Ergod. Th. & Dynam. Sys. (2008), **28**, 1465–1478
doi:10.1017/S0143385707000880

Laminated currents

JOHN ERIK FORNÆSS†, YINXIA WANG‡ and ERLEND FORNÆSS WOLD§

† *Mathematics Department, The University of Michigan, East Hall, Ann Arbor, MI 48109, USA*
(e-mail: fornaess@umich.edu)

‡ *Department of Mathematics, Henan Polytechnic University, Jiaozuo, 454000, China*
(e-mail: yinxiawang@gmail.com)

§ *Mathematisches Institut, Universität Bern, Sidlerstr. 5, CH-3012 Bern, Switzerland*
(e-mail: erlendfw@math.uio.no)

(*Received* 9 *March* 2007 *and accepted in revised form* 26 *October* 2007)

Abstract. In this paper we prove the equivalence of two definitions of laminated currents.

1. *Introduction*

Let K be a relatively-closed subset of the bidisc $\Delta^2(z, w) = \{(z, w); |z|, |w| < 1\}$. We suppose that K is a disjoint union of holomorphic graphs, $w = f_\alpha(z)$, where f_α is a holomorphic function on the unit disc with $f_\alpha(0) = \alpha$ and $|f_\alpha(z)| < 1$. We let $\mathcal{L}$ denote the lamination of K.

There are two notions of laminated currents that we will discuss. Let T be a positive closed (1, 1)-current supported on K. We assume that T is the restriction of a positive closed current defined on a neighborhood of $\overline{\Delta}^2$. We denote by $[V_\alpha]$ the current of integration along the graph of f_α. Let λ denote a continuous (1, 0)-form which at $(z, f_\alpha(z))$ equals a non-zero multiple of $dw - f'_\alpha(z)\, dz$.

Definition 1. We say that T is a *laminated current directed by* $\mathcal{L}$ if $\lambda \wedge T = 0$ for any such λ.

These are the same as Sullivan's *structure currents* [**10**]. The present terminology was introduced by Berndtsson and Sibony in [**1**], and such currents were treated further in [**4**]. In accordance with Dujardin [**3**] we also define the following.

Definition 2. We say that T is a *laminated current subordinate to* $\mathcal{L}$ if there is a positive measure μ such that $T = \int_\alpha [V_\alpha]\, d\mu(\alpha)$.

Our main result is the following.

MAIN THEOREM. *The current T is subordinate to $\mathcal{L}$ if and only if it is directed by $\mathcal{L}$.*

We note that this is a result by Sullivan in the case of the lamination being smooth, i.e. the graphs vary smoothly with α [**10**]. In the continuous setting Dujardin has shown that if a current T is dominated by a current subordinate to $\mathcal{L}$ then T is subordinate to $\mathcal{L}$.

The part of Sullivan's proof that does not go through automatically in the non-smooth case is a certain approximation step, and so in the present article we are concerned with approximation of partially-smooth functions. In [**5**] the authors proved such an approximation theorem in the case of laminations in $\mathbb{R}^2$ and in $\mathbb{R}^3$. In the last section we show that the main theorem breaks down for Riemann-surface laminations in higher dimension.

For related material on laminated currents the reader may consult the paper of Bedford *et al* [**2**].

2. *Holomorphic motions and preliminary estimates for slopes of holomorphic graphs*

We need to know how the lamination $\mathcal{L}$ defined above varies with the parameter λ, and we use the fact that it defines a holomorphic motion. Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in $\mathbb{C}$. A holomorphic motion is a subset E of the complex plane $\mathbb{C}$ (or the Riemann sphere $\widehat{\mathbb{C}}$) and a map $f : \Delta \times E \to \mathbb{C}$ (or $\widehat{\mathbb{C}}$) such that $f(0, \cdot) = \mathrm{id}$, $f(\lambda, \cdot)$ is injective for each λ, and $f(\cdot, z)$ is holomorphic for each z. The lamination $\mathcal{L}$ defines a holomorphic motion.

Let us briefly recall some facts. It is known [**9**] that any holomorphic motion has an extension to a holomorphic motion $f : \Delta \times \mathbb{C} \to \mathbb{C}$. This means that we may regard K as a subset of a lamination of $\Delta \times \mathbb{C}$. From [**8**] we have that f is automatically jointly continuous in (λ, z); in fact the map $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ is a homeomorphism onto $\Delta \times \mathbb{C}$. Moreover, $f(\lambda, \cdot)$ is quasi-conformal for each λ, and $f(\lambda, \cdot)$ distorts cross-ratios by a bounded amount depending on $|\lambda|$. In particular we have the following. If C is compact in $\mathbb{C}^*$ and x, y, z are three distinct points in $\mathbb{C}$ with $c_0 = (x - y)/(z - y) \in C$, then $(f_\lambda(x) - f_\lambda(y))/(f_\lambda(z) - f_\lambda(y))$ is close to c_0 depending only on $|\lambda|$ (for a fixed C). To see this one can consider the map $\lambda \mapsto (f_\lambda(x) - f_\lambda(y))/(f_\lambda(z) - f_\lambda(y))$, a map from the unit disk to $\mathbb{C} \setminus \{0, 1\}$, and use the fact that it has to be distance-decreasing in the Poincaré metric. Finally we recall that $f(\lambda, \cdot)$ is Hölder continuous with exponent $1 + \epsilon(|\lambda|)$.

Next we need a basic estimate on slopes of the graphs. For the benefit of the reader we include the details of this well-known fact. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω. Let $\|\cdot\|_\infty$ denote the sup norm. Set

$$H^\infty = H^\infty(\Delta) = \{f \in \mathcal{O}(\Delta) : \|f\|_\infty < \infty\}.$$

Also, if $0 < C < \infty$ we set

$$H^\infty_C = H^\infty_C(\Delta) = \{f \in \mathcal{O}(\Delta) : \|f\|_\infty < C\}.$$

LEMMA 1. *If $f \in H^\infty_1(\Delta)$ and $f(z) \neq 0$ for all $z \in \Delta$, then*

$$|f'(0)| \leq 2|f(0)| \log(1/|f(0)|).$$

Proof. Pick a holomorphic function $f(z)$ on the unit disc such that $0 \neq |f(z)| < 1$ for all $z \in \Delta$. We can replace $f(z)$ by $e^{i\theta} f(z)$ for any real θ. This does not change $|f(0)|$ or $|f'(0)|$. Hence we can assume that $f(0) > 0$.

We set $h(z) := \log f(z)$. Then $h(z)$ is a holomorphic function on the unit disc and $\mathrm{Re}(h(z)) < 0$. We can also choose a branch of the logarithm so that $\log(f(0)) = -a < 0$. If $k(z) = h(z)/a$, then $k(z)$ is a holomorphic function on the unit disc and $k(0) = -1$, $\mathrm{Re}(k(z)) < 0$. We define $L(w) = (w+1)/(w-1)$. Then $L(-1) = 0$ and if $\mathrm{Re}(w) < 0$ then $|L(w)| < 1$. Then $\Gamma(z) := L(k(z))$ is a holomorphic function from the unit disc to the unit disc. Moreover $\Gamma(0) = L(k(0)) = L(-1) = 0$. Since $\Gamma(0) = 0$ and $|\Gamma(z)| < 1$ we can apply the Schwarz lemma. So we can conclude that $|\Gamma'(0)| \leq 1$. By the chain rule, $\Gamma'(0) = L'(k(0))k'(0) = L'(-1)k'(0)$. Since $L'(w) = -2/(w-1)^2$ we get $\Gamma'(0) = -2/(-1-1)^2 k'(0)$ and therefore $k'(0) = -2\Gamma'(0)$. Hence we get $|k'(0)| \leq 2$. Since $k(z) = h(z)/a$, we can conclude next that $|k'(0)| = |h'(0)|/a$. Hence $|h'(0)| = a|k'(0)| \leq a \cdot 2$, so $|h'(0)| \leq 2a$. Next recall that $h(z) = \log f(z)$, so $f(z) = e^{h(z)}$. Hence $f'(z) = e^{h(z)}h'(z)$. Therefore $f'(0) = e^{h(0)}h'(0) = f(0)h'(0)$. Hence $|f'(0)| \leq |f(0)||h'(0)|$. This implies that $|f'(0)| \leq 2a|f(0)|$. Now recall that $\log f(0) = -a$. But we have set this up so that $\log f(0) = \log|f(0)| + i \arg f(0)$ is real-valued. So $\log|f(0)| = -a$, i.e. $\log(1/|f(0)|) = a$. Therefore $|f'(0)| \leq 2a|f(0)| = 2|f(0)| \log(1/|f(0)|)$. This concludes the proof of the lemma. $\square$

COROLLARY 1. *Suppose that we have two functions f and g holomorphic on the unit disc with $f - g \in H_1^\infty(\Delta)$. Suppose that $f(z) \neq g(z)$ for each $z \in \Delta$. We then have the estimate $|f'(z) - g'(z)| \leq 4|f(z) - g(z)| \log(1/|f(z) - g(z)|)$ for all $z \in \Delta$, $|z| < 1/2$.*

Proof. Pick z, $|z| < 1/2$. We define $G(w) = f(z + w/2) - g(z + w/2)$. Then $G(w)$ satisfies the conditions of Lemma 1. Hence $|G'(0)| \leq 2|G(0)| \log(1/|G(0)|)$. Therefore,

$$\frac{1}{2}|f'(z) - g'(z)| \leq 2|f(z) - g(z)| \log \frac{1}{|f(z) - g(z)|}. \qquad \square$$

3. *Approximation for complex curves in $\mathbb{C}^2$*

We assume that for every $c = (a, b) = (a + ib) \in \mathbb{C}$ we have a holomorphic graph Γ_c given by $w = y_1 + iy_2 = f_c(z)$, $z = x_1 + ix_2 \in \Delta$. We assume that all surfaces are disjoint and that there is a surface through every point in $\Delta \times \mathbb{C}$. We assume that $f_c(0) = c$.

Let $\pi : \Delta \times \mathbb{C} \to \mathbb{C}$ be defined by $\pi(z, f_c(z)) = c$. By the discussion in the previous section the function π is continuous.

Fix a positive constant R. By Corollary 1 there exists a positive real number $\delta_0 > 0$ such that if $z \in (1/2)\Delta$ and if $c, c' \in R\Delta$ with $|c - c'| < \delta_0$ then

$$\left|\frac{\partial}{\partial z} f_{c'}(z) - \frac{\partial}{\partial z} f_c(z)\right| \leq 4 \cdot |f_{c'}(z) - f_c(z)| \log \frac{1}{|f_{c'}(z) - f_c(z)|}. \tag{1}$$

We define a class of partially-smooth functions:

$$\begin{aligned}\mathcal{A} := \bigg\{&\phi \in \mathcal{C}(\Delta \times \mathbb{C}) : \phi(z, f_c(z)) \in \mathcal{C}^1(\Gamma_c),\\ &\Phi(x_1, x_2, w) := \frac{\partial}{\partial x_1}\phi(x_1, x_2, f_c(x_1, x_2)),\ w = f_c(x_1, x_2) \in \mathcal{C}(\Delta \times \mathbb{C}),\\ &\Psi(x_1, x_2, w) := \frac{\partial}{\partial x_2}\phi(x_1, x_2, f_c(x_1, x_2)),\ w = f_c(x_1, x_2) \in \mathcal{C}(\Delta \times \mathbb{C})\bigg\}.\end{aligned}$$

THEOREM 1. *Let $\phi \in \mathcal{A}$, let R be a positive real number and let $\epsilon > 0$. Then there exists a function $\psi \in \mathcal{C}^1(\Delta \times R\Delta)$ such that for every point $(x_1, x_2, w) = (x_1, x_2, f_c(x_1, x_2)) \in \Delta \times R\Delta$:*

$$|\psi(x_1, x_2, w) - \phi(x_1, x_2, w)| < \epsilon,$$
$$\left|\frac{\partial}{\partial x_1}[\psi(x_1, x_2, f_c(x_1, x_2))] - \frac{\partial}{\partial x_1}[\phi(x_1, x_2, f_c(x_1, x_2))]\right| < \epsilon,$$
$$\left|\frac{\partial}{\partial x_2}[\psi(x_1, x_2, f_c(x_1, x_2))] - \frac{\partial}{\partial x_2}[\phi(x_1, x_2, f_c(x_1, x_2))]\right| < \epsilon.$$

We will prove the theorem using the following result.

PROPOSITION 1. *Let $g \in \mathcal{A}$, $g(x_1, x_2, f_{a+ib}(x_1, x_2)) = a$, and let R be a positive real number. There exists a positive real number t_0 such that the following holds. For all $\epsilon > 0$ there exists a function $h \in \mathcal{C}^1(t_0\Delta \times R\Delta)$ such that for every point $(x_1, x_2, w) = (x_1, x_2, f_c(x_1, x_2)) \in t_0\Delta \times R\Delta$:*

$$|h(x_1, x_2, w) - g(x_1, x_2, w)| < \epsilon,$$
$$\left|\frac{\partial}{\partial x_1}[h(x_1, x_2, f_c(x_1, x_2))]\right| < \epsilon,$$
$$\left|\frac{\partial}{\partial x_2}[h(x_1, x_2, f_c(x_1, x_2))]\right| < \epsilon.$$

The same result holds if we replace a by b in the definition of g.

Proof of Theorem 1 from Proposition 1.

LEMMA 2. *Let $p \in \Delta$ be a point, and let R, t_0 be positive real numbers such that $\Delta_{t_0}(p) \subset\subset \Delta$. Consider the lamination restricted to $\Delta_{t_0}(p) \times \mathbb{C}$. If the conclusion of Proposition 1 holds on $\Delta_{t_0}(p) \times R\Delta$ (with respect to projection onto $\{p\} \times \mathbb{C}$), then the conclusion of Theorem 1 holds on $\Delta_{t_0}(p) \times R\Delta$.*

Proof. Let $\pi = (\pi_1, \pi_2)$ denote the projection onto $\{p\} \times \mathbb{C}$. For each $j, k \in \mathbb{Z}$ and $\delta > 0$ we let $c^\delta(j, k)$ denote the point $(p, j\delta + k\delta i)$. Let Λ_j^δ denote the $\mathcal{C}^1$-smooth function defined by $\Lambda_j^\delta(t) = \cos^2[\pi/2\delta(t - j\delta)]$ when $(j-1)\delta \le t \le (j+1)\delta$ and 0 otherwise. For each $c^\delta(j, k)$ we first define a function

$$\psi_{jk}^\delta(z) := \phi(z, f_{c^\delta(j,k)}(z)),$$

and then we define a preliminary approximation

$$\psi^\delta(z, w) = \sum_{j,k} \psi_{jk}^\delta(z)\Lambda_j(\pi_1(z, w))\Lambda_k(\pi_2(z, w)).$$

Let $(z_0, w_0) \in \Delta_{t_0}(p) \times R\Delta$. Then $\pi(z_0, w_0)$ is contained in a square with corners $c^\delta(j, k)$, $c^\delta(j+1, k)$, $c^\delta(j, k+1)$ and $c^\delta(j+1, k+1)$, and

$$\psi^\delta(z_0, w_0) = \sum_{m=j,j+1,n=k,k+1} \psi_{mn}^\delta(z_0)\Lambda_m(\pi_1(z_0, w_0))\Lambda_n(\pi_2(z_0, w_0)).$$

We have

$$|\psi^\delta(z_0, w_0) - \phi(z_0, w_0)| = \bigg| \sum_{m=j,j+1,n=k,k+1} [\psi^\delta_{mn}(z_0) - \phi(z_0, w_0)] \times \Lambda^\delta_m(\pi_1(z_0, w_0)) \cdot \Lambda^\delta_n(\pi_2(z_0, w_0)) \bigg| \leq \max_{m=j,j+1,n=k,k+1} |\psi^\delta_{mn}(z_0) - \phi(z_0, w_0)|.$$

Since the map from $\overline{\Delta}_{t_0}(p) \times \mathbb{C}$ defined by $(z, \alpha) \mapsto (z, f_\alpha(z))$ is a homeomorphism it follows that $\psi^\delta \to \phi$ uniformly as $\delta \to 0$.

Next we approximate derivatives along leaves. Let α be such that $(z_0, w_0) = (z_0, f_\alpha(z_0))$. Since the functions $\Lambda^\delta_j \circ \pi_i$ are constant along leaves,

$$\begin{aligned} &\left| \frac{\partial}{\partial x_i} [\psi^\delta(z_0, f_\alpha(z_0)) - \phi(z_0, f_\alpha(z_0))] \right| \\ &= \Bigg| \sum_{m=j,j+1,n=k,k+1} \left[\frac{\partial}{\partial x_i} [\psi^\delta_{mn}(z_0) - \phi(z_0, f_\alpha(z_0))] \right] \\ &\quad \times \Lambda^\delta_m(\pi_1(z_0, f_\alpha(z_0))) \cdot \Lambda^\delta_n(\pi_2(z_0, f_\alpha(z_0))) \Bigg| \\ &\leq \max_{m=j,j+1,n=k,k+1} \left| \frac{\partial}{\partial x_i} [\psi^\delta_{mn}(z_0) - \phi(z_0, f_\alpha(z_0))] \right|. \end{aligned}$$

It follows that $\psi^\delta \to \phi$ also in $\mathcal{C}^1$-norm on leaves.

Now the conclusion of Lemma 2 follows because the functions π_j can be approximated uniformly and in $\mathcal{C}^1$-norm on leaves. □

For each point $p \in \Delta$ there exists by Proposition 1 a positive real number t_p such that constant approximation is possible on $\Delta_{t_p}(p) \times R\Delta$. Hence by Lemma 2 approximation of functions in $\mathcal{A}$ is possible.

We may then choose a locally-finite cover $\{U_\alpha\}_{\alpha \in \mathbb{N}}$ of Δ by disks such that approximation by functions in $\mathcal{A}$ is possible on each $U_\alpha \times R\Delta$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. For each α let $C_\alpha = \|\nabla \varphi_\alpha\|$.

For a given ϵ_α let g_{ϵ_α} be an ϵ_α-approximating function of ϕ on $U_\alpha \times R\triangle$. We will show that there is a sequence $\{\epsilon_\alpha\}$ such that the function

$$\psi = \sum_\alpha \varphi_\alpha \cdot g_{\epsilon_\alpha}$$

satisfies the claims of the theorem.

Let $z_0 \in U_\alpha$, and let $\{\alpha_1, \dots, \alpha_m\}$ be the finite set of α's, such that the support of ϕ_α intersects U_α. Then

$$\psi(z, f_c(z)) = \sum_{i=1}^m \varphi_{\alpha_i}(z) \cdot g_{\epsilon_{\alpha_i}}(z, f_c(z)),$$

for all z near z_0. Then

$$\begin{aligned}
&|\psi(z_0, f_c(z_0)) - \phi(z_0, f_c(z_0))| \\
&\quad = \left| \left[\sum_{i=1}^{m} \varphi_{\alpha_i}(z_0) \cdot \ g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0)) \right] - \phi(z_0, f_c(z_0)) \right| \\
&\quad \leq \sum_{i=1}^{m} \varphi_{\alpha_i}(z_0) \cdot |g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0)) - \phi(z_0, f_c(z_0))| \\
&\quad \leq \max\{\epsilon_{\alpha_i}\}.
\end{aligned}$$

Further

$$\begin{aligned}
&\left| \frac{\partial}{\partial x_1} [\psi(z_0, f_c(z_0)) - \phi(z_0, f_c(z_0))] \right| \\
&\quad = \left| \frac{\partial}{\partial x_1} \left[\left[\sum_{i=1}^{m} \varphi_{\alpha_i}(z) \cdot \ g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0)) \right] - \phi(z_0, f_c(z_0)) \right] \right| \\
&\quad = \left| \sum_{i=1}^{m} \frac{\partial}{\partial x_1} [\varphi_{\alpha_i}(z_0) \cdot (g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0)) - \phi(z_0, f(z_0)))] \right| \\
&\quad = \left| \sum_{i=1}^{m} \frac{\partial}{\partial x_1} [\varphi_{\alpha_i}(z_0)] \cdot (g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0))) - (\phi(z_0, f(z_0))) \right. \\
&\qquad \left. + \sum_{i=1}^{m} \varphi_{\alpha_i}(z_0) \cdot \frac{\partial}{\partial x_1} [g_{\epsilon_{\alpha_i}}(z_0, f_c(z_0)) - \phi(z_0, f(z_0))] \right| \\
&\quad \leq m \cdot \max\{C_{\alpha_i}\} \cdot \max\{\epsilon_{\alpha_i}\} + \max\{\epsilon_{\alpha_i}\}.
\end{aligned}$$

Similarly we get that

$$\left| \frac{\partial}{\partial x_2} [\psi(z_0, f_c(z_0)) - \phi(z, f_c(z_0))] \right| \leq m \cdot \max\{C_{\alpha_i}\} \cdot \max\{\epsilon_{\alpha_i}\} + \max\{\epsilon_{\alpha_i}\}.$$

It is clear that we may choose ϵ_{α_i} for $i = 1, \ldots, m$ to get the desired estimate for all points $z_0 \in U_\alpha$ for this particular α. Running through all α we find that any particular α_i will only come under consideration a finite number of times. Hence we may choose the sequence $\{\epsilon_\alpha\}$. $\square$

We proceed to prove the proposition.

Fix δ_0 to get the estimate (1) (in the beginning of §3) for all $|c - c'| < \delta_0$ with $|c|, |c'| \leq 2R$. For any δ with $0 < \delta < \delta_0$ we let $c^\delta(j, k) = (j + k \cdot i) \cdot \delta$ for $j, k \in \mathbb{Z}$. Let $\chi : [0, 1] \to \mathbb{R}$ be a smooth function such that $\chi(t) = 0$ for $0 \leq t \leq 1/4$ and $\chi(t) = 1$ for $3/4 \leq t \leq 1$. Let C be a constant such that $|\chi'(t)| \leq C$ for all $t \in [0, 1]$.

We first define a function h_δ on the surfaces $\Gamma_{c^\delta(j,k)}$ simply by $h_\delta|_{\Gamma_{c^\delta(j,k)}} \equiv j\delta$. We want to interpolate this function between the surfaces.

For a fixed z consider the sets of points

$$Q_{c^\delta(j,k)}(z) := \{f_{c^\delta(j,k)}(z), f_{c^\delta(j+1,k)}(z), f_{c^\delta(j,k+1)}(z), f_{c^\delta(j+1,k+1)}(z)\}.$$

We first show that these sets move nicely with z for small enough $|z|$ and independent of δ. In particular we want to know that we may define quadrilateral regions $R_{\delta,j,k}(z)$, with straight edges and corners $Q_{c^\delta(j,k)}(z)$, and that these sets have disjoint interior.

We make the change of coordinates in the w variable, by setting

$$\tilde{w}(z, w) = \tilde{w}_{jk}(z, w) = \frac{w - f_{c^\delta(j,k)}(z)}{f_{c^\delta(j+1,k)} - f_{c^\delta(j,k)}(z)}.$$

We get

$$\tilde{w}(z, f_{c^\delta(j,k)}(z)) \equiv 0,$$
$$\tilde{w}(z, f_{c^\delta(j+1,k)}(z)) \equiv 1.$$

From the discussion on holomorphic motions in §2 we get the following.

LEMMA 3. *Fix N. Then there exists a real number $t_0 > 0$ independent of δ such that if $|l|, |m| < N$ then $|\tilde{w}_{jk}(z, f_{c^\delta(j+l,k+m)}(z)) - \tilde{w}_{jk}(z, f_{c^\delta(j+l,k+m)}(0))| < 1/10$ for all $|z| < t_0$ and any j, k.*

From now on we assume that $|z| \leq t_0$.

LEMMA 4. *The quadrilaterals have disjoint interiors.*

Proof. Pick (j, k). We use the linear change of coordinates in the w direction for fixed z:

$$\tilde{w}_{jk}(z, w) = \frac{w - f_{c^\delta(j,k)}(z)}{f_{c^\delta(j+1,k)}(z) - f_{c^\delta(j,k)}(z)}.$$

This sends $f_{c^\delta(j+l,k+m)}(z)$ close to $(j + l, k + m)$ on a small disc in the z direction for uniformly bounded (l, m). Hence it is clear that the quadrilaterals are disjoint. □

Next we define preliminary functions h^δ_{jk} on the respective quadrilaterals. First we define a function $t_z(y_1, y_2)$ to be constant equal to 0 on the line between $f_{c^\delta(j,k)}(z)$ and $f_{c^\delta(j,k+1)}(z)$, and constant equal to 1 on the line between $f_{c^\delta(j+1,k)}(z)$ and $f_{c^\delta(j+1,k+1)}(z)$. We extend t_z continuously to be affine on the two other edges, and then we extend t_z to be constant equal to v on the line between $f_{c^\delta(j,k)}(z) + v \cdot (f_{c^\delta(j+1,k)}(z) - f_{c^\delta(j,k)}(z))$ and $f_{c^\delta(j,k+1)}(z) + v \cdot (f_{c^\delta(j+1,k+1)}(z) - f_{c^\delta(j,k+1)}(z))$. Finally we define h^δ_{jk} by

$$h^\delta_{jk}(z, y_1, y_2) = j\delta + \delta \cdot (\chi \circ t_z)\,(y_1, y_2).$$

The h^δ_{jk} patch up smoothly along the vertical sides of the quadrilaterals where the functions are constant. To be able to patch them together in the 'horizontal' directions we first extend each h^δ_{jk} across the 'horizontal' edges.

To do this we use the coordinates defined by $\tilde{w}$. Consider the normalization

$$\tilde{w}_{jk}(z, w) = \frac{w - f_{c^\delta(j,k)}(z)}{f_{c^\delta(j+1,k)}(z) - f_{c^\delta(j,k)}(z)}.$$

Let $\tilde{h}^\delta_{jk}$ be defined by $\tilde{h}^\delta_{jk} \circ \tilde{w} = h^\delta_{jk}$. We want to glue together the two functions on the quadrilaterals sharing (in the new coordinates) the line segment γ between (0, 0) and (1, 0), i.e. the function $\tilde{h}^\delta_{jk}$ defined above γ and the function $\tilde{h}^\delta_{j(k-1)}$ below γ.

We start by extending the function $\tilde{h}^\delta_{jk}$. Note first that by Lemma 3 the quadrilaterals $R_{\delta,j,k}$ and $R_{\delta,j,k-1}$ in the new coordinates – henceforth denoted $\tilde{R}_{\delta,j,k}$ and

$\tilde{R}_{\delta,j,k-1}$ – have corners within $(1/10)$-distance from the points (l, m) for $l, m \in \{0, 1, -1\}$. Note also that if we define a function $\tilde{t}_z(\tilde{y}_1, \tilde{y}_2)$ $(\tilde{w} = \tilde{y}_1 + i\tilde{y}_2)$ along lines in the quadrilateral $\tilde{R}_{\delta,j,k}(z)$ in the new coordinates as we did when we defined $t_z(y_1, y_2)$ above, then $h^\delta_{jk} = (j\delta + \delta(\chi \circ \tilde{t})) \circ \tilde{w}$. Because of the placing of the corners we see that there exists a constant K independent of δ, j, k such that $\|\nabla_{\tilde{w}}(j\delta + \delta(\chi \circ \tilde{t}))\| \leq K\delta$.

Continue the lines in $\tilde{R}_{\delta,j,k}$ that pass through the interval $[(1/8), 1 - (1/8)]$ and extend $\tilde{h}^\delta_{jk}$ to be constant on these lines. By the placing of the corners there is a constant μ – independent of δ and j, k – such that these lines can be extended to the line between $(0, -\mu)$ and $(1, -\mu)$. Let $\tilde{P}_{\delta,j,k}$ denote the extended set $\tilde{R}_{\delta,j,k} \cup (\tilde{R}_{\delta,j,k-1} \cap \{y_2 \geq -\mu\})$; we see that $\tilde{h}^\delta_{jk}$ extends to be constant on the part of $\tilde{P}_{\delta,j,k}$ where it is not already defined. Extend $\tilde{h}^\delta_{j(k-1)}$ similarly in the other direction.

To glue the functions together we choose a smooth function $\varphi(z, \tilde{y}_1, \tilde{y}_2) = \varphi(\tilde{y}_2)$ such that $\varphi(\tilde{y}_2) = 1$ if $y_2 \geq \mu$ and such that $\varphi(\tilde{y}_2) = 0$ if $y_2 \leq -\mu$. We define our final function

$$h_\delta(z, w) := (\varphi \circ \tilde{w}_{jk})\,(z, w) \cdot h^\delta_{jk}(z, w) + (1 - \varphi \circ \tilde{w}_{jk})\,(z, w) \cdot h^\delta_{j(k-1)}(z, w). \quad (2)$$

Fix a constant M such that $\|\partial\varphi/\partial\tilde{y}_2\| = M$.

LEMMA 5. *There are constants N_1 and N_2 such that for each j, k, δ we have $h^\delta_{jk}(z, w) = j\delta$ if $|w - f_{c^\delta(j,k)}(z)| \leq N_1|f_{c^\delta(j+1,k)}(z) - f_{c^\delta(j,k)}(z)|$. Moreover there is a smooth function $\tilde{g}^\delta_{jk}(z, \tilde{y}_1, \tilde{y}_2)$ such that $h^\delta_{jk} = \tilde{g}^\delta_{jk} \circ \tilde{w}$ and $\|\nabla_{\tilde{w}}\tilde{g}^\delta_{jk}\| \leq N_2\delta$.*

Proof. The existence of the constant N_1 can be seen by our description of the function in local coordinates where we used Lemma 3. To see the rest let us give the function $\tilde{g}^\delta_{jk}$ explicitly.

Fix z. Let (a_1, a_2) denote the corner of $\tilde{R}^\delta_{j,k}$ that is close to $(0, 1)$, and define a map $A_z(\tilde{y}_1, \tilde{y}_2) := (\tilde{y}_1 - \tilde{y}_2(a_1/a_2), \tilde{y}_2(1/a_2))$. Then A_z changes smoothly with z and $\|A_z\| < 2$ for all the possibilities of (a_1, a_2) we are considering.

Next we define a function $\widehat{t}$ on the quadrilateral $A_z(\tilde{R}^\delta_{j,k})$ along lines as above. Let (b_1, b_2) denote the corner close to $(1, 1)$ and fix $\widehat{y} = (\widehat{y}_1, \widehat{y}_2)$. We have that the two vertical sides of $A_z(\tilde{R}^\delta_{j,k})$ meet at the point $(0, -L)$ where $L = b_2/(b_1 - 1)$. Calculating the slope of the line from the point $\widehat{y}$ to the point $(\widehat{t}(\widehat{y}), 0)$, we get that $\widehat{y}_1/(L + \widehat{y}_2) = \tilde{t}(y)/L$, which gives us

$$\widehat{t}(\widehat{y}) = \frac{\widehat{y}_1 \cdot L}{L + \widehat{y}_2} = \frac{\widehat{y}_1 \cdot b_2}{b_2 + \widehat{y}_2(b_1 - 1)}.$$

We have that $\widehat{t}$ varies smoothly with (b_1, b_2) and we see that $\widehat{t}$ has bounded derivatives for the cases of (b_1, b_2) we are considering. Define $\tilde{g}^\delta_{jk}$ by

$$\tilde{g}^\delta_{jk} = j\delta + \delta(\chi \circ \widehat{t} \circ A_z),$$

and the function h^δ_{jk} is given by $h^\delta_{jk} = \tilde{g}^\delta_{jk} \circ \tilde{w}$. □

LEMMA 6. *$h_\delta \to g$ in sup norm on $\Delta_{t_0} \times R\Delta$.*

Proof. It is clear that $h_\delta(0, \cdot) \to g(0, \cdot)$ uniformly. The claim then follows from Lemma 8 below. □

LEMMA 7. *If t_0 and δ are small enough, then $|f_{c^\delta(j,k)}(z) - f_{c^\delta(j+1,k)}(z)| \geq \delta^2$ for all z with $|z| \leq t_0$ and all j, k such that $|c^\delta(j,k)| \leq 2R$.*

Proof. This follows from the Hölder continuity of the holomorphic motion. □

LEMMA 8. *Let $c \in R\overline{\Delta}$. The function $h_\delta(z, f_c(z))$ is small in $\mathcal{C}^1$-norm along the graph Γ_c.*

Proof. We need to estimate the derivatives of the function $h_\delta(z, f_c(z))$ at an arbitrary point $(z_0, f_c(z_0))$, and this point is contained in some extended quadrilateral $P_{\delta,j,k}$. We estimate $\partial/\partial x = \partial/\partial x_1$ – the case of $\partial/\partial x_2$ is similar. Since we are working on lines we use the notation (x, y_1, y_2) for coordinates.

If the point is close to the vertical edges, then the function h_δ is locally constant, so we are done. We can assume that also $(z_0, f_c(z_0)) \in P_{\delta,j,k} \setminus P_{\delta,j,k+1}$. We divide the proof into two cases. Assume first that $(z_0, f_c(z_0))$ is not in $P_{\delta,j,k-1}$. Then the function h_δ is simply equal to the function h^δ_{jk} (see (2)).

We have that

$$\begin{aligned}\frac{\partial}{\partial x}(h^\delta_{jk}(x, f(x))) &= \left(\frac{\partial h^\delta_{jk}}{\partial x}, \frac{\partial h^\delta_{jk}}{\partial y_1}, \frac{\partial h^\delta_{jk}}{\partial y_2}\right)(x, f(x)) \cdot \left(1, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}\right)(x)\\ &= \frac{\partial h^\delta_{jk}}{\partial x}(x, f(x)) + \left(\frac{\partial h^\delta_{jk}}{\partial y_1}, \frac{\partial h^\delta_{jk}}{\partial y_2}\right)(x, f(x)) \cdot \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}\right)(x).\end{aligned} \tag{3}$$

For fixed s, v we may define a curve $(x, g(x))$:

$$\begin{aligned}g(x) = (1-s)\,&[(1-v) f_{c^\delta(j,k)}(x) + v f_{c^\delta(j+1,k)}(x)]\\ &+ s[(1-v) f_{c^\delta(j,k+1)}(x) + v f_{c^\delta(j+1,k+1)}(x)].\end{aligned}$$

Then $h^\delta_{jk}(x, g(x)) \equiv j\delta + \chi(v)\delta$. Choose s and v so that $(x_0, g(x_0)) = (x_0, f_c(x_0))$. We get that

$$\begin{aligned}0 &= \frac{\partial}{\partial x}(h^\delta_{jk}(x, g(x)))\\ &= \frac{\partial h^\delta_{jk}}{\partial x}(x, g(x)) + \left(\frac{\partial h^\delta_{jk}}{\partial y_1}, \frac{\partial h^\delta_{jk}}{\partial y_2}\right)(x, g(x)) \cdot \left(\frac{\partial g_1}{\partial x}, \frac{\partial g_2}{\partial x}\right)(x),\end{aligned} \tag{4}$$

and so substracting (4) from (3) we get

$$\frac{\partial}{\partial x}(h^\delta_{jk}(x_0, f(x_0))) = \left(\frac{\partial h^\delta_{jk}}{\partial y_1}, \frac{\partial h^\delta_{jk}}{\partial y_2}\right)(x_0, g(x_0)) \cdot \left(\frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial g_2}{\partial x}\right)(x_0).$$

Using Lemma 3 we see that $\|f_c(x_0) - f_{c^\delta(j+l,k+m)}(x_0)\| \leq 2\|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)\|$ for $l, m \in \{0, 1\}$, and so

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x}(f_c - f_{c^\delta(j+l,k+m)})\,(x_0)\right\| \\
&\quad\le 4\|(f_c - f_{c^\delta(j+l,(k+m)})\,(x_0))\|\log\frac{1}{\|(f_c - f_{c^\delta(j+l,k+m)})\,(x_0)\|} \\
&\quad\le 8\|(f_{c^\delta(j+1,k)} - f_{c^\delta(j,k)})\,(x_0)\|\log\frac{1}{2\|(f_{c^\delta(j+1,k)} - f_{c^\delta(j,k)})\,(x_0)\|}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\|\frac{\partial}{\partial x}(h_\delta(x,\,f(x)))\right\| \le 8\cdot\left\|\left(\frac{\partial h_\delta}{\partial y_1},\,\frac{\partial h_\delta}{\partial y_2}\right)\right\|\cdot\|(f_{c^\delta(j+1,k)} - f_{c^\delta(j,k)})\,(x_0)\| \\
\times\ \log\frac{1}{2\|(f_{c^\delta(j+1,k)} - f_{c^\delta(j,k)})\,(x_0)\|}.
\end{aligned}$$

We proceed to estimate $\|(\partial h_\delta/\partial y_1,\,\partial h_\delta/\partial y_2)\|$. We change coordinates according to Lemma 5 and write h_δ as a composition $\tilde{g}_\delta\circ\tilde{w}(y)$. We get $\|D_w\tilde{w}\| = 1/(\|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)\|)$, and we have that $\|\nabla_{\tilde{w}}\tilde{g}_\delta\|\le N_2\delta$. This shows that

$$\left\|\left(\frac{\partial h_\delta}{\partial y_1},\,\frac{\partial h_\delta}{\partial y_2}\right)\right\|\le N_2\delta\frac{1}{\|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)\|}.$$

This gives

$$\left\|\frac{\partial}{\partial x}(h_\delta(x,\,f(x)))\right\|\le 8N_2\delta\log\frac{1}{\|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)\|}.$$

We have by Lemma 7 that $\|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)\|\ge\delta^2$, and so

$$\left\|\frac{\partial}{\partial x}(h_\delta(x_0,\,f(x_0)))\right\|\le 8N_2\delta\log\frac{1}{2\delta^2}\to 0\quad\text{as }\delta\to 0.$$

The other case we have to consider is when $(z_0,\,f_c(z_0))$ is contained in an overlap where we glued our functions together. In that case we may assume that $(z_0,\,f_c(z_0))$ is also contained in $P^\delta_{j(k-1)}$ (see (2)).

Let $\vec{v}$ denote the vector $\vec{v} = \partial/\partial x(x_0,\,f_c(x_0))$. We have that

$$\begin{aligned}
\nabla h_\delta(x_0,\,f_c(x_0))\cdot\vec{v} &= \nabla[\varphi\circ\tilde{w}\cdot h^\delta_{jk}]\,(x_0,\,f_c(x_0))\cdot\vec{v} \\
&\quad + \nabla[(1-\varphi)\circ\tilde{w}\cdot h^\delta_{j(k-1)}]\,(x_0,\,f_c(x_0))\cdot\vec{v} \\
&= h^\delta_{jk}(x_0,\,f_c(x_0))\cdot\nabla[\varphi\circ\tilde{w}]\,(x,\,f_c(x_0))\cdot\vec{v} \\
&\quad + (\varphi\circ\tilde{w})\,(x_0,\,f_c(x_0))\cdot\nabla[h^\delta_{jk}]\,(x_0,\,f_c(x_0))\cdot\vec{v} \\
&\quad + h^\delta_{j(k-1)}(x_0,\,f_c(x_0))\cdot\nabla[(1-\varphi)\circ\tilde{w}]\,(x,\,f_c(x_0))\cdot\vec{v} \\
&\quad + ((1-\varphi)\circ\tilde{w})\,(x_0,\,f_c(x_0))\cdot\nabla[h^\delta_{j(k-1)}]\,(x_0,\,f_c(x_0))\cdot\vec{v}.
\end{aligned}$$

By the above calculations we need not worry about the second and fourth term in this sum so we have to check that

$$(h^\delta_{jk}(x_0,\,f_c(x_0)) - h^\delta_{j(k-1)}(x_0,\,f_c(x_0)))\cdot\nabla[\varphi\circ\tilde{w}]\,(x_0,\,f_c(x_0))\cdot\vec{v}\to 0$$

as $\delta\to 0$.

First of all we have that $|h^\delta_{jk}(x_0, f_c(x_0)) - h^\delta_{j(k-1)}(x_0, f_c(x_0))| \le 2\delta$. Further, $|\nabla[\varphi \circ \tilde{w}]\,(x_0, f_c(x_0)) \cdot \vec{v}| \le M \cdot \|D[\tilde{w}]\,(x_0, f_c(x_0))\,(\vec{v})\|$.

Now

$$D[\tilde{w}]\,(x_0, f_c(x_0))\,(\vec{v}) = \frac{\partial}{\partial x}\left[\left(x, \frac{f_c(x) - f_{c^\delta(j,k)}(x)}{f_{c^\delta(j+1,k)}(x) - f_{c^\delta(j,k)}(x)}\right)\right](x_0).$$

Ignoring the constant term (it gets killed by δ), we get that

$$\begin{aligned}
\|D[\tilde{w}]\,(x_0, f_c(x_0))\,(\vec{v})\| &\le \frac{|f'_c(x_0) - f'_{c^\delta(j,k)}(x_0)|}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|} \\
&\quad + \frac{|f_c(x_0) - f_{c^\delta(j,k)}(x_0)| \cdot |f'_{c^\delta(j+1,k)}(x_0) - f'_{c^\delta(j,k)}(x_0)|}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|^2} \\
&\le \frac{|f_c(x_0) - f_{c^\delta(j,k)}(x_0)|}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|} \log \frac{1}{|f_c(x_0) - f_{c^\delta(j,k)}(x_0)|} \\
&\quad + \frac{|f_c(x_0) - f_{c^\delta(j,k)}(x_0)| \cdot |f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|^2} \\
&\quad \times \log \frac{1}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|}.
\end{aligned}$$

By Lemma 3, $|f_c(x_0) - f_{c^\delta(j,k)}(x_0)|/|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)| \le 2$, and so

$$\begin{aligned}
\|D[\tilde{w}]\,(x_0, f_c(x_0))\,(\vec{v})\| &\le 2 \cdot \log \frac{1}{|f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)|} \\
&\quad + 2 \log \frac{1}{|f_c(x_0) - f_{c^\delta(j,k)}(x_0)|}.
\end{aligned}$$

By Lemma 5, our function is constant unless $|f_c(x_0) - f_{c^\delta(j,k)}(x_0)| \ge N_1 |f_{c^\delta(j+1,k)}(x_0) - f_{c^\delta(j,k)}(x_0)| \ge N_1 \delta^2$ (by Lemma 7), and so we may assume that

$$\|D[\tilde{w}]\,(x_0, f_c(x_0))\,(\vec{v})\| \le 2 \log \frac{1}{\delta^2} + 2 \log \frac{1}{N_1 \delta^2}.$$

All in all:

$$\begin{aligned}
&|(h^\delta_{jk}(x_0, f_c(x_0)) - h^\delta_{j(k-1)}(x_0, f_c(x_0))) \cdot \nabla[\varphi \circ \tilde{w}]\,(x_0, f_c(x_0)) \cdot \vec{v}| \\
&\le 4M\delta\left(\log \frac{1}{\delta^2} + \log \frac{1}{N_1 \delta^2}\right) \to 0 \quad \text{as } \delta \to 0. \qquad \square
\end{aligned}$$

4. *Proof of the main theorem*

We are ready to prove the main theorem. As pointed out in §2, by the theorem of Slodkowski [**9, 11**], we can assume that $\mathcal{L}$ is a lamination of $\Delta \times \mathbb{C}$ as in the previous section.

Proof of the main theorem. Suppose that T is a positive closed $(1, 1)$-current on $\Delta^2(0, 1)$, supported on the laminated set K described in the introduction. We assume that T is subordinate to the lamination $\mathcal{L}$ of K. Hence there is a positive measure μ such that

$T = \int [V_\alpha]\, d\mu(\alpha)$. Suppose that $\lambda = dw - f'_\alpha(z)\, dz$. We want to show that $\lambda \wedge T = 0$. Let ϕ be any smooth (1, 0) test form. We need to show that $\langle \lambda \wedge T, \phi\rangle = 0$. This follows since

$$\begin{aligned}\langle \lambda \wedge T, \phi\rangle &= \int (\lambda \wedge T) \wedge \phi \\ &= \int T \wedge (\lambda \wedge \phi) \\ &= \int_\alpha \left(\int_{V_\alpha} \lambda \wedge \phi \right) d\mu(\alpha) \\ &= \int_\alpha 0 = 0.\end{aligned}$$

Assume next that T is directed by $\mathcal{L}$. Since $\mathcal{L}$ is a lamination of $\Delta \times \mathbb{C}$ we may invoke the approximation result from the previous section. With the approximation result at hand the implication follows from Sullivan's proof of the smooth case [**10**]. We include the proof for the benefit of the reader.

Step 1 is to show that there exists a family of probability measures σ_α such that σ_α is supported on Γ_α, and a measure μ' on the α-plane such that for all test forms ω,

$$T(\omega) = \int \left(\int_{\Gamma_\alpha} \omega \, d\sigma_\alpha \right) d\mu'.$$

Let ω be a (1, 1) test form and let $\lambda(z, w) = dw - f'_\alpha(z)\, dz$ for $w = f_\alpha(z)$. Let $\vec{v_1}(z, w) = (1, f'_\alpha(z))$ and let $\vec{v_2}(z, w) = (i, i \cdot f'_\alpha(z))$ for $w = f_\alpha(z)$, and define the 2-tangent field $v(z, w) = (\vec{v_1}(z, w), \vec{v_2}(z, w))$.

Switching basis,

$$\omega = \psi_1\, dz \wedge d\overline{z} + \psi_2\, dz \wedge \overline{\lambda} + \psi_3\, d\overline{z} \wedge \lambda + \psi_4 \lambda \wedge \overline{\lambda},$$

for some functions ψ_i, and by assumption, $T(\omega) = T(\psi_1\, dz \wedge d\overline{z})$. The function ψ_1 is given by $\psi_1 = (1/2i)\omega(v)$, and so

$$T(\omega) = T\left(\frac{1}{2i} \omega(v)\, dz \wedge d\overline{z} \right).$$

On the other hand we may use T to define a linear functional L on $\mathcal{C}_0(\Delta \times \mathbb{C})$ by $L(\psi) = T(\psi\, dz \wedge d\overline{z})$, and so by the Riesz representation theorem there is a measure ν such that

$$L(\psi) = \int \psi \, d\nu.$$

This means that

$$T(\omega) = \int \frac{1}{2i} \omega(v)\, d\nu.$$

Now the measure ν disintegrates [**6**]: there exists a family of probability measures σ_α such that σ_α is supported on Γ_α, and a measure μ' on the α-plane such that for all $\psi \in \mathcal{C}_0(\Delta \times \mathbb{C})$,

$$\int \psi \, d\nu = \int \left(\int_{\Gamma_\alpha} \psi \, d\sigma_\alpha \right) d\mu'.$$

We define currents T_α by $T_\alpha(\omega) = \int_{\Gamma_\alpha} (1/2i)\omega(v)\, d\sigma_\alpha$, and we get that

$$T(\omega) = \int T_\alpha(\omega)\, d\mu'.$$

The next step is to show that T_α is closed for μ'-almost all α. Let $\{\omega_j\}$ be a dense set of $\mathcal{C}^1$-smooth (0, 1) test forms and fix a $j \in \mathbb{N}$. Let g be a continuous function in the α-variable and extend g constantly along leaves. We want to show that

$$\int g \cdot T_\alpha(\partial\omega)\, d\mu' = 0,$$

because this would imply that $\partial T_\alpha = 0$ for μ'-almost all α (since g is arbitrary).

By Theorem 1 there exists a sequence g_i of smooth functions such that $g_i \to g$ uniformly and in $\mathcal{C}^1$-norm on leaves. Since T is closed,

$$0 = \int T_\alpha(\partial(g\omega_j))\, d\mu' = \int T_\alpha(\partial g_i \wedge \omega_j)\, d\mu' + \int g_i \cdot T_\alpha(\partial\omega_j)\, d\mu'.$$

Since $T_\alpha(\partial g_i \wedge \omega) \to 0$ we get that

$$\int g \cdot T_\alpha(\partial\omega_j)\, d\mu' = \lim_{i\to\infty} \int g_i \cdot T_\alpha(\partial\omega_j)\, d\mu' = 0.$$

Running through all ω_j we see that T_α is closed for μ'-almost all α. The only possibility then is that the measures σ_α are constant multiples of $dz \wedge d\bar{z}$, i.e. $\sigma_\alpha = \varphi(\alpha)\, dz \wedge d\bar{z}$ where φ is a measurable function [**7**]. Define $\mu := \varphi \cdot \mu'$.

5. *Two counterexamples*

In [**5**] the authors proved versions of the main theorem for real laminations in $\mathbb{R}^2$ and $\mathbb{R}^3$. In those results we added an extra slope condition on the laminations which is analogous to the estimate in Corollary 1. We give here a simple example of a lamination of curves in $\mathbb{R}^2$ where the slope condition is not satisfied. Also, the conclusion of the main theorem fails. The analogue of Theorem 1, i.e. approximation of partially-smooth functions, fails as well.

For each $t \in \mathbb{R}$, we let γ_t be the curve $y = f_t(x) = (x - t)^3$ in $\mathbb{R}^2$. Clearly this gives a continuous lamination of $\mathbb{R}^2$ by curves. The curves are all tangent to the x-axis. This implies that the current of integration of the x-axis is annihilated by the 1-form λ defined by $dy - f_t'(x)\, dx$ on γ_t. However, this current is not an integral of currents $[\gamma_t]$. We also observe that the function $a(x, y)$ defined by $a(x, f_t(x)) = t$ cannot be approximated by $\mathcal{C}^1$ functions, because any such approximation will have to have a small derivative along the x-axis.

We can also modify this example so that we have a Riemann surface lamination in $\mathbb{C}^3$. For $t \in \mathbb{C}$, let γ_t be the complex curve $\gamma_t(s) = (z, w, \tau) = (s, (s - t)^2, (s - t)^3)$. These curves laminate $\mathbb{C}^3$, and γ_t is tangent to the z-axis at $(t, 0, 0)$. Hence the z-axis is annihilated by any continuous 1-forms defining the lamination. Hence the current of integration of the z-axis is directed. But clearly it is not subordinate to the lamination. Again the function $a(z, w, \tau)$ defined by $a|_{\gamma_t} = t$ cannot be approximated by $\mathcal{C}^1$ functions.

Acknowledgements. The first author is supported by an NSF grant. The third author is supported by Schweizerische Nationalfonds grant 200021-116165/1.

References

[1] B. Berndtsson and N. Sibony. The $\bar{\partial}$-equation on a positive closed current. *Invent. Math.* **147** (2002), 371–428.

[2] E. Bedford, M. Lyubich and J. Smillie. Polynomial diffeomorphisms of $\mathbb{C}^2$, IV. *Invent. Math.* **112** (1993), 77–125.

[3] R. Dujardin. Approximation des fonctions lisses sur certaines laminations. *Indiana Univ. Math. J.* **55** (2006), 579–592.

[4] J. E. Fornæss and N. Sibony. Harmonic currents and finite energy of laminations. *Geom. Funct. Anal.* **15** (2005), 962–1003.

[5] J. E. Fornæss, Y. Wang and E. F. Wold. Approximation of partially smooth functions. *Proceedings of Qikeng Lu Conference (June 2006). Sci. China A: Math.* **51** (4) (2008), 553–561.

[6] P. R. Halmos. *Measure Theory*. Van Nostrand, New York, 1950.

[7] P. Lelong. *Fonctions Plurisubharmoniques et Formes Différentielles Positives*. Gordon and Breach, Paris, London, New York, 1968.

[8] P. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. École Norm. Sup.* **16** (1983), 193–217.

[9] Z. Słodkowski. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.* **111** (1991), 347–355.

[10] D. Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.* **36** (1976), 225–255.

[11] K. Astala and G. J. Martin. Holomorphic motions. Papers on Analysis (A volume dedicated to Olli Martio on the occasion of his 60th birthday). *Report*, No. 83, University of Jyväskylä, Jyväskylä, 2001, pp. 27–40.