 / Published online: 21 September 2007

Abstract Since a paper by Rosay and Rudin (Trans. Am. Math. Soc. **310**, 47–86, 1988) there has been an open question whether all Fatou–Bieberbach domains are Runge. We give an example of a Fatou–Bieberbach domain Ω in $\mathbb{C}^2$ which is not Runge. The domain Ω provides (yet) a negative answer to a problem of Bremermann.

Mathematics Subject Classification (2000) 32E20 · 32E30 · 32H02

1 Introduction

We give a negative answer to the problem, initially posed by Rosay and Rudin [6] and later in [4], as to whether all Fatou–Bieberbach domains are Runge:

Theorem 1 *There is a Fatou–Bieberbach domain Ω in $\mathbb{C}^* \times \mathbb{C}$ which is Runge in $\mathbb{C}^* \times \mathbb{C}$ but not in $\mathbb{C}^2$.*

A Fatou–Bieberbach domain is a proper subdomain of $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$, and a domain $\Omega \subset \mathbb{C}^n$ is said to be Runge (in $\mathbb{C}^n$) if any holomorphic function $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compacts in Ω by polynomials.

It should be noted that although the domain Ω is not Runge it still has the property that the intersection of Ω with any complex line L is simply connected: Let V be a connected component of $\Omega \cap L$, let $\Gamma \subset V$ be a simple closed curve, and let D denote the disk in L bounded by Γ. Since Γ is null-homotopic in Ω we have that D is contained in $\mathbb{C}^* \times \mathbb{C}$ and so the claim follows from the fact that Ω is Runge in $\mathbb{C}^* \times \mathbb{C}$.

Supported by Schweizerische Nationalfonds grant 200021-116165/1.

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By intersecting Ω with a suitable bounded subset of $\mathbb{C}^2$ this gives a negative answer to the problem of Bremermann: "Suppose that D is a Stein domain in $\mathbb{C}^n$ such that for every complex line l in $\mathbb{C}^n$, $l \setminus D$ is connected. Is it true that D is Runge in $\mathbb{C}^n$?". Negative answers to this problem have also recently been given in [1] and [5]. On can infact show, using an argument as above together with the argument principle, that if $\mathcal{R}$ is a smoothly bounded planar domain and if $\varphi(\mathcal{R})$ is a holomorphic embedding of $\mathcal{R}$ into $\mathbb{C}^2$ with $\varphi(\partial\mathcal{R}) \subset \Omega$, then $\varphi(\mathcal{R}) \subset \Omega$.

The idea of the proof is the following: Observe first that if Ω is a Fatou–Bieberbach domain in $\mathbb{C}^2$ which is Runge, then Ω has the property that if $Y \subset \Omega$ is compact then its polynomially convex hull

$$\widehat{Y} := \{(z, w) \in \mathbb{C}^2; |P(z, w)| \leq \|P\|_Y \quad \forall P \in \mathcal{P}(\mathbb{C}^2)\}$$

is contained in Ω. To prove the theorem we will construct a domain Ω such that $\widehat{Y} \setminus \Omega \neq \varnothing$ for a certain compact set Y. For a compact subset $Y \subset \mathbb{C}^* \times \mathbb{C}$ let $\widehat{Y}_*$ denote the set

$$\widehat{Y}_* := \{(z, w) \in \mathbb{C}^2; |P(z, w)| \leq \|P\|_Y \quad \forall P \in \mathcal{O}(\mathbb{C}^* \times \mathbb{C})\}.$$

We say that the set Y is holomorphically convex if $\widehat{Y}_* = Y$. We will first construct (a construction by Stolzenberg) a holomorphically convex compact set $Y \subset \mathbb{C}^* \times \mathbb{C}$ having the property that $\widehat{Y} \cap (\{0\} \times \mathbb{C}) \neq \varnothing$. Y is the disjoint union of two disks is $\mathbb{C}^* \times \mathbb{C}$. We will then use the fact that $\mathbb{C}^* \times \mathbb{C}$ has the *density property* to construct a Fatou–Bieberbach domain $\Omega \subset \mathbb{C}^* \times \mathbb{C}$ such that $Y \subset \Omega$. The domain Ω cannot be Runge.

A few words about the density property and approximation by automorphisms. As defined in [9], *a complex manifold M is said to have the density property if every holomorphic vector field on M can be approximated locally uniformly by Lie combinations of complete vector fields on M*. It was proved in [9] that $\mathbb{C}^* \times \mathbb{C}$ has the density property. In Andersén–Lempert theory the density property corresponds to the fact that in $\mathbb{C}^n$ every entire vector field can be approximated by sums of complete vector fields. This has been studied also in [10].

Using the density property of $\mathbb{C}^* \times \mathbb{C}$ one gets as in [4] (by copying their arguments): Let Ω be an open set in $\mathbb{C}^* \times \mathbb{C}$. For every $t \in [0, 1]$, let φ_t be a biholomorphic map from Ω into $\mathbb{C}^* \times \mathbb{C}$, of class $\mathcal{C}^2$ in $(t, z) \in [0, 1] \times \Omega$. Assume that $\varphi_0 = \mathrm{Id}$, and assume that each domain $\Omega_t = \varphi_t(\Omega)$ is Runge in $\mathbb{C}^* \times \mathbb{C}$. Then for every $t \in [0, 1]$ the map φ_t can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^* \times \mathbb{C}$. In the proof of Theorem 1 we will construct such an isotopy.

We will let π denote the projection onto the first coordinate in $\mathbb{C}^* \times \mathbb{C}$ and in $\mathbb{C}^2$, and we will let $B_\varepsilon(p)$ denote the open ball of radius ε centered at a point p.

2 Construction of the set Y

We start by defining a certain rationally convex subset Y of $\mathbb{C}^2$. The set will be a union of two disjoint polynomially convex disks in $\mathbb{C}^* \times \mathbb{C}$, but the polynomial hull of the

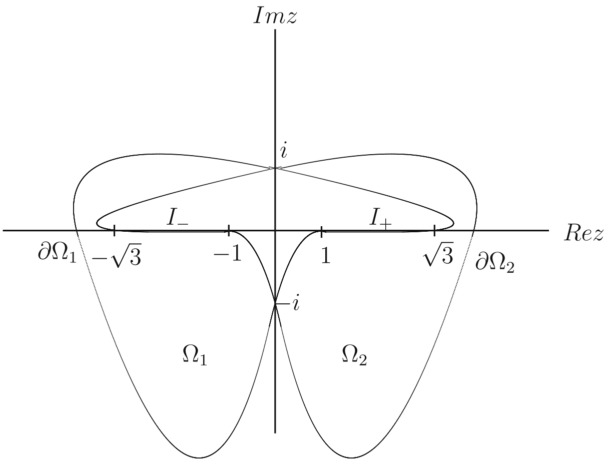

Fig. 1 Two smoothly bounded simply connected domains Ω_j with $\partial\Omega_1 \cap \partial\Omega_2 = \pm i$

union will contain the origin. This construction is taken from [8], page 392–396, and is due to Stolzenberg [7].

Let Ω_1 and Ω_2 be simply connected domains in $\mathbb{C}$, as in Fig. 1. above, with smooth boundary, such that if $I_+ = [1, \sqrt{3}]$, $I_- = [-\sqrt{3}, -1]$, then $I_+ \subset \partial\Omega_1$, $I_- \subset \partial\Omega_2$. Require that $\partial\Omega_1$ and $\partial\Omega_2$ meet only twice, that $I_- \subset \Omega_1$, $I_+ \subset \Omega_2$, and, finally, that $\partial\Omega_1 \cup \partial\Omega_2$ be the union of the boundary of the unbounded component of $\mathbb{C} \setminus (\partial\Omega_1 \cup \partial\Omega_2)$, together with the boundary of the component of this set that contains the origin. Let the intersections of the boundaries be the points i and $-i$.

We define

$$V_1 = \{(z, w) \in \mathbb{C}^2;\ z^2 - w \text{ is real and lies in } [0, 1]\},$$
$$V_2 = \{(z, w) \in \mathbb{C}^2;\ w \text{ is real and lies in } [1, 2]\},$$
$$X_1 = \{(z, w) \in V_1;\ z \in \partial\Omega_2\},$$
$$X_2 = \{(z, w) \in V_2;\ z \in \partial\Omega_1\},$$

Note that X_1 and X_2 are totally real annuli, that they are disjoint, and that the origin is contained in the polynomial hull of X_1. Next we want to remove pieces from X_1 and X_2 to create two disks.

Define

$$\tilde{V}_1 = V_1 \cap \pi^{-1}(I_+),$$
$$\tilde{V}_2 = V_2 \cap \pi^{-1}(I_-),$$
$$Y_1 = \overline{X_1 \setminus \tilde{V}_2},$$
$$Y_2 = \overline{X_2 \setminus \tilde{V}_1}.$$

The set Y will be defined as $Y = Y_1 \cup Y_2$. Note that

$$(*)\ \tilde{V}_1 \subset \widehat{X}_1, \tilde{V}_2 \subset \widehat{X}_2.$$

Let us describe what Y_1 and Y_2 looks like over I_- and I_+, respectively. By the equations we see that these sets are contained in $\mathbb{R}^2$. Let (x, y) denote the real parts of (z, w).

Over I_- we have that Y_1 is the union of the two sets defined by

(a) $2 \leq y \leq x^2$ if $-\sqrt{3} \leq x \leq -\sqrt{2}$,
(b) $x^2 - 1 \leq y \leq 1$ if $-\sqrt{2} \leq x \leq -1$.

Over I_+ we have that Y_2 is the union of the sets defined by

(c) $x^2 \leq y \leq 2$ if $1 \leq x \leq \sqrt{2}$,
(d) $1 \leq y \leq x^2 - 1$ if $\sqrt{2} \leq x \leq \sqrt{3}$.

From these equations we see that Y_1 and Y_2 are disks.

We have that

$$(**)\ \widehat{Y} \text{ contains the origin}$$

because of the following: We already noted that the origin is contained in $\widehat{X}_1$, so the claim follows from $(*)$ and the following simpler version of Lemma 29.31, [8], p. 392: Let X_1 and X_2 be disjoint compact sets in $\mathbb{C}^N$, and let S_1 and S_2 be relatively open subsets of X_1 and X_2, respectively such that $S_1 \subset \widehat{X}_2$, $S_2 \subset \widehat{X}_1$. Then $\widehat{X_1 \cup X_2} = (X_1 \setminus \widehat{S_1) \cup (X_2} \setminus S_2)$. The reason for this, which was pointed out by the referee, is simply that neither S_1 nor S_2 can contain peak points for the algebra generated by the polynomials on $X_1 \cup X_2$.

3 Proof of Theorem 1

It is proved in [8] that the set Y is rationally convex, and that the sets Y_j are polynomially convex separately. For our construction we need to know that Y is holomorphically convex, so we prove the following:

Lemma 3.1 *We have that Y is holomorphically convex in $\mathbb{C}^* \times \mathbb{C}$.*

Proof For $j = 1, 2$, let Y_j^+ and Y_j^- denote the sets $Y_j \cap \{\mathrm{Re}(z) \geq 0\}$ and $Y_j \cap \{\mathrm{Re}(z) \leq 0\}$, respectively. Let $Y^+ = Y_1^+ \cup Y_2^+$ and $Y^- = Y_1^- \cup Y_2^-$.

Observe first that Y^+ and Y^- are polynomially convex separately: assume to get a contradiction that $\widehat{Y^-}$ contains nontrivial points. In that case there exists a graph $G(f)$ of a bounded holomorphic function defined on the topological disk U bounded by $\pi(Y^-)$, such that $G(f) \subset \widehat{Y^-}$, and such that $(z, f(z)) \in Y^-$ for a.a. (in terms of radial limits if we regard U as a proper disk) $z \in \pi(Y^-)$ (Theorem 20.2. in [2], p. 172, holds by the discussion on p. 171 even though the fibers over $\pm i$ are not convex). Then for continuity reasons $\overline{G(f)}$ would have to contain nontrivial points of $\widehat{Y^-}$ in the fibers $\{\pm i\} \times \mathbb{C}$—but as this clearly cannot be the case, we have our contradiction. The case of Y^+ is similar.

Next assume to get a contradiction that there is a point $(z_0, w_0) \in \widehat{Y}_* \setminus Y$ with $\mathrm{Re}(z_0) < 0$. The function $f(z)$ defined to be $(z+i)(z-i)$ on $\pi(Y^-) \cup \{z_0\}$ and zero on $\pi(Y^+)$ can be uniformly approximated on $\pi(Y) \cup \{z_0\}$ by polynomials in z and $\frac{1}{z}$, and so any representing Jensen measure (see [8], Chap. 2) for the functional $g \mapsto g(z_0, w_0)$ would have to be supported on Y^-. But then the point (z_0, w_0) would have to be in the hull of Y^- which is a contradiction. The corresponding conclusion holds for $\mathrm{Re}(z_0) > 0$.

Finally, Rossi's local maximum principle excludes the possibility of there being nontrivial points in the hull contained in $\{\pm i\} \times \mathbb{C}$. $\square$

Lemma 3.2 *Let $p = (z_0, w_0) \in \mathbb{C}^* \times \mathbb{C}$ and let $\varepsilon > 0$. Then there exists an automorphism ψ of $\mathbb{C}^* \times \mathbb{C}$ such that $\psi(Y) \subset B_\varepsilon(p)$.*

Proof We need to argue that there exists an isotopy as described in the introduction, and we content ourselves by demonstrating that there exist isotopies mapping Y_1 and Y_2 into separate arbitrarily small balls - the rest is trivial. Let $q_j \in Y_j$ be a point for $j = 1, 2$, and let $\delta > 0$. Since Y_j is a smooth disk there clearly exists a smooth map $f^j : [0, 1] \times Y_j \to Y_j$ such that for each fixed t the map $f_t^j : Y_j \to Y_j$ is a smooth diffeomorphism, such that f_0^j is the identity, and such that $f_1^j(Y_j) \subset B_\delta(q_j)$. Since Y_j is totally real there exists, by [3] Corollary 3.2, for each $\varepsilon > 0$ a real analytic map $\Phi^j : [0, 1] \times \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Phi_t^j \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^2)$ for each t, Φ_0^j is the identity, and $\|f^j - \Phi^j\|_{[0,1]\times Y_j} < \varepsilon$. For small enough ε we restrict Φ^j to a sufficiently small Runge neighborhood of Y_j. $\square$

Proof of Theorem 1 Let G be an automorphism of $\mathbb{C}^* \times \mathbb{C}$ with an attracting fixed point $p \in \mathbb{C}^* \times \mathbb{C}$. It is well known that the basin of attraction of the point p is a Fatou–Bieberbach domain. This domain is clearly contained in $\mathbb{C}^* \times \mathbb{C}$. Denote this domain by $\Omega(G)$. Let ε be a positive real number such that $B_\varepsilon(p) \subset \Omega(G)$. By Lemma 3.2 there is an automorphism ψ of $\mathbb{C}^* \times \mathbb{C}$ such that $\psi(Y) \subset B_\varepsilon(p)$. Then $Y \subset \psi^{-1}(\Omega(G))$. The set $\psi^{-1}(\Omega(G))$ is biholomorphic to $\mathbb{C}^2$, and from (∗∗) in Sect. 2 we have that $\widehat{Y}$ contains the origin. On the other hand it is clear that $\Omega(G)$ is Runge in $\mathbb{C}^* \times \mathbb{C}$, and so $\psi^{-1}(\Omega(G))$ is Runge in $\mathbb{C}^* \times \mathbb{C}$. $\square$

Acknowledgments The author would like to thank the referee for comments; in particular for suggesting a simplified proof of Lemma 3.1.

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