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Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems II: strongly monotone quasi-Newtonian flows

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In this article, we develop the *a priori* and *a posteriori* error analysis of *hp*-version interior penalty discontinuous Galerkin finite element methods for strongly monotone quasi-Newtonian fluid flows in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. In the latter case, computable upper and lower bounds on the error are derived in terms of a natural energy norm, which are explicit in the local mesh size and local polynomial degree of the approximating finite element method. A series of numerical experiments illustrate the performance of the proposed *a posteriori* error indicators within an automatic *hp*-adaptive refinement algorithm.

Keywords: *hp*-version finite element methods; discontinuous Galerkin methods; *hp*-adaptivity; quasi-linear PDEs; quasi-Newtonian flows.

1. Introduction

In this article, we develop the *a priori* and *a posteriori* error analysis, with respect to a mesh-dependent energy norm, of *hp*-version discontinuous Galerkin finite element methods (DGFEMs) for the quasi-Newtonian fluid flow problem

$$-\nabla \cdot \{\mu(\mathbf{x}, |\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u})\} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma. \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded polygonal Lipschitz domain with boundary $\Gamma = \partial\Omega$, $\mathbf{f} \in \mathrm{L}^2(\Omega)^d$ is a given source term, $\mathbf{u} = (u_1, \ldots, u_d)^\top$ is the velocity vector, p is the pressure, and $\underline{e}(\mathbf{u})$ is the symmetric

$d \times d$ rate-of-strain tensor defined by

$$e_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d.$$

Furthermore, $|\underline{e}(\mathbf{u})|$ is the Frobenius norm of $\underline{e}(\mathbf{u})$ and μ is assumed to satisfy the structural hypothesis stated in Assumption 2.1.

Quasi-Newtonian fluid flow models of this kind arise as steady-state equations governing creeping flow of incompressible, homogeneous, non-Newtonian liquids in two and three space dimensions ($d = 2, 3$), with a Carreau-law-type dependence of the viscosity on the rate-of-strain tensor (cf. Barrett & Liu, 1994; Bao & Barrett, 1998, for example). For a detailed survey of mathematical models for non-Newtonian flow problems and their numerical approximation, the reader is referred to the monograph of Owens & Phillips (2002). The mathematical analysis of classical (continuous) Galerkin finite element methods for quasi-Newtonian fluid flow models is already well developed (see, for example, the recent work of Diening *et al.,* (2012) and the references therein). The status of the subject in the case of DGFEMs is much less satisfactory; it is fair to say that the field is still very much in its infancy.

In recent years there has been considerable interest in DGFEMs for the numerical solution of a wide range of partial differential equations (PDEs); for an extensive survey of this area of research we refer the reader to Cockburn *et al.* (2000) and the references cited therein. DGFEMs have several important advantages over well-established finite volume methods: the concept of higher-order discretization is inherent to the DGFEM; the stencil is minimal in the sense that each element communicates only with its direct neighbours; in particular, in contrast to the increasing stencil size needed to increase the accuracy of classical finite volume methods, the stencil of DGFEMs is the same for any order of accuracy, which has important advantages for the implementation of boundary conditions and for the parallel efficiency of the method; moreover, because of the simple communication at element interfaces, elements with so-called hanging nodes can be easily treated, a fact that simplifies local mesh refinement (*h*-refinement); additionally, the communication at element interfaces is identical for any order of the method, which simplifies the use of methods with different polynomial degrees p in adjacent elements. This allows for the variation of the degrees of polynomials over the computational domain (*p*-refinement), which in combination with *h*-refinement leads to so-called *hp*-adaptivity.

In the present article, we formulate a class of *hp*-version interior penalty DGFEMs for the numerical approximation of the quasi-Newtonian fluid flow problem (1.1–1.3). This article represents the continuation of the work initiated in Houston *et al.* (2005a, 2008), where the *a priori* and *a posteriori* error analysis, respectively, of DGFEMs was developed for quasilinear, elliptic, boundary value problems, in the case of a single equation; here, we focus on quasilinear elliptic systems. In particular, we establish the existence and uniqueness of both the analytical solution to (1.1–1.3) and of its DGFEM counterpart, and we undertake the *a priori* error analysis of the class of DGFEMs under consideration, with respect to the associated natural energy norm. We then derive computable upper and lower bounds on the error, again measured in terms of the energy norm, which are explicit in the local mesh size and the local polynomial degree of the approximating finite element method. At the expense of a slight suboptimality with respect to the polynomial degree of the approximating finite element method, this upper bound holds on general 1-irregular meshes. In particular, this means that elements can be divided into smaller elements without the need for connecting the resulting hanging nodes. This feature clearly improves both the feasibility and the flexibility of an *hp*-adaptive process. In addition, we note that the use of irregular meshes is very natural and quite easily realizable in the context of DGFEM schemes because of the discontinuous character of the corresponding finite element spaces. The proof of the upper bound is based on employing a suitable DGFEM space decomposition, together with an *hp*-version projection operator. This general approach was pursued in the series of articles by Karakashian & Pascal (2003) and Houston *et al.* (2004a, 2005b, 2007, 2008). The proof of the local lower error bounds (efficiency) is based on the techniques presented in Melenk & Wohlmuth (2001), subject to the treatment of the nonlinearity. On the basis of these *a posteriori* error indicators, we design and implement the corresponding *hp*-adaptive algorithm to ensure reliable and efficient control of the discretization error. Numerical experiments are presented, which demonstrate the performance of the proposed algorithm. For related work on *h*-version local DGFEMs for quasilinear PDEs, we refer to Bustinza & Gatica (2004), González & Meddahi (2004) and Bustinza *et al.* (2005), for example.

The article is organized as follows. In Section 2, we state the weak formulation of (1.1–1.3) and prove its well-posedness. In Section 3, we formulate the interior penalty *hp*-DGFEM for the numerical approximation of the boundary value problem (1.1–1.3), and show that the proposed scheme is also well-posed. Section 4 is devoted to the *a priori* error analysis of the underlying *hp*-DGFEM. In Section 5, we establish the upper and lower *a posteriori* error bounds. Section 6 contains a series of numerical experiments, which illustrate our theoretical results; in particular, we demonstrate the performance of an *hp*-adaptive algorithm based on the *hp*-error indicators. Finally, in Section 7, we summarize the main results of this article and draw some conclusions.

2. Weak formulation

In this section, we will present a weak formulation for (1.1–1.3) and prove its well-posedness.

2.1 *Notation*

We shall use the following standard notation throughout the paper. For a bounded Lipschitz domain $D \subset \mathbb{R}^d$, $d \geqslant 1$, we write $\mathrm{H}^t(D)$ to denote the usual Sobolev space of real-valued functions, of order $t \geqslant 0$, with norm $\|\cdot\|_{t,D}$. In the case when $t = 0$, we set $\mathrm{L}^2(D) = \mathrm{H}^0(D)$. We define $\mathrm{H}^1_0(D)$ to be the subspace of functions in $\mathrm{H}^1(D)$ with zero trace on ∂D. Additionally, we set $\mathrm{L}^2_0(D) := \{q \in \mathrm{L}^2(D) : \int_D q \,\mathrm{d}\mathbf{x} = 0\}$. For a function space $X(D)$, we let $X(D)^d$ and $X(D)^{d\times d}$ denote the spaces of vector and tensor fields, respectively, whose components belong to $X(D)$. These spaces are equipped with the usual product norms which, for simplicity, we denote in the same way as the norm in $X(D)$.

For the d-component vector-valued functions $\mathbf{v}, \mathbf{w}$ and $d \times d$ matrix-valued functions $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{d\times d}$, we define the operators

$$(\nabla \mathbf{v})_{ij} := \frac{\partial v_i}{\partial x_j}, \quad (\nabla \cdot \underline{\sigma})_i := \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad (\mathbf{v} \otimes \mathbf{w})_{ij} := v_i w_j, \quad \underline{\sigma} : \underline{\tau} := \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij}.$$

For matrix-valued functions the Frobenius norm can be written as $|\underline{\tau}|^2 = \underline{\tau} : \underline{\tau}$.

2.2 *Variational form*

By introducing the forms

$$A(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) : \underline{e}(\mathbf{v}) \,\mathrm{d}\mathbf{x}, \quad B(\mathbf{v}, q) := -\int_{\Omega} q \nabla \cdot \mathbf{v} \,\mathrm{d}\mathbf{x},$$

a natural weak formulation of the quasi-Newtonian fluid flow problem (1.1–1.3) is to find $(\mathbf{u}, p)$ in $\mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}, \tag{2.1}$$

$$-B(\mathbf{u}, q) = 0 \tag{2.2}$$

for all $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$. We note that the bilinear form B satisfies the following inf–sup condition: there exists a constant $\kappa > 0$ such that

$$\inf_{0 \neq q \in \mathrm{L}_0^2(\Omega)} \sup_{0 \neq \mathbf{v} \in \mathrm{H}_0^1(\Omega)^d} \frac{B(\mathbf{v}, q)}{\|q\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega}} \geqslant \kappa; \tag{2.3}$$

see, for example, Brezzi & Fortin (1991). We shall assume throughout this article that the function μ satisfies the following structural hypothesis.

ASSUMPTIONS 2.1 We assume that the nonlinearity μ satisfies the following conditions.

(A1) $\mu \in \mathrm{C}(\bar{\Omega} \times [0, \infty))$.

(A2) There exist constants $m_\mu, M_\mu > 0$ such that

$$m_\mu(t - s) \leqslant \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leqslant M_\mu(t - s), \quad t \geqslant s \geqslant 0, \quad \text{for all } \mathbf{x} \in \bar{\Omega}. \tag{2.4}$$

From Barrett & Liu (1994, Lemma 2.1), we note that as μ satisfies (2.4), there exist positive constants C_1 and C_2 such that, for all $\underline{\tau}, \underline{\omega} \in \mathbb{R}^{d \times d}$ and all $\mathbf{x} \in \bar{\Omega}$,

$$|\mu(\mathbf{x}, |\underline{\tau}|)\underline{\tau} - \mu(\mathbf{x}, |\underline{\omega}|)\underline{\omega}| \leqslant C_1 |\underline{\tau} - \underline{\omega}|, \tag{2.5}$$

$$C_2 |\underline{\tau} - \underline{\omega}|^2 \leqslant (\mu(\mathbf{x}, |\underline{\tau}|)\underline{\tau} - \mu(\mathbf{x}, |\underline{\omega}|)\underline{\omega}) : (\underline{\tau} - \underline{\omega}). \tag{2.6}$$

For the ease of notation we shall suppress the dependence of μ on $\mathbf{x}$ and write $\mu(t)$ instead of $\mu(\mathbf{x}, t)$.

2.3 *Well-posedness*

We will now show that the weak formulation (2.1, 2.2) admits a unique solution in the given spaces.

An operator $A : D(A) \subset X \to X'$ on a normed linear space X, where X' is the dual space of X, and $D(A)$ signifies the domain of A, is called *monotone* if

$$\langle Au - Av, u - v \rangle \geqslant 0 \quad \forall\, u, v \in D(A),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X'. An operator $A : D(A) \subset X \to X'$ is said to be *hemicontinuous* if the map $t \mapsto \langle A(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$. Finally, A is said to be *coercive* if $\lim_{\|u\|_X \to +\infty} \langle Au, u \rangle / \|u\|_X = +\infty$.

We shall require the following classical result from the theory of monotone operators.

THEOREM 2.2 (Browder–Minty theorem) Suppose that X is a reflexive Banach space and that the operator $A : X \to X'$ is monotone, hemicontinuous and coercive; then, A is surjective.

We are now ready to state and prove the following general theorem.

THEOREM 2.3 Suppose that X and M are reflexive Banach spaces. Furthermore, consider forms $a : X \times X \to \mathbb{R}$, $b : X \times M \to \mathbb{R}$ and $\ell : X \to \mathbb{R}$ such that the following conditions are satisfied.

(a) a is strongly monotone, hemicontinuous and coercive in the sense that, respectively,

- $a(u, u - v) - a(v, u - v) \geqslant \|u - v\|_X \gamma(\|u - v\|_X)$ for all $u, v \in X$, where $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a function with $\gamma(t) \to +\infty$ for $t \to +\infty$, and with $\gamma(t) = 0$ if and only if $t = 0$;
- $t \mapsto a(u + tv, w)$ is continuous on $[0, 1]$ for all $u, v, w \in X$;
- $\sup_{\|u\|_X \to +\infty} a(u, u)/\|u\|_X = +\infty$.

Suppose further that the functional $v \mapsto a(w, v)$ is linear and continuous on X for any fixed element $w \in X$.

(b) b is bilinear and continuous on $X \times M$; furthermore, b is inf–sup stable in the sense that there exists a constant $\kappa > 0$ such that

$$\inf_{0 \neq q \in M} \sup_{0 \neq v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_M} \geqslant \kappa. \tag{2.7}$$

(c) ℓ is linear and continuous on X.

Then, there exists a unique solution $(u, p) \in X \times M$ to the variational equation

$$a(u, v) + b(v, p) - b(u, q) = \ell(v) \quad \forall (v, q) \in X \times M. \tag{2.8}$$

Proof. Let us begin by defining $V := \{v \in X : b(v, q) = 0 \; \forall\, q \in M\}$. By taking $(v, q) = (0, q) \in X \times M$ in (2.8), we deduce that if there exists a pair $(u, p) \in X \times M$ satisfying (2.8), then $b(u, q) = 0$ for all $q \in M$; hence, $u \in V$. Similarly, by taking $(v, q) = (v, 0) \in V \times M$ in (2.8), we deduce that $b(v, p) = \ell(v) - a(u, v)$. Thus, we have shown that if a pair $(u, p) \in X \times M$ is a solution to (2.8), then

$$u \in V, \text{ and it satisfies} \quad a(u, v) = \ell(v) \quad \forall\, v \in V, \tag{2.9}$$

$$\text{and } p \in M \text{ satisfies} \quad b(v, p) = \ell(v) - a(u, v) \quad \forall\, v \in X. \tag{2.10}$$

Clearly, the converse of this statement is also true; thereby, problems (2.8) and (2.9, 2.10) are equivalent.

As the bilinear functional b is continuous on $X \times M$, it follows that V is a closed linear subspace of the Banach space X. Therefore V itself is a Banach space when equipped with the norm of X. Introduce the operator $\mathsf{A} : V \to V'$ with $\langle \mathsf{A}w, v\rangle = a(w, v)$ for $w, v \in V$. Owing to (a), the operator $\mathsf{A} : V \to V'$ is monotone, hemicontinuous and coercive. Thus, by the Browder–Minty theorem, $\mathsf{A} : V \to V'$ is surjective. Furthermore, the strong monotonicity of a implies that $\mathsf{A} : V \to V'$ is injective. Thus, we deduce that $\mathsf{A} : V \to V'$ is bijective. Hence there exists a unique $u \in V$ such that $\mathsf{A}u = \ell$; equivalently, there exists a unique $u \in V$ such that $a(u, v) = \ell(v)$ for all $v \in V$.

Now, let $\mathsf{B} : X \to M'$ denote the bounded linear operator defined by $\langle \mathsf{B}w, q\rangle = b(w, q)$ for $w \in X$ and $q \in M$, where M' denotes the dual space of M. Let $\mathsf{B}' : M \to X'$ be the (bounded linear) dual operator of B; i.e., $\langle \mathsf{B}'q, w\rangle = \langle \mathsf{B}w, q\rangle = b(w, q)$ for $q \in M$ and $w \in X$. Clearly, $V = \mathrm{Ker}(\mathsf{B})$; denote by $V^\circ := \{g \in X' : \langle g, v\rangle = 0 \; \forall\, v \in V\}$ the polar set of V. Owing to Girault & Raviart (1986, Lemma 4.1), B' is an isomorphism from M onto V°; we note that Girault & Raviart (1986, Lemma 4.1) is stated for Hilbert spaces X and M, but the equivalence of the statements (i) and (ii) in that lemma, which is all that we require here, is valid for reflexive Banach spaces X and M. Now, note that the right-hand side $v \in X \mapsto g(v) := \ell(v) - a(u,v) \in \mathbb{R}$ of (2.10) belongs to V°. Thus, (2.10) is equivalent to finding $p \in M$ such that $\mathsf{B}'p = g \in V^\circ$. As B' is an isomorphism from M onto V°, the existence of a unique such $p \in M$ follows.

Thus, we have shown the existence of a unique solution to (2.9, 2.10) and thereby also to (2.8). This completes the proof. □

We will now apply the above result to (2.1, 2.2). To this end, we consider the form

$$\mathscr{A}((\mathbf{u},p);(\mathbf{v},q)) := A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) - B(\mathbf{u},q)$$

on the space $(\mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)) \times (\mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega))$, and the norm $|||(\cdot,\cdot)|||$, defined by

$$|||(\mathbf{u},p)|||^2 := \|\underline{e}(\mathbf{u})\|^2_{0,\Omega} + \|p\|^2_{0,\Omega}.$$

We are now ready to prove the following result.

THEOREM 2.4 There exists exactly one solution $(\mathbf{u},p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ to the weak formulation (2.1, 2.2).

Proof. We note that (2.1, 2.2) is equivalent to finding $(\mathbf{u},p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ such that

$$\mathscr{A}((\mathbf{u},p);(\mathbf{v},q)) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall (\mathbf{v},q) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega).$$

To complete the proof, it remains to show that the forms $A(\cdot,\cdot) : \mathrm{H}^1_0(\Omega)^d \times \mathrm{H}^1_0(\Omega)^d \to \mathbb{R}$, $B(\cdot,\cdot) : \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega) \to \mathbb{R}$, and $\ell : \mathrm{H}^1_0(\Omega)^d \to \mathbb{R}$, defined by

$$\ell(v) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \qquad \mathbf{v} \in \mathrm{H}^1_0(\Omega)^d,$$

satisfy the hypotheses of Theorem 2.3 with $X = \mathrm{H}^1_0(\Omega)^d$ and $M = \mathrm{L}^2_0(\Omega)$.

We begin by considering $A(\cdot,\cdot)$. Owing to (2.5) and (2.6), we have that

$$|A(\mathbf{u},\mathbf{w}) - A(\mathbf{v},\mathbf{w})| \leqslant C_1\|\mathbf{u}-\mathbf{v}\|_X\|\mathbf{w}\|_X \quad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathrm{H}^1_0(\Omega)^d,$$
$$A(\mathbf{u},\mathbf{u}-\mathbf{v}) - A(\mathbf{v},\mathbf{u}-\mathbf{v}) \geqslant C_2\|\mathbf{u}-\mathbf{v}\|^2_X \quad \forall\, \mathbf{u},\mathbf{v} \in \mathrm{H}^1_0(\Omega)^d,$$

and

$$A(\mathbf{u},\mathbf{u}) \geqslant C_2\|\mathbf{u}\|^2_X \quad \forall\, \mathbf{u} \in \mathrm{H}^1_0(\Omega)^d.$$

Thus, we have verified hypothesis (a) of Theorem 2.3. The validity of hypothesis (b) directly follows from the definition of the bilinear form $B(\cdot,\cdot)$ and the inf–sup condition (2.3). Finally, the validity of hypothesis (c) of Theorem 2.3 follows from the definition of ℓ, the Cauchy–Schwarz inequality and Korn's inequality, according to which there exists a positive constant C_* such that $\|\mathbf{v}\|_{1,\Omega} \leqslant C_*\|\underline{e}(\mathbf{v})\|_{0,\Omega}$ for all $\mathbf{v} \in \mathrm{H}^1_0(\Omega)^d$. This completes the proof. □

We shall also require the following result.

PROPOSITION 2.5 There exist two constants $L, c > 0$ such that the following hold.

(a) Continuity: for any $(\mathbf{u}, p)$, $(\mathbf{v}, q)$, $(\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$, we have

$$|\mathscr{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}((\mathbf{w}, r); (\mathbf{v}, q))| \leqslant L |||(\mathbf{u} - \mathbf{w}, p - r)||| \, |||(\mathbf{v}, q)|||.$$

(b) Inf–sup stability: for any $(\mathbf{u}, p)$, $(\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ there exists $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$\mathscr{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}((\mathbf{w}, r); (\mathbf{v}, q)) \geqslant c |||(\mathbf{u} - \mathbf{w}, p - r)|||, \quad |||(\mathbf{v}, q)||| \leqslant 1.$$

(c) For any $0 \neq (\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$,

$$\sup_{(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathscr{A}((\mathbf{u}, p); (\mathbf{v}, q)) > 0.$$

Proof. We prove (a–c) separately.

Proof of (a): applying the triangle inequality, we have that

$$|\mathscr{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}((\mathbf{w}, r); (\mathbf{v}, q))| \leqslant |A(\mathbf{u}, \mathbf{v}) - A(\mathbf{w}, \mathbf{v})| + |B(\mathbf{v}, p - r)| + |B(\mathbf{u} - \mathbf{w}, q)|.$$

Then, recalling (2.5) leads to

$$\begin{aligned} |A(\mathbf{u}, \mathbf{v}) - A(\mathbf{w}, \mathbf{v})| &\leqslant \int_\Omega |\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})| |\underline{e}(\mathbf{v})| \, \mathrm{d}\mathbf{x} \\ &\leqslant C_1 \int_\Omega |\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})| |\underline{e}(\mathbf{v})| \, \mathrm{d}\mathbf{x} \leqslant C_1 \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega}. \end{aligned}$$

Furthermore,

$$|B(\mathbf{v}, p - r)| \leqslant \int_\Omega |p - r| |\nabla \cdot \mathbf{v}| \, \mathrm{d}\mathbf{x} \leqslant \|p - r\|_{0,\Omega} \|\nabla \mathbf{v}\|_{0,\Omega}.$$

According to Korn's inequality, there exists a positive constant C_* such that $\|\mathbf{v}\|_{1,\Omega} \leqslant C_* \|\underline{e}(\mathbf{v})\|_{0,\Omega}$ for all $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$; thus, we arrive at

$$|B(\mathbf{v}, p - r)| \leqslant C_* \|p - r\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega}.$$

Similarly,

$$|B(\mathbf{u} - \mathbf{w}, q)| \leqslant C_* \|q\|_{0,\Omega} \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega}.$$

Combining these estimates, we obtain

$$\begin{aligned} |\mathscr{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}((\mathbf{w}, r); (\mathbf{v}, q))| &\leqslant C_1 \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w}))\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega} \\ &\quad + C_* \|p - r\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega} + C_* \|q\|_{0,\Omega} \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega}. \end{aligned}$$

Thence, using the Cauchy–Schwarz inequality, we deduce (a).

Proof of (b): let $p - r \in \mathrm{L}_0^2(\Omega)$; then, from the inf–sup condition (2.3), there exists $\boldsymbol{\xi} \in \mathrm{H}_0^1(\Omega)^d$ such that

$$-\int_\Omega (p-r)\nabla\cdot\boldsymbol{\xi}\,\mathrm{d}\mathbf{x} \geqslant \kappa\|p-r\|_{0,\Omega}^2, \quad \|\underline{e}(\boldsymbol{\xi})\|_{0,\Omega} \leqslant \|p-r\|_{0,\Omega}. \tag{2.11}$$

Now, we choose

$$\hat{\mathbf{v}} := \alpha(\mathbf{u}-\mathbf{w}) + \beta\boldsymbol{\xi}, \quad \hat{q} := \alpha(p-r),$$

with

$$\alpha := C_2^{-1}(1 + C_1^2\kappa^{-2}), \quad \beta := 2\kappa^{-1},$$

where C_1 and C_2 are the constants from (2.5) and (2.6). Noting (2.5), (2.6), (2.11) and the arithmetic–geometric mean inequality, we deduce that

$$\begin{aligned}
&\mathscr{A}((\mathbf{u},p);(\hat{\mathbf{v}},\hat{q})) - \mathscr{A}((\mathbf{w},r);(\hat{\mathbf{v}},\hat{q}))\\
&\quad = \int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\hat{\mathbf{v}})\,\mathrm{d}\mathbf{x} - \int_\Omega (p-r)\nabla\cdot\hat{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \int_\Omega \hat{q}\nabla\cdot(\mathbf{u}-\mathbf{w})\,\mathrm{d}\mathbf{x}\\
&\quad = \alpha\int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\mathbf{u}-\mathbf{w})\,\mathrm{d}\mathbf{x}\\
&\qquad + \beta\int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{x} - \beta\int_\Omega (p-r)\nabla\cdot\boldsymbol{\xi}\,\mathrm{d}\mathbf{x}\\
&\quad \geqslant \alpha C_2\int_\Omega |\underline{e}(\mathbf{u}-\mathbf{w})|^2\,\mathrm{d}\mathbf{x} - \frac{1}{2}\kappa\beta\int_\Omega |\underline{e}(\boldsymbol{\xi})|^2\,\mathrm{d}\mathbf{x} + \beta\kappa\|p-r\|_{0,\Omega}^2\\
&\qquad - \frac{1}{2}\kappa^{-1}\beta\int_\Omega |\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})|^2\,\mathrm{d}\mathbf{x}\\
&\quad \geqslant \left(\alpha C_2 - \frac{1}{2}\kappa^{-1}\beta C_1^2\right)\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + \frac{1}{2}\beta\kappa\|p-r\|_{0,\Omega}^2 = |||(\mathbf{u}-\mathbf{w}, p-r)|||^2.
\end{aligned}$$

Using the triangle inequality, we deduce that

$$\begin{aligned}
|||(\hat{\mathbf{v}},\hat{q})|||^2 &= \|\underline{e}(\hat{\mathbf{v}})\|_{0,\Omega}^2 + \|\hat{q}\|_{0,\Omega}^2\\
&\leqslant 2\alpha^2\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + 2\beta^2\|\underline{e}(\boldsymbol{\xi})\|_{0,\Omega}^2 + \alpha^2\|p-r\|_{0,\Omega}^2\\
&\leqslant 2\alpha^2\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + (\alpha^2 + 2\beta^2)\|p-r\|_{0,\Omega}^2\\
&\leqslant \max(2\alpha^2, \alpha^2 + 2\beta^2)|||(\mathbf{u}-\mathbf{w}, p-r)|||^2.
\end{aligned}$$

Setting $(\mathbf{v}, q) = \max(2\alpha^2, \alpha^2 + 2\beta^2)^{-1/2}|||(\mathbf{u}-\mathbf{w}, p-r)|||^{-1}(\hat{\mathbf{v}},\hat{q})$ completes the proof.

Proof of (c): let $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega) \setminus \{(\mathbf{0}, 0)\}$. Then, for $\mathbf{v} \neq 0$, we have that

$$\sup_{(\mathbf{u},p)\in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathscr{A}((\mathbf{u},p);(\mathbf{v},q)) \geqslant \mathscr{A}((\mathbf{v},q);(\mathbf{v},q)) = A(\mathbf{v},\mathbf{v}),$$

and noting (2.6) yields

$$A(\mathbf{v},\mathbf{v}) = \int_{\Omega} \mu(|\underline{e}(\mathbf{v})|)\underline{e}(\mathbf{v}) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \geqslant C_2 \|\underline{e}(\mathbf{v})\|_{0,\Omega}^2 > 0.$$

If $\mathbf{v} = \mathbf{0}$, $q \neq 0$, we use the inf–sup condition (2.3) to find $\mathbf{v}_q \in \mathrm{H}_0^1(\Omega)^d$ such that

$$\sup_{(\mathbf{u},p)\in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathscr{A}((\mathbf{u},p);(\mathbf{0},q)) \geqslant \mathscr{A}(-(\mathbf{v}_q,0);(\mathbf{0},q)) = B(\mathbf{v}_q,q) \geqslant \kappa \|q\|_{0,\Omega} > 0.$$

This completes the proof. □

3. DGFEM approximation of non-Newtonian flows

In this section, we present the discretization of (1.1–1.3) based on employing the *hp*-version of a family of interior penalty (IP) DGFEMs, which includes the symmetric, nonsymmetric and incomplete IP schemes. To this end, we first introduce the necessary notation.

3.1 *Meshes, spaces and trace operators*

Let $\mathscr{T}_h$ be a subdivision of Ω into disjoint open-element domains K such that $\bar{\Omega} = \bigcup_{K\in\mathscr{T}_h} \bar{K}$. We assume that the family of subdivisions $\{\mathscr{T}_h\}_{h>0}$ is shape regular (Braess, 2001, pp. 61, 118 and Remark 2.2, p. 114) and each $K \in \mathscr{T}_h$ is an affine image of a fixed master element $\hat{K}$; i.e., for each $K \in \mathscr{T}_h$, there exists an affine mapping $T_K : \hat{K} \to K$ such that $K = T_K(\hat{K})$, where $\hat{K}$ is the open cube $(-1,1)^3$ in $\mathbb{R}^3$ or the open square $(-1,1)^2$ in $\mathbb{R}^2$. By h_K we denote the element diameter of $K \in \mathscr{T}_h$, $h = \max_{K\in\mathscr{T}_h} h_K$, and $\mathbf{n}_K$ signifies the unit outward normal vector to K. We allow the meshes $\mathscr{T}_h$ to be 1-*irregular*, i.e., each face of any one element $K \in \mathscr{T}_h$ contains at most one hanging node (which, for simplicity, we assume to be at the centre of the corresponding face) and each edge of each face contains at most one hanging node (yet again assumed to be at the centre of the edge). Here, we suppose that $\mathscr{T}_h$ is *regularly reducible* (see Ortner & Süli, 2007), i.e., there exists a shape-regular conforming mesh $\tilde{\mathscr{T}}_h$ such that the closure of each element in $\mathscr{T}_h$ is a union of closures in $\tilde{\mathscr{T}}_h$, and that there exists a constant $C > 0$, independent of mesh sizes, such that, for any two elements $K \in \mathscr{T}_h$ and $\tilde{K} \in \tilde{\mathscr{T}}_h$ with $\tilde{K} \subseteq K$, we have that $h_K/h_{\tilde{K}} \leqslant C$. Note that these assumptions imply that the family $\{\mathscr{T}_h\}_{h>0}$ is of *bounded local variation*, i.e., there exists a constant $\rho_1 \geqslant 1$, independent of the element sizes, such that

$$\rho_1^{-1} \leqslant h_K/h_{K'} \leqslant \rho_1 \tag{3.1}$$

for any pair of elements $K, K' \in \mathscr{T}_h$ that share a common face $F = \partial K \cap \partial K'$. We store the element sizes in the vector $\mathbf{h} := \{h_K : K \in \mathscr{T}_h\}$.

For a non-negative integer k, we denote by $\mathscr{Q}_k(\hat{K})$ the set of all tensor-product polynomials on $\hat{K}$ of degree k in each coordinate direction. To each $K \in \mathscr{T}_h$ we assign a polynomial degree $k_K \geqslant 1$ (local approximation order) and store these in a vector $\mathbf{k} = \{k_K : K \in \mathscr{T}_h\}$. We suppose that $\mathbf{k}$ is also of bounded local variation, i.e., there exists a constant $\rho_2 \geqslant 1$, independent of the element sizes and $\mathbf{k}$, such that, for any pair of neighbouring elements $K, K' \in \mathscr{T}_h$,

$$\rho_2^{-1} \leqslant k_K/k_{K'} \leqslant \rho_2. \tag{3.2}$$

With this notation we introduce the finite element spaces

$$\mathbf{V}_h := \{\mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \circ T_K \in \mathscr{Q}_{k_K}(\hat{K})^d, K \in \mathscr{T}_h\},$$
$$Q_h := \{q \in \mathrm{L}^2_0(\Omega) : q|_K \circ T_K \in \mathscr{Q}_{k_K-1}(\hat{K}), K \in \mathscr{T}_h\}.$$

We define an interior face F of $\mathscr{T}_h$ as the intersection of two neighbouring elements $K, K' \in \mathscr{T}_h$, i.e., $F = \partial K \cap \partial K'$. Similarly, we define a boundary face $F \subset \Gamma$ as the entire face of an element K on the boundary. We denote by $\mathscr{F}_{\mathscr{I}}$ the set of all interior faces, $\mathscr{F}_{\mathscr{B}}$ the set of all boundary faces and $\mathscr{F} = \mathscr{F}_{\mathscr{I}} \cup \mathscr{F}_{\mathscr{B}}$ the set of all faces.

We shall now define suitable face operators that are required for the definition of the proceeding DGFEM. Let q, $\mathbf{v}$ and $\underline{\tau}$ be scalar-, vector- and matrix-valued functions, respectively, which are smooth inside each element $K \in \mathscr{T}_h$. Given two adjacent elements, $K^+, K^- \in \mathscr{T}_h$, which share a common face $F \in \mathscr{F}_{\mathscr{I}}$, i.e., $F = \partial K^+ \cap \partial K^-$, we write $q^\pm$, $\mathbf{v}^\pm$ and $\underline{\tau}^\pm$ to denote the traces of the functions q, $\mathbf{v}$ and $\underline{\tau}$, respectively, on the face F, taken from the interior of $K^\pm$, respectively. With this notation, the averages of q, $\mathbf{v}$, and $\underline{\tau}$ at $\mathbf{x} \in F$ are given by

$$\{q\} := \tfrac{1}{2}(q^+ + q^-), \quad \{\mathbf{v}\} := \tfrac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \quad \{\underline{\tau}\} := \tfrac{1}{2}(\underline{\tau}^+ + \underline{\tau}^-),$$

respectively. Similarly, the jumps of q, $\mathbf{v}$ and $\underline{\tau}$ at $\mathbf{x} \in F$ are given by

$$[\![q]\!] := q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}, \quad [\![\mathbf{v}]\!] := \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-},$$
$$\underline{[\![\mathbf{v}]\!]} := \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \quad [\![\underline{\tau}]\!] := \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\mathbf{n}_{K^-}.$$

On a boundary face $F \in \mathscr{F}_{\mathscr{B}}$, we set $\{q\} := q$, $\{\mathbf{v}\} := \mathbf{v}$, $\{\underline{\tau}\} := \underline{\tau}$, $[\![q]\!] := q\mathbf{n}$, $[\![\mathbf{v}]\!] := \mathbf{v} \cdot \mathbf{n}$, $\underline{[\![\mathbf{v}]\!]} := \mathbf{v} \otimes \mathbf{n}$ and $[\![\underline{\tau}]\!] := \underline{\tau}\mathbf{n}$, with $\mathbf{n}$ denoting the unit outward normal vector on the boundary Γ.

With this notation, we have the following elementary identities for any scalar-, vector- and matrix-valued functions q, $\mathbf{v}$ and $\underline{\tau}$, respectively:

$$\begin{aligned}\sum_{K \in \mathscr{T}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n}_K \,\mathrm{d}s &= \sum_{F \in \mathscr{F}} \int_F [\![q]\!] \cdot \{\mathbf{v}\} \,\mathrm{d}s + \sum_{F \in \mathscr{F}_{\mathscr{I}}} \int_F \{q\}[\![\mathbf{v}]\!] \,\mathrm{d}s, \\ \sum_{K \in \mathscr{T}_h} \int_{\partial K} \underline{\tau} : (\mathbf{v} \otimes \mathbf{n}_K) \,\mathrm{d}s &= \sum_{F \in \mathscr{F}} \int_F \underline{[\![\mathbf{v}]\!]} : \{\underline{\tau}\} \,\mathrm{d}s + \sum_{F \in \mathscr{F}_{\mathscr{I}}} \int_F \{\mathbf{v}\} \cdot [\![\underline{\tau}]\!] \,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Here, $\mathbf{n}_K$ denotes the unit outward normal vector to the element $K \in \mathscr{T}_h$.

3.2 *DGFEM discretization*

Given a partition $\mathscr{T}_h$ of Ω, together with the corresponding polynomial degree vector $\mathbf{k}$, the IP DGFEM formulation is defined as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}), \tag{3.4}$$
$$-B_h(\mathbf{u}_h, q) = 0 \tag{3.5}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, where

$$A_h(\mathbf{u},\mathbf{v}) := \int_\Omega \mu(|\underline{e}_h(\mathbf{u})|)\underline{e}_h(\mathbf{u}) : \underline{e}_h(\mathbf{v})\,\mathrm{d}\mathbf{x} - \sum_{F\in\mathscr{F}} \int_F \{\mu(|\underline{e}_h(\mathbf{u})|)\underline{e}_h(\mathbf{u})\} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s$$
$$+ \theta \sum_{F\in\mathscr{F}} \int_F \{\mu(h_F^{-1}|\underline{[\![\mathbf{u}]\!]}|)\underline{e}_h(\mathbf{v})\} : \underline{[\![\mathbf{u}]\!]}\,\mathrm{d}s + \sum_{F\in\mathscr{F}} \int_F \sigma \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s,$$
$$B_h(\mathbf{v}, q) := -\int_\Omega q \nabla_h \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathscr{F}} \int_F \{q\}[\![\mathbf{v}]\!]\,\mathrm{d}s$$

and

$$F_h(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

Here, $\underline{e}_h(\cdot)$ and ∇_h denote the elementwise rate-of-strain tensor and gradient operator, respectively, and $\theta \in [-1, 1]$. The *interior penalty parameter* σ is defined as

$$\sigma := \gamma \frac{k_F^2}{h_F}, \tag{3.6}$$

where $\gamma \geqslant 1$ is a constant, which must be chosen sufficiently large (independent of the local element sizes and the polynomial degree). For a face $F \in \mathscr{F}$, we define h_F as the diameter of the face and the face polynomial degree k_F as

$$k_F := \begin{cases} \max(k_K, k_{K'}) & \text{if } F = \partial K \cap \partial K' \in \mathscr{F}_{\mathscr{I}}, \\ k_K & \text{if } F = \partial K \cap \Gamma \in \mathscr{F}_{\mathscr{B}}. \end{cases}$$

REMARK 3.1 We note that the formulation (3.4, 3.5) corresponds to the symmetric interior penalty (SIP) method when $\theta = -1$, the nonsymmetric interior penalty method when $\theta = 1$ and the incomplete interior penalty method when $\theta = 0$.

We introduce the energy norms $\|\cdot\|_{1,h}$ and $\|(\cdot,\cdot)\|_{\mathrm{DG}}$, respectively, by

$$\|\mathbf{v}\|_{1,h}^2 := \|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F\in\mathscr{F}} \int_F \sigma |\underline{[\![\mathbf{v}]\!]}|^2\,\mathrm{d}s$$

and

$$\|(\mathbf{v}, q)\|_{\mathrm{DG}}^2 := \|\mathbf{v}\|_{1,h}^2 + \|q\|_{0,\Omega}^2. \tag{3.7}$$

LEMMA 3.2 The following inequality holds:

$$\|\underline{e}_h(\mathbf{v})\|_{0,\Omega} \leqslant \|\nabla_h \mathbf{v}\|_{0,\Omega}.$$

Furthermore, there exists a constant $C_{\mathscr{K}} > 0$, independent of $\mathbf{h}$ and $\mathbf{k}$, such that

$$\|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla_h \mathbf{v}\|_{0,\Omega}^2 \leqslant C_{\mathscr{K}} \left(\|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F\in\mathscr{F}} \int_F h_F^{-1} |\underline{[\![\mathbf{v}]\!]}|^2\,\mathrm{d}s \right)$$

for all $\mathbf{v} \in \mathrm{H}^1(\Omega, \mathscr{T}_h)$, where $\mathrm{H}^1(\Omega, \mathscr{T}_h) = \{\mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \in \mathrm{H}^1(K)^d, K \in \mathscr{T}_h\}$.

Proof. The proof of the first bound follows from elementary manipulations and the application of the Cauchy–Schwarz inequality. The second estimate is a discrete Korn inequality for piecewise H^1 vector fields; see Brenner (2004, Equation 1.19) and the first inequality of Brenner (2004, p. 1071). □

3.3 *Well-posedness of the DGFEM formulation*

In this section, we will prove that the DGFEM formulation (3.4) and (3.5) admits a unique solution. To this end, let us assume that the bilinear form B_h satisfies the following discrete inf–sup condition:

$$\inf_{0 \neq q \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,h} \|q\|_{0,\Omega}} \geqslant c \left(\max_{K \in \mathcal{T}_h} k_K \right)^{-1}. \tag{3.8}$$

We note that this inf–sup condition holds

- for $k_K \geqslant 2$, $K \in \mathcal{T}_h$; and
- for $k \geqslant 1$ if $\mathcal{T}_h$ is conforming and $k_K = k$ for all $K \in \mathcal{T}_h$;

see Schötzau *et al.* (2002, Theorem 6.2 and Theorem 6.12, respectively).

THEOREM 3.3 Provided that the penalty parameter γ featuring in (3.6) is chosen sufficiently large, there is exactly one solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of the *hp*-DGFEM (3.4, 3.5).

Proof. We set

$$\mathscr{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) := A_h(\mathbf{u}, \mathbf{v}) + B_h(\mathbf{v}, p) - B_h(\mathbf{u}, q),$$

which allows the DGFEM defined in (3.4) and (3.5) to be written in the following compact form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\mathscr{A}_h((\mathbf{u}_h, p_h); (\mathbf{v}, q)) = F_h(\mathbf{v}) \tag{3.9}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

The proof now proceeds analogously to that of Theorem 2.4. Specifically, we will check conditions (a–c) of Theorem 2.3, except that now, instead of $\mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$, we shall work on the finite element space $\mathbf{V}_h \times Q_h$, equipped with the norm $\|\cdot\|_{\mathrm{DG}}$ defined above.

We begin by considering $A_h(\cdot, \cdot)$. Owing to arguments analogous to those in Houston *et al.* (2005a, Lemmas 2.2 and 2.3), we have that

$$|A_h(\mathbf{u}, \mathbf{w}) - A_h(\mathbf{v}, \mathbf{w})| \leqslant C_3 \|\mathbf{u} - \mathbf{v}\|_{1,h} \|\mathbf{w}\|_{1,h} \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_h,$$

$$A_h(\mathbf{u}, \mathbf{u} - \mathbf{v}) - A_h(\mathbf{v}, \mathbf{u} - \mathbf{v}) \geqslant C_4 \|\mathbf{u} - \mathbf{v}\|^2_{1,h} \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}_h$$

and

$$A_h(\mathbf{u}, \mathbf{u}) \geqslant C_4 \|\mathbf{u}\|^2_{1,h} \quad \forall\, \mathbf{u} \in \mathbf{V}_h, \tag{3.10}$$

where C_3 and C_4 are positive constants which are independent of the discretization parameters $\boldsymbol{h}$ and $\boldsymbol{k}$. Thus, we have verified hypothesis (a) of Theorem 2.3. The validity of hypothesis (b) directly follows from the definition of the bilinear form $B_h(\cdot, \cdot)$ and the discrete inf–sup condition (3.8). Finally, the validity of hypothesis (c) of Theorem 2.3 follows from the definition of ℓ, the Cauchy–Schwarz inequality and the discrete version of Korn's inequality stated in Lemma 3.2. This completes the proof. □

Next we shall state the discrete analogue of Proposition 2.5. Its proof is very similar to that of Proposition 2.5 and is therefore omitted.

PROPOSITION 3.4 There exist two constants $L, c > 0$, independent of $\boldsymbol{h}$ and $\boldsymbol{k}$, such that the following hold.

(a) Continuity: for any $(\mathbf{u}, p)$, $(\mathbf{v}, q)$, $(\mathbf{w}, r) \in \mathbf{V}_h \times Q_h$, we have

$$|\mathscr{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}_h((\mathbf{w}, r); (\mathbf{v}, q))| \leqslant L \|(\mathbf{u} - \mathbf{w}, p - r)\|_{\mathrm{DG}} \|(\mathbf{v}, q)\|_{\mathrm{DG}}.$$

(b) Inf–sup stability: for any $(\mathbf{u}, p)$, $(\mathbf{w}, r) \in \mathbf{V}_h \times Q_h$ there exists $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ such that

$$\mathscr{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) - \mathscr{A}_h((\mathbf{w}, r); (\mathbf{v}, q)) \geqslant c \left(\max_{K \in \mathscr{T}_h} k_K \right)^{-2} \|(\mathbf{u} - \mathbf{w}, p - r)\|_{\mathrm{DG}}, \quad \|(\mathbf{v}, q)\|_{\mathrm{DG}} \leqslant 1. \tag{3.11}$$

(c) For any $0 \neq (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$,

$$\sup_{(\mathbf{u}, p) \in \mathbf{V}_h \times Q_h} \mathscr{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) > 0.$$

After these preparatory considerations we are now ready to embark on the error analysis of the DGFEM defined by (3.4) and (3.5). We begin by developing the *a priori* error analysis of the method, followed by its *a posteriori* error analysis.

4. *A priori* error analysis

The goal of this section is to derive an *a priori* error bound for the *hp*-DGFEM proposed in this paper. To this end, we state the following result.

THEOREM 4.1 Let the penalty parameter γ be sufficiently large and the solution $(\mathbf{u}, p)$ of (1.1–1.3) belong to $(\mathrm{C}^1(\Omega) \cap \mathrm{H}^2(\Omega))^d \times (\mathrm{C}^0(\Omega) \cap \mathrm{H}^1(\Omega))$, and let $\mathbf{u}|_K \in \mathrm{H}^{s_K+1}(K)^d$, $p|_K \in \mathrm{H}^{s_K}(K)$, $s_K \geqslant 1$, $K \in \mathscr{T}_h$. Then, provided that the discrete inf–sup condition (3.8) is valid, the following estimate holds:

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}}^2 \leqslant C \max_{K \in \mathscr{T}_h} k_K^4 \sum_{K \in \mathscr{T}_h} \left(\frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|_{s_K+1,K}^2 + \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2 \right),$$

where $(\mathbf{u}_h, p_h)$ is the DGFEM solution defined in (3.4) and (3.5), and the constant $C > 0$ is independent of the mesh size and the polynomial degrees.

Proof. Let us consider two interpolants $\Pi_{\mathbf{u}}$ and Π_p satisfying

$$\begin{aligned} \|\mathbf{u} - \Pi_{\mathbf{u}} \mathbf{u}\|_{1,h}^2 &\leqslant C \sum_{K \in \mathscr{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|_{s_K+1,K}^2, \\ \sum_{K \in \mathscr{T}_h} (\|p - \Pi_p p\|_{0,K}^2 + h_K k_K^{-1} \|p - \Pi_p p\|_{0,\partial K}^2) &\leqslant C \sum_{K \in \mathscr{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2; \end{aligned} \tag{4.1}$$

see Houston *et al.* (2005a, Equation (3.2)), and Houston *et al.* (2002), respectively. Thence, defining

$$\mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \Pi_{\mathbf{u}}\mathbf{u}) + (\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}_h) =: \eta_{\mathbf{u}} + \boldsymbol{\xi}_{\mathbf{u}},$$
$$p - p_h = (p - \Pi_p p) + (\Pi_p p - p_h) =: \eta_p + \xi_p,$$

we have $(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p) \in \mathbf{V}_h \times Q_h$. Next, by the inf–sup stability (3.11), we find $(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p) \in \mathbf{V}_h \times Q_h$ with $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)\|_{\mathrm{DG}} \leqslant 1$ and

$$c\left(\max_{K\in\mathcal{T}_h} k_K\right)^{-2} \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}} \leqslant \mathscr{A}_h((\Pi_{\mathbf{u}}\mathbf{u}, \Pi_p p); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)) - \mathscr{A}_h((\mathbf{u}_h, p_h); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)).$$

Then, owing to our regularity assumptions, the DGFEM (3.4, 3.5) is consistent, and thus,

$$\begin{aligned} c\left(\max_{K\in\mathcal{T}_h} k_K\right)^{-2} \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}} &\leqslant \mathscr{A}_h((\Pi_{\mathbf{u}}\mathbf{u}, \Pi_p p); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)) - \mathscr{A}_h((\mathbf{u}, p); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)) \\ &\leqslant |A_h(\Pi_{\mathbf{u}}\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}})| + |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| + |B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

For term T_1, we apply Houston *et al.* (2005a, Lemma 3.2) to obtain

$$T_1 = |A_h(\Pi_{\mathbf{u}}\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}})| \leqslant C\left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2\right)^{1/2} \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h}.$$

For term T_2, by applying the Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned} T_2 = |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| &\leqslant \|\nabla\cdot\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{0,\Omega}\|\Pi_p p - p\|_{0,\Omega} \\ &\quad + \left(\sum_{F\in\mathcal{F}} \int_F \sigma^{-1}|\{\Pi_p p - p\}|^2\,\mathrm{d}s\right)^{1/2} \left(\sum_{F\in\mathcal{F}} \int_F \sigma|\underline{[\![\hat{\boldsymbol{\xi}}_{\mathbf{u}}]\!]}|^2\,\mathrm{d}s\right)^{1/2}. \end{aligned}$$

By applying Korn's inequality and recalling (3.6) we have that

$$|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leqslant C\|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h}\left(\sum_{K\in\mathcal{T}_h}(\|p - \Pi_p p\|_{0,K}^2 + h_K k_K^{-2}\|p - \Pi_p p\|_{0,\partial K}^2)\right)^{1/2}.$$

Invoking (4.1) results in

$$|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leqslant C\|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h}\left(\sum_{K\in\mathcal{T}_h} \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2\right)^{1/2}.$$

Similarly,

$$T_3 = |B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \leqslant C\|\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}\|_{1,h} \left(\sum_{K \in \mathscr{T}_h} (\|\hat{\xi}_p\|_{0,K}^2 + h_K k_K^{-2} \|\hat{\xi}_p\|_{0,\partial K}^2) \right)^{1/2}.$$

Applying an inverse estimate to the boundary term (see, for example, Schwab, 1998, Theorem 4.76) and scaling, and using (4.1), leads to

$$|B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \leqslant C\|\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}\|_{1,h}\|\hat{\xi}_p\|_{0,\Omega} \leqslant C\|\hat{\xi}_p\|_{0,\Omega} \left(\sum_{K \in \mathscr{T}_h} \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2 \right)^{1/2}.$$

Finally, recalling that $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)\|_{\mathrm{DG}} \leqslant 1$, noting that

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}} \leqslant \|(\boldsymbol{\eta}_{\mathbf{u}}, \eta_p)\|_{\mathrm{DG}} + \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}},$$

and combining the bounds on T_1, T_2 and T_3, completes the proof. □

5. *A posteriori* error analysis

In this section, we develop the *a posteriori* error analysis of the DGFEM defined by (3.4) and (3.5). We define, for an element $K \in \mathscr{T}_h$ and face $F \in \mathscr{F}_{\mathscr{I}}$, the data-oscillation terms

$$\mathscr{O}_K^{(1)} := h_K^2 k_K^{-2} \|(\mathbb{I} - \Pi_{\mathscr{T}_h})|_K(\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\})\|_{0,K}^2$$

and

$$\mathscr{O}_F^{(2)} := h_K k_K^{-1} \|(\mathbb{I} - \Pi_{\mathscr{F}})|_F([\![\mu(\underline{e}_h(\mathbf{u}_h))\underline{e}_h(\mathbf{u}_h)]\!])\|_{0,F}^2,$$

respectively, which depend on the right-hand side $\mathbf{f}$ in (1.1) and the numerical solution $\mathbf{u}_h$ from (3.4) and (3.5). Here, $\mathbb{I}$ represents a generic identity operator, $\Pi_{\mathscr{T}_h}$ is an elementwise L^2 projector onto the finite element space with polynomial degree vector $\{k_K - 1 : K \in \mathscr{T}_h\}$ and $\Pi_{\mathscr{F}}|_F$ is the L^2 projector onto $\mathscr{Q}_{k_F-1}(F)$.

5.1 *Upper bounds*

We now state the following *a posteriori* upper bound for the DGFEM defined by (3.4, 3.5).

THEOREM 5.1 Let $(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ be the analytical solution to the problem (1.1–1.3) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be its DGFEM approximation obtained from (3.4, 3.5). Then, the following *hp*-version *a posteriori* error bound holds:

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}} \leqslant C \left(\sum_{K \in \mathscr{T}_h} \eta_K^2 + \mathscr{O}(\mathbf{f}, \mathbf{u}_h) \right)^{1/2}, \tag{5.1}$$

where the *local error indicators* $\eta_K, K \in \mathscr{T}_h$, are defined by

$$\eta_K^2 := h_K^2 k_K^{-2} \|\Pi_{\mathscr{T}_h}(\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u_h})|)\underline{e}(\mathbf{u}_h)\}) - \nabla p_h\|_{0,K}^2 + \|\nabla \cdot \mathbf{u}_h\|_{0,K}^2 \\ + h_K k_K^{-1} \|[\![p_h]\!] - \Pi_{\mathscr{F}}([\![\mu(|\underline{e}_h(\mathbf{u_h})|)\underline{e}_h(\mathbf{u}_h)]\!])\|_{0,\partial K \setminus \Gamma}^2 + \gamma^2 h_K^{-1} k_K^3 \|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K}^2 \tag{5.2}$$

and

$$\mathscr{O}(\mathbf{f}, \mathbf{u}_h) := \sum_{K \in \mathscr{T}_h} \mathscr{O}_K^{(1)} + \sum_{F \in \mathscr{F}_\mathscr{I}} \mathscr{O}_F^{(2)}. \tag{5.3}$$

Here, the constant $C > 0$ is independent of $\mathbf{h}$, the polynomial degree vector $\mathbf{k}$ and the parameter γ, and only depends on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

The proof of this result will follow in Section 5.3.

Remark 5.2 We observe a slight suboptimality with respect to the polynomial degree in the last term of the local error indicator η_K in (5.2). This results from the use of a nonconforming interpolant in the proof of Theorem 5.1 to deal with the possible presence of hanging nodes in $\mathscr{T}_h$. For conforming meshes, i.e., meshes without hanging nodes, a conforming *hp*-version Clément interpolant, as constructed in Melenk (2005), can be employed, which results in an *a posteriori* error bound of the form (5.1) with the final term in the local error indicators (5.2) replaced by the improved expression

$$\gamma h_K^{-1} k_K^2 \|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K}^2;$$

cf. Houston *et al.* (2005b).

5.2 *Local lower bounds*

For simplicity we shall restrict ourselves to local lower bounds on conforming meshes $\mathscr{T}_h$; the extension to nonconforming 1-irregular regularly reducible meshes follows analogously; cf., for example, Houston *et al.* (2008, Remark 3.9). The following result can be proved along the lines of the analyses contained in Houston *et al.* (2004a, 2008); for details, see Congreve (in preparation).

Theorem 5.3 Let K and K' be any two neighbouring elements in $\mathscr{T}_h$, $F = \partial K \cap \partial K'$ and $\omega_F = (\bar{K} \cup \bar{K}')^\circ$. Then, for all $\delta \in (0, \frac{1}{2}]$, the following *hp*-version *a posteriori* local bounds on the error between the analytical solution $(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ satisfying (1.1–1.3) and the numerical solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ obtained by (3.4) and (3.5) hold:

(a)

$$\|\Pi_{\mathscr{T}_h}|_K(\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\}) - \nabla p\|_{0,K} \\ \leqslant C h_K^{-1} k_K^2 \left(\|\underline{e}(\mathbf{u} - \mathbf{u}_h)\|_{0,K} + \|p - p_h\|_{0,K} + k_K^{\delta - 1/2} \sqrt{\mathscr{O}_K^{(1)}} \right);$$

(b)

$$\|\nabla \cdot \mathbf{u}_h\|_{0,K} \leqslant C \|\underline{e}(\mathbf{u} - \mathbf{u}_h)\|_{0,K};$$

(c)

$$\|[\![p_h]\!] - \Pi_{\mathscr{F}}|_F([\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!])\|_{0,F} \leqslant C h_K^{-1/2} k_K^{\delta+3/2} \Bigg(\|\underline{e}(\mathbf{u}-\mathbf{u}_h)\|_{0,\omega_F} + \|p - p_h\|_{0,\omega_F}$$

$$+ k_K^{\delta-1/2} \sum_{\tau\in\{K,K'\}} \sqrt{\mathscr{O}_\tau^{(1)}} + k_K^{-1/2}\sqrt{\mathscr{O}_F^{(2)}} \Bigg);$$

(d)

$$\|\underline{[\![\mathbf{u}_h]\!]}\|_{0,F} \leqslant C\gamma^{-1/2} h_K^{1/2} k_K^{-1} \|\sigma^{1/2}\underline{[\![\mathbf{u}-\mathbf{u}_h]\!]}\|_{0,F}.$$

Here, the generic constant $C > 0$ depends on δ, but is independent of $\mathbf{h}$ and $\mathbf{k}$.

5.3 *Proof of Theorem* 5.1

The proof of Theorem 5.1 is based on the techniques developed in Houston *et al.* (2004a, 2008); cf. also Karakashian & Pascal (2003).

5.3.1 *DGFEM decomposition.* In order to admit 1-irregular meshes, we consider a subdivision $\mathscr{T}_h$ which is regularly reducible, i.e., $\mathscr{T}_h$ may be refined to create a conforming mesh $\tilde{\mathscr{T}}_h$ as outlined in Section 3.1; cf. Ortner & Süli (2007) and Houston *et al.* (2008). We point out that an analogous hierarchical construction, based on employing enriched 1-irregular partitions $\tilde{\mathscr{T}}_h$, has been studied in two and three dimensions in Zhu & Schötzau (2010) and Zhu *et al.* (2011), respectively. We denote by $\tilde{\mathbf{V}}_h$ and $\tilde{Q}_h$ the corresponding DGFEM finite element spaces with polynomial degree vector $\tilde{\mathbf{k}}$ defined by $\tilde{k}_{\tilde{K}} := k_K$ for any $\tilde{K} \in \tilde{\mathscr{T}}_h$ with $\tilde{K} \subseteq K$, for some $K \in \mathscr{T}_h$. We note that $\mathbf{V}_h \subseteq \tilde{\mathbf{V}}_h$, $Q_h \subseteq \tilde{Q}_h$ and owing to the assumptions in Section 3.1, the energy norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{\widetilde{1,h}}$ corresponding to the spaces $\mathbf{V}_h$ and $\tilde{\mathbf{V}}_h$, respectively, are equivalent on $\mathbf{V}_h$; in particular, there exist constants $N_1, N_2 > 0$, independent of $\mathbf{h}$ and $\mathbf{k}$, such that

$$N_1 \sum_{F\in\mathscr{F}} \int_F \sigma |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s \leqslant \sum_{\tilde{F}\in\tilde{\mathscr{F}}} \int_{\tilde{F}} \tilde{\sigma} |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s \leqslant N_2 \sum_{F\in\mathscr{F}} \int_F \sigma |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s; \tag{5.4}$$

cf. Ortner & Süli (2007) and Houston *et al.* (2008). Here, $\tilde{\mathscr{F}}$ denotes the set of all faces in the mesh $\tilde{\mathscr{T}}_h$, and $\tilde{\sigma}$ is the discontinuous penalization parameter on $\tilde{\mathbf{V}}_h$ which is defined analogously to σ on $\mathbf{V}_h$.

An important step in the proof is to decompose the DGFEM space $\tilde{\mathbf{V}}_h$ into two orthogonal subspaces: a conforming part $\tilde{\mathbf{V}}_h^{\mathrm{c}} = \tilde{\mathbf{V}}_h \cap \mathrm{H}_0^1(\Omega)^d$ and a nonconforming part $\tilde{\mathbf{V}}_h^{\perp}$, which is defined as the orthogonal complement of $\tilde{\mathbf{V}}_h^{\mathrm{c}}$ with respect to the energy inner product $(\cdot,\cdot)_{\widetilde{1,h}}$ (inducing the norm $\|\cdot\|_{\widetilde{1,h}}$), i.e.,

$$\tilde{\mathbf{V}}_h = \tilde{\mathbf{V}}_h^{\mathrm{c}} \oplus_{\|\cdot\|_{\widetilde{1,h}}} \tilde{\mathbf{V}}_h^{\perp}.$$

Based on this setting the DGFEM solution $\mathbf{u}_h$ may be split accordingly:

$$\mathbf{u}_h = \mathbf{u}_h^{\mathrm{c}} + \mathbf{u}_h^{\perp}, \tag{5.5}$$

where $\mathbf{u}_h^{\mathrm{c}} \in \tilde{\mathbf{V}}_h^{\mathrm{c}}$ and $\mathbf{u}_h^{\perp} \in \tilde{\mathbf{V}}_h^{\perp}$. Furthermore, we define the error in the velocity vector as

$$\mathbf{e_u} := \mathbf{u} - \mathbf{u}_h, \tag{5.6}$$

and the error in the pressure as

$$e_p := p - p_h, \tag{5.7}$$

and let

$$\mathbf{e}_{\mathbf{u}}^{\mathrm{c}} := \mathbf{u} - \mathbf{u}_h^{\mathrm{c}} \in \mathrm{H}_0^1(\Omega)^d. \tag{5.8}$$

5.3.2 *Auxiliary results.* In order to prove Theorem 5.1, we require the following auxiliary results.

PROPOSITION 5.4 Under the foregoing assumptions on the subdivision $\tilde{\mathscr{T}}_h$, the following bound holds over the space $\tilde{\mathbf{V}}_h^{\perp}$:

$$\tilde{C}\|\mathbf{v}\|_{\widetilde{1,h}}^2 \leqslant \sum_{\tilde{F}\in\tilde{\mathscr{F}}} \int_{\tilde{F}} \tilde{\sigma}\,|\underline{[\![\mathbf{v}]\!]}|^2\,\mathrm{d}s \quad \forall\,\mathbf{v} \in \tilde{\mathbf{V}}_h^{\perp},$$

where the constant $\tilde{C} > 0$ depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

Proof. The proof follows, for the case when $d = 2$, by first applying Lemma 3.2 and then extending Houston *et al.* (2007, Proposition 4.1) and Houston *et al.* (2008, Proposition 3.5) to vector-valued functions. The case when $d = 3$ can be similarly derived from Zhu *et al.* (2011, Theorem 4.1). □

COROLLARY 5.5 With $\mathbf{u}_h^{\perp}$ defined by (5.5), the following bound holds:

$$\|\mathbf{u}_h^{\perp}\|_{\widetilde{1,h}} \leqslant D\left(\sum_{F\in\mathscr{F}} \int_F \sigma\,|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2},$$

where the constant $D > 0$ is independent of γ, $\mathbf{h}$ and $\mathbf{k}$, and depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

Proof. Owing to the fact that Proposition 5.4 holds we can simply extend the proof from Houston *et al.* (2008, Corollary 3.6). □

We now state the following approximation result.

LEMMA 5.6 For any $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$ a there exists $\mathbf{v}_h \in \mathbf{V}_h$ such that

$$\sum_{K\in\mathscr{T}_h} \left(\frac{k_K^2}{h_K}\|\mathbf{v}-\mathbf{v}_h\|_{0,K}^2 + \|\underline{e}(\mathbf{v}-\mathbf{v}_h)\|_{0,K}^2 + \frac{k_K}{h_K}\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K}^2\right) \leqslant C_I\|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2,$$

with an interpolation constant $C_I > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$, which depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

Proof. This follows from applying Houston *et al.* (2008, Lemma 3.7) componentwise to the vector field $\mathbf{v}$. □

5.3.3 *Proof of Theorem 5.1.* We now complete the proof of Theorem 5.1. To this end we recall the compact formulation (3.9) as well as the definition of the error in (5.6) and (5.7). Then, by (5.4), Corollary 5.5 and the facts that $\gamma \geqslant 1$ and $k_K \geqslant 1$, we have

$$\begin{aligned}
\|(\mathbf{e}_{\mathbf{u}}, e_p)\|_{\mathrm{DG}} &\leqslant \|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} + \|\mathbf{u}_h^{\perp}\|_{1,h} \\
&= \|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} + \left(\sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \|\underline{e}(\mathbf{u}_h^{\perp})\|_{0,\tilde{K}}^2 + \sum_{F \in \mathscr{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h^{\perp}]\!]}|^2 \,\mathrm{d}s \right)^{1/2} \\
&\leqslant \|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} + \max(1, N_1^{-1/2}) \|\mathbf{u}_h^{\perp}\|_{\widetilde{1,h}} \\
&\leqslant \|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} + \max(1, N_1^{-1/2}) D \left(\sum_{F \in \mathscr{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h]\!]}|^2 \,\mathrm{d}s \right)^{1/2} \\
&\leqslant \|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} + \max(1, N_1^{-1/2}) D \left(\sum_{K \in \mathscr{T}_h} \eta_K^2 \right)^{1/2}. \qquad (5.9)
\end{aligned}$$

To bound the term $\|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}}$, we invoke the result from Proposition 2.5(b) which gives a function $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$c\|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} \leqslant \mathscr{A}_h(\mathbf{u}, p, \mathbf{v}, q) - \mathscr{A}_h(\mathbf{u}_h^{\mathrm{c}}, p_h, \mathbf{v}, q), \quad \|(\mathbf{v}, q)\|_{\mathrm{DG}} \leqslant 1. \qquad (5.10)$$

Note here that, since $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$, we have that $\underline{[\![\mathbf{v}]\!]} = \underline{0}$ on $\mathscr{F}$. Therefore, from (5.5), we deduce that

$$\begin{aligned}
c\|(\mathbf{e}_{\mathbf{u}}^{\mathrm{c}}, e_p)\|_{\mathrm{DG}} &\leqslant \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h^{\mathrm{c}})|)\underline{e}(\mathbf{u}_h^{\mathrm{c}})\} : \underline{e}(\mathbf{v}) \,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} (p - p_h) \nabla \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} q \nabla \cdot (\mathbf{u} - \mathbf{u}_h^{\mathrm{c}}) \,\mathrm{d}\mathbf{x} \\
&= \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} : \underline{e}(\mathbf{v}) \,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) - \mu(|\underline{e}(\mathbf{u}_h^{\mathrm{c}})|)\underline{e}(\mathbf{u}_h^{\mathrm{c}})\} : \underline{e}(\mathbf{v}) \,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} (p - p_h) \nabla \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} q \nabla \cdot (\mathbf{u} - \mathbf{u}_h) \,\mathrm{d}\mathbf{x} + \sum_{\tilde{K} \in \tilde{\mathscr{T}}_h} \int_{\tilde{K}} q \nabla \cdot \mathbf{u}_h^{\perp} \,\mathrm{d}\mathbf{x} \\
&\equiv T_1 + T_2, \qquad (5.11)
\end{aligned}$$

where

$$
\begin{aligned}
T_1 &= \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h} \int_{\tilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h} \int_{\tilde{K}} (p-p_h)\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h} \int_{\tilde{K}} q\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\mathbf{x}, \\
T_2 &= \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h} \int_{\tilde{K}} \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) - \mu(|\underline{e}(\mathbf{u}_h^{\mathrm{c}})|)\underline{e}(\mathbf{u}_h^{\mathrm{c}})\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} + \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h} \int_{\tilde{K}} q\nabla\cdot\mathbf{u}_h^{\perp}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We start by bounding T_1. To this end, employing integration by parts and equations (1.1) and (1.2), we obtain

$$
\begin{aligned}
T_1 &= \sum_{K\in\mathscr{T}_h} \int_K (-\nabla\cdot\{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u})\} + \nabla p)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathscr{T}_h} \int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{K\in\mathscr{T}_h} \int_K p_h\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{K\in\mathscr{T}_h} \int_K q\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\mathbf{x} \\
&= \sum_{K\in\mathscr{T}_h} \int_K \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathscr{T}_h} \int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{K\in\mathscr{T}_h} \int_K p_h\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathscr{T}_h} \int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We let $\mathbf{v}_h \in \mathbf{V}_h$ be the elementwise interpolant of $\mathbf{v}$, which satisfies Lemma 5.6. Then, noting from (3.9) that $\mathscr{A}_h(\mathbf{u}_h, p_h, \mathbf{v}_h, 0) - F_h(\mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{V}_h$, gives

$$
\begin{aligned}
T_1 &= \sum_{K\in\mathscr{T}_h} \int_K \mathbf{f}\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathscr{T}_h} \int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{F\in\mathscr{F}} \int_F (\{\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)\} : \underline{[\![\mathbf{v}_h]\!]} - \theta\{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\} : \underline{[\![\mathbf{u}_h]\!]})\,\mathrm{d}s \\
&\quad + \sum_{K\in\mathscr{T}_h} \int_K p_h\nabla\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathscr{F}} \int_F \{p_h\}[\![\mathbf{v}_h]\!]\,\mathrm{d}s - \sum_{K\in\mathscr{T}_h} \int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{F\in\mathscr{F}} \int_F \sigma\,\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s.
\end{aligned}
$$

Integration by parts yields

$$
\begin{aligned}
T_1 &= \sum_{K\in\mathscr{T}_h} \int_K (\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{K\in\mathscr{T}_h} \int_{\partial K} (p_h(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}_K - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : (\mathbf{v}-\mathbf{v}_h)\otimes\mathbf{n}_K)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{F\in\mathscr{F}} \int_F (\{\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)\} : \underline{[\![\mathbf{v}_h]\!]} - \theta\{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\} : \underline{[\![\mathbf{u}_h]\!]})\,\mathrm{d}s \\
&+ \sum_{F\in\mathscr{F}} \int_F \{p_h\}[\![\mathbf{v}_h]\!]\,\mathrm{d}s - \sum_{K\in\mathscr{T}_h} \int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathscr{F}} \int_F \sigma\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s.
\end{aligned}
$$

Since $\mathbf{v}\in\mathrm{H}_0^1(\Omega)^d$, we have that $\underline{[\![\mathbf{v}]\!]} = \underline{0}$, which implies that $|\underline{[\![\mathbf{v}_h]\!]}| = |\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}|$ on $\mathscr{F}$. Thereby, using this result, together with the application of (3.3), gives

$$
\begin{aligned}
T_1 &= \sum_{K\in\mathscr{T}_h} \int_K (\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{F\in\mathscr{F}_{\mathscr{I}}} \int_F ([\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!])\cdot\{\mathbf{v}-\mathbf{v}_h\}\,\mathrm{d}s - \sum_{K\in\mathscr{T}_h} \int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} \\
&\quad + \theta \sum_{F\in\mathscr{F}} \int_F \{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\} : \underline{[\![\mathbf{u}_h]\!]}\,\mathrm{d}s + \sum_{F\in\mathscr{F}} \int_F \sigma\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h - \mathbf{v}]\!]}\,\mathrm{d}s \\
&\leqslant \sum_{K\in\mathscr{T}_h} \|\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\|_{0,K}\|\mathbf{v}-\mathbf{v}_h\|_{0,K} + \sum_{K\in\mathscr{T}_h} \|q\|_{0,K}\|\nabla\cdot\mathbf{u}_h\|_{0,K} \\
&\quad + C\sum_{K\in\mathscr{T}_h} \|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma}\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K\setminus\Gamma} \\
&\quad + M_\mu|\theta| \left(\sum_{F\in\mathscr{F}} \int_F h_F^{-1}k_F^2|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2} \left(\sum_{F\in\mathscr{F}} \int_F h_F k_F^{-2}|\{|\underline{e}_h(\mathbf{v}_h)|\}|^2\,\mathrm{d}s\right)^{1/2} \\
&\quad + \left(\sum_{F\in\mathscr{F}} \int_F \sigma k_F|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2} \left(\sum_{F\in\mathscr{F}} \int_F \sigma k_F^{-1}|\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2}.
\end{aligned}
$$

Exploiting the trace inequalities in Schwab (1998, Theorem 4.76) and Schötzau *et al.* (2002, Lemma 7.1), and noting that $k_F \geqslant 1$, we obtain

$$
\begin{aligned}
T_1 &\leqslant \sum_{K\in\mathscr{T}_h} h_K k_K^{-1}\|\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\|_{0,K} h_K^{-1}k_K\|\mathbf{v}-\mathbf{v}_h\|_{0,K} \\
&\quad + \sum_{K\in\mathscr{T}_h} \|q\|_{0,K}\|\nabla\cdot\mathbf{u}_h\|_{0,K} \\
&\quad + C\sum_{K\in\mathscr{T}_h} h_K^{1/2}k_K^{-1/2}\|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma} h_K^{-1/2}k_K^{1/2}\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K\setminus\Gamma} \\
&\quad + C|\theta| \left(\sum_{F\in\mathscr{F}} \int_F h_F^{-1}k_F^2|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2} \left(\sum_{K\in\mathscr{T}_h} \|\underline{e}(\mathbf{v}_h)\|_{0,K}^2\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\gamma^{1/2}\left(\sum_{F\in\mathscr{F}}\int_F \sigma k_F|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{1/2}\left(\sum_{K\in\mathscr{T}_h} h_K^{-1}k_K\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K}^2\right)^{1/2}\\
&\leqslant C\Bigg\{\sum_{K\in\mathscr{T}_h}(h_K^2k_K^{-2}\|\mathbf{f}+\nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\}-\nabla p_h\|_{0,K}^2+\|\nabla\cdot\mathbf{u}_h\|_{0,K}^2\\
&\quad + h_Kk_K^{-1}\|[\![p_h]\!]-[\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma}^2)+\gamma\sum_{F\in\mathscr{F}}\int_F\sigma k_F|\underline{[\![\mathbf{u_h}]\!]}|^2\,\mathrm{d}s\Bigg\}^{1/2}\\
&\quad\times\left\{\sum_{K\in\mathscr{T}_h}(h_K^{-2}k_K^2\|\mathbf{v}-\mathbf{v}_h\|_{0,K}^2+h_K^{-1}k_K\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K}^2+|\theta|\|\underline{e}(\mathbf{v}_h)\|_{0,K}^2+\|q\|_{0,K}^2)\right\}^{1/2}.
\end{aligned}
$$

For $K\in\mathscr{T}_h$ we write

$$
\begin{aligned}
\tilde{\eta}_K^2 &= h_K^2k_K^{-2}\|\mathbf{f}+\nabla\cdot\{\mu(|\underline{e}(\mathbf{u_h})|)\underline{e}(\mathbf{u}_h)\}-\nabla p_h\|_{0,K}^2+\|\nabla\cdot\mathbf{u}_h\|_{0,K}^2\\
&\quad + h_Kk_K^{-1}\|[\![p_h]\!]-[\![\mu(|\underline{e}_h(\mathbf{u_h})|)\underline{e}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma}^2+\gamma^2h_K^{-1}k_K^3\|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K}^2.
\end{aligned}
$$

Then, noting that $\gamma\geqslant 1\geqslant|\theta|\geqslant 0$, $\|\underline{e}(\mathbf{v}_h)\|_{0,K}^2\leqslant\|\underline{e}(\mathbf{v}-\mathbf{v}_h)\|_{0,K}^2+\|\underline{e}(\mathbf{v})\|_{0,K}^2$, applying Lemma 5.6 and (5.10) gives

$$
\begin{aligned}
T_1 &\leqslant C\left(\sum_{K\in\mathscr{T}_h}\tilde{\eta}_K^2\right)^{1/2}\left(\sum_{K\in\mathscr{T}_h}\{\|\underline{e}(\mathbf{v})\|_{0,K}^2+\|q\|_{0,K}^2\}\right)^{1/2}\\
&\leqslant C\left(\sum_{K\in\mathscr{T}_h}\tilde{\eta}_K^2\right)^{1/2}\|(\mathbf{v},q)\|_{\mathrm{DG}}\leqslant C\left(\sum_{K\in\mathscr{T}_h}\tilde{\eta}_K^2\right)^{1/2}.
\end{aligned}
$$

By the application of the triangle inequality we deduce the following bound for T_1:

$$
T_1\leqslant C\left(\sum_{K\in\mathscr{T}_h}\eta_K^2+\mathscr{O}(\mathbf{f},\mathbf{u}_h)\right)^{1/2}. \tag{5.12}
$$

We now consider the T_2 term. By using the bound (2.5) and the trace inequality, we obtain that

$$
\begin{aligned}
T_2 &\leqslant \sum_{\tilde{K}\in\tilde{\mathscr{T}}_h}\int_{\tilde{K}}|\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)-\mu(|\underline{e}(\mathbf{u}_h^{\mathrm{c}})|)\underline{e}(\mathbf{u}_h^{\mathrm{c}})||\underline{e}(\mathbf{v})|\,\mathrm{d}\mathbf{x}+\sum_{\tilde{K}\in\tilde{\mathscr{T}}_h}\int_{\tilde{K}}|q||\nabla\cdot\mathbf{u}_h^{\perp}|\,\mathrm{d}\mathbf{x}\\
&\leqslant C_1\sum_{\tilde{K}\in\tilde{\mathscr{T}}_h}\int_{\tilde{K}}|\underline{e}(\mathbf{u}_h^{\perp})||\underline{e}(\mathbf{v})|\,\mathrm{d}\mathbf{x}+\sum_{\tilde{K}\in\tilde{\mathscr{T}}_h}\int_{\tilde{K}}|q||\nabla\cdot\mathbf{u}_h^{\perp}|\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

$$\leqslant C_1 \sum_{\tilde{K}\in\tilde{\mathcal{T}}_h} (\|\underline{e}(\mathbf{u}_h^\perp)\|_{0,\tilde{K}} \|\underline{e}(\mathbf{v})\|_{0,\tilde{K}} + \|q\|_{0,\tilde{K}} \|\nabla\cdot\mathbf{u}_h^\perp\|_{0,\tilde{K}})$$

$$\leqslant C \left\{ \sum_{\tilde{K}\in\tilde{\mathcal{T}}_h} (\|\underline{e}(\mathbf{u}_h^\perp)\|_{0,\tilde{K}}^2 + \|\nabla\cdot\mathbf{u}_h^\perp\|_{0,\tilde{K}}^2) \right\}^{1/2} \left\{ \sum_{\tilde{K}\in\tilde{\mathcal{T}}_h} (\|\underline{e}(\mathbf{v})\|_{0,\tilde{K}}^2 + \|q\|_{0,\tilde{K}}^2) \right\}^{1/2}.$$

We note that, because of Lemma 3.2, we have that

$$\sum_{\tilde{K}\in\tilde{\mathcal{T}}_h} \|\nabla\cdot\mathbf{u}_h^\perp\|_{0,\tilde{K}}^2 \leqslant d \sum_{\tilde{K}\in\tilde{\mathcal{T}}_h} \|\nabla\mathbf{u}_h^\perp\|_{0,\tilde{K}}^2 \leqslant dC_{\mathcal{K}} \|\mathbf{u}_h^\perp\|_{\widetilde{1,h}}^2.$$

Therefore, applying Corollary 5.5 gives

$$T_2 \leqslant C((1+dC_{\mathcal{K}})\|\mathbf{u}_h^\perp\|_{\widetilde{1,h}}^2)^{1/2} \|(\mathbf{v},q)\|_{\mathrm{DG}} \leqslant C \left(\sum_{F\in\mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h]\!]}|^2 \,\mathrm{d}s \right)^{1/2} \|(\mathbf{v},q)\|_{\mathrm{DG}}.$$

Recalling (5.10), we deduce that

$$T_2 \leqslant C \left(\sum_{K\in\mathcal{T}_h} \eta_K^2 \right)^{1/2}. \tag{5.13}$$

Substituting (5.11), (5.12) and (5.13) into (5.9) completes the proof.

6. Numerical experiments

In this section, we present a series of numerical experiments to verify the *a priori* error estimate derived in Section 4, as well as to demonstrate the performance of the *a posteriori* error bound derived in Theorem 5.1 within an automatic *hp*-adaptive refinement procedure based on 1-irregular quadrilateral elements for $\Omega \subset \mathbb{R}^2$. Throughout this section the DGFEM solution (u_h, p_h) defined by (3.4, 3.5) is computed with $\theta = -1$, i.e., we employ the SIP DGFEM. Additionally, we set the constant γ appearing in the interior penalty parameter σ defined by (3.6) equal to 10. The resulting system of nonlinear equations is solved by a damped Newton method; for each inner (linear) iteration, we employ the Multifrontal Massively Parallel Solver (MUMPS); see Amestoy *et al.* (2000, 2001, 2006).

The *hp*-adaptive meshes are constructed by first marking the elements for refinement/derefinement according to the size of the local error indicators η_K; this is achieved via a fixed fraction strategy where the refinement and derefinement fractions are set to 25% and 5%, respectively. We employ the *hp*-adaptive strategy developed by Houston & Süli (2005) to decide whether *h*- or *p*-refinement/derefinement should be performed on an element $K \in \mathcal{T}_h$ marked for refinement/derefinement. We note here that we start with a polynomial degree of $k_K = 3$ for all $K \in \mathcal{T}_h$.

The purpose of these experiments is to demonstrate that the *a posteriori* error indicator in Theorem 5.1 converges to zero at the same asymptotic rate as the actual error in the DGFEM energy norm $\|(\cdot,\cdot)\|_{\mathrm{DG}}$, on a sequence of nonuniform *hp*-adaptively refined meshes. We also demonstrate that the *hp*-adaptive strategy converges at a higher rate than an *h*-adaptive refinement strategy, which uses the same 25% and 5% refinement/derefinement fixed fraction strategy, but only undertakes mesh subdivision for a fixed (uniform) polynomial degree distribution. As in Becker *et al.* (2003) and Houston *et al.* (2008) we set the constant C arising in Theorem 5.1 equal to 1 for simplicity; in gen-

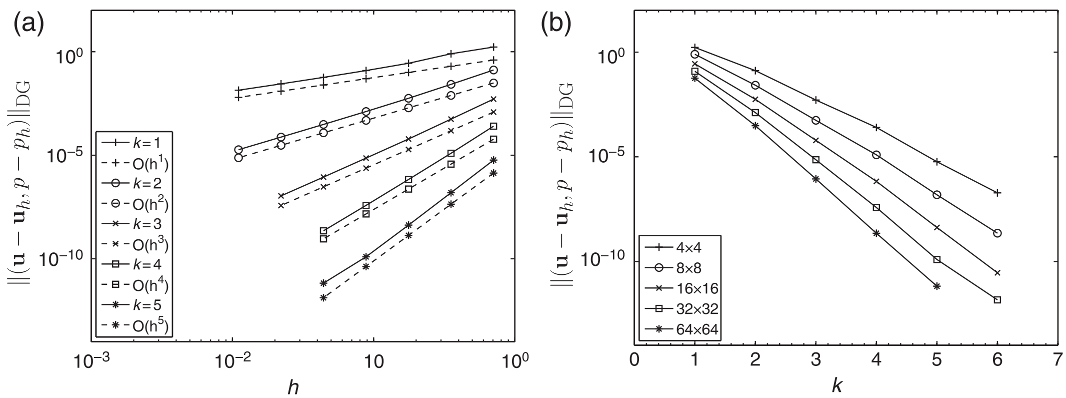

FIG. 1. Example 1. Convergence of the DGFEM with (a) h-refinement; (b) p-refinement.

eral this constant must be determined numerically from the underlying problem to ensure the reliability of the error estimator; cf. Eriksson *et al.* (1995). We are then able to check that the effectivity indices, defined as the ratio of the *a posteriori* error bound and the DGFEM energy norm of the true error, is roughly constant. We also ignore in all our experiments the data-oscillation terms arising in Theorem 5.1.

6.1 *Example 1: smooth solution*

In this first example, we let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and consider the nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = 2 + \frac{1}{1+|e(\mathbf{u})|^2}.$$

In addition, we select $\mathbf{f}$ so that the analytical solution to (1.1–1.3) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} -\mathrm{e}^x(y\cos(y)+\sin(y)) \\ \mathrm{e}^x y\sin(y) \end{pmatrix},$$
$$p(x,y) = 2\mathrm{e}^x\sin(y) - \frac{2(1-\mathrm{e})(\cos(1)-1)}{3}.$$

Here, we investigate the convergence of the DGFEM defined by (3.4) and (3.5) on a sequence of hierarchically and uniformly refined square meshes for different (fixed) values of the polynomial degree k. To this end, in Fig. 1(a) we present a comparison of the DGFEM energy norm $\|(\cdot,\cdot)\|_{\mathrm{DG}}$ with the mesh function h for k ranging between 1 and 5. Here, we clearly see that $\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}}$ converges like $\mathscr{O}(h^k)$ as h tends to zero for each (fixed) k, which is in complete agreement with Theorem 4.1. Secondly, we investigate the convergence of the DGFEM with p-enrichment for fixed h. Since the analytical solution to this problem is a real analytic function, we expect to observe exponential rates of convergence. Indeed, Fig. 1(b) clearly illustrates this behaviour: on the linear–log

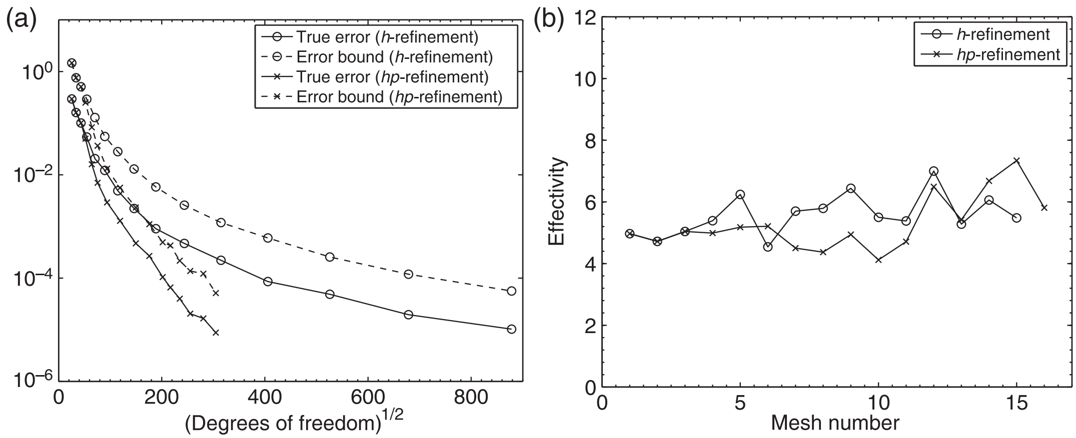

FIG. 2. Example 2. (a) Comparison of the error in the DGFEM norm employing both h- and hp-refinement, with respect to the number of degrees of freedom; (b) the effectivity index using both h- and hp-refinement.

scale, the convergence plots for each mesh become straight lines as the degree of the approximating polynomial is increased.

6.2 *Example 2: cavity problem*

In this example we consider the cavity-like problem from Berrone & Süli (2008, Section 6.1) using the Carreau-law nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = k_\infty + (k_0 - k_\infty)(1 + \lambda|\underline{e}(\mathbf{u})|^2)^{(\theta-2)/2},$$

with $k_\infty = 1, k_0 = 2, \lambda = 1$ and $\theta = 1.2$. We let Ω be the unit square $(0,1)^2 \subset \mathbb{R}^2$ and select the forcing function $\mathbf{f}$ so that the analytical solution to (1.1–1.3) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} \left(1 - \cos\left(2\dfrac{\pi(\mathrm{e}^{\theta x}-1)}{\mathrm{e}^{\theta}-1}\right)\right)\sin(2\pi y) \\ -\theta\,\mathrm{e}^{\theta x}\sin\left(2\dfrac{\pi(\mathrm{e}^{\theta x}-1)}{\mathrm{e}^{\theta}-1}\right)\dfrac{1-\cos(2\pi y)}{\mathrm{e}^{\theta}-1} \end{pmatrix},$$

$$p(x,y) = 2\pi\theta\,\mathrm{e}^{\theta x}\sin\left(2\frac{\pi(\mathrm{e}^{\theta x}-1)}{\mathrm{e}^{\theta}-1}\right)\frac{\sin(2\pi y)}{\mathrm{e}^{\theta}-1}.$$

In this example, we now turn our attention to the performance of the proposed hp-adaptive refinement algorithm. To this end, in Fig. 2(a) we present a comparison of the actual error measured in the DGFEM norm and the *a posteriori* error bound versus the square root of the number of degrees of freedom on a linear–log scale for the sequence of meshes generated by both the h- and hp-adaptive algorithm; in each case the initial value of the polynomial degree k is set equal to 3. We observe that the error bound overestimates the true error by roughly a consistent factor; this is confirmed in Fig. 2(b), where the effectivity indices for the sequence of meshes which, although slightly oscillatory, all lie

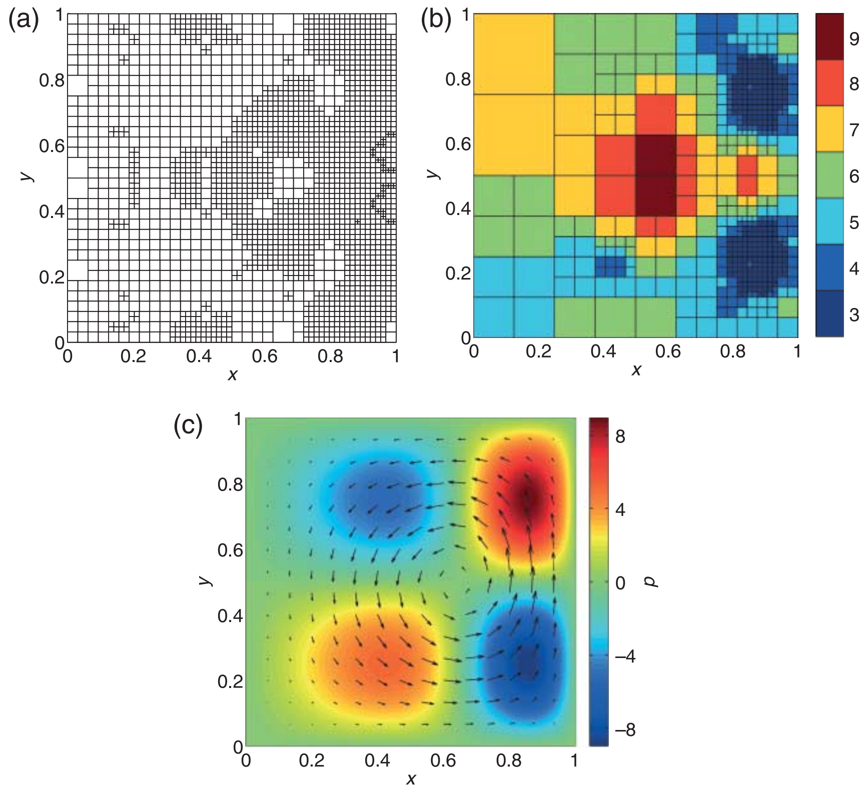

FIG. 3. Example 2. Finite element mesh after 10 adaptive refinements: (a) h-adaptivity; (b) hp-adaptivity; (c) analytical solution.

roughly in the range 4–7. From Fig. 2(a), we can also see that the DGFEM norm of the error converges to zero at an exponential rate when hp-adaptivity is employed. Consequently, we observe the superiority of the grid adaptation algorithm based on employing hp-refinement in comparison to a standard h-version method; on the final mesh the DGFEM norm of the discretization error is over an order of magnitude smaller when the former algorithm is employed, in comparison to the latter, for a fixed number of degrees of freedom.

In (Figure 3a, b), we show the meshes generated after 10 mesh refinements using the h- and hp-adaptive mesh refinement strategies, respectively. Figure 3(c) displays the analytical solution to this example for comparison with the meshes; as noted in Berrone & Süli (2008), the flow exhibits a counter-clockwise vortex around the point $((1/\theta)\log((e^{\theta}+1)/2), 1/2)$, though the analytical solution is relatively smooth. We can see that the h-adaptive refinement strategy performs nearly uniform h-refinement as we would expect for such a smooth analytical solution, with more refinement around the vortex centre and the hill and valley on the right side of the vortex. With the hp-refinement strategy, we note that mostly p-refinement has occurred, which is as expected for a smooth analytical solution, with the main p-refinement occurring around the vortex centre and more h-refinement occurring around the centre of the hills and valleys in the pressure function; further h-refinement has also occurred in the 'tighter' hill and valley on the right caused by the off-centre vortex.

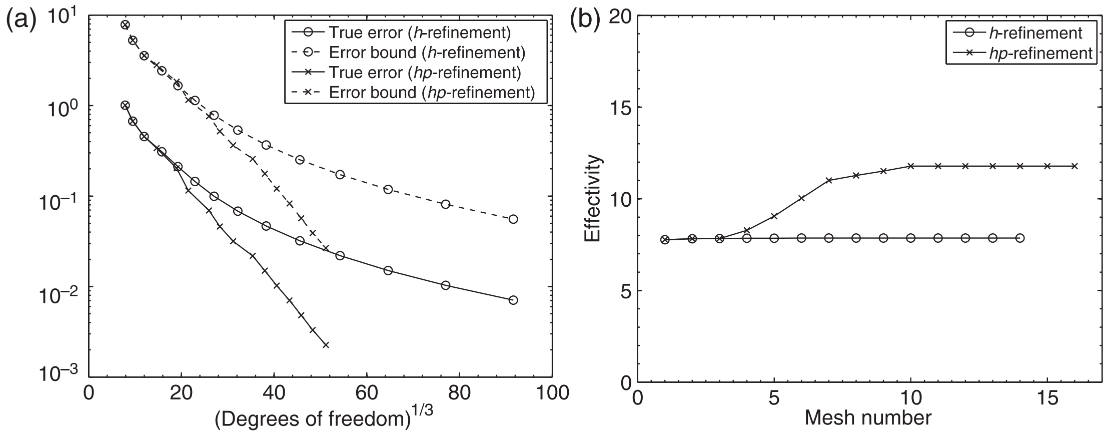

FIG. 4. Example 3. (a) Comparison of the error in the DGFEM norm employing both h- and hp-refinement, with respect to the number of degrees of freedom; (b) the effectivity index using both h- and hp-refinement.

6.3 *Example 3: singular solution*

For this example we consider a nonlinear version of the singular solution from Verfürth (1996, p. 113) (see also Houston *et al.*, 2004a) using the nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = 1 + \mathrm{e}^{-|\underline{e}(\mathbf{u})|}.$$

We let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$ and select $\mathbf{f}$ so that the analytical solution to (1.1–1.3), where (r,φ) denotes the system of polar coordinates, is given by

$$\mathbf{u}(r,\phi) = r^{\lambda}\begin{pmatrix}(1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi)\\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi)\end{pmatrix},$$

$$p(r,\phi) = -r^{\lambda-1}\frac{(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)}{(1-\lambda)},$$

where

$$\Psi(\varphi) = \frac{\sin((1+\lambda)\varphi)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\varphi) - \frac{\sin((1-\lambda)\varphi)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\varphi),$$

and $\omega = 3\pi/2$. Here, the exponent λ is the smallest positive solution of $\sin(\lambda\omega) + \lambda\sin(\omega) = 0$; thereby, $\lambda \approx 0.54448373678246$. We note that $(\mathbf{u}, p)$ is analytic in $\bar{\Omega} \setminus \{\mathbf{0}\}$, but both $\nabla\mathbf{u}$ and p are singular at the origin; indeed, $\mathbf{u} \notin \mathrm{H}^2(\Omega)^2$ and $p \notin \mathrm{H}^1(\Omega)$.

Figure 4(a) presents the comparison of the actual error in the DGFEM norm and the *a posteriori* error bound versus the third root of the number of degrees of freedom on a linear–log scale for the sequence of meshes generated by the h- and hp-adaptive algorithms. We remark that the choice of the third root of the number of degrees of freedom is based on the *a priori* analysis performed in Schötzau & Wihler (2002) for the linear Stokes problem; cf. Houston *et al.* (2004b). We again observe that the error bound overestimates the true error by a roughly consistent factor, although the hp-refinement has

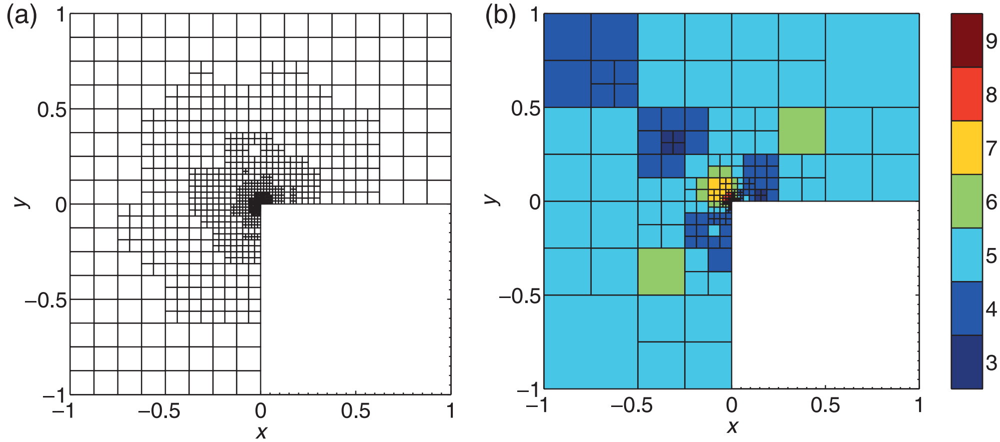

FIG. 5. Example 3. Finite element mesh after 8 adaptive refinements: (a) *h*-adaptivity; (b) *hp*-adaptivity.

some initial increase before stabilizing at a higher value than for *h*-refinement; this is confirmed again by the effectivity indices for the sequence of meshes; cf. Fig. 4(b). From Fig. 4(a), we can also see that yet again the error in the DGFEM norm converges to zero at an exponential rate when the *hp*-adaptive algorithm is employed, leading to a greater reduction in the error for a given number of degrees of freedom when compared with the corresponding quantity computed using *h*-refinement.

Figure (5a, b) shows the meshes generated after 8 mesh refinements using the *h*- and *hp*-adaptive mesh refinement strategies, respectively. We can see that both refinement strategies perform mostly *h*-refinement in the region of the singularity at the origin. However, the *hp*-adaptive strategy is able to perform less *h*-refinement around the origin as it only performs enough to isolate the singularity; then it performs mostly uniform *p*-refinement, with a larger *p*-refinement to the immediate top left of the singularity.

7. Concluding remarks

In this article, we have studied the numerical approximation of a quasi-Newtonian flow problem of strongly monotone type by means of *hp*-interior penalty discontinuous Galerkin methods. We have established well-posedness for both the given PDE system as well as for the proposed *hp*-DGFEM. In addition, *a priori* and *a posteriori* error bounds in the discontinuous Galerkin energy norm (3.7) have been derived. In the latter case, both global upper and local lower residual-based *a posteriori* error bounds have been given. The proof of the upper bound is based on employing a suitable DGFEM space decomposition, together with an *hp*-version projection operator. At the expense of a slight suboptimality with respect to the polynomial degree of the approximating finite element method, this upper bound holds on general 1-irregular meshes. The numerical experiments undertaken in this article demonstrate the theoretical results. In particular, we have shown that the *a posteriori* upper bound converges to zero at the same asymptotic rate as the true error measured in the DGFEM energy norm on sequences of *hp*-adaptively refined meshes.

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