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Nash implementation with lottery mechanisms

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Abstract Consider the problem of exact Nash Implementation of social choice correspondences. Define a lottery mechanism as a mechanism in which the planner can randomize on alternatives out of equilibrium while pure alternatives are always chosen in equilibrium. When preferences over alternatives are strict, we show that Maskin monotonicity (Maskin in *Rev Econ stud* 66: 23–38, 1999) is both necessary and sufficient for a social choice correspondence to be Nash implementable. We discuss how to relax the assumption of strict preferences. Next, we examine social choice correspondences with private components. Finally, we apply our method to the issue of voluntary implementation (Jackson and Palfrey in *J Econ Theory* 98: 1–25, 2001).

1 Introduction

The goal of implementation theory is to design institutions that eliminate strategic manipulations, on the part of the agents, in order to implement desirable *social choice correspondences* (henceforth SCCs). *Maskin monotonicity* (Maskin 1999) is a necessary condition for (exact) Nash implementation. When there are at least

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three players, it is also sufficient if coupled with the assumption of *no-veto power*.¹ In economic environments with a perfectly divisible good (e.g. money), where there is typically conflict of interest, *no-veto power* is vacuously satisfied since agents will never agree on a best alternative. For more general environments, the gap between necessity and sufficiency was closed by Moore and Repullo (1990), Sjöström (1991) and Danilov (1992), among others.² The necessary and sufficient condition is complex and may be hard to interpret.

We suggest closing the gap between necessity and sufficiency by looking at a different class of mechanisms than the canonical one used in Maskin (1999). We call it the class of *lottery mechanisms*. While equilibrium messages should still deliver (pure) alternatives selected by the SCC, we allow the planner to use non-degenerate lotteries out of equilibrium. We construct an alternative version of the canonical mechanism in which agents can obtain a lottery on two alternatives for some message profiles. Unlike in Maskin (1999), preferences of agents should also be extended to preference over lotteries. A consequence is that, under some assumptions on the environment, every *Maskin monotonic* SCC is Nash implementable by a *lottery mechanism*. The theorem closes the gap between Maskin monotonicity and Nash implementability. It does not require agents to have preferences satisfying the Von-Neumann and Morgenstern axioms.

In an independent study, Benoit and Ok (2004) address the same question. The mechanism they use belongs to our class of *lottery mechanisms*. Using a domain restriction weaker than ours called top-coincidence, they show that if there are at least three agents, then any *unanimous* and *Maskin monotonic* SCC is implementable by a *lottery mechanism*.³

In the main theorem, for simplicity, we restrict our attention to linear orderings over (pure) alternatives. Next, we first discuss how we can easily relax the assumption of strict preferences. The domain restriction we introduce is *top strict difference*. It says that if an alternative is ranked top by at least $(n - 1)$ agents, then the set of top alternatives should be a singleton for at least two agents. Moreover, our result does not require that the SCC be *unanimous*.⁴ Second, we examine cases where alternatives are vectors with private components.⁴ Finally, we consider the case of voluntary implementation (Jackson and Palfrey 2001) and show that their *h-no-veto power* axiom can also be dispensed with by considering a *lottery mechanism*.

2 The set-up

There is a fixed finite set of agents $N \equiv \{1, \dots, n\}$, with $n \geq 3$, and a fixed finite set of alternatives A , with $|A| \equiv \ell \geq 2$. Let $\mathcal{L} \equiv \Delta^{\ell-1}$ be the set of lotteries over A .

¹ *No-veto power* states that if at least $n - 1$ agents agree on a best alternative at a preference profile, it should be selected by the SCC at that profile. This condition is restrictive. For example, the individually rational SCC, both in problems of indivisible goods assignment and in voting settings, fail to satisfy it.

² Danilov (1992) derives an elegant necessary and sufficient condition for Nash implementation in the case of linear orders on alternatives called *essential monotonicity*. In such an environment, his condition is equivalent to the condition of Moore and Repullo (1990) or Sjöström (1991).

³ They also discuss implementation using mechanisms with awards and implementation with a renegotiation function.

⁴ E.g., the assignment of indivisible goods without monetary transfers, matching problems, etc.

In lottery $x = (x_a)_{a \in A} \in \mathcal{L}$, alternative a occurs with probability x_a . Abusing notation, we write a both for the pure alternative $a \in A$ and for the lottery $x \in \mathcal{L}$ with $x_a = 1$. The support of a lottery $x \in \mathcal{L}$ is the set of pure alternatives receiving a strictly positive probability in x : $\text{supp}x \equiv \{a \in A \mid x_a > 0\}$.

The set of admissible preference profiles over A and over \mathcal{L} are, respectively, Θ and Γ . For any $\theta \in \Theta$, $R_i(\theta)$ stands for the (weak) preference of agent $i \in N$ over alternatives in A ; that is, $R_i(\theta)$ is a complete and transitive binary relation. We denote by $P_i(\theta)$ and $I_i(\theta)$ the associated strict and indifference binary relations, respectively. Similarly, define for any $\gamma \in \Gamma$, $R_i(\gamma)$, $P_i(\gamma)$ and $I_i(\gamma)$ in the same fashion. Given $\theta \in \Theta$, let $\Sigma(\theta) \subseteq \Gamma$ be the set of preference profiles over \mathcal{L} that agree on the ranking over A . We make two assumptions on preferences.

Strictness: For each $\theta \in \Theta$, each $a, b \in A$ and each $i \in N$, if $aR_i(\theta)b$, then $aP_i(\theta)b$ or $a = b$.

Monotonicity in probabilities: Preferences over lotteries are *monotonic in probabilities*, that is, shifts in probability to strictly preferred alternatives yield strictly preferred lotteries. For each $i \in N$, and each $k \in \{1, \dots, \ell\}$, let $p_{ik} : \Theta \rightarrow A$ be defined by,

$$a \in p_{ik}(\theta) \iff |\{b \in A \mid bR_i(\theta)a\}| = k + |\{b \in A, b \neq a \mid bI_i(\theta)a\}|.$$

That is, $p_{i1}(\theta)$ is the set of preferred alternatives of agent i at profile θ , $p_{i2}(\theta)$ his set of second preferred alternative, etc.⁵ Define $x_{p_{ik}(\theta)} \equiv \sum_{a \in p_{ik}(\theta)} x_a$. Then, if two lotteries $x \equiv (x_a)_{a \in A}$ and $y \equiv (y_a)_{a \in A}$ are such that for each $k^* \in \{1, \dots, \ell\}$, $\sum_{k \leq k^*} x_{p_{ik}(\theta)} \geq \sum_{k \leq k^*} y_{p_{ik}(\theta)}$, then $xR_i(\gamma)y$ for any $\gamma \in \Sigma(\theta)$, and whenever at least one such inequality is strict, then $xP_i(\gamma)y$ for all $\gamma \in \Sigma(\theta)$.⁶

Denote by $LC_i(\theta, a)$ the lower contour set of agent $i \in N$ at profile $\theta \in \Theta$ and alternative $a \in A$, i.e. $LC_i(\theta, a) \equiv \{b \in A : aR_i(\theta)b\}$.

For each $\theta \in \Theta$, and each $i \in N$, let the set of top alternatives be defined by,

$$TOP_i(\theta) \equiv \{a \in A : aR_i(\theta)b \text{ for each } b \in A\}.$$

A *social choice correspondence* is a mapping $f : \Theta \rightarrow A$ that associates to each preference profile a non-empty subset of alternatives.

Unanimity: A SCC f is *unanimous* if and only if for each pair $(a, \theta) \in A \times \Theta$,

$$[aR_i(\theta)b \text{ for each } i \in N \text{ and each } b \in A] \implies [a \in f(\theta)].$$

Notice that *unanimity* is implied by *no-veto power*, defined next.

No-Veto power: A SCC f satisfies *no-veto power* if and only if for each pair $(a, \theta) \in A \times \Theta$, and each $i \in N$,

$$[aR_j(\theta)b \text{ for each } j \neq i \text{ and each } b \in A] \implies [a \in f(\theta)].$$

⁵ If preferences satisfy *strictness*, then any such $p_{ik}(\theta)$ is a singleton. However, we choose this more general definition because we will relax *strictness* in Sect. 4 of the paper.

⁶ Note that this assumption implies that if for some $i \in N$ and $a, b \in A$ we have $aI_i(\theta)b$, then agent i is also indifferent between any lottery over a and b .

Maskin monotonicity: A SCC f is *Maskin monotonic* (Maskin 1999) if and only if for each pair $(\theta, \phi) \in \Theta^2$ and each $a \in f(\theta)$,

$$[LC_i(\theta, a) \subseteq LC_i(\phi, a) \text{ for each } i \in N] \implies [a \in f(\phi)].$$

For each agent $i \in N$, each $\theta \in \Theta$, and each $a \in A$, define

$$C_i(\theta, a) \equiv \left\{ \begin{array}{l} c \in LC_i(\theta, a) : \text{for all } \phi \in \Theta, \text{ if } LC_i(\theta, a) \subseteq LC_i(\phi, c) \text{ and,} \\ \text{for each } j \neq i, LC_j(\phi, c) = A, \text{ then } c \in f(\phi). \end{array} \right\}$$

Strong monotonicity: A SCC f is *strongly monotonic* if and only if for each pair $(\theta, \phi) \in \Theta^2$ and $a \in f(\theta)$,

$$[C_i(\theta, a) \subseteq LC_i(\phi, a) \text{ for each } i \in N] \implies [a \in f(\phi)].^7$$

A *deterministic mechanism* (or game form) is a pair $G = (M, g)$ where $M \equiv \prod_{i \in N} M_i$, (M_i is the message space of agent $i \in N$) and $g : M \rightarrow A$ is an outcome function that associates an alternative to every profile of messages.

A mechanism G is a *lottery mechanism* if for each $\theta \in \Theta$ and each $m \in M$, $g(m) \in \mathcal{L}$. Hence, $g : M \rightarrow \mathcal{L}$. That is, any outcome of the mechanism is a lottery—whether this lottery is degenerate or non-degenerate. Because our focus is on exact implementation of deterministic SCCs, the Nash equilibrium outcomes of the mechanism should be degenerate lotteries. Let \mathcal{G} be the class of *lottery mechanisms*.

A *game* for G is a pair (G, γ) for some $\gamma \in \Gamma$. We will restrict our attention to *pure-strategy* Nash equilibria of the game (G, γ) , denoted $NE(G, \gamma)$. A *lottery mechanism* $G = (M, g)$ is *ordinal* if the set of Nash equilibria only depends on agents' preferences over pure alternatives; that is, for each $\theta \in \Theta$, each $m \in M$ and all $\gamma, \delta \in \Sigma(\theta)$, $NE(G, \gamma) = NE(G, \delta)$. We confine our attention to *ordinal* game forms. Therefore, abusing notation, for any $\gamma \in \Sigma(\theta)$, let $NE(G, \theta)$ denote the *pure-strategy* Nash equilibria of the game (G, γ) .

A SCC f is *Nash implementable via a lottery mechanism* if there exists a mechanism $G \in \mathcal{G}$ such that the Nash equilibrium outcomes of each game coincides with the outcomes chosen by f . That is, for each $\theta \in \Theta$ and each $\gamma \in \Sigma(\theta)$, $f(\theta) = g(NE(G, \theta))$.

3 Enlarging the class of mechanisms

To understand why enlarging the class of admissible mechanisms could help dispense with the *no-veto power* assumption, it is useful to recall Maskin's Theorem 3 (Maskin 1999) and his canonical mechanism. The necessity of *Maskin monotonicity* for Nash implementation with *deterministic mechanisms* is omitted from the Theorem statement because the sufficiency part is the focus of our paper.

Theorem 1 (Maskin 1999) *If $n \geq 3$, any Maskin monotonic SCC that satisfies no-veto power is Nash implementable via a deterministic mechanism.*

⁷ When *strictness* is satisfied, the *essential monotonicity* condition of Danilov (1992) coincides with *strong monotonicity*.

Proof For each $i \in N$, let $M_i \equiv A \times \Theta \times \mathbb{N}$. A typical message is $m_i \equiv (m_i^1, m_i^2, m_i^3) \equiv (a, \theta, n)$. The outcome function is described as follows.

Rule 1 If $m_i = (a, \theta, \cdot)$ for all $i \in N$ and $a \in f(\theta)$, then $g(m) = a$.

Rule 2 If for some $i \in N$, $m_j = (a, \theta, \cdot)$ for each $j \neq i$, $a \in f(\theta)$ and $m_i = (b, \phi, \cdot)$ with $(b, \phi) \neq (a, \theta)$, then the outcome is

$$g(m) = \begin{cases} b & \text{if } b \in LC_i(\theta, a) \\ a & \text{otherwise.} \end{cases}$$

Rule 3 In all other cases, the outcome is $m_{i^*}^1$, where i^* is defined as $i^* \equiv \min \{i \in N : n_i \geq n_j \forall j \in N\}$.

The proof of Theorem 1 is well-known, but we sketch it here for the sake of completeness.

Suppose that the true state is $\theta \in \Theta$. We have different cases to consider.

1. $g(m)$ is given by *Rule 3*.

Note that any agent $j \neq i^*$ could obtain his top-ranked outcome by announcing $n_j > n_{i^*}$. If $g(m)$ is a Nash equilibrium outcome of the mechanism, then $LC_i(\theta, g(m)) = A$ for each $i \in N$. By *no-veto power*, $g(m) \in f(\theta)$.

2. $g(m)$ is given by *Rule 2*.

Note that any agent $j \neq i$ could deviate, trigger the integer game and obtain his top-ranked outcome, say a , by announcing $n_j > n_k$ for all $k \neq j$. If $g(m)$ is a Nash equilibrium outcome, then $LC_j(\theta, g(m)) = A$ for each $j \neq i$. By *no-veto power*, $g(m) \in f(\theta)$.

3. $g(m)$ is given by *Rule 1*.

If θ is announced truthfully, then $g(m) \in f(\theta)$. So, suppose instead that $m_i = (a, \phi, \cdot)$ and $a \in f(\phi)$. Any agent i could deviate and obtain any alternative b such that $b \in LC_i(\phi, a)$. If $g(m)$ is a Nash equilibrium outcome, then for each $i \in N$, $aR_i(\phi)b \implies aR_i(\theta)b$. By *Maskin monotonicity*, $a \in f(\theta)$.

To conclude the proof, suppose the true state is $\theta \in \Theta$ and fix $a \in f(\theta)$. We show that each agent $i \in N$ reporting $m_i = (a, \theta, 0)$ forms a Nash equilibrium of the mechanism with respect to θ . Given $m = (m_i)_{i \in N}$, *Rule 1* applies and the outcome is $a \in f(\theta)$. Any agent $j \in N$ can trigger rule 2 by announcing $m_j = (b, \phi, \cdot)$ and obtain $b \in LC_j(\theta, a)$. By the definition of the lower contour set, such a deviation cannot be profitable. Therefore, m is a Nash equilibrium. \square

Note that in the proof, *no-veto power* is in fact used only to rule out undesirable equilibria in *Rule 2*. In *Rule 3*, only *unanimity*—obviously implied by *no-veto power*—is needed. What happens if *no-veto power* is not satisfied?⁸ Suppose that the true state is θ and that messages reported are for each $j \neq i$, $m_j = (a, \phi, \cdot)$ with $a \in f(\phi)$, and $m_i = (c, \theta, \cdot)$ with $c \in LC_i(\phi, a)$. The outcome is $g(m) = c$ given by *Rule 2*. If *no-veto power* is not satisfied and $(c, \theta) \neq (a, \phi)$, it could be the

⁸ A similar discussion can be found in the excellent survey of Maskin and Sjöström (2002, Sect. 3.3). We follow here their terminology.

case that for each $j \neq i$, $LC_j(\theta, c) = A$, $LC_i(\phi, a) \subseteq LC_i(\theta, c)$ and $c \notin f(\theta)$. In such a case, no profitable deviations from m are possible: any agent $j \neq i$ could trigger the integer game but c is top-ranked for any such agent. Furthermore, by changing his message, agent i can only obtain an outcome in $LC_i(\phi, a)$, which can be no better than c according to $R_i(\theta)$. Thus, message m is a Nash equilibrium. Using the terminology of Maskin and Sjöström (2002), c is an *awkward* outcome for agent i in $LC_i(\phi, a)$. In the canonical mechanism, given message $m \in M$ with $m_i = (a, \theta, \cdot)$ for all $i \in N$ and $a \in f(\theta)$, the attainable set from *Rule 1* is $LC_i(\theta, a)$ for each $i \in N$. When f violates *no-veto power*, the planner should restrict the attainable sets by removing *awkward* alternatives. For all $i \in N$, the (personalized) attainable sets $C_i(\theta, a)$, as defined in Sect. 3, should be constructed. In *Rule 2*, for each $\theta \in \Theta$ and each $a \in f(\theta)$, one should replace $LC_i(\theta, a)$ by $C_i(\theta, a)$ for each $i \in N$. In the environment considered by Maskin, a necessary and sufficient condition for Nash implementation via a *deterministic mechanism* is *strong monotonicity*.⁹ Indeed, *strong monotonicity* implies *Maskin monotonicity* but the converse is not true. However, *Maskin monotonicity* together with *no-veto power* imply *strong monotonicity*.

By considering a larger class of mechanisms that we call *lottery mechanisms*, each agent i 's attainable set need only be $LC_i(\theta, a)$. The planner does not need to construct personalized attainable sets by removing every *awkward* outcome.

To illustrate our approach, we consider an important example from Maskin (1985, 1999) of a SCC that is *Maskin monotonic*, but does not satisfy *no-veto power*. We use it here to show that this particular SCC is implementable in Nash equilibrium if the planner uses a *lottery mechanism*. Thus, checking whether the SCC satisfies *Maskin monotonicity* is enough to know if it is Nash implementable.

Example 1 (Maskin 1985, 1999):

$N \equiv \{1, 2, 3\}$, $\Theta \equiv \{\theta, \phi\}$ and $A \equiv \{a, b, c\}$. The preferences are described below.

	θ	ϕ
	1 2 3	1 2 3
b	a	b
a	c	c
c	b	a

The SCC is described as follows. Given $\theta' \in \Theta$, (i) $a \in f(\theta')$ if and only if a majority prefers a to b ; (ii) $b \in f(\theta')$ if and only if a majority prefers b to a ; and (iii) $c \in f(\theta')$ if and only if $LC_i(\theta', c) = A$ for each $i \in N$. This SCC is *Maskin monotonic* but does not satisfy *no-veto power*. It is thus not Nash implementable by Maskin's mechanism. In this example, $f(\theta) = \{a\} = f(\phi)$. Observe that $LC_1(\theta, a) = LC_1(\phi, c) = \{a, c\}$ and $LC_i(\phi, c) = A$ for each $i \in N \setminus \{1\}$. Hence, alternative c is an *awkward* outcome in $LC_1(\theta, a)$. In Maskin's mechanism, if the true state is ϕ , then the profile of messages $m \in M$, with $m_j = (a, \theta, \cdot)$ for $j \neq 1$ and $m_1 = (c, \phi, \cdot)$ is a Nash equilibrium. *Rule 2* prescribes the outcome $c \notin f(\phi)$.

Clearly, what needs to be modified is *Rule 2*. Instead of the outcome being c , suppose *Rule 2* gives a lottery $(1 - \varepsilon)a + \varepsilon c$, with $\varepsilon \in (0, 1)$. By *monotonicity*

⁹ See Maskin and Sjöström (2002) for a more detailed discussion of *strong monotonicity*.

in probabilities, when the true state is ϕ , agents 2 and 3 would rather get c with probability one than a lottery on a and c . Therefore, when the true state is ϕ , the profile $m \in M$, with $m_j = (a, \theta, \cdot)$ for $j \neq 1$ and $m_1 = (c, \phi, \cdot)$ is no longer a Nash equilibrium.

We have just informally shown that we can, on the one hand, dispense with *no-veto power*, and on the other hand, that we do not need to construct restricted attainable sets. In fact, when *strictness* is satisfied, we only need to check whether a SCC is *Maskin monotonic* to know if it is Nash implementable. Having possibly non-degenerate lotteries in *Rule 2* eliminates the need for *no-veto power*. Moreover, in the necessary and sufficient condition for Nash implementation, only *Maskin monotonicity* will have bite.¹⁰

Before stating the main result, it may be worth mentioning that the next theorem can be understood using the necessary and sufficient conditions for Nash implementation applied to the feasible set \mathcal{L} . Notice that no completely mixed lottery in the probability simplex can be *awkward* because $(n - 1)$ players cannot have a top alternative there. The only difference between the attainable sets being LC_i or C_i is only on the boundary of the simplex. But then continuity of preferences implies the equivalence between *strong monotonicity* and *Maskin monotonicity*. We are now ready to state our result.

Theorem 2 *If $n \geq 3$ and preferences satisfy strictness and monotonicity in probabilities, any SCC is implementable in Nash equilibrium by a lottery mechanism if and only if it is Maskin monotonic.*¹¹

Proof We first prove the necessity part of the theorem. If f is implementable in Nash equilibrium by a *lottery mechanism*, there exists a mechanism (M, g) that implements it. Consider $\theta \in \Theta$ and $a \in f(\theta)$. Since f is implemented, there is $m \in NE(G, \theta)$ such that $g(m) = a$. Hence, for each $i \in N$, $a R_i(\gamma) g(m'_i, m_{-i})$ for all $m'_i \in M_i$ and for all $\gamma \in \Sigma(\theta)$. Because the equilibrium condition should hold for all $\gamma \in \Sigma(\theta)$, it follows that for each $i \in N$, for each $m'_i \neq m_i$, $a R_i(\theta) b$ for all $b \in \text{supp}(m'_i, m_{-i})$. Suppose there exists $\phi \in \Theta$ with $a \notin f(\phi)$. Implementation of f requires $m \notin NE(G, \phi)$. Hence, there is $i \in N$ and $m'_i \in M_i$ such that $g(m'_i, m_{-i}) P_i(\gamma) a$ for some $\gamma \in \Sigma(\phi)$. By *monotonicity in probabilities*, there exists at least one $b \in \text{supp}(m'_i, m_{-i})$ such that $b P_i(\phi) a$. Therefore, f is *Maskin monotonic*.

We now prove the sufficiency part. We construct the following (*ordinal*) *lottery mechanism*.¹² The message space of each agent $i \in N$ is $M_i \equiv A \times \Theta \times A \times \mathbb{N}$.

¹⁰ It is clear from the proof of Theorem 1 that *unanimity* is needed in *Rule 3* of Maskin's mechanism. In Theorem 2, we show that *unanimity* can also be dropped by modifying *Rule 3* of Maskin's mechanism.

¹¹ Alternatively, we could also drop the restriction to *ordinal* game forms. By dropping this assumption, one can look at a richer set of SCCs that uses cardinal information; i.e. $f: \Gamma \rightarrow \mathcal{L}$. In that case, any cardinal SCC is implementable in Nash equilibrium by a *lottery mechanism* if and only if it is *Maskin monotonic* (in the simplex).

¹² The mechanism uses an integer game. This has been the object of many criticisms—see for instance Jackson (1991). The device in *Rule 3* works because the set of profitable deviations is open. If such constructions are to be avoided, then the theorem may hold only for *unanimous* SCCs. Moreover, Jackson (1991) shows that restricting the class of mechanisms may have a severe impact on the class of implementable SCCs. There is no presumption that the sufficiency result here or in Maskin (1999) can be replicated using “well-behaved” mechanisms.

A typical message is $m_i \equiv (m_i^1, m_i^2, m_i^3, m_i^4) \equiv (x, \theta, a, n_i)$. Fix a number $\varepsilon \in (0, 1)$.

Rule 1 If $m_i = (\bar{x}, \bar{\theta}, \bar{a}, \cdot)$ for all $i \in N$ and $\bar{x} \in f(\bar{\theta})$, then $g(m) = \bar{x}$.

Rule 2 If $m_j = (\bar{x}, \bar{\theta}, \bar{a}, \cdot)$ for each $j \neq i$, $\bar{x} \in f(\bar{\theta})$ and $m_i = (c, \phi, b, \cdot)$ with $(c, \phi, b) \neq (\bar{x}, \bar{\theta}, \bar{a})$, then the outcome is

$$g(m) = \begin{cases} (1 - \varepsilon)\bar{x} + \varepsilon b & \text{if } b \in LC_i(\bar{\theta}, \bar{x}) \\ \bar{x} & \text{otherwise} \end{cases}$$

Rule 3 In all other cases, the outcome is $(1 - \frac{1}{1+n_{i^*}})a + \frac{1}{1+n_{i^*}}b$ if $m_{i^*} = (a, \cdot, b, n_{i^*})$, where $i^* = \min \{i \in N : n_i \geq n_j \forall j \in N\}$, and $a \neq b$. Otherwise if $a = b$, the outcome is the lottery that assigns equal weights on all the alternatives in A .

We show that this game form Nash implements any SCC f that is *Maskin monotonic*.

First, suppose that the true profile is θ and that $x \in f(\theta)$. The message profile $m \in M$ with $m_i = (x, \theta, x, 0)$ for each $i \in N$ is a Nash Equilibrium of G . By unilaterally deviating, an agent $i \in N$ can only trigger *Rule 2* and obtain either x with probability 1 or a lottery on x and another alternative $b \in LC_i(\theta, x)$. In that case, the deviation decreases the probability of $x \in f(\theta)$ and increases the probability of a worse alternative $b \in A$. By *monotonicity in probabilities*, this deviation is not profitable. Therefore, m is a Nash equilibrium.

Second, suppose the true state is $\theta \in \Theta$. We now show that for each $\gamma \in \Sigma(\theta)$, $g(NE(G, \theta)) \subseteq f(\theta)$. We have three cases to consider.

1. $g(m)$ is given by *Rule 3*.

We show that there is no equilibrium. If $a \neq b$, the outcome is $g(m) = (1 - \frac{1}{1+n_{i^*}})a + \frac{1}{1+n_{i^*}}b$. By *strictness* and *monotonicity in probabilities*, even if agents agree on the ranking of a and b —with a being top-ranked—any agent $i \in N$ could announce $m'_i = (a, \cdot, b, n'_i)$ with $n'_i > n_{i^*}$ in order to give more weight to a and relatively less weight to b . Next, if the outcome $g(m)$ is the uniform distribution over alternatives, again by *strictness* and *monotonicity in probabilities*, any agent $i \in N$ would deviate to $m'_i = (a, \cdot, b, n'_i)$, with $a \neq b$, $n'_i > n_{i^*}$, $aP_i(\theta)b$ and $bP_i(\theta)c$ for each $c \neq a$, in order to obtain a lottery restricted to his two best alternatives.

2. $g(m)$ is given by *Rule 2*

If $g(m)$ is obtained by *Rule 2*, note that any agent $j \neq i$ could deviate by announcing m'_j with $(m_j^1, m_j^2, m_j^3) \neq (\bar{x}, \bar{\theta}, \bar{a})$ and $n'_j > n_k$ for all $k \neq j$, trigger *Rule 3* and obtain a lottery on his two top alternatives under θ .

We have two cases to consider. In the first case, $g(m) \in \mathcal{L}$ and $\text{supp}(g(m))$, is not a singleton. Because preferences are strict, for each agent, any non-degenerate lottery is dominated by a lottery on the two top alternatives with sufficiently high weight placed on the top alternative. Under *Rule 3*, for any $j \neq i$ this is obtained by reporting a sufficiently large n'_j . Hence, $g(m)$ is not a Nash equilibrium outcome.

In the second case, $g(m) = \bar{x}$ is a degenerate lottery. Suppose that \bar{x} is a Nash equilibrium outcome for θ chosen by *Rule 2*. Agent i can profitably deviate by reporting $(c', \phi', b', \cdot) \neq (\bar{x}, \bar{\theta}, \bar{a})$ if $b' \in LC_i(\bar{\theta}, \bar{x})$ and $((1 - \epsilon)\bar{x} + \epsilon b') P_i(\gamma)\bar{x}$ for some $\gamma \in \Sigma(\theta)$. Because preferences over lotteries are monotone in probabilities, such a profitable deviation does not exist if $\bar{x} R_i(\theta)b'$ for all $b' \in LC_i(\bar{\theta}, \bar{x})$. Thus, $LC_i(\bar{\theta}, \bar{x}) \subseteq LC_i(\theta, \bar{x})$. Moreover, if no agent $j \neq i$ has a profitable deviation, then $LC_j(\theta, \bar{x}) = A$. Thus, $LC_j(\bar{\theta}, \bar{x}) \subseteq LC_j(\theta, \bar{x})$ for all $j \neq i$. By *Maskin monotonicity*, $\bar{x} \in f(\theta)$.

3. $g(m)$ is given by *Rule 1*.

If θ is announced truthfully, then the outcome lies in $f(\theta)$ by *Rule 1*. Suppose instead that each $i \in N$ announces $m_i = (x, \phi, a, \cdot)$ with $\phi \neq \theta$ and $x \in f(\phi)$. To complete the proof, it is sufficient to show that $LC_i(\phi, x) \subseteq LC_i(\theta, x)$ for all $i \in N$ because it would then follow from *Maskin monotonicity* that $x \in f(\theta)$.

On the contrary, suppose that there exists an $i \in N$ and $b \in LC_i(\phi, x)$ such that $b P_i(\theta)x$. By deviating, agent i can obtain $(1 - \epsilon)x + \epsilon b$. In order for this deviation not to be profitable, it must be the case that $x R_i(\gamma)((1 - \epsilon)x + \epsilon b)$ for all $\gamma \in \Sigma(\theta)$. By *monotonicity in probabilities*, this contradicts the assumption that $b P_i(\theta)x$. Thus, $x \in f(\theta)$. \square

Remark Given the results on virtual implementation in Abreu and Sen (1990) or Abreu and Matsushima (1992), one could hope that the use of lotteries would relax the necessary condition for Nash implementation. Theorem 2 shows that enlarging the range of the mechanism from A to \mathcal{L} closes the gap between *Maskin monotonicity* and *Nash implementability*, provided that preferences satisfy *strictness* and *monotonicity in probabilities*. Since the focus is not on approximate but on exact Nash implementation, *Maskin monotonicity* remains necessary.

Abreu and Matsushima (1994) show that any stochastic social choice function can be exactly implemented in iteratively weakly undominated strategies. However, the existence of small fines that can be levied on players and their assumption that agents have a finite set of preferences satisfying the Von-Neumann Morgenstern axioms are crucial to their result.

4 Discussion

4.1 Relaxing strictness of preferences

If *strictness* is not satisfied, the mechanism used to establish Theorem 2 may generate bad equilibria. For instance, suppose that in *Rule 3*, the outcome is $\left(1 - \frac{1}{1+n_i^*}\right)a + \frac{1}{1+n_i^*}b$ and alternatives a and b are both top-ranked for every agent. Since f is deterministic, $\left(1 - \frac{1}{n_i^*}\right)a + \left(\frac{1}{n_i^*}\right)b$ would be an undesirable equilibrium outcome. The same thing happens if the outcome is $(1 - \epsilon)\bar{x} + \epsilon b$ determined by *Rule 2*.

In order to circumvent this problem, we need an additional restriction on the environment. One such restriction is the following.

Top strict difference: For each $\theta \in \Theta$, and for each $a \in A$ such that $LC_i(\theta, a) = A$ for at least $(n - 1)$ agents $i \in N$, there exist $j, k \in N$ for whom $TOP_j(\theta) = TOP_k(\theta) = \{a\}$.

This rules out the undesirable equilibrium outcomes in *Rule 2* and *Rule 3* of the mechanism used in Theorem 2. To illustrate why this is so, suppose that $\theta \in \Theta$ is the true state, and that $(1 - \epsilon)x + \epsilon b$ is the outcome of the game following the report of the message profile $m \in M$. If $LC_j(\theta, b) = A$ for each j but possibly $i \in N$, *top strict difference* guarantees that for at least one such j , $TOP_j(\theta) = \{b\}$, and therefore guarantees that a profitable deviation exists: agent j can trigger the integer game by announcing the highest integer and obtain a preferred lottery. Moreover, if an outcome is determined by *Rule 3*, again *top strict difference* rules it out as an equilibrium outcome: there, it is enough to have one agent for whom the set of top alternatives is a singleton.

By assuming *top strict difference*, we can establish the following proposition.

Proposition 1 *If $n \geq 3$ and preferences satisfy top strict difference and monotonicity in probabilities, any SCC is Nash implementable if and only if it is Maskin monotonic.*

4.2 SCCs with private components

SCCs with private components violating *no-veto power* include—among others—the core correspondence, the individually rational correspondence (henceforth the IR correspondence) in the assignment of indivisible goods and the stable rule in matching problems. For all these rules, Theorem 1 does not allow one to check whether or not they are Nash implementable.

The message of Theorem 2 and the discussion that followed is that, given some mild domain restrictions, any *Maskin monotonic* SCC is Nash implementable by a *lottery mechanism*. When *strictness* is satisfied, this result extends to environments where alternatives are vectors with private components.¹³ However, when indifferences are introduced, it is important to note that the *lottery mechanism* constructed in Theorem 2 will not work. First, *top strict difference* needs to be redefined. From now on, for each $a \in A$, let $a = (a_1, \dots, a_n)$.

Top strict difference for private components: For each $\theta \in \Theta$, and each $a \in A$ such that $a_i \in TOP_i(\theta)$ for at least $(n - 1)$ agents, there exists $j, k \in N$ for whom $TOP_j(\theta) = \{a' : a'_j = a_j\}$ and $TOP_k(\theta) = \{a' : a'_k = a_k\}$.

If *top strict difference for private components* is satisfied and preferences satisfy *monotonicity in probabilities*, then when there are private components, *Maskin monotonic* SCCs are not implementable by the *lottery mechanism* we have constructed in Theorem 2. To see this, let us look at an example for the assignment of indivisible goods without monetary transfers.

Example 2 Assignment of indivisible objects and the individually rational rule.

¹³ Notice that in *Rule 2* the requirement $b \in LC_i(\theta, a)$ should be read as $b_i \in LC_i(\theta, a_i)$. In *Rule 3*, the requirement $a \neq b$ should be read as $a_{i^*} \neq b_{i^*}$.

$N \equiv \{1, 2, 3, 4\}$. There are four objects a_1, a_2, a_3 and a_4 . The initial endowment ω_i of each agent $i \in N$ is the object a_i . An allocation is an assignment of indivisible objects, one to each agent. Formally, an assignment is a bijection $\sigma : N \rightarrow A$. Let Z be the set of assignments. The IR rule is a correspondence such that for each $\theta \in \Theta$, $f(\theta) = \{\sigma \in Z : \sigma_i R_i(\theta)\omega_i \text{ for each } i \in N\}$. Consider the following preferences over objects.

θ				ϕ			
1	2	3	4	1	2	3	4
a_1	a_2	$a_3 a_4$	$a_3 a_4$	a_1	a_2	a_1	a_3
a_2	a_1	a_1	a_2	a_2	a_1	$a_3 a_2$	a_4
a_3	a_3	a_2	a_1	a_4	a_3	a_4	a_1
a_4	a_4			a_3	a_4		a_3

The selection operated by f is $f(\theta) = \{(a_1, a_2, a_3, a_4), (a_1, a_2, a_4, a_3)\}$, and $f(\phi) = \{(a_1, a_2, a_3, a_4)\}$. Notice that f violates *no-veto power* because the allocation (a_1, a_2, a_4, a_3) is excluded when the state is ϕ . Moreover, f is not Nash implementable by a version of the mechanism in Theorem 2 even though *top strict difference for private components* is satisfied. Let $a = (a_1, a_2, a_3, a_4)$ and $a' = (a_1, a_2, a_4, a_3)$. Consider the case where the true state is θ and the message m with $m_j = (a, \phi, a', \cdot)$ for $j = 2, 3, 4$ and $m_1 = (\cdot, \theta, a', \cdot)$ has been reported. The outcome is $g(m) = (1 - \epsilon)a + \epsilon a'$. By monotonicity in probabilities, individuals 3 and 4 are indifferent between $g(m)$ and both a and a' . Thus, $g(m)$ is a Nash equilibrium outcome under θ , in contradiction with the Nash implementation of f . The problem does not come from the violation of *no-veto power* but merely from the fact that *top strict difference* is vacuous. Agents 1 and 2 meet the restriction imposed by *top strict difference for private components* but they receive the same object in both allocations. Observe that it is crucial that the number of agents be at least four. With three agents, this problem is never encountered.

As a consequence, the mechanism has to be further complicated. For instance, in Rule 2, if $g(m) \in \mathcal{L}$, then among the $(n - 1)$ agents $j \neq i$, it should be the case that at least one is not indifferent between both components he receives.¹⁴

Rule 2.1 If $m_j = (\bar{x}, \bar{\theta}, \bar{a}, \cdot)$ for each $j \neq i$, $\bar{x} \in f(\bar{\theta})$ and $m_i = (c, \phi, b, \cdot)$ with $(c, \phi, b) \neq (\bar{x}, \bar{\theta}, \bar{a})$, then the outcome is $g(m) = (1 - \epsilon)\bar{x} + \epsilon b$ if

- 1) $b_i \neq \bar{x}_i, b_i \in LC_i(\bar{\theta}, \bar{x}_i)$ and
- 2) there is $j \neq i$ such that $\{\bar{x}_j, b_j\} \not\subseteq TOP_j(\phi)$

Otherwise, $g(m) = \bar{x}$.

Using this modified form of the canonical mechanism, we can establish the following proposition.

¹⁴ Rule 3 has to be modified in a similar fashion. One such modification is the following:
Rule 3.1 In all other cases, the outcome is $\left(1 - \frac{1}{1+n_i^*}\right)a + \frac{1}{1+n_i^*}b$ if $m_{i^*} = (a, \cdot, b, n_{i^*})$, where $i^* = \min \{i \in N : n_i \geq n_j \forall j \in N\}$, $a_{i^*} \neq b_{i^*}$ and $a_j \neq b_j$ for each $j \neq i^*$. Otherwise, the outcome is the lottery that assigns equal weights on all the alternatives in A .

Proposition 2 *If $n \geq 3$ and preferences satisfy top strict difference for private components and monotonicity in probabilities, any SCC with private components is Nash implementable by a lottery mechanism if and only if it is Maskin monotonic.*

Once specific forms of indifferences are allowed, *lottery mechanisms* become more complex. In *Rule 2.1*, the attainable set of agent i is restricted because some alternatives may be removed from $LC_i(\theta, x_i)$. Because of *top strict difference for private components*, this restriction imposes no constraints on deviations.

5 Application: voluntary implementation

Our approach is not restricted only to Maskin's Theorem. In a recent article, Jackson and Palfrey (2001) consider voluntary implementation. The problem related to the enforceability of the outcome function out of equilibrium is studied. Agents are not forced to accept the outcome of the mechanism, and can veto some subset of the set of alternatives. For instance, a state-contingent participation constraint defines a mapping from outcomes vetoed by agents into individually rational outcomes.

First, we need to introduce some additional definitions.¹⁵ Let \mathcal{F} be the set of all social choice functions over A . A reversion function is a mapping $h : \Theta \rightarrow A$ that indicates what the outcome is in the case of a veto by at least one individual. A reversion function h induces a mapping $H : A \times \Theta \times \mathcal{F}$ by

$$H(a, \theta, h) = \begin{cases} a & \text{if } a R_i(\theta) h(\theta) \text{ for each } i \in N \\ h(\theta) & \text{otherwise.} \end{cases}$$

Given a game form (M, g) , a message profile m is an h -Nash equilibrium of (M, g) at θ if for each agent $i \in N$,

$$H(g(m), \theta, h) R_i(\theta) H(g(m'_i, m_{-i}), \theta, h) \quad \forall m'_i \in M_i.$$

A SCC f is h -Nash implementable if there exists a mechanism (M, g) such that, for all $\theta \in \Theta$:

1. For each $a \in f(\theta)$, there exists an h -Nash equilibrium $m \in M$ such that $H(g(m), \theta, h) = a$.
2. If $m \in M$ is an h -Nash equilibrium at θ , then $H(g(m), \theta, h) \in f(\theta)$.

An analog to *Maskin monotonicity* is derived. A SCC f is *reversion-monotonic relative to h* if for each $\theta \in \Theta$ and each $a \in f(\theta)$, there exists $z \in A$ such that

1. $H(z, \theta, h) = a$.
2. For all $\phi \in \Theta$ such that $H(z, \phi, h) \notin f(\phi)$, there exists $y \in A$ and $i \in N$ such that $H(z, \theta, h) R_i(\theta) H(y, \theta, h)$ and $H(y, \phi, h) P_i(\phi) H(z, \phi, h)$.

Reversion monotonicity is indeed necessary for h -Nash implementation. Coupled with *h-no-veto power*, defined next, it is also sufficient provided that $n \geq 3$.

A SCC f satisfies *h-no-veto power* if for each $(a, \theta) \in A \times \Theta$, and each $i \in N$,

$$\begin{aligned} & [H(a, \theta, h) R_j(\theta) H(b, \theta, h) \text{ for each } j \neq i \text{ and each } b \in A] \\ & \implies [H(a, \theta, h) \in f(\theta)]. \end{aligned}$$

¹⁵ We follow the notation introduced by Jackson and Palfrey (2001).

A SCC f satisfies h -unanimity if for each $(a, \theta) \in A \times \Theta$,

$$[H(a, \theta, h)R_i(\theta)H(b, \theta, h) \text{ for each } i \in N \text{ and each } b \in A] \\ \implies [H(a, \theta, h) \in f(\theta)].$$

Again, h -no-veto power is not necessary for h -Nash implementation. An interesting feature of the voluntary implementation approach is that it is possible to construct *non-Maskin monotonic* SCCs that are, given h , *reversion-monotonic relative to h* . We construct an example of such a SCC that also violates h -no-veto power. It is a variant of an example of Jackson and Palfrey (2001).¹⁶ The example is as follows:

Example 3 $N \equiv \{1, 2, 3\}$, $A \equiv \{a, b, c, d\}$ and $\Theta \equiv \{\theta, \phi\}$. The status-quo is c . The reversion function is constant across states and equal to the status-quo, that is $h(\theta) = c$ for each $\theta \in \Theta$. The function H maps outcomes that are not individually rational to the status-quo. Hence, we want to perform *IR-Nash* implementation. The preferences over alternatives are described as follows.

θ			ϕ		
1	2	3	1	2	3
a	b	d	d	b	d
d	d	a	a	d	a
b	a	c	b	a	b
c	c	b	c	c	c

Consider the *non-Maskin monotonic* SCC f , with $f(\theta) = \{a, d\}$ and $f(\phi) = \{a\}$. This SCC is *reversion-monotonic relative to h* . We show that agent 2 experiences a *preference reversal relative to h* when going from θ to ϕ . Since alternative d is individually rational for every agent, $H(d, \theta, h) = H(d, \phi, h) = d$. However, for alternative b , by $cP_3(\theta)b$, we obtain that $H(b, \theta, h) = c$. Moreover, $H(b, \phi, h) = b$. Thus,

$$H(d, \theta, h)R_2(\theta)H(b, \theta, h) \text{ and } H(b, \phi, h)P_2(\phi)H(d, \phi, h);$$

or, equivalently,

$$dR_2(\theta)c \text{ and } bP_2(\phi)d.$$

Finally, it is easy to see that f does not satisfy h -no-veto power. For agents 1 and 3, $LC_1(\phi, H(d, \phi, h)) = LC_3(\phi, H(d, \phi, h)) = A$ but $H(d, \phi, h) = d \notin f(\phi)$.

We can state the following theorem.

Theorem 3 *If $n \geq 3$ and preferences satisfy strictness and monotonicity in probabilities, any SCC satisfying h -unanimity is h -implementable by a lottery mechanism if and only if it is reversion monotonic relative to h .*

¹⁶ The SCC in their example satisfies h -no-veto power.

Proof The necessity part is omitted and can be adapted from Jackson and Palfrey (2001).

For the sufficiency part, we construct the following mechanism. The message space of each agent $i \in N$ is $M_i \equiv A \times \Theta \times A \times \mathbb{N}$. A typical message will be denoted $m_i \equiv (m_i^1, m_i^2, m_i^3, m_i^4) \equiv (x, \theta, a, n_i)$. Fix a number $\varepsilon \in (0, 1)$. The outcome function is described as follows:

Rule 1 If $m_i = (\bar{x}, \bar{\theta}, \bar{a}, \cdot)$ for all $i \in N$, $H(\bar{x}, \bar{\theta}, h) \in f(\bar{\theta})$ and \bar{x} satisfies (2) in the definition of *reversion monotonicity*, then $g(m) = \bar{x}$.

Rule 2 If $m_j = (\bar{x}, \bar{\theta}, \bar{a}, \cdot)$ for each $j \neq i$, $H(\bar{x}, \bar{\theta}, h) \in f(\bar{\theta})$, \bar{x} satisfies (2) in the definition of *reversion monotonicity* and $m_i = (c, \phi, b, \cdot)$ with $(c, \phi, b) \neq (\bar{x}, \bar{\theta}, \bar{a})$, then the outcome is

$$g(m) = \begin{cases} (1 - \varepsilon)\bar{x} + \varepsilon b & \text{if } H(b, \bar{\theta}, h) \in LC_i(\bar{\theta}, H(\bar{x}, \bar{\theta}, h)) \\ \bar{x} & \text{otherwise.} \end{cases}$$

Rule 3 In all other cases, the outcome is a determined by $m_{i^*} = (a, \cdot, \cdot, n_{i^*})$, where $i^* = \min \{i \in N : n_i \geq n_j \forall j \in N\}$.

The proof is similar to the proof of Theorem 2 and is therefore omitted. \square

The assumption of strict preferences can be relaxed, as before. If the SCC is on alternatives with private components, the discussion in Section 4.2 applies.

6 Concluding remarks

1. Extending the class of admissible mechanisms is useful for (exact) Nash implementation. Theorem 2 closes the gap between *Maskin monotonicity* and Nash implementability provided that preferences are strict and satisfy monotonicity in probabilities. Moreover, no restriction on the attainable sets has to be constructed.
2. Theorem 2 extends to environments where *strictness* is relaxed to *top strict difference*. Nevertheless, for environments with private consumption sets or in matching problems, *top strict difference for private components* is vacuous. In such a case, restricted attainable sets may have to be constructed for each agent. But as seen in Proposition 2, this imposes no further constraints.
3. *Lottery mechanisms* can also be useful for alternative implementation approaches. Vartiainen (2003) shows that if preferences are strict, one can randomize out of equilibrium and drop the assumption of *no-veto power* from the theorem of Abreu and Sen (1990).¹⁷ We conjecture that similar results could be obtained in incomplete information settings, for instance with Bayesian implementation.¹⁸
4. A possible extension of this work would be to design simple lottery mechanisms to implement SCCs violating *no-veto power*.

¹⁷ I thank an anonymous referee for pointing out this paper.

¹⁸ See, for instance, Jackson (1991) for a discussion of Bayesian implementation.

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