Electron. Commun. Probab. **18** (2013), no. 67, 1–10.
DOI: 10.1214/ECP.v18-2837
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Continuum percolation for Gibbs point processes*

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Abstract

We consider percolation properties of the Boolean model generated by a Gibbs point process and balls with deterministic radius. We show that for a large class of Gibbs point processes there exists a critical activity, such that percolation occurs a.s. above criticality.

Keywords: Gibbs point process; Percolation; Boolean model; Conditional intensity.
AMS MSC 2010: 60G55; 60K35.
Submitted to ECP on May 29, 2013, final version accepted on August 1, 2013.
Supersedes arXiv:1305.0492v1.

1 Introduction

Let Ξ be a point process in $\mathbb{R}^d$, $d \geq 2$, and fix a $R > 0$. Consider the random set $Z_R(\Xi) = \bigcup_{x\in\Xi} \mathbb{B}_R(x)$, where $\mathbb{B}_R(x)$ denotes the open ball with radius R around x. Each connected component of $Z_R(\Xi)$ is called a *cluster*. We say that Ξ *percolates* (or R-*percolates*) if $Z_R(\Xi)$ contains with positive probability an infinite cluster. In the terminology of [9] this is a Boolean percolation model driven by Ξ and deterministic radius distribution R.

It is well-known that for Poisson processes there exists a critical intensity β_c such that a Poisson processes with intensity $\beta > \beta_c$ percolates a.s. and if $\beta < \beta_c$ there is a.s. no percolation, see e.g. [17], or [14] for a more general Poisson percolation model.

For Gibbs point processes the situation is less clear. In [13] it is shown that for some two-dimensional pairwise interacting models with an attractive tail, percolation occurs if the activity parameter is large enough, see also [1] for a similar result concerning the Strauss hard core process in two dimensions. There is a related work [7], which states conditions on the intensity instead of the activity. Furthermore in [11, 16] it is shown that for finite-range pairwise interacting model there is no percolation at low activity.

The aim of this work is to extend the results of [13, 1] to any dimension $d \geq 2$ and to very general Gibbs point processes. We give a percolation condition on the conditional intensity of a Gibbs process, which is easily understandable and which is satisfied for most Gibbs processes, provided the percolation radius R is large enough. Our main result on percolation, Theorem 3.1 then states, that there exists a critical activity, such that the Gibbs point processes percolate a.s. above criticality. The main idea of the proof is the contour method from statistical physics.

Furthermore we state a result on the absence of percolation for locally stable Gibbs processes. Here the idea of the proof is that a locally stable Gibbs process can be

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dominated by a Poisson process, and then we use the percolation results available for Poisson processes.

The plan of the paper is as follows. In Section 2 we introduce the Gibbsian formalism and give the necessary notations. Our main results are stated in Section 3, and in Section 4 they are applied to pairwise interaction processes. Finally, Section 5 contains the proofs of our main results.

2 Preliminaries

Our state space is $\mathbb{R}^d$, for some $d \geq 2$, with the Borel σ-algebra. Let $|\Lambda|$ denote Lebesgue measure of a measurable $\Lambda \subset \mathbb{R}^d$, and let $\alpha_d = |\mathbb{B}_1(0)|$ be the volume of the unit ball. For two measurable sets $\Lambda, \Lambda' \subset \mathbb{R}^d$ denote $\operatorname{dist}(\Lambda, \Lambda') = \inf_{x\in\Lambda,\, y\in\Lambda'} \|x - y\|$, where $\|\cdot\|$ denotes the Euclidean norm. If $\Lambda = \{x\}$ we write $\operatorname{dist}(x, \Lambda')$ instead of $\operatorname{dist}\big(\{x\}, \Lambda'\big)$.

Let $(\mathfrak{N}, \mathcal{N})$ denote the space of locally finite point measures on $\mathbb{R}^d$ equipped with the σ-algebra generated by the evaluation maps $[\mathfrak{N} \ni \xi \mapsto \xi(\Lambda) \in \mathbb{Z}_+]$ for bounded Borel sets $\Lambda \subset \mathbb{R}^d$. A point process is just a $\mathfrak{N}$-valued random element. We assume the point processes to be *simple*, i.e. do not allow multi-points. Thus we can use set notation, e.g. $x \in \xi$ means that the point x lies in the support of the measure ξ and $|\xi| = \xi(\mathbb{R}^d)$ denotes the total number of points in ξ. For a measurable $\Lambda \subset \mathbb{R}^d$ let $(\mathfrak{N}_\Lambda, \mathcal{N}_\Lambda)$ be the space of locally finite point measures restricted to Λ and its canonical σ-algebra, respectively. Denote $\xi|_\Lambda$ for the restriction of a point configuration $\xi \in \mathfrak{N}$ onto Λ.

To define Gibbs processes on $\mathbb{R}^d$ we use the *Dobrushin-Lanford-Ruelle*-approach of local specifications, see e.g. [5, 12, 15]. Fix a bounded measurable $\Lambda \subset \mathbb{R}^d$ and a configuration $\omega \in \mathfrak{N}_{\Lambda^c}$, called the *boundary condition*, where $\Lambda^c = \mathbb{R}^d \setminus \Lambda$. Furthermore fix a measurable function $\Phi\colon \mathfrak{N} \to \mathbb{R} \cup \{\infty\}$, called the *potential*. For a $\xi \in \mathfrak{N}_\Lambda$ let

$$u_{\Lambda,\omega}(\xi) = \exp\bigg(- \sum_{\xi' \subset \xi,\, \xi' \neq \emptyset,\, \omega' \subset \omega} \Phi\big(\xi' \cup \omega'\big)\bigg). \tag{2.1}$$

Define a probability measure on $\mathfrak{N}$ by

$$\mu_{\Lambda,\omega}(A) = \frac{1}{c_{\Lambda,\omega}} \bigg(\mathbb{1}\{\omega \in A\} + \sum_{k=1}^{\infty} \frac{1}{k!} \int_\Lambda \cdots \int_\Lambda \mathbb{1}\big\{\{x_1, \dots, x_k\} \cup \omega \in A\big\} \times u_{\Lambda,\omega}\big(\{x_1, \dots, x_k\}\big)\, dx_1 \cdots dx_k \bigg), \tag{2.2}$$

where

$$c_{\Lambda,\omega} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_\Lambda \cdots \int_\Lambda u_{\Lambda,\omega}\big(\{x_1, \dots, x_k\}\big)\, dx_1 \cdots dx_k \tag{2.3}$$

is called the *partition function*. Note that the sum in (2.1) and the partition function may not exist. However we tacitly assume that the potential Φ is chosen such that $\mu_{\Lambda,\omega}$ is well-defined for all bounded measurable $\Lambda \subset \mathbb{R}^d$ and for all $\omega \in \mathfrak{N}_{\Lambda^c}$; we refer the reader to [15] or [4] for such and related questions. A probability measure μ on $\mathfrak{N}$ is called a *Gibbs measure*, if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\mu(A) = \int_{\mathfrak{N}} \mu_{\Lambda, \xi|_{\Lambda^c}}(A)\, \mu(d\xi), \tag{2.4}$$

for all $A \in \mathcal{N}$ and for all bounded measurable $\Lambda \subset \mathbb{R}^d$. Let $\mathcal{G}(\Phi)$ denote the set of all Gibbs measures corresponding to the potential Φ. It may happen that $\mathcal{G}$ contains more than one Gibbs measure; such an event is called a *phase transition*. A point process Ξ is a *Gibbs point process* with potential Φ if it has a distribution $\mu \in \mathcal{G}(\Phi)$. The measures in (2.2) are the *local specifications* of μ. They are nothing else than conditional probabilities, i.e. if $\Xi \sim \mu$, then we have for all bounded measurable $\Lambda \subset \mathbb{R}^d$, for all boundary conditions $\omega \in \mathfrak{N}_{\Lambda^c}$ and for all $A \in \mathcal{N}$

$$\mathbb{P}(\Xi \in A \mid \Xi|_{\Lambda^c} = \omega) = \mu_{\Lambda,\omega}(A).$$

For $x \in \mathbb{R}^d$ and $\xi \in \mathfrak{N}$ define the *conditional intensity* λ, see [12], as

$$\lambda(x \mid \xi) = \exp\Big(- \sum_{\xi' \subset \xi} \Phi\big(\{x\} \cup \xi'\big)\Big). \tag{2.5}$$

Fix a bounded domain $\Lambda \subset \mathbb{R}^d$ and a boundary condition $\omega \in \mathfrak{N}_{\Lambda^c}$. Then we get from (2.1)

$$\lambda(x \mid \xi \cup \omega) = \frac{u_{\Lambda,\omega}\big(\{x\} \cup \xi\big)}{u_{\Lambda,\omega}\big(\xi\big)}, \tag{2.6}$$

for all $x \in \Lambda$ and for all $\xi \in \mathfrak{N}_\Lambda$ such that $x \notin \xi$ and $u_{\Lambda,\omega}(\xi) > 0$. Let $\mu \in \mathcal{G}(\Phi)$ and define $N_\Lambda = \{\xi \in \mathfrak{N} \colon u_{\Lambda,\xi|_{\Lambda^c}}(\xi|_\Lambda) = 0\}$. Obviously $\mu_{\Lambda,\omega}(N_\Lambda) = 0$ for all $\omega \in \mathfrak{N}_{\Lambda^c}$, and by (2.4) we get $\mu(N_\Lambda) = 0$ as well. Thus (2.6) holds for μ-a.e. $\xi \in \mathfrak{N}$ with $x \notin \xi$.

Note that if we are interested in Gibbs point processes on a bounded domain with empty boundary condition ($\omega = \emptyset$), Equation (2.6) coincides, up to the null-set N_Λ, with the definition of the conditional intensity commonly used in spatial statistics, see e.g. [10, Def. 6.1]. Roughly speaking, the conditional intensity is the infinitesimal probability that Ξ has a point at x, given that Ξ coincides with the configuration ξ everywhere else.

For the rest of this paper we assume the potential Φ to be constant for one-point configurations, i.e. $\Phi\big(\{x\}\big) = \Phi\big(\{0\}\big)$ for all $x \in \mathbb{R}^d$. Let $\beta = \exp\big(- \Phi\big(\{0\}\big)\big)$ and for $x \in \mathbb{R}^d$ and $\xi \in \mathfrak{N}$ denote

$$\tilde{\lambda}(x \mid \xi) = \frac{\lambda(x \mid \xi)}{\beta} = \exp\Big(- \sum_{\xi' \subset \xi,\, \xi' \neq \emptyset} \Phi\big(\{x\} \cup \xi'\big)\Big).$$

The constant β is called *activity parameter* or simply *activity*. To be able to keep track of β in our main results, we will mainly use the notation $\beta\tilde{\lambda}$ for the conditional intensity.

In order to describe Gibbs processes by the DLR-approach, one can equivalently characterize them through the conditional intensity, see [12]. Therefore, denote $\mathcal{G}(\beta, \tilde{\lambda})$ as the set of Gibbs measures corresponding to the conditional intensity $\beta\tilde{\lambda}$.

A Gibbs point process $\Xi \sim \mu \in \mathcal{G}(\Phi)$ is called a *pairwise interaction process* if for every configuration $\xi \in \mathfrak{N}$ with $|\xi| \geq 3$ we have $\Phi(\xi) = 0$. By denoting $\varphi(x, y) = \exp\big(- \Phi\big(\{x, y\}\big)\big)$ the conditional intensity simplifies to

$$\lambda(x \mid \xi) = \beta \prod_{y \in \xi} \varphi(x, y),$$

for all $x \in \mathbb{R}^d$ and for all $\xi \in \mathfrak{N}$ with $x \notin \xi$. With a slight abuse of notation, we shall use $\mathcal{G}(\beta, \varphi)$ for the set of the corresponding Gibbs measures.

One of the most important point processes is surely the *Poisson process*. Here, we concentrate only on homogeneous Poisson processes, which can be defined as pairwise interaction processes with $\varphi \equiv 1$. For Poisson processes the parameter β is called *intensity*. The more common definition, however, is the following. A point process Π is called a Poisson process with intensity β if for bounded and pairwise disjoint sets $\Lambda_1, \ldots, \Lambda_n$ the random variables $\Pi(\Lambda_1), \ldots, \Pi(\Lambda_n)$ are independent and Poisson distributed with mean $\beta|\Lambda_i|$, for $i = 1, \ldots, n$. It is an easy exercise to show the equivalence of the two definitions using (2.2).

For Poisson processes we have the following percolation result, see [17], or also [6, Sec. 12.10].

Proposition 2.1. *Let Π be a Poisson process with intensity β. For all $R > 0$ there exists a critical intensity $0 < \beta_c < \infty$ such that $Z_R(\Pi)$ contains a.s. only finite clusters if $\beta < \beta_c$ and $Z_R(\Pi)$ has an infinite cluster a.s. if $\beta > \beta_c$.*

3 Main results

For our main result on percolation we need the following definition. Let $\Xi \sim \mu \in \mathcal{G}(\beta, \tilde{\lambda})$. We say Ξ satisfies *condition* **(P)** with constants $r, \delta > 0$ if

$$\tilde{\lambda}(x \mid \xi) \geq \delta \quad \text{for all } x \in \mathbb{R}^d \text{ and for } \mu\text{-a.e. } \xi \in \mathfrak{N} \text{ with } \operatorname{dist}(x, \xi) \geq r.$$

If for all $\beta > 0$, all Gibbs processes with distribution in $\mathcal{G}(\beta, \tilde{\lambda})$ satisfy condition **(P)**, then we say that $\tilde{\lambda}$ satisfies condition **(P)** itself.

A physical interpretation of condition **(P)** could be the following. Let $\xi \in \mathfrak{N}$ and choose a $x \in \mathbb{R}^d$ such that its nearest point in ξ is at a distance at least r. Assume that the point process Ξ coincides with ξ everywhere except at x. Then the condition **(P)** states that the *energy cost* of adding a point at x is bounded from above by an universal constant which does not depend on the location x nor on the configuration ξ.

The next theorem is our main result on percolation.

Theorem 3.1. *Let $\tilde{\lambda}$ satisfy condition **(P)** with constants r and δ. Then for all $R > r$ there exist a $\beta_+ < \infty$ such that for all $\beta > \beta_+$ and for all Gibbs processes Ξ with distribution in $\mathcal{G}(\beta, \tilde{\lambda})$, the set $Z_R(\Xi)$ contains a.s. an infinite cluster.*

Remark 3.2. *In fact, the proof of Theorem 3.1 yields a slightly stronger percolation result. Let Ξ be as in Theorem 3.1, and let $\mathbb{H} \subset \mathbb{R}^d$ be any affine subspace of dimension at least two. Then $Z_R(\Xi) \cap \mathbb{H}$ contains a.s. an infinite cluster.*

For our result on non-percolation we need the following stability assumption. Let Ξ be a Gibbs process with distribution μ and conditional intensity λ. Then Ξ is called *locally stable* if there exists a constant c^* such that $\lambda(x \mid \xi) \leq c^*$ for all $x \in \mathbb{R}^d$ and for μ-a.e. $\xi \in \mathfrak{N}$. If all Gibbs point processes corresponding to the conditional intensity λ are locally stable, we call λ itself locally stable. Most Gibbs point processes considered in spatial statistics are locally stable, see [10, p. 84] and [8, p. 850]. However the most important example in statistical physics, the *Lennard–Jones process*, is not locally stable.

Theorem 3.3. *Let $\beta\tilde{\lambda}$ be locally stable. Then for all $R > 0$ there exists a $\beta_- > 0$ such that for all $\beta < \beta_-$ and for all Gibbs processes Ξ with distribution in $\mathcal{G}(\beta, \tilde{\lambda})$, the set $Z_R(\Xi)$ contains a.s. only finite clusters.*

Remark 3.4. *The proof of Theorem 3.3 relies on the fact that a locally stable Gibbs process can be dominated by a Poisson process, and then Proposition 2.1 is applied. However for Poisson processes there are more general percolation results available, e.g. the balls could be replaced by some random geometric objects. Thus Theorem 3.3 may be generalized along the lines of [14].*

The next example combines Theorem 3.1 and Theorem 3.3 to get a similar result as in Proposition 2.1.

Example 3.5. *A Gibbs point process is called an* area interaction process*, see [2], if its conditional intensity is given by*

$$\lambda(x \mid \xi) = \beta\gamma^{-|\mathbb{B}_{r_0}(x) \setminus \cup_{y \in \xi} \mathbb{B}_{r_0}(y)|}$$

for some $\gamma, r_0 > 0$. One easily gets the estimates

$$1 \leq \tilde{\lambda}(x \mid \xi) \leq \gamma^{-\alpha_d {r_0}^d} \quad \text{if} \quad 0 \leq \gamma \leq 1,$$
$$\gamma^{-\alpha_d {r_0}^d} \leq \tilde{\lambda}(x \mid \xi) \leq 1 \quad \text{if} \quad \gamma \geq 1.$$

*Thus λ satisfies condition **(P)** for all $r > 0$ with $\delta = \min\{1, \gamma^{-\alpha_d {r_0}^d}\}$ and it is locally stable with constant $c^* = \beta \max\{1, \gamma^{-\alpha_d {r_0}^d}\}$. Then the Theorems 3.1 and 3.3 yield that for all $R > 0$ there exists two constants $0 < \beta_- \leq \beta_+ < \infty$ such that all area interaction processes with $\beta < \beta_-$ do not R-percolate a.s. and all area interaction processes with $\beta > \beta_+$ do R-percolate a.s. It remains open whether $\beta_- = \beta_+$, as for the Poisson process, see Proposition 2.1.*

4 Pairwise interaction processes

In this section we consider pairwise interaction processes and compare the results of [13] with ours. The next proposition shows which pairwise interaction processes satisfy condition **(P)**.

Proposition 4.1. *Let $r > 0$. Assume that the interaction function φ satisfies one of the following conditions.*

(i) For all $x, y \in \mathbb{R}^d$ with $\|x - y\| \geq r$ we have $\varphi(x, y) \geq 1$.
(ii) There exist constants $\tilde{\delta} > 0$ and $r_{\max} < \infty$ such that

$$\varphi(x, y) = 0 \quad \text{if} \quad 0 \leq \|x - y\| < r,$$
$$\varphi(x, y) \geq \tilde{\delta} \quad \text{if} \quad r \leq \|x - y\| < r_{\max},$$
$$\varphi(x, y) \geq 1 \quad \text{if} \quad r_{\max} \leq \|x - y\|.$$

*Then there exists a $\delta > 0$ such that all Gibbs processes with distribution in $\mathcal{G}(\beta, \varphi)$ satisfy condition **(P)** with constants r and δ.*

Proof. (i) Since $\tilde{\lambda}(x \mid \xi) = \prod_{y \in \xi} \varphi(x, y)$, condition **(P)** is trivially satisfied with $\delta = 1$.

(ii) Assume $\tilde{\delta} < 1$, otherwise condition (i) is satisfied. Let $\mu \in \mathcal{G}(\beta, \varphi)$. Fix a $x \in \mathbb{R}^d$ and consider the event

$$N_x = \big\{\xi \in \mathfrak{N}\colon \text{ There exist } y, y' \in \xi|_{\mathbb{B}_{r_{\max}}(x)} \text{ with } \|y - y'\| < r\big\}.$$

For a bounded $\Lambda \supset \mathbb{B}_{r_{\max}}(x)$ and any boundary condition $\omega \in \mathfrak{N}_{\Lambda^c}$ we get $\mu_{\Lambda,\omega}(N_x) = 0$ and by (2.4) also $\mu(N_x) = 0$. Furthermore there exists a constant $m < \infty$ such that for all $\xi \in \mathfrak{N} \setminus N_x$ we have $\xi\big(\mathbb{B}_{r_{\max}}(x)\big) \leq m$, and m does not depend on x (e.g. m can be chosen as the maximal number of balls with radius $r/2$ which can be placed in a ball with radius $r_{\max} + r/2$). Thus condition **(P)** is satisfied with $\delta = \tilde{\delta}^m$ for all $x \in \mathbb{R}^d$ and for μ-a.e. $\xi \in \mathfrak{N}$.

□

In [13] there are various assumptions on φ including our condition (i) of Proposition 4.1 and an attraction condition. Namely, φ is said to have an attractive tail if there exists two constants $r_a < r'_a$ such that $\varphi(x, y) > 1$, whenever $r_a \leq \|x - y\| \leq r'_a$.

Furthermore the authors percolation radius R has to be greater than $\sqrt{2}r$. We could reduce this bound by a factor $\sqrt{2}$, but in some cases, e.g. for a *Strauss hard core* process ($\varphi(x, y) = \mathbb{1}\{\|x - y\| \geq r\}$), one would expect a lower bound on the percolation radius of $r/2$. The main difference is however, that the percolation result of [13] is valid only in two dimension, whereas our result holds in any dimension $d \geq 2$.

A pairwise interaction process is locally stable in the following two cases. Firstly if the interaction function $\varphi(x, y)$ is bounded by one; such a process is called *inhibitory*. Secondly if it has a *hard core radius* r, i.e. $\varphi(x, y) = 0$ whenever $\|x - y\| \leq r$, and $\varphi(x, y) \to 1$ fast enough as $\|x - y\| \to \infty$. The non-percolation result in [13] treats only the hard core case. However, unless for the percolation result, the authors proof is quite different from ours.

5 Proofs

The main idea for the proof of percolation is based on techniques close to the contour method in lattice models. In particular, it is a modification of the arguments in [13] and [1]. Let condition **(P)** be satisfied with constants $r, \delta > 0$, and let $R > r$. Choose a $m \in \mathbb{N}$ such that $m > \sqrt{d}/(R - r)$. Divide $\mathbb{R}^d$ into cubes of length $1/m$. For this sake define

$$Q_z = \Big\{x \in \mathbb{R}^d \,:\, \|x - z\|_{\max} \leq \frac{1}{2m}\Big\},$$

where we use the maximum norm $\|x\|_{\max} = \max_{i=1,\dots,d}|x_i|$. Then $\{Q_z,\, z \in \frac{1}{m}\mathbb{Z}^d\}$ covers the whole space $\mathbb{R}^d$. Two cubes Q_z and $Q_{z'}$ are called *neighbours* if $\|z - z'\|_{\max} = 1/m$. A set $S \subset \frac{1}{m}\mathbb{Z}^d$ is called *connected* if for each pair $\{z, z'\} \subset S$ there exists a sequence of neighbouring cubes $Q_{z_1}, Q_{z_2}, \dots, Q_{z_n}$ with $z = z_1$ and $z' = z_n$. The cardinality of S is denoted by $|S|$.

Lemma 5.1. *There exists a constant $c > 0$ such that for all $S \subset \frac{1}{m}\mathbb{Z}^d$ with $|S| < \infty$, there exists a $S' \subset S$ with $|S'| \geq c|S|$ and for each pair $z, z' \in S'$ we have $\operatorname{dist}(Q_z, Q_{z'}) \geq r$.*

Proof. We can assume $|S| \geq 1$ and then obviously there exists a S' with cardinality one. To identify the constant c, consider the following inductive procedure. Choose an arbitrary point $z_1 \in S$ and draw a ball around z_1 of radius $r + \sqrt{d}/m$. Exclude all points of S which are contained in this ball and denote by S_1 the remaining points of S. Note that $\operatorname{dist}(Q_{z_1}, Q_z) \geq r$ for all $z \in S_1$. Continue this procedure and define $S' = \{z_1, \dots, z_n\}$, the set of the chosen points. At each step we exclude at most $\alpha_d(r + 3\sqrt{d}/(2m))^d m^d$ points. Thus

$$n \geq \max\left(\left\lfloor \frac{|S|}{\alpha_d(r + 3\sqrt{d}/(2m))^d m^d} \right\rfloor, 1\right),$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Choosing $c = 1/(2\alpha_d(r + 3\sqrt{d}/(2m))^d m^d)$ yields the claim. □

The following Lemma is the key ingredient for the proof of Theorem 3.1.

Lemma 5.2. *Assume $\Xi \sim \mu \in \mathcal{G}(\beta, \tilde{\lambda})$ satisfies condition* **(P)** *with constants r and δ. Let $S \subset \frac{1}{m}\mathbb{Z}^d$ with $|S| < \infty$. Then if $\beta \geq m^d/\delta$ we have,*

$$\mathbb{P}\Big(\operatorname{dist}\big(\Xi, \textstyle\bigcup_{z\in S} Q_z\big) \geq r\Big) \leq \left(\frac{\beta\delta}{m^d}\right)^{-c|S|}, \tag{5.1}$$

where c is the same constant as in Lemma 5.1.

Proof. By Lemma 5.1 choose a $S' \subset S$ with $|S'| \geq c|S|$ and such that for each pair $z, z' \in S'$ we have $\operatorname{dist}(Q_z, Q_{z'}) \geq r$. Set $n = |S'|$ and denote $S' = \{z_1, \dots, z_n\}$. Consider the events

$$A_{S,r} = \big\{\xi \in \mathfrak{N} \,:\, \operatorname{dist}\big(\xi, \textstyle\bigcup_{z\in S} Q_z\big) \geq r\big\} \quad \text{and}$$
$$B_{S,r} = \big\{\xi \cup \{x_1, \dots, x_n\} \in \mathfrak{N}\colon\, \xi \in A_{S,r},\, x_i \in Q_{z_i}, \text{ for } i = 1, \dots, n\big\}.$$

Choose a bounded $\Lambda \subset \mathbb{R}^d$, such that the distance between S and the boundary of Λ is at least $r+\sqrt{d}/m$ and fix a boundary condition $\omega \in \mathfrak{N}_{\Lambda^c}$. Then for μ-a.e. $\xi \cup \{x_1, \dots, x_n\} \in B_{S,r}$ we have by (2.6) and condition **(P)** that

$$\begin{aligned} u_{\Lambda,\omega}\big(\xi|_\Lambda \cup \{x_1,\dots,x_n\}\big) &= u_{\Lambda,\omega}\big(\xi|_\Lambda\big)\lambda\big(x_1 \mid \xi|_\Lambda \cup \omega\big)\times \\ &\quad \lambda\big(x_2 \mid \xi|_\Lambda \cup \{x_1\} \cup \omega\big)\cdots\lambda\big(x_n \mid \xi|_\Lambda \cup \{x_1,\dots,x_{n-1}\}\cup\omega\big) \\ &\geq (\beta\delta)^n u_{\Lambda,\omega}(\xi|_\Lambda). \end{aligned} \tag{5.2}$$

The partition function can then be bounded from below as

$$\begin{aligned} c_{\Lambda,\omega} &\geq \sum_{k=0}^{\infty}\frac{1}{k!}\int_\Lambda\cdots\int_\Lambda \mathbb{1}\big\{\{x_1,\dots,x_k\}\in B_{S,r}\big\}\times \\ &\qquad u_{\Lambda,\omega}\big(\{x_1,\dots,x_k\}\big)\,dx_1\cdots dx_k \\ &= \sum_{k=n}^{\infty}\frac{1}{k!}\frac{k!}{(k-n)!}\int_\Lambda\cdots\int_\Lambda \mathbb{1}\big\{x_1\in Q_{z_1},\dots,x_n\in Q_{z_n}\big\}\times \\ &\qquad \mathbb{1}\big\{\{x_{n+1},\dots,x_k\}\in A_{S,r}\big\}u_{\Lambda,\omega}\big(\{x_1,\dots,x_k\}\big)\,dx_1\cdots dx_k \\ &\geq \Big(\frac{\beta\delta}{m^d}\Big)^n\sum_{j=0}^{\infty}\frac{1}{j!}\int_\Lambda\cdots\int_\Lambda \mathbb{1}\big\{\{x_1,\dots,x_j\}\in A_S,r\big\}\times \\ &\qquad u_{\Lambda,\omega}\big(\{x_1,\dots,x_j\}\big)\,dx_1\cdots dx_j, \end{aligned}$$

where $k!/(k-n)!$ is the number of possibilities to choose a set of $k-n$ elements and n sets of one elements out of a set of k elements, and the last inequality follows by (5.2), by integrating over the cubes $\{Q_{z_i}, i=1,\dots,n\}$ and by the change of variable $j=k-n$. Thus, by (2.2)

$$\mu_{\Lambda,\omega}(A_{S,r}) \leq \Big(\frac{\beta\delta}{m^d}\Big)^{-n}. \tag{5.3}$$

Since (5.3) does not depend on ω, Equation (2.4) yields

$$\mathbb{P}\Big(\operatorname{dist}\big(\Xi, \textstyle\bigcup_{z\in S} Q_z\big) \geq r\Big) = \mathbb{P}(\Xi \in A_{S,r}) \leq \Big(\frac{\beta\delta}{m^d}\Big)^{-n} \leq \Big(\frac{\beta\delta}{m^d}\Big)^{-c|S|},$$

where the last step follows by $n \geq c|S|$. □

Remainder of the proof of Theorem 3.1. We call a set $L \subset \frac{1}{m}\mathbb{Z}^d$ a *loop* if $|L|<\infty$, L is connected and each cube $\{Q_z,\, z\in L\}$ has exactly two neighbours. We say $L \subset \frac{1}{m}\mathbb{Z}^d$ is a *loop around the origin* if the origin is contained in the convex hull of L. Denote by $\mathcal{L}_0$ the set of all loops around the origin.

Consider the loop $L=\{z_1,\dots,z_k\}\in\mathcal{L}_0$ such that Q_{z_1} and Q_{z_k} are neighbours and for $i=1,\dots,k-1$ the cubes Q_{z_i} and $Q_{z_{i+1}}$ are neighbours. Since the origin is contained in the convex hull of L, the point z_1 lies necessarily in the big cube $[-k/m,k/m]^d$, which gives $(2k+1)^d$ possibilities. For $i=1,\dots,k-1$, the cube $Q_{z_{i+1}}$ is a neighbour of Q_{z_i}, thus there are at most 2^d possibilities to place the cube $Q_{z_{i+1}}$. We conclude, that there are at most $(2k+1)^d 2^{d(k-1)}$ loops in $\mathcal{L}_0$ with length k. Consider the event

$$A_k = \big\{\xi\in\mathfrak{N}\colon \text{There exists a } L\in\mathcal{L}_0 \text{ with } |L|=k \text{ and } \operatorname{dist}\big(\xi, \textstyle\bigcup_{z\in L}Q_z\big)\geq r\big\}.$$

Then for $\beta \geq m^d/\delta$ Lemma 5.2 yields

$$\begin{aligned} \mathbb{P}(\Xi\in A_k) &= \mathbb{P}\Big(\bigcup_{L\in\mathcal{L}_0,\,|L|=k}\big\{\operatorname{dist}\big(\Xi,\textstyle\bigcup_{z\in L}Q_z\big)\geq r\big\}\Big) \\ &\leq \sum_{L\in\mathcal{L}_0,\,|L|=k}\mathbb{P}\Big(\operatorname{dist}\big(\Xi,\textstyle\bigcup_{z\in L}Q_z\big)\geq r\Big) \leq (2k+1)^d 2^{d(k-1)}\Big(\frac{\beta\delta}{m^d}\Big)^{-ck}. \end{aligned}$$

Set $\beta_+ = (2^{1/c}m)^d/\delta$ and choose a $\beta > \beta_+$. Then,

$$\sum_{k=1}^{\infty} \mathbb{P}(\Xi \in A_k) \le \sum_{k=1}^{\infty} (2k+1)^d 2^{d(k-1)} \Big(\frac{\beta\delta}{m^d}\Big)^{-ck} < \infty. \tag{5.4}$$

The first Borel-Cantelli Lemma together with (5.4) then yield that there are a.s. only finitely many loops $L \in \mathcal{L}_0$ with $\operatorname{dist}\big(\Xi, \bigcup_{z\in L} Q_z\big) \ge r$.

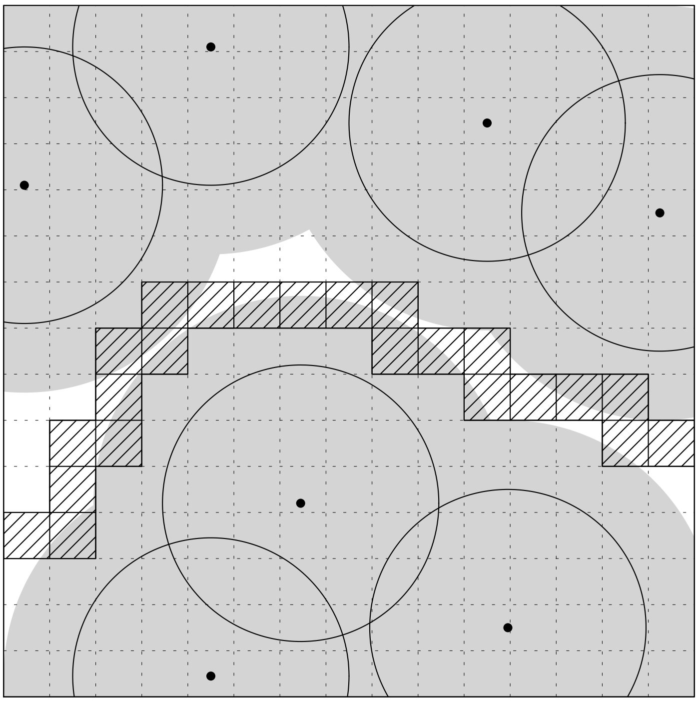

Figure 1: Here we choose $d = 2$, $r = 0.2$, $R = 0.3$ and $m = 15$. The grey area corresponds to $Z_R(\Xi)$ and the chain of shaded boxes shows one possibility to separate the two components of $Z_R(\Xi)$. Note that the boxes do not intersect with the circles of radius r.

The following trick allows us to reduce the problem to the two-dimensional case. Consider the set $Z_R^{(2)}(\Xi) = Z_R(\Xi) \cap \mathbb{R}^2$, where we identify $\mathbb{R}^2$ as the plane spanned by the first two canonical basis vectors in $\mathbb{R}^d$. A loop is said to be a $\mathbb{R}^2$-*loop* if the centres of its cubes lie in $\mathbb{R}^2$.

The parameter m is chosen such that if two points x and x' lie in the same connected component of $\mathbb{R}^2 \setminus Z_R^{(2)}(\Xi)$, then there exists a sequence of neighbouring cubes with centres $S = \{z_1, \ldots, z_n\} \subset \frac{1}{m}\mathbb{Z}^d \cap \mathbb{R}^2$, with $x \in Q_{z_1}$, $x' \in Q_{z_n}$ and $\operatorname{dist}\big(\Xi, \bigcup_{z\in S} Q_z\big) \ge r$. To see that this is possible consider a path in $\mathbb{R}^2 \setminus Z_R^{(2)}(\Xi)$ which connects x with x' and choose all cubes which intersect with this path. Obviously the cubes are connected and the length of the diagonal of a cube is less than $R - r$. Thus, by the reverse triangle inequality, the selected cubes and Ξ are separated by a distance at least r. Figure 1 shows a graphical illustration of this procedure.

Assume that $Z_R(\Xi)$ does not percolate, i.e. there exists a.s. only finite clusters. Obviously the clusters in $Z_R^{(2)}(\Xi)$ are also finite, and each cluster can be surrounded by a $\mathbb{R}^2$-loop $L \in \mathcal{L}_0$ with $\operatorname{dist}\big(\Xi, \bigcup_{z\in L} Q_z\big) \geq r$. For large β this eventually leads to a contradiction.

Note that the choice of a basis in $\mathbb{R}^d$ is by no means important. Thus, the statement of Remark 3.2. □

Proof of Theorem 3.3. Let Ξ be a locally stable Gibbs process with conditional intensity $\beta\tilde{\lambda} \leq c^*$. Denote $\Lambda_n = [-n, n]^d$, for $n \in \mathbb{N}$. Note that conditioned on $\Xi|_{\Lambda_n^c} = \omega$, the law of $\Xi|_{\Lambda_n}$ is uniquely characterized by the conditional intensity $\beta\tilde{\lambda}(\cdot \mid \cdot \cup \omega)$.

It is a known fact that every locally stable Gibbs process on a bounded domain can be obtained as a dependent random thinning of a Poisson process; see [8, Remark 3.4]. In particular, there exists a Poisson process Π_n with intensity c^* and a Gibbs process $\tilde{\Xi}_n \sim \mathcal{G}\big(\beta\tilde{\lambda}(\cdot \mid \cdot \cup \omega)\big)$ on Λ_n, such that $\tilde{\Xi}_n \subset \Pi_n$ a.s. Let Ξ_n be the point process obtained replacing $\Xi|_{\Lambda_n}$ with $\tilde{\Xi}_n$, i.e. $\Xi_n = \Xi|_{\Lambda_n^c} \cup \tilde{\Xi}_n$. Since $\Xi|_{\Lambda_n}$ and $\tilde{\Xi}_n$ have the same conditional distribution $\mu_{\Lambda_n,\omega}$, Equation (2.4) yields that Ξ and Ξ_n have the same distribution for all $n \in \mathbb{N}$. A standard result ([3, Theorem 11.1.VII]) then yields that as $n \to \infty$ the sequence Ξ_n has a weak limit Ξ' with the same distribution as Ξ. By the same result $\Pi_n \to \Pi$ as $n \to \infty$, where Π is a Poisson process on $\mathbb{R}^d$ with intensity c^*. Furthermore, since $\tilde{\Xi}_n \subset \Pi_n$ a.s. for all $n \in \mathbb{N}$, we conclude $\Xi' \subset \Pi$ a.s.

Obviously $Z_R(\Xi') \subset Z_R(\Pi)$. Thus if $Z_R(\Pi)$ does not percolate, neither does $Z_R(\Xi')$, and Proposition 2.1 finishes the proof. □

References

[1] David Aristoff. Percolation of hard disks. *Preprint*, 2012. arXiv:1207.2136.

[2] Adrian J. Baddeley and Marie-Colette N. M. van Lieshout. Area-interaction point processes. *Ann. Inst. Statist. Math.*, 47(4):601–619, 1995. MR-1370279

[3] Daryl J. Daley and David Vere-Jones. *An introduction to the theory of point processes. Vol. II.* Probability and its Applications (New York). Springer, New York, second edition, 2008. MR-2371524

[4] David Dereudre, Remy Drouilhet, and Hans-Otto Georgii. Existence of Gibbsian point processes with geometry-dependent interactions. *Probab. Theory Related Fields*, 153(3-4):643–670, 2012. MR-2948688

[5] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1988. MR-0956646

[6] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999. MR-1707339

[7] Sabine Jansen. Continuum percolation for Gibbsian point processes with attractive interactions. *Preprint*, 2012. arXiv:1208.1223.

[8] Wilfrid S. Kendall and Jesper Møller. Perfect simulation using dominating processes on ordered spaces, with application to locally stable point processes. *Adv. in Appl. Probab.*, 32(3):844–865, 2000. MR-1788098

[9] Ronald Meester and Rahul Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. MR-1409145

[10] Jesper Møller and Rasmus P. Waagepetersen. *Statistical inference and simulation for spatial point processes*, volume 100 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL, 2004. MR-2004226

[11] Michael G. Mürmann. Equilibrium distributions of physical clusters. *Comm. Math. Phys.*, 45(3):233–246, 1975. MR-0413957

[12] Xuan-Xanh Nguyen and Hans Zessin. Integral and differential characterizations of the Gibbs process. *Math. Nachr.*, 88:105–115, 1979. MR-0543396

[13] Eugene Pechersky and Anatoly Yambartsev. Percolation properties of the non-ideal gas. *J. Stat. Phys.*, 137(3):501–520, 2009. MR-2564287

[14] Mathew D. Penrose. On a continuum percolation model. *Adv. in Appl. Probab.*, 23(3):536–556, 1991. MR-1122874

[15] David Ruelle. *Statistical mechanics: Rigorous results.* W. A. Benjamin, Inc., New York–Amsterdam, 1969. MR-0289084

[16] Hans Zessin. A theorem of Michael Mürmann revisited. *Izv. Nats. Akad. Nauk Armenii Mat.*, 43(1):69–80, 2008. MR-2465000

[17] Sergey A. Zuyev and Alexander F. Sidorenko. Continuous models of percolation theory. I. *Teoret. Mat. Fiz.*, 62(1):76–86, 1985. MR-0782099

Acknowledgments. The Author thanks Ilya Molchanov and Dominic Schuhmacher for stimulating discussions about the topic.