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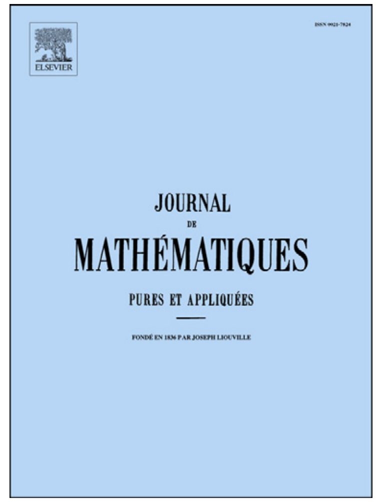

Frequency of Sobolev and quasiconformal dimension distortion

Zoltán M. Balogh, Roberto Monti, Jeremy T. Tyson

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Frequency of Sobolev and quasiconformal dimension distortion☆

Zoltán M. Balogh[a,1], Roberto Monti[b,2], Jeremy T. Tyson[c,3,*]

[a] *Mathematisches Institut, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland.*
[b] *Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste, 63 35121 Padova, Italy*
[c] *Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, IL 61801, USA.*

Abstract

We study Hausdorff and Minkowski dimension distortion for images of generic affine subspaces of Euclidean space under Sobolev and quasiconformal maps. For a supercritical Sobolev map f defined on a domain in $\mathbb{R}^n$, we estimate from above the Hausdorff dimension of the set of affine subspaces parallel to a fixed m-dimensional linear subspace, whose image under f has positive $\mathcal{H}^\alpha$ measure for some fixed $\alpha > m$. As a consequence, we obtain new dimension distortion and absolute continuity statements valid for almost every affine subspace. Our results hold for mappings taking values in arbitrary metric spaces, yet are new even for quasiconformal maps of the plane. We illustrate our results with numerous examples.

Keywords: Hausdorff dimension, Sobolev mapping, potential theory, quasiconformal mapping, space-filling mapping,
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☆Dedicated to Robert Kaufman

*Corresponding author
Email addresses: zoltan.balogh@math.unibe.ch (Zoltán M. Balogh), monti@math.unipd.it (Roberto Monti), tyson@math.uiuc.edu (Jeremy T. Tyson)

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1. Introduction

Every continuous Sobolev mapping is ACL, i.e., its components are absolutely continuous when restricted on almost every line. In particular, almost every line parallel to any fixed vector is mapped onto a locally rectifiable curve, and hence onto a curve of Hausdorff dimension one. Moreover, every supercritical Sobolev mapping satisfies Lusin's condition N, i.e., sets of Lebesgue measure zero are mapped to sets of measure zero.

It is natural to investigate similar regularity properties of Sobolev maps on subspaces of intermediate dimension. For a fixed set this was done by Kaufman [29] and earlier by Astala [2] and Gehring–Väisälä [16] for quasiconformal maps. In this paper, we study absolute continuity and dimension distortion properties for the restriction of Sobolev and quasiconformal maps to *generic* affine subspaces. Our main results are Theorems 1.3, 1.4 and 1.6.

The literature on generic dimension estimates is extensive. A rich line of inquiry into dimensions of generic projections of Euclidean sets was initiated by Marstrand in his fundamental paper [33] and furthered by Kaufman, Mattila, Falconer and many others. We refer to Mattila's book [36] for a history of these developments and a for a list of references. Mattila [34], [35] later proved an important series of results on dimensions of generic intersections of translates or rigid motions of Euclidean sets. These results gave signficant impetus and visibility to the subject of generic dimension estimates. Recently, Falconer [12], [14] investigated the dimensions of invariant sets for generic elements in parameterized families of self-affine iterated function systems. See also the papers by Solomyak [43] and Falconer–Miao [11] for further work on this subject. Ideas from these papers were taken up by the authors in [5], [6] and [4] for the study of dimensions of generic invariant sets associated to sub-Riemannian iterated function systems.

Our goal in this paper is to apply techniques from geometric measure theory used in the proof of such theorems towards the understanding of the generic dimension distortion behavior of Sobolev maps on affine subspaces. Our main results suggest many extensions and generalizations. Section 6 contains open problems and questions motivated by this study.

We consider the foliation of $\mathbb{R}^n$ by m-dimensional affine subspaces

$$V_a := V + a,$$

where V is an m-dimensional linear subspace of $\mathbb{R}^n$, i.e., an element of the Grassmannian $G(n,m)$, and a ranges over the orthogonal complement $V^\perp$ of V. We assume throughout this paper that m and n are integers satisfying

$$1 \le m \le n-1. \tag{1.1}$$

The notion of genericity is measured by suitable Hausdorff measures on $V^\perp$. For instance, the ACL property of a Sobolev map $f : \Omega \to \mathbb{R}^m$ asserts that, for a given $V \in G(n,1)$,

$$\begin{aligned} f|_{V_a\cap\Omega} : (V_a \cap \Omega, \mathcal{H}^1) \to (f(V_a \cap \Omega), \mathcal{H}^1) \text{ is absolutely continuous} \\ \text{for } \mathcal{H}^{n-1} \text{ almost every point } a \text{ in } V^\perp \in G(n, n-1). \end{aligned} \tag{1.2}$$

Since $f(V_a \cap \Omega)$ has locally finite Hausdorff 1-measure at such points a, we also conclude

$$\dim f(V_a \cap \Omega) \le 1 \text{ for } \mathcal{H}^{n-1} \text{ almost every } a \in V^\perp. \tag{1.3}$$

Throughout this paper, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and by dim the Hausdorff dimension.

In this paper, we shall prove a sweeping generalization of (1.2) and (1.3) for families of affine subspaces of arbitrary dimension.

We take advantage of recent developments in analysis in metric spaces to formulate our results for Sobolev maps taking values in arbitrary metric spaces. The notion of a metric space-valued Sobolev map has been introduced by Ambrosio [1] and Reshetnyak [39]. It was used in [46] and [26] to provide an analytic characterization of quasisymmetric maps in metric spaces, and in [3] to investigate properties of quasiconformal maps with Sobolev boundary values from the perspective of conformal densities.

Despite this general framework, we stress that our results are already new for Sobolev and quasiconformal maps between Euclidean domains, even domains in the plane.

Definition 1.1. Let Ω be a domain in some Euclidean space and let B be a Banach space. A map $f : \Omega \to B$ is said to lie in $W^{1,p}(\Omega, B)$ if $\langle b^*, f\rangle \in W^{1,p}(\Omega)$ for every b^* in the dual space B^*, and if the weak gradients of the functions $\langle b^*, f\rangle$, $||b^*|| \le 1$, are uniformly dominated in $L^p(\Omega)$.

Let Y be a separable metric space. A map $f : \Omega \to Y$ is said to lie in $W^{1,p}(\Omega, Y)$ if $\iota \circ f$ lies in the Sobolev space $W^{1,p}(\Omega, \ell^\infty)$, where $\iota : Y \to \ell^\infty$ denotes an isometric embedding.

Fix n and m as in (1.1). Let Ω and Y be as in Definition 1.1, and let f be in $W^{1,p}(\Omega, Y)$. For the moment we restrict our attention to the case of *supercritical* mappings, i.e., the case $p > n$. The Sobolev embedding theorem in this case implies that f is Hölder continuous. The following proposition gives an a priori estimate for the distortion of dimension of an m-dimensional affine subspace under a supercritical Sobolev map. Kaufman [29] proved a more general statement covering subsets of arbitrary Hausdorff dimension. See Proposition 2.5. Although Kaufman's paper is the first place where we have seen this explicit result in print, the underlying principle (increased Sobolev regularity implies improved dimension distortion bounds), had apparently already been recognized for some time. In the quasiconformal category, it was used by both Astala [2] and Gehring–Väisälä [16].

Proposition 1.2 (Kaufman)**.** *Let $f \in W^{1,p}(\Omega, Y)$ for $p > n$ and let $V \in G(n,m)$. Then $f(V_a \cap \Omega)$ has zero $\mathcal{H}^{pm/(p-n+m)}$ measure for each $a \in V^\perp$. In particular,*

$$\dim f(V_a \cap \Omega) \le \frac{pm}{p-n+m}. \tag{1.4}$$

Note that a naive application of the $(1-n/p)$-Hölder regularity of f would yield the weaker estimate

$$\dim f(V_a \cap \Omega) \le \frac{pm}{p-n}.$$

Proposition 1.2 provides an upper bound, strictly smaller than n, for the dimension of the image of an arbitrary m-dimensional subspace under a supercritical $W^{1,p}$ mapping f. How frequently can the intermediate values

$$m < \alpha < \frac{pm}{p-n+m}$$

be exceeded? Our first main theorem provides a quantitative measurement of this frequency.

Fix n and m satisfying (1.1). For $p \ge 1$ and $m \le \alpha \le \frac{pm}{p-n+m}$, set

$$\beta(p,\alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right)p. \tag{1.5}$$

The following theorem, which is the primary result of this paper, asserts an $\mathcal{H}^\beta$-almost everywhere upper bound on the dimensions of images of affine subspaces parallel to a fixed m-dimensional linear subspace of $\mathbb{R}^n$ under a supercritical Sobolev map.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in W^{1,p}(\Omega, Y)$, $p > n$, $V \in G(n,m)$, and*

$$m < \alpha \le \frac{pm}{p-n+m}. \tag{1.6}$$

Then $f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$, where $\beta = \beta(p,\alpha)$.

Since $\beta(p,\alpha) = 0$ if and only if $\alpha = \frac{pm}{p-n+m}$, Theorem 1.3 includes Proposition 1.2 as a special case. Theorem 1.3 implies both the dimension estimate

$$\dim f(V_a \cap \Omega) \le \alpha \tag{1.7}$$

as well as the absolute continuity of

$$f|_{V_a \cap \Omega} : (V_a \cap \Omega, \mathcal{H}^m) \to (f(V_a \cap \Omega), \mathcal{H}^\alpha) \tag{1.8}$$

for $\mathcal{H}^\beta$-a.e. $a \in V^\perp$.

Theorem 1.3 is sharp. In the following theorem, we construct a $W^{1,p}$ map which increases from m to α the dimension of *each* element in a $\beta(p,\alpha)$-dimensional set of parallel affine m-dimensional subspaces of $\mathbb{R}^n$. In order to describe precisely the class of sets to which the theorem applies, we fix some useful notation. For a bounded set $E \subset \mathbb{R}^n$ and for $r > 0$, we denote by $\mathbf{N}(E,r)$ the smallest number of balls of radius r needed to cover E.

Theorem 1.4. *Let $p \geq 1$, let α satisfy $m < \alpha \leq \frac{pm}{p-n+m}$ for $p > n-m$ and $m < \alpha$ for $p \leq n-m$, and define $\beta(p,\alpha)$ by the formula (1.5). Let $E \subset \mathbb{R}^{n-m}$ be any bounded Borel set for which*

$$\limsup_{r \to 0} r^{\beta} \mathbf{N}(E, r) < \infty \tag{1.9}$$

with $\beta = \beta(p, \alpha)$. Then for any integer $N > \alpha$, there exists $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ so that $f(\{a\} \times \mathbb{R}^m)$ has Hausdorff dimension at least α, for $\mathcal{H}^{\beta}$-a.e. $a \in E$.

Note that we only assume $p \geq 1$ in the statement of Theorem 1.4. Choosing $p > n$ and a set $E \subset \mathbb{R}^{n-m}$ with positive and finite Hausdorff $\mathcal{H}^{\beta}$ measure which satisfies the assumptions of the theorem shows that Theorem 1.3 is sharp. Sets of this type exist in abundance. For instance, we may take any compact subset $E \subset \mathbb{R}^{n-m}$ which is Ahlfors regular of dimension $\beta(p, \alpha)$, e.g., a self-similar Cantor set.

The map in Theorem 1.4 is obtained by a random construction. We exhibit a large family of $W^{1,p}$ maps and show that almost every map in this family has the desired property.

Theorem 1.3 holds in particular for Euclidean quasiconformal maps. We obtain almost sure dimension estimates for the size of the exceptional set of points a in $V^{\perp}$ for which the quasiconformal m-manifold $f(V_a \cap \Omega)$ has positive $\mathcal{H}^{\alpha}$ measure. By Gehring's higher integrability theorem [15], quasiconformal maps in $\mathbb{R}^n$ lie in $W^{1,p}$ for some $p > n$. Since

$$\beta(p, \alpha) < \beta(n, \alpha) = m\left(\frac{n}{\alpha} - 1\right)$$

for all $p > n$, we obtain the following

Corollary 1.5. *Let $f : \Omega \to \Omega'$ be a quasiconformal map between domains in $\mathbb{R}^n$, let $V \in G(n, m)$, and let $m < \alpha < n$. Then $\mathcal{H}^{\alpha}(f(V_a \cap \Omega)) = 0$ for $\mathcal{H}^{m\left(\frac{n}{\alpha}-1\right)}$-a.e. $a \in V^{\perp}$. In particular,*

$$\dim f(V_a \cap \Omega) \leq \alpha \tag{1.10}$$

for $\mathcal{H}^{m\left(\frac{n}{\alpha}-1\right)}$-a.e. $a \in V^{\perp}$.

Estimates for quasiconformal dimension distortion are often obtained via conformal modulus techniques. Our proof makes no explicit use of modulus, although it is motivated by modulus arguments used in estimates of conformal dimension (Remark 3.4). Quasiconformal and quasisymmetric dimension distortion is a classical subject ([16], [45], [2]), but we are unaware of prior theorems yielding simultaneous dimension estimates for the images of a large family of parallel subspaces. See Remark 5.8 for more details.

Remarkably, even Corollary 1.5 is sharp, provided we replace Hausdorff dimension by upper Minkowski dimension in (1.10). To simplify the exposition here in the introduction, we only state the following theorem in the case $m = 1$, i.e., for parameterized families of lines and their images. A similar result holds for higher dimensional subspaces, but only for a restricted choice of image dimensions α. See Remark 5.6.

Theorem 1.6. *Let $n \geq 2$. For each $\alpha \in (1, n)$ and each $\epsilon > 0$, there exists a Borel set $E \subset \mathbb{R}^{n-1}$ of Hausdorff dimension at least $\left(\frac{n}{\alpha} - 1\right) - \epsilon$ and a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\{a\} \times \mathbb{R})$ has upper Minkowski dimension at least α, for all $a \in E$.*

We recall that the upper Minkowski dimension of E is the infimum of those values $\beta > 0$ for which (1.9) is satisfied.

Theorem 1.6 provides the first example of which we are aware of a quasiconformal map which simultaneously increases the (Minkowski) dimension of a family of parallel subspaces of optimal size. We do not know any example of a quasiconformal map which simultaneously increases the Hausdorff dimension of such a large family of subspaces, although previous examples of Bishop [7], David–Toro [10] and Kovalev–Onninen [32] should be noted. We review the examples of Bishop, David–Toro and Kovalev–Onninen in Remark 5.8.

Theorem 1.3 refers to supercritical Sobolev maps, i.e., $W^{1,p}$ maps with $p > n$. The situation for weaker integrability criteria is more intriguing. Recently, Hencl and Honzík [28] extended Theorem 1.3 to certain subcritical Sobolev spaces by proving that the conclusion of Theorem 1.3 continues to hold on the level of Hausdorff dimension for the p-quasicontinuous representatives of $W^{1,p}$ mappings when $m < \alpha < p \leq n$. See Theorem 5.9 for a precise statement. The method of proof in [28] is rather new and relies on pointwise estimates, similar in spirit to but more intricate than the Sobolev embedding theorem, for such mappings. They also provide an elaboration on Theorem 1.4 which further highlights the necessity of the restriction $p > \alpha$.

When $p \leq m$ the situation is still not completely clear. In Example 5.11 we construct mappings in

$$W^{1,m}([0,1]^n, \ell^2),$$

for any $2 \leq m < n$, with the property that **every** image $f(V_a \cap [0,1]^n)$, $a \in V^\perp$, is infinite-dimensional. In fact, every such image coincides with a fixed infinite-dimensional cube. The construction makes use of space-filling Sobolev mappings with metric space targets, as constucted by Hajłasz–Tyson [24] and Wildrick–Zürcher [47], [48]. The methods can be adapted to construct a mapping in $W^{1,p}$ for

$$m < p < n \tag{1.11}$$

with similar properties, but at present, a complete understanding of the generic dimension distortion behavior of m-dimensional affine subspaces by $W^{1,p}$ maps from $\mathbb{R}^n$ for arbitrary p remains a challenging open problem.

Outline of the paper. In Section 2 we review the Ambrosio/Reshetnyak framework for metric space-valued Sobolev maps, emphasizing dimension distortion and absolute continuity properties. Section 3 contains the proof of Theorem 1.3. We use the technique of energy integrals to obtain generic lower bounds on dimension.

In section 4 we prove Theorem 1.4. The desired Sobolev map is obtained via a random method, as a generic representative in a parameterized family of mappings. The idea goes back to Kaufman [29].

Section 5 is devoted to examples. Here we prove Theorem 1.6. The quasiconformal map in Theorem 1.6 is constructed in a piecewise fashion on a Whitney decomposition of the complement of a codimension one subspace. The construction is a refined version of an earlier one by Heinonen and Rohde [27], who constructed a quasiconformal map of the unit ball in $\mathbb{R}^n$ sending an $(n-1)$-dimensional family of radial segments onto curves of infinite length.

In Section 5 we also discuss subcritical Sobolev mappings. The space-filling constructions of Hajłasz–Tyson [24] yield an example of a $W^{1,m}$ mapping f from $\mathbb{R}^n$ to the Hilbert space ℓ^2 for which $f(V_a)$ is infinite-dimensional for every $a \in V^\perp \in G(n, n-m)$. In subsection 5.2 we generalize the constructions from [24] to build similar maps in $W^{1,p}$ for $m < p < n$.

Section 6 is reserved for open problems and questions arising out of this study.

Remark. An earlier draft of this paper included versions of Theorem 1.3 in certain borderline Sobolev spaces such as the Sobolev–Lorentz class $W^{1,n,1}$ and the space of continuous pseudomonotone $W^{1,n}$ maps. As these results of ours have now been superseded by the work of Hencl and Honzík (see Theorem 5.9) we do not include them in this final version.

Conventions. We denote by $\#S$ the cardinality of a finite set S. The Lebesgue measure in $\mathbb{R}^n$ will be written $\mathcal{L}^n$. We denote unspecified positive constants by C or c. We write $C = C(a, b, \ldots)$ to mean that C depends on the data $a, b, \ldots$. We employ the following convention: we write C if we wish to emphasize that a certain constant is *finite*, and we write c if we wish to emphasize that it is *positive*.

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We extend our grateful appreciation to the referee, whose detailed comments and corrections greatly improved the paper. In particular, we would like to thank the referee for providing the elegant statement and proof of Lemma 4.4; our original version of this lemma was significantly more technical. We also thank the referee for directing our attention to the work of Hencl and Honzík [28], and we thank Stanislav Hencl for his valuable comments on this related work.

2. Sobolev maps valued in metric spaces

Our results are naturally phrased in the modern language of metric space-valued Sobolev mappings (see Definition 1.1). This notion was introduced by Ambrosio [1] in 1990 and later studied by Reshetnyak [39]. For the reader's convience, we repeat the definition.

Let B be a Banach space, let $1 \le p < \infty$, and let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$. The Bochner–Lebesgue space $L^p(\Omega, B)$ consists of all weakly measurable, essentially separably valued maps $f : \Omega \to B$ satisfying $\int_\Omega ||f(x)||^p \, dx < \infty$.

Definition 2.1. A map $f : \Omega \to B$ in the Bochner–Lebesgue space $L^p(\Omega, B)$ belongs to the *Ambrosio–Reshetnyak–Sobolev space* $W^{1,p}(\Omega, B)$ if there exists $g \in L^p(\Omega)$ so that for every $b^* \in B^*$ with $||b^*|| \le 1$, we have $\langle b^*, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle b^*, f\rangle| \le g$ a.e.

A function g as in the definition will be called an *upper gradient* for f. Thus $W^{1,p}(\Omega, B)$ consists of those functions in $L^p(\Omega, B)$ which admit an L^p upper gradient.

We may equip $W^{1,p}(\Omega, B)$ with the norm

$$||f||_{1,p} := ||f||_{L^p(\Omega,B)} + \inf_g ||g||_{L^p(\Omega)}. \tag{2.1}$$

Here the infimum is taken over all upper gradients $g \in L^p(\Omega)$ for f. Endowed with this norm, $W^{1,p}(\Omega, B)$ is a Banach space. See, for example Theorem 3.13 in [26].

Furthermore, when $1 < p < \infty$ there exists an upper gradient $g_f \in L^p(\Omega)$ so that

$$||f||_{1,p} = ||f||_{L^p(\Omega,B)} + ||g_f||_{L^p(\Omega)}.$$

Moreover, g_f is unique up to modification on a set of measure zero. The existence of such a *minimal upper gradient* g_f follows by a standard convexity argument.

The space $W^{1,p}(\Omega, B)$ admits the following weak characterization.

Proposition 2.2. *Let B be the dual of a separable Banach space. Then $W^{1,p}(\Omega, B)$ coincides with the space of all functions $f \in L^p(\Omega, B)$ which have weak partial derivatives in $L^p(\Omega, B)$.*

As usual, we say that $f : \Omega \to B$ has $g_i : \Omega \to B$ as a *weak i-th partial derivative* if

$$\int_\Omega (\partial_i \varphi) f \, dx = -\int_\Omega \varphi g_i \, dx \tag{2.2}$$

for all C^∞ functions φ which are compactly supported in Ω. Here $i \in \{1, \dots, n\}$ and the identity (2.2) is understood in the sense of the Bochner integral, as an equality between elements of B. For a proof of Proposition 2.2, see for example [24].

Now suppose that (Y, d) is a separable metric space. Fix an isometric embedding ι of Y into ℓ^∞. In this case, we say that $f : \Omega \to Y$ is in the *Ambrosio–Reshetnyak–Sobolev space* $W^{1,p}(\Omega, Y)$ if $\iota \circ f \in W^{1,p}(\Omega, \ell^\infty)$. Since ℓ^∞ is the dual of the separable Banach space ℓ^1, the membership of $\iota \circ f$ in $W^{1,p}(\Omega, \ell^\infty)$ can be understood in the weak sense via Proposition 2.2. When $1 < p < \infty$ we write $g_f = g_{\iota \circ f}$ and call this the *minimal upper gradient* of f.

The existence of isometric embeddings of separable metric spaces in ℓ^∞ is well known. For instance, we may use the Kuratowski embedding [25, Chapter 12].

The space $W^{1,p}(\Omega, Y)$ is naturally equipped with a metric by the rule

$$d(f_1, f_2) = ||\iota \circ f_1 - \iota \circ f_2||_{1,p},$$

where $|| \cdot ||_{1,p}$ denotes the norm in (2.1). We emphasize that this metric depends on the choice of the isometric embedding ι. While membership in the class $W^{1,p}(\Omega, Y)$ turns out to be independent of the choice of ι, the metric structure of the space is highly dependent on that choice. This fact has been explored in detail by Hajłasz [19], [20], [22] who has shown, for example, the surprising result that the question of density of Lipschitz mappings in the Sobolev space can admit a different answer depending on the choice of ι.

For additional information on this notion of metric space-valued Sobolev space, we recommend the clear and readable survey [21] by Hajłasz.

Sobolev maps from Ω to Y are absolutely continuous along almost every line, and restrict to Sobolev maps on almost every affine subspace of dimension at least two. We record this fact in the following proposition. It is easily deduced from Proposition 2.2 by standard arguments. See Theorem 2.1.4 and Remark 2.1.5 in [49].

Proposition 2.3. *Let $f \in W^{1,p}(\Omega, Y)$, $p \geq 1$. Then f has an ACL representative $\overline{f}$. In particular, for any $V \in G(n,1)$, the set of $a \in V^\perp$ for which $\overline{f}|_{V_a \cap \Omega}$ is not absolutely continuous from $(V_a \cap \Omega, \mathcal{H}^1)$ to $(\overline{f}(V_a \cap \Omega), \mathcal{H}^1)$ has zero $\mathcal{H}^{n-1}$-measure. Moreover, for any $V \in G(n,m)$, $m \geq 2$, the set of $a \in V^\perp$ for which $\overline{f}|_{V_a \cap \Omega} \notin W^{1,p}(V_a \cap \Omega, Y)$ has zero $\mathcal{H}^{n-m}$-measure.*

By the Morrey–Sobolev embedding theorem, each supercritical mapping $f \in W^{1,p}(\Omega, Y)$, $p > n$, has a representative which is locally $(1 - n/p)$-Hölder continuous. In the remainder of the paper we always work with this representative. In the following proposition, we summarize several basic properties of supercritical Sobolev mappings.

Proposition 2.4. *Let Y be a separable metric space, $\Omega \subset \mathbb{R}^n$, and $f \in W^{1,p}(\Omega, Y)$, $p > n$, represented as above. Let g_f denote the minimal upper gradient for f. Then for all cubes Q compactly contained in Ω, we have*

$$\operatorname{diam} f(Q) \leq C(n,p)(\operatorname{diam} Q)^{1-n/p} \left(\int_Q g_f^p \, dx \right)^{1/p}. \tag{2.3}$$

Also, f satisfies the following quantitative version of Lusin's condition N:

$$\mathcal{H}^n(f(E)) \leq C(n,p)\mathcal{L}^n(E)^{1-n/p} ||g_f||^n_{L^p(\Omega)} \tag{2.4}$$

for all measurable sets $E \subset \Omega$.

The local Hölder continuity and the estimate in (2.3) are established by standard arguments as in the Euclidean case, beginning from the Sobolev–Poincaré inequality for supercritical Sobolev functions. For details, we refer to Ziemer [49, Theorem 2.4.4] or Hajłasz–Koskela [23]. We prove the quantitative Lusin property (2.4). While this argument is also standard, it serves as a model for other proofs which occur in this paper.

We make repeated use of the fact that Hausdorff dimension can be computed using dyadic coverings. By a *dyadic cube* of size 2^{-j}, $j \in \mathbb{Z}$, we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, with side length 2^{-j} and vertices in the set $2^{-j} \cdot \mathbb{Z}^n$. The *$s$-dimensional dyadic Hausdorff measure* $\mathcal{H}^s_{dyadic}$ is defined by the usual Carathéodory procedure to be

$$\mathcal{H}^s_{dyadic}(E) = \lim_{\delta \to 0} \mathcal{H}^s_{dyadic,\delta}(E)$$

where $\mathcal{H}^s_{dyadic,\delta}(E)$ is the infimum of the expressions $\sum_j (\operatorname{diam} Q_j)^s$ over all coverings $\{Q_j\}$ of E by dyadic cubes of diameter no more than δ. The inequalities

$$\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_{dyadic,\delta}(E) \leq (4\sqrt{n})^s \mathcal{H}^s_\delta(E), \qquad E \subset \mathbb{R}^n, 0 \leq \delta \leq \infty, \tag{2.5}$$

show that the dyadic Hausdorff measures generate the same dimension value as do the standard Hausdorff measures. See Mattila [36, §5.2] for details. We recall that the dyadic cubes of a fixed size form a nonoverlapping decomposition of $\mathbb{R}^n$ (that is, they have disjoint interiors).

To prove (2.4), let $\epsilon > 0$, choose $\delta > 0$ sufficiently small relative to ϵ, and consider an arbitrary covering $\{Q_i\}$ of E by nonoverlapping dyadic cubes with side length $r_i < \delta$. Then $f(E)$ is covered by the sets $\{f(Q_i)\}$, and

$$\operatorname{diam} f(Q_i) \le C(n,p) r_i^{1-n/p} (\int_{Q_i} g_f^p \, dx)^{1/p} \le C(n,p,||g_f||_{L^p(\Omega)}) \delta^{1-\frac{n}{p}} \tag{2.6}$$

which is less than ϵ by (2.3), provided δ is chosen appropriately. Summing the n-th powers of (2.6) over i, applying Hölder's inequality together with the essential disjointedness of the family $\{Q_i\}$, and taking the infimum over all such coverings $\{Q_i\}$ yields

$$\mathcal{H}^n_\epsilon(f(E)) \le C(n,p) ||g_f||^n_{L^p(\Omega)} \mathcal{H}^n_{dyadic,\delta}(E)^{1-n/p}. \tag{2.7}$$

Letting δ and ϵ tend to zero and recalling the equivalence of $\mathcal{H}^s$ and $\mathcal{H}^s_{dyadic}$ completes the proof of (2.4).

Kaufman [29] generalized the preceding proposition to cover the full range of Hausdorff measures $\mathcal{H}^s$, $0 < s < n$. Proposition 1.2 is a special case of the following theorem.

Proposition 2.5 (Kaufman)**.** *Let $E \subset \Omega$ be a set of σ-finite $\mathcal{H}^\alpha$ measure for some $0 < \alpha < n$. Let $f \in W^{1,p}(\Omega, Y)$ for some $p > n$. Then $f(E)$ has zero $\mathcal{H}^{p\alpha/(p-n+\alpha)}$ measure.*

The proof of Proposition 2.5 proceeds along exactly the same lines as that of Proposition 2.4 with one additional modification. Since $\alpha < n$, we have that E is a null set for the Lebesgue measure in Ω. Instead of (2.7) we obtain

$$\mathcal{H}^{p\alpha/(p-n+\alpha)}_\epsilon(f(E)) \le C(n,p,\alpha) ||g_f||^{\frac{p\alpha}{p-n+\alpha}}_{L^p(U)} \mathcal{H}^\alpha_{dyadic,\delta}(E)^{\frac{p-n}{p-n+\alpha}}$$

for each open set U containing E. Taking the infimum over all such open sets and using the outer regularity of the Lebesgue measure yields the desired conclusion.

3. Exceptional sets for Sobolev maps

In this section, we prove Theorem 1.3.

For $\delta > 0$ we denote by $\mathcal{H}^\alpha_\delta$ the α-dimensional Hausdorff premeasure at scale δ. In particular, $\mathcal{H}^\alpha_\infty$ denotes the α-dimensional Hausdorff content. See [36, Chapter 4] for definitions.

Using countable stability of Hausdorff measure and the invariance of Hausdorff measure under rigid motions of $\mathbb{R}^n$, it suffices to assume that Ω is bounded and $V = \{0\} \times \mathbb{R}^m$. Since the null sets for $\mathcal{H}^\alpha$ and $\mathcal{H}^\alpha_\infty$ coincide [36, Lemma 4.6], the exceptional set of points from the statement of Theorem 1.3 consists of those points $a \in V^\perp$ for which

$$\mathcal{H}^\alpha_\infty(f(V_a \cap \Omega)) > 0.$$

Let us denote this set by $\mathrm{Exc}_f(\alpha)$.

Our first task is to show that $\mathrm{Exc}_f(\alpha)$ is a Borel set. This will permit us to use Frostman's lemma in later proofs.

Lemma 3.1. *For each $\alpha \in [m,n)$, $\mathrm{Exc}_f(\alpha)$ is a Borel set.*

For a linear subspace $W \subset \mathbb{R}^n$, let $P_W : \mathbb{R}^n \to W$ denote the orthogonal projection onto W.

Proof. As described above, we may assume that Ω is bounded. Exhaust Ω with an increasing sequence of compact sets $\{K_i\}$. For $\delta > 0$, let $E(\alpha, i, \delta)$ be the set of points $a \in V^\perp$ with the following property: *whenever $f(V_a \cap K_i)$ is covered by a countable family of open sets, $\{A_k\}$, then $\sum_k (\operatorname{diam} A_k)^\alpha > \delta$.* Then

$$\mathrm{Exc}_f(\alpha) = \bigcup_i \bigcup_{\delta > 0} E(\alpha, i, \delta).$$

We will prove that $E(\alpha, i, \delta)$ is a closed set.

Let (a_j) be a sequence of points in $E(\alpha, i, \delta)$ with $\lim_{j\to\infty} a_j = a$. Let $\{A_k\}$ be a countable family of open sets covering $f(V_a \cap K_i)$. For each k, let $B_k = f^{-1}(A_k)$. Since f is continuous and $V_a \cap K_i$ is compact, it follows from the Tube Lemma [37, Lemma 5.8] that there exists a neighborhood U of a in $V^\perp$ so that $P^{-1}_{V^\perp}(U) \cap K_i \subset \cup_k B_k$. For sufficiently large j, $a_j \in U$ and hence $f(V_{a_j} \cap K_i) \subset \bigcup_k A_k$. Since $\sum_k (\operatorname{diam} A_k)^\alpha > \delta$ we conclude that $a \in E(\alpha, i, \delta)$. This completes the proof.

Denote by $B_{V^\perp}(a,r)$ the ball in $V^\perp$ with center a and radius $r > 0$. We will deduce Theorem 1.3 from the following proposition.

Proposition 3.2. *Let α satisfy (1.6), let $p > n$, and define $\beta = \beta(p, \alpha)$ by the formula (1.5). Let $E \subset V^\perp$ be a set of finite $\mathcal{H}^\beta$ measure and assume that μ is a positive Borel measure supported on E and satisfying the growth condition*

$$\mu(B_{V^\perp}(a,r)) \le r^\beta \qquad \text{for all } a \in V^\perp \text{ and } r > 0. \tag{3.1}$$

Finally, let $f \in W^{1,p}(\Omega, Y)$. Then $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for μ-a.e. $a \in E$.

Proof. We may assume without loss of generality that $\Omega = (0,1)^n$ and that $E \subset P_{V^\perp}(\Omega)$. Fix $\delta > 0$. Since $\beta < n - m$, E can be included in an open set $U_\delta \subset \mathbb{R}^{n-m}$ of $\mathcal{H}^{n-m}$ measure at most δ. Let g_f denote the minimal L^p upper gradient for f. Since $g_f \in L^p(\Omega)$,

$$\lim_{\delta \to 0} \int_{U_\delta \times (0,1)^m} g_f^p \, dx = 0. \tag{3.2}$$

Consider a nonoverlapping collection of dyadic cubes, $\{R_i\}$, contained in U_δ and covering E, for which

$$\sum_i r_i^\beta < \mathcal{H}^\beta_{dyadic,\delta}(E) + \delta.$$

Here r_i denotes the side length of R_i; we assume without loss of generality that $r_i < \delta$ for all i. For each i, let $\{Q_{ij}\}_{j=1}^{N_i}$ be a family of nonoverlapping dyadic cubes in $\mathbb{R}^n$, each of which has side length r_i, with the property that $\bigcup_j Q_{ij} = R_i \times (0,1)^m$. For fixed i, the number N_i of cubes Q_{ij} is r_i^{-m}.

By Proposition 2.4(i),

$$\operatorname{diam} f(Q_{ij}) \le C r_i^{1-n/p} \left(\int_{Q_{ij}} g_f^p \, dx \right)^{1/p} \le C||g_f||_{L^p(Q_{ij})} \delta^{1-n/p} =: \epsilon. \tag{3.3}$$

For each $a \in E$, we have

$$\mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \le \sum_{j=1}^{N_i} (\operatorname{diam} f(Q_{ij}))^\alpha$$

for each i so that $a \in R_i$. For fixed i and $a \in E$, let

$$\chi(i,a) = \begin{cases} 1, & \text{if } a \in R_i, \\ 0, & \text{else.} \end{cases}$$

Then $\mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \le \sum_i \chi(i,a) \sum_{j=1}^{N_i} (\operatorname{diam} f(Q_{ij}))^\alpha$ and so

$$\begin{aligned} \int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \, d\mu(a) &\le \int^*_{V^\perp} \sum_i \chi(i,a) \sum_{j=1}^{N_i} (\operatorname{diam} f(Q_{ij}))^\alpha \, d\mu(a) \\ &= \sum_i \mu(R_i) \sum_j (\operatorname{diam} f(Q_{ij}))^\alpha \\ &\le C(n,p) \sum_i r_i^\beta r_i^{\alpha(1-n/p)} \sum_j \left(\int_{Q_{ij}} g_f^p \, dx \right)^{\alpha/p}, \end{aligned}$$

where we used (2.3) and (3.1). (Here we employed the upper integral $\int^*$ to avoid the difficult issue of measurability of the integrand $a \mapsto \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))$.)

Applying Hölder's inequality to the inner sum, we obtain

$$\begin{aligned} &\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \, d\mu(a) \\ &\quad \le C(n,p) \sum_i r_i^{\beta + \alpha(1-n/p)} (N_i)^{1-\alpha/p} \left(\sum_{j=1}^{N_i} \int_{Q_{ij}} g_f^p \, dx \right)^{\alpha/p} \\ &\quad \le C(n,p) \sum_i r_i^{\beta + \alpha(1-n/p) - m(1-\alpha/p)} \left(\int_{R_i \times (0,1)^m} g_f^p \, dx \right)^{\alpha/p}. \end{aligned}$$

Applying Hölder's inequality again yields

$$\begin{aligned} &\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \, d\mu(a) \\ &\quad \le C(n,p) \left(\sum_i \int_{R_i \times (0,1)^m} g_f^p \, dx \right)^{\frac{\alpha}{p}} \left(\sum_i r_i^{(\beta + \alpha(1-\frac{n}{p}) - m(1-\frac{\alpha}{p}))\frac{p}{p-\alpha}} \right)^{1-\frac{\alpha}{p}}. \end{aligned}$$

Since $\beta = \beta(p,\alpha)$,

$$\left(\beta + \alpha(1-\tfrac{n}{p}) - m(1-\tfrac{\alpha}{p})\right)\left(\frac{p}{p-\alpha}\right) = \beta.$$

Thus

$$\begin{aligned}\int_{V^\perp}^* \mathcal{H}^\alpha_\epsilon(f(V_a\cap\Omega))\,d\mu(a) &\le C(n,p)\left(\int_{U_\delta\times(0,1)^m} g_f^p\,dx\right)^{\frac{\alpha}{p}}\left(\sum_i r_i^\beta\right)^{1-\frac{\alpha}{p}} \\ &\le C(n,p)||g_f||^\alpha_{L^p(U_\delta\times(0,1)^m)}\left(\mathcal{H}^\beta_{dyadic,\delta}(E)+\delta\right)^{1-\frac{\alpha}{p}}.\end{aligned} \tag{3.4}$$

Letting $\delta \to 0$ and using the Monotone Convergence Theorem, the equivalence of $\mathcal{H}^s$ and $\mathcal{H}^s_{dyadic}$, and (3.2), we conclude that $\int_{V^\perp}^* \mathcal{H}^\alpha(f(V_a\cap\Omega))\,d\mu(a)$ is equal to zero. This completes the proof of the proposition.

Remark 3.3. The reader may have noticed that we only used the condition $\alpha < p$ in the preceding proof, while the hypotheses include the stronger restriction

$$\alpha < \frac{pm}{p-n+m}. \tag{3.5}$$

The reason for (3.5) is implicit in the proof: recall that (3.5) holds if and only if $\beta > 0$. In practice, the desired measure μ will be obtained by an application of Frostman's lemma, which requires the growth exponent β to be positive.

Remark 3.4. Some aspects of the preceding proof are modelled on a lemma of Bourdon [9] (see also Pansu [38]) which provides lower estimates for the conformal dimension of a metric space. This formal similarity is not surprising. Lower bounds on the conformal dimension of a metric space indicate that a large family of (quasisymmetrically equivalent) spaces have uniformly large dimension, while Theorem 1.3 indicates restrictions on the set of parameters a for which the dimensions of the fiber images $f(V_a\cap\Omega)$ are all uniformly large.

Proof of Theorem 1.3. Let $\beta = \beta(p,\alpha)$. Suppose $\mathrm{Exc}_f(\alpha)$ has positive $\mathcal{H}^\beta$ measure. By Lemma 3.1 and Theorem 8.13 in [36], there exists a compact set $E \subset \mathrm{Exc}_f(\alpha)$ so that $0 < \mathcal{H}^\beta(E) < \infty$. By Frostman's lemma ([36, Theorem 8.9]), there exists a positive Borel measure $\mu \ne 0$ supported on E such that $\mu(B_{V^\perp}(a,r)) \le r^\beta$ for all $a \in E$ and $r > 0$. Then μ is absolutely continuous with respect to $\mathcal{H}^\beta \llcorner E$, so $\mu(E) < \infty$. By Proposition 3.2, $\mathcal{H}^\alpha(f(V_a\cap\Omega)) = 0$ for μ-a.e. $a \in E$. This contradicts the definition of $\mathrm{Exc}_f(\alpha)$.

Remark 3.5. Quasiconformal self-maps of $\mathbb{R}^n$, $n \ge 2$, lie in $W^{1,p}$ for some $p > n$. This is Gehring's higher integrability theorem [15]. Corollary 1.5 follows from this fact and Theorem 1.3. More precisely, if f is K-quasiconformal then

$$\dim \mathrm{Exc}_f(\alpha) \le (n-m) - \left(1-\frac{m}{\alpha}\right)p(n,K) \tag{3.6}$$

where $p(n,K) > n$ denotes the sharp exponent of higher integrability for the partial derivatives of a K-quasiconformal mapping. We say that f is K-*quasiconformal* if $H_f(x) \le K$ for all $x \in \Omega$, where

$$H_f(x) = \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)| : |x-y| = r\}}{\inf\{|f(x)-f(z)| : |x-z| = r\}}$$

denotes the *metric dilatation* of a homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$. A celebrated theorem of Astala [2] asserts that

$$p(2,K) = \frac{2K}{K-1}; \tag{3.7}$$

the corresponding value $p(n,K) = \frac{nK}{K-1}$ remains a conjecture when $n \ge 3$.

Astala's theorem yields sharp bounds on dimension distortion by planar quasiconformal maps. If f is a K-quasiconformal map between planar domains Ω, Ω' and $E \subset \Omega$, then

$$\frac{1}{K}\left(\frac{1}{\dim E} - \frac{1}{2}\right) \le \frac{1}{\dim f(E)} - \frac{1}{2} \le K\left(\frac{1}{\dim E} - \frac{1}{2}\right). \tag{3.8}$$

We deduce from (3.6) and (3.7) that

$$\dim \mathrm{Exc}_f(\alpha) \le \frac{2K - (K+1)\alpha}{\alpha(K-1)} \tag{3.9}$$

whenever f is a K-quasiconformal map between planar domains, $V \in G(2,1)$, and $\alpha \in [1,2)$. Note that the right hand side of (3.9) is equal to zero precisely when

$$\alpha = \frac{2K}{K+1} = 1 + \left(\frac{K-1}{K+1}\right).$$

This agrees with the upper bound in (3.8) for the dimension of the image of any 1-dimensional set under a planar K-quasiconformal map. In fact, the proof of (3.8) given in [2] uses only the higher Sobolev integrability of f.

We discuss the case of quasiconformal mappings further in Problem 6.2.

4. Sobolev maps which increase the dimension of many affine subspaces

In this section we prove Theorem 1.4. Our proof is modelled closely on that of an analogous result of Kaufman [29, Theorem 3], which exhibits Sobolev maps which increase maximally the dimension of a fixed subset. Our situation is complicated by the fact that we work with the orthogonal splitting of $\mathbb{R}^n$ into $V = \{0\} \times \mathbb{R}^m$ and $V^\perp = \mathbb{R}^{n-m} \times \{0\}$ and look for a map which simultaneously increases the dimension of many fibers.

Recall that our goal is to construct a $W^{1,p}$ map of $\mathbb{R}^n$ which increases the dimensions of all of the fibers V_a over the points a in a certain set $E \subset V^\perp$ from m to α. To achieve this, we will use a random construction. We will define a family of maps (f_ξ) parameterized by sequences ξ of independent and identically distributed random variables. All of these maps will lie in the Sobolev class $W^{1,p}$, and we will show that, almost surely with respect to ξ, such maps have the desired property. We do not know whether a deterministic construction can be given.

Recall also that in the statement of Theorem 1.4 we assume that the set E satisfies the growth condition

$$\mathbf{N}(E,r) \le Cr^{-\beta} \tag{4.1}$$

for all $r < r_0$, for some constants C and $r_0 > 0$. Here $\beta = \beta(p,\alpha)$ is the value given in (1.5). In particular, $\mathcal{H}^\beta(E) < \infty$ and so

$$\dim E \le \beta. \tag{4.2}$$

We fix an integer $N > \alpha$; this value will be the dimension of the target space. When $p > n$, we may set $N = n$.

Proof of Theorem 1.4. Let E be a bounded subset of $\mathbb{R}^{n-m}$ satisfying (4.1) for all $0 < r < r_0$, for suitable constants C and r_0. By applying a preliminary homothety, we may assume that $E \subset [0,1]^{n-m}$. The maps $f_\xi \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ which we shall construct will satisfy

$$\mathcal{H}^{\alpha'}(f_\xi(V_a \cap [0,1]^n)) = \infty \tag{4.3}$$

for $\mathcal{H}^\beta$-almost every $a \in E$ and almost surely in ξ, for each $\alpha' < \alpha$. This clearly suffices to obtain the desired conclusion $\dim f_\xi(V_a) \ge \alpha$ for $\mathcal{H}^\beta$-a.e. $a \in E$, almost surely in ξ.

Before continuing with the proof, we pause to review terminology from symbolic dynamics.

Let $W = \{1,\dots,2^n\}$, let W^j be the set of (ordered) j-tuples of elements of W, and let

$$W^* = \bigcup_{j \ge 0} W^j$$

be the set of all finite sequences of elements of W (including the empty sequence). We call the elements of W^* *words* comprised of the *letters* in W. If $v = (v_1,\dots,v_j)$ and $w = (w_1,\dots,w_k)$ are words with $j \ge k$, we say that w is a *subword* of v if $v_i = w_i$ for all $i = 1,\dots,k$. The *length* $|w|$ of a word $w \in W^j$ is equal to j.

We use W^* to index the cubes in the standard dyadic decomposition

$$\mathcal{D} = \{Q_w\}_{w \in W^*}$$

of $Q = [0,1]^n$. We choose this indexing in such a way that the side length $s(Q_w)$ of Q_w is equal to 2^{-j} if w has length j, and also that $Q_w \subset Q_v$ if v is a subword of w. For each j, the cubes $\{Q_w\}_{w \in W^j}$ form a nonoverlapping decomposition of Q.

We also introduce a second collection of cubes, obtained by dilating the elements of $\mathcal{D}$. For each $w \in W^*$, let $Q'_w = 100Q_w$. It is important to note that, for fixed j, the collection $\{Q'_w\}_{w \in W^j}$ has bounded overlap: no points of $\mathbb{R}^n$ lies in more than C of the cubes in this collection, where C is a constant depending only on the dimension n.

We project these cubes into the subspaces V and $V^\perp$. In order to maintain a consistent notation we write

$$Q_w^{V^\perp} = P_{V^\perp}(Q_w) \qquad \text{and} \qquad Q_w^V = P_V(Q_w)$$

for such projections. We view these as cubes in $\mathbb{R}^{n-m}$ and $\mathbb{R}^m$ respectively. Similarly, we define $(Q_w^{V^\perp})'$ and $(Q_w^V)'$ to be the corresponding dilated cubes. Note that Q_w, $Q_w^{V^\perp}$ and Q_w^V all have the same side length $2^{-|w|}$. Similarly, Q'_w, $(Q_w^{V^\perp})'$ and $(Q_w^V)'$ all have the same side length $100 \cdot 2^{-|w|}$. In particular, we denote by $Q^V = P_V(Q)$ the unit cube $[0,1]^m$ and by $Q^{V^\perp} = P_{V^\perp}(Q)$ the unit cube $[0,1]^{n-m}$.

For each $w \in W^*$, let ψ_w be a function in $C_0^\infty(\mathbb{R}^n)$ satisfying the following conditions:

(i) $0 \le \psi_w \le 1$,
(ii) $\psi_w \equiv 1$ on Q_w,
(iii) $\psi_w \equiv 0$ on the complement of $\frac{5}{4}Q_w$,
(iv) $|\nabla \psi_w| \le \frac{C}{s(Q_w)} = C2^{|w|}$.

Let $\xi = (\xi_w)$ be a countable sequence of elements, indexed by the words w in W^*, each lying in the unit ball $B \subset \mathbb{R}^N$. We define the mappings f_ξ. For each $j \ge 0$, we first define mappings $f_{\xi,j} : \mathbb{R}^n \to \mathbb{R}^N$ by the formula

$$f_{\xi,j}(a,x) = 2^{-jm/\alpha} \sum_{\substack{w \in W^j \\ Q_w^{V^\perp} \cap E \neq \emptyset}} \psi_w(a,x)\xi_w, \qquad x \in V, a \in V^\perp.$$

Note that

$$\mathcal{H}^m((Q_w^V)') = C(m)2^{-jm} \tag{4.4}$$

whenever $w \in W^j$, for some fixed constant $C(m)$.

Lemma 4.1. *For all ξ as above and all $j \ge 0$, the map $f_{\xi,j}$ is in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$, with $||f_{\xi,j}||_{1,p}$ bounded above by a finite constant independent of ξ and j.*

We now define $f_\xi : \mathbb{R}^n \to \mathbb{R}^N$ by the formula

$$f_\xi(a,x) = \sum_{j \ge 0} (1+j)^{-2} f_{\xi,j}(a,x). \tag{4.5}$$

Corollary 4.2. *For all ξ as above, f_ξ is in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$, with $||f_\xi||_{1,p}$ bounded above by a finite constant which is independent of ξ.*

To simplify the notation, we henceforth write

$$W^j(E) := \{w \in W^j : Q_w^{V^\perp} \cap E \neq \emptyset\}$$

and $W^*(E) = \bigcup_{j \ge 0} W^j(E)$.

Proof of Lemma 4.1. It is easy to see that the functions $f_{\xi,j}$ are uniformly bounded, so it suffices to check the integrability of the gradient

$$\nabla f_{\xi,j}(a,x) = 2^{-jm/\alpha} \sum_{w \in W^j(E)} \nabla \psi_w(a,x)\xi_w.$$

Since the cubes $\{\frac{5}{4}Q_w\}$ have bounded overlap, we obtain

$$\begin{aligned}\int_Q |\nabla f_{\xi,j}|^p &\le C \int_Q 2^{-jmp/\alpha} \sum_{w\in W^j(E)} |\nabla \psi_w(a,x)|^p \, da\, dx \\ &\le C 2^{j(p-n-mp/\alpha)} \# W^j(E).\end{aligned} \tag{4.6}$$

Let $T = \{1, \dots, 2^{n-m}\}$, $T^* = \bigcup_{j\ge 0} T^j$, and let $\{R_t\}_{t\in T^*}$ denote the usual dyadic decomposition in $Q^{V^\perp}$. Then we have

$$\# W^j(E) = \sum_{\substack{t\in T^j \\ R_t\cap E\neq\emptyset}} \#\{w \in W^j : Q_w^{V^\perp} = R_t\},$$

where $\#\{w \in W^j : Q_w^{V^\perp} = R_t\}$ is bounded by 2^{jm} times a constant independent of t. So we obtain $\# W^j(E) \le C 2^{j(m+\beta)}$, by (4.1), and, returning to (4.6), we find

$$\int_Q |\nabla f_{\xi,j}|^p \le C 2^{j(\beta+p-n+m-\frac{mp}{\alpha})} = C$$

with C independent of ξ and j. This completes the proof of the lemma.

In the second part of the proof, we show that a generic choice of ξ yields a map f_ξ with the desired property. To this end, we now view $\xi = (\xi_w)$ as a sequence of independent random variables, identically distributed according to the uniform probability distribution on B.

For $\alpha > 0$, denote by

$$I_\alpha(\mu) := \iint |x-y|^{-\alpha} \, d\mu(x)\, d\mu(y)$$

the *α-energy* of a finite Borel measure μ in $\mathbb{R}^N$. The Riesz s-capacity, $s > 0$, of a set $A \subset \mathbb{R}^n$ is defined by

$$C_s = \sup\left\{I_s(\mu)^{-1} : \mu \in \mathcal{M}(A) \text{ and } \mu(\mathbb{R}^n) = 1\right\},$$

where $\mathcal{M}(A)$ is the set of Radon measures in $\mathbb{R}^n$ with compact support contained in A. We need the following version of Frostman's Lemma [36, Theorem 8.9(1)]:

Lemma 4.3. *If $s > 0$ and $\mathcal{H}^s(A) < \infty$, then $C_s(A) = 0$.*

For each $a \in E$, consider the measure $(f_\xi)_\#(\mathcal{H}^m \llcorner V_a)$, i.e., the pushforward of the Hausdorff m-measure on the affine subspace V_a via the map f_ξ. We claim that the expectation

$$\mathbb{E}_\xi \left(\int_E I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a))\, d\mathcal{H}^\beta(a) \right) \tag{4.7}$$

is finite for each $\alpha' < \alpha$. If we can prove this claim, then almost surely with respect to ξ, we have

$$\int_E I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a))\, d\mathcal{H}^\beta(a) < \infty$$

and hence $I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a))$ is finite for $\mathcal{H}^\beta$-a.e. $a \in E$. By considering a sequence $\alpha'_n \nearrow \alpha$ and using the countable stability of the Hausdorff measures and Frostman's lemma, we reach our desired conclusion (4.3).

It remains to verify the finiteness of the value in (4.7). By Tonelli's theorem, (4.7) equals

$$\int_{[0,1]^m} \int_{[0,1]^m} \int_E \mathbb{E}_\xi(|f_\xi(a,x) - f_\xi(a,y)|^{-\alpha'})\, d\mathcal{H}^\beta(a)\, d\mathcal{H}^m(x)\, d\mathcal{H}^m(y).$$

To estimate the integrand, we write

$$f_\xi(a,x) - f_\xi(a,y) = \sum_{w \in W^*(E)} c_w(a,x,y)\xi_w$$

where the coefficients are given by

$$c_w(a,x,y) := (1+j)^{-2} 2^{-jm/\alpha} \Big(\psi_w(a,x) - \psi_w(a,y) \Big), \qquad w \in W^j. \tag{4.8}$$

For $a \in V^\perp$ and $x, y \in V$, we let $\mathbf{c}(a,x,y) = (c_w(a,x,y))$ be the sequence of coefficients defined in (4.8). Clearly, $\mathbf{c}(a,x,y)$ is a summable sequence. We denote by $||\mathbf{c}(a,x,y)||_\infty$ the supremum of the terms in $\mathbf{c}(a,x,y)$.

We require the following elementary lemma from probability theory.

Lemma 4.4. *Let $\{X_i\}$ be a countable sequence of independent random variables, identically distributed according to the uniform distribution on the unit ball B in $\mathbb{R}^N$. Let $\mathbf{c} = (c_i) \in \ell^1$. Finally, let $0 < \alpha < N$. Then there exists a constant $C = C(N, \alpha)$ so that*

$$\mathbb{E}\left(\left| \sum_i c_i X_i \right|^{-\alpha} \right) \le C ||\mathbf{c}||_\infty^{-\alpha}. \tag{4.9}$$

Proof. In view of the homogeneity of (4.9) it suffices to prove that

$$\mathbb{E}\left(\left| \sum_i c_i X_i \right|^{-\alpha} \right) \le C$$

when $||\mathbf{c}||_\infty = 1$. Without loss of generality, assume that $||\mathbf{c}||_\infty = |c_1|$. We introduce the multiple random variable $\hat{X} = (X_2, X_3, \ldots)$ and the random variable $Y = -\sum_{i=2}^\infty c_i X_i$. Since X_1 and Y are independent,

$$\mathbb{E}\left(\left| \sum_i c_i X_i \right|^{-\alpha} \right) = \mathbb{E}_{\hat{X}} \left(\mathbb{E}_{X_1} \left(\left| X_1 - Y \right|^{-\alpha} \right) \right)$$

by the law of iterated expectations. Since $\alpha < N$, a simple symmetrization argument yields

$$\mathbb{E}_{X_1} \left(\left| X_1 - y \right|^{-\alpha} \right) \le \frac{1}{|B|} \int_B |x|^{-\alpha}\, dx = C(N, \alpha) < \infty$$

for every $y \in \mathbb{R}^N$. The expectation over $\hat{X}$ does not increase this bound any further. The proof is complete.

Using this lemma, we finish the proof of Theorem 1.4. Applying Lemma 4.4 to the sequences ξ and $\mathbf{c}(a,x,y)$, and noting that $\mathcal{H}^\beta(E)$ is finite, we observe by another application of Tonelli's theorem that it suffices to prove the estimate

$$\int_{[0,1]^m} ||\mathbf{c}(a,x,y)||_\infty^{-\alpha'}\, d\mathcal{H}^m(y) \le C < \infty,$$

where C denotes a constant which is independent of $a \in E$ and $x \in [0,1]^m$.

Fix $a \in E$ and $x \in [0,1]^m = Q^V$. For $y \in Q^V$, let $j(y)$ be the largest integer $j \ge 0$ with the property that x and y lie in identical or adjacent dyadic cubes Q^V_w of level j. It follows from the construction that there exists a word w_0 in $W^{j(y)+1}$ so that $x \in Q^V_{w_0}$ and $y \in (Q^V_{w_0})'$, but $y \notin \frac{5}{4}Q^V_{w_0}$. Furthermore, we may choose the word w_0 so that $Q^{V^\perp}_{w_0} \cap E \neq \emptyset$, i.e., $w_0 \in W^*(E)$. Observe that

$$||\mathbf{c}(a,x,y)||_\infty \ge |c_{w_0}(a,x,y)| = (2+j(y))^{-2}(100 \cdot 2^{-j(y)-1})^m.$$

Let F_j denote the set of points $y \in Q^V$ for which $j(y) = j$. Note that $F_j \subset (Q^V_{w_0})'$. We have

$$\begin{aligned}\int_{Q^V} ||\mathbf{c}(a,x,y)||_\infty^{-\alpha'}\, d\mathcal{H}^m(y) &= \sum_{j\ge 0}\int_{F_j} ||\mathbf{c}(a,x,y)||_\infty^{-\alpha'}\, d\mathcal{H}^m(y)\\ &\le C\sum_{j\ge 0}(2+j)^{2\alpha'}2^{-jm(1-\alpha'/\alpha)}\end{aligned}$$

by (4.4). Since $\alpha' < \alpha$, the series converges. The proof of Theorem 1.4 is complete.

5. Examples

5.1. Quasiconformal maps which increase the Minkowski dimension of many lines

Theorem 1.3 applies in particular to quasiconformal maps. It is natural to ask how sharp the theorem is in that category.

In this section, we prove Theorem 1.6. We construct a quasiconformal mapping for which the exceptional set associated to upper Minkowski dimension distortion has close-to-optimal dimension. We do not have a corresponding example asociated to Hausdorff dimension distortion.

Let us recall the definition of the Minkowski dimension.

Definition 5.1. Let S be a bounded subset of $\mathbb{R}^n$. The *upper Minkowski dimension* of S is

$$\overline{\dim}_M S := \limsup_{r\to 0}\frac{\log \mathbf{N}(S,r)}{\log 1/r}.$$

The *lower Minkowski dimension* of S, denoted $\underline{\dim}_M S$, is defined similarly, with lim inf replacing lim sup. In case the limit exists, the corresponding value is called the *Minkowski dimension*.

Theorem 1.6 corresponds to the case $m = 1$ in the following more general theorem. As we will see in the proof, we may choose

$$\delta_{n,1} = 1 - \frac{1}{n}$$

and so the full range $1 < \alpha < n$ is allowed. Note that Minkowski dimension is only defined for bounded sets, which explains the reason why we only consider the compact set $f(\{a\} \times [0,1]^m)$ in the conclusion of the theorem.

Theorem 5.2. *Let $n \geq 2$ and $1 \leq m \leq n-1$ be integers. Then there exists a positive constant $\delta_{n,m}$ so that for each α satisfying $m < \alpha < m/(1-\delta_{n,m})$ and for each $\epsilon > 0$, there exists a compact set $E \subset \mathbb{R}^{n-m}$ of Hausdorff dimension at least $m\left(\frac{n}{\alpha} - 1\right) - \epsilon$ and a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ so that $\overline{\dim}_M f(\{a\} \times [0,1]^m) \geq \alpha$ for all $a \in E$.*

To simplify the exposition, we will only prove the case $n = 2$, $m = 1$ in what follows. In Remarks 5.5 and 5.6 we comment on the changes required to cover the general situation.

Recall that

$$\dim E \leq \underline{\dim}_M E \leq \overline{\dim}_M E$$

for bounded sets E, with equality throughout if E is nice, for instance, if E is Ahlfors regular. While Hausdorff dimension is countably stable (the dimension of any countable union is the supremum of the dimensions of the pieces), Minkowski dimension is only finitely stable (the dimension of any finite union is the maximum of the dimensions of the pieces).

We begin with a lemma of Heinonen and Rohde. The quasiconformal map g_T in the following lemma maps an interior segment of the unit square in the xy-plane onto a nonrectifiable arc of von Koch snowflake type. The image of this segment under g_T has an increased (Minkowski or Hausdorff) dimension. Nearby segments are mapped onto smooth arcs, hence we realize no increase in their Hausdorff dimension. However, such nearby segments are stretched significantly by the mapping (due to local quasisymmetry), which increases their contribution to the covering number $\mathbf{N}(g_T(\{a\} \times \mathbb{R}), \epsilon)$. To complete the proof of Theorem 5.2, we sum these contributions over all squares in a Whitney-style decomposition of the x-axis.

In the following lemma, we write $A \simeq B$ to indicate that two quantities A and B are comparable up to an absolute multiplicative constant.

For an arbitrary square $T \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, we use the following notation: $\varphi_T : Q \to T$ denotes the unique homothety of positive ratio from the unit square $Q = [0,1]^2$ onto T, s_T denotes the side length of T, and $M_T = \varphi_T(\{\frac{1}{2}\} \times [\frac{1}{4}, \frac{3}{4}])$ denotes a vertical segment in the middle of T of length $\frac{1}{2} s_T$. For $a \in \mathbb{R}$, we denote by γ_a the set $\{a\} \times \mathbb{R}$.

Lemma 5.3 (Heinonen–Rohde)**.** *Fix a real number D, $1 < D < 2$. Let T be any square in the plane. Then there exists a homeomorphism $g_T : T \to T$ with the following properties:*

(i) *g_T is quasiconformal on the interior of T,*
(ii) *$g_T|_{\partial T}$ is the identity,*

(iii) if $p, q \in T$ are within distance $\frac{1}{8}s_T$ from M_T and

$$|p - q| \geq \max\{\operatorname{dist}(p, M_T), \operatorname{dist}(q, M_T)\},$$

then

$$|g_T(p) - g_T(q)| \simeq |p - q|^{1/D} s_T^{1-1/D}. \tag{5.1}$$

(iv) if $a \in \mathbb{R}$ satisfies $d := \operatorname{dist}(\gamma_a, M_T) \leq \frac{1}{8}s_T$, then

$$\mathbf{N}(g_T(\gamma_a \cap T), cd^{1/D} s_T^{1-1/D}) \geq \frac{s_T}{d} \tag{5.2}$$

for some positive constant c.

We remark that the quantities $\mathbf{N}(g_T(\gamma_a \cap T), cd^{1/D} s_T^{1-1/D})$ and s_T/d from (5.2) are in fact comparable, in view of the local quasisymmetry of g_T. However, we only need the stated lower bound in what follows.

Proof. Parts (i), (ii) and (iii) of this lemma coincide with Lemma 3.2 on page 401 in [27]; see also the discussion on page 402. Briefly, the map g_T is constructed as follows. Choose a quasiconformal map h of $\mathbb{R}^2$ which sends M_T onto a D-dimensional snowflake curve of von Koch type contained in the interior of T. Such a map can be chosen so that the estimate in (5.1) holds for all $p, q \in M_T$. For a construction of such a map h, see for instance [44, p. 151]. Next, by a standard technique from quasiconformal function theory, we may choose a map $g_T : \mathbb{R}^2 \to \mathbb{R}^2$ which is equal to the identity on the complement of T, and which agrees with h on a neighborhood of M_T. This is the desired map.

To complete the proof, we need only verify part (iv). Let a be a point satisfying the stated conditions, choose an integer N satisfying

$$\frac{s_T}{N} \geq d > \frac{s_T}{N+1},$$

and choose $N+1$ points $p_0, \ldots, p_N$ on $\gamma_a \cap T$ so that $|p_i - p_{i-1}| = s_T/N$ for all $i = 1, \ldots, N$. If $i \neq j$, then $|p_i - p_j| \geq \frac{s_T}{N} \geq d$ and hence (by part (iii)),

$$|g_T(p_i) - g_T(p_j)| \geq \frac{1}{C}|p_i - p_j|^{1/D} s_T^{1-1/D} \geq \frac{1}{C}\left(\frac{s_T}{N}\right)^{1/D} s_T^{1-1/D}$$

for some constant C. Hence we require at least $N+1$ balls of radius $c\left(\frac{s_T}{N}\right)^{1/D} s_T^{1-1/D}$ to cover $g_T(\gamma_a \cap T)$, where $c = \frac{1}{3C}$. A fortiori, we require at least $N+1$ balls of radius $cd^{1/D} s_T^{1-1/D}$ to cover $g_T(\gamma_a \cap T)$. We conclude the proof by observing that $N + 1 > \frac{s_T}{d}$.

In the proof of Theorem 5.2 we will use the following calculation of the Hausdorff dimension of certain Cantor sets. See, e.g., Example 4.6 in [13].

Proposition 5.4. *Let $W_1, W_2, \ldots$ be finite sets with $M_j := \#W_j \geq 2$ for each j, let $W^* = \bigcup_{j \geq 0}(W_1 \times \cdots \times W_j)$, and let $\{I_w\}_{w \in W^*}$ be a family of closed intervals satisfying the following conditions:*

(i) $I_w \subset I_v$ whenever v is a subword of w,

(ii) $\max\{|I_w| : w \in W_1 \times \cdots \times W_j\} \to 0$ *as* $j \to \infty$, *and*

(iii) *there exists a decreasing sequence* (ϵ_j) *of positive real numbers so that* $\operatorname{dist}(I_v, I_w) \geq \epsilon_j$ *whenever* $v, w \in W_1 \times \cdots \times W_j$ *are distinct.*

Let $E = \bigcap_{j\geq 1} \bigcup_{w\in W_1\times\cdots\times W_j} I_w$. *Then*

$$\dim E \geq \liminf_{j\to\infty} \frac{\sum_{i=1}^{j} \log M_i}{-\log(\epsilon_{j+1} M_{j+1})}. \tag{5.3}$$

Proof of Theorem 5.2. Let $\alpha \in (1,2)$ and $\epsilon > 0$ be fixed. Without loss of generality, we may assume that $\epsilon < \frac{2}{\alpha} - 1$. Choose a rational number $b > 1$ satisfying

$$\frac{\alpha}{2-\alpha} < b < \frac{\alpha}{2-(1+\epsilon)\alpha}$$

and define

$$D := \alpha \left(\frac{b-1}{b-\alpha} \right).$$

Observe that

$$\frac{1}{b} > \left(\frac{2}{\alpha} - 1 \right) - \epsilon$$

and also that $\alpha < D < 2$.

Let $(n_j)_{j\geq 1}$ be any increasing sequence of positive integers with the following properties:

(i) $n_{j+1} - bn_j$ is an integer for each $j \geq 1$, and

(ii) the limit of $\frac{\sum_{i=1}^{j} n_i}{n_{j+1}}$ as $j \to \infty$ is equal to zero.

For instance, if $b = \frac{P}{Q}$ in lowest terms, we may choose $n_j = QP^j 2^{2^j}$.

We associate to the sequence (n_j) a *sub-Whitney decomposition* $\mathcal{W}$ of the upper half plane, or more precisely, of the domain $\Omega = (0,1) \times (-2,2)$ relative to the x-axis. This means that we begin with the standard Whitney decomposition of Ω relative to the x-axis, and subdivide all squares in this decomposition with size between 2^{-n_j} and $2^{-n_{j+1}}$ into subsquares of size $2^{-n_{j+1}}$. Note that the resulting squares T have the property that $\operatorname{diam} T$ is bounded above by a constant multiple of the distance d from T to the x-axis, however, $\operatorname{diam} T$ may be significantly smaller than d.

Define a map $f : \Omega \to \Omega$ by setting $f|_T = g_T$ for each $T \in \mathcal{W}$. Since g_T is the identity on the boundary of T, this map is well-defined and continuous. Extend it to a map f of $\mathbb{R}^2$ to itself by the identity. Then f is quasiconformal.

We now define a Cantor set on the x-axis by an iterative procedure. For each $j \geq 1$ and each square $T \in \mathcal{W}$ with $s_T = 2^{-n_j}$ and $T \cap \{(x,y) : y = 2^{-n_j}\} \neq \emptyset$, the projection P of the set $T \cap \{(x,y) : y = 2^{-n_j}\}$ onto the x-axis consists of $2^{n_{j+1}-n_j}$ nonoverlapping closed intervals, each of length $2^{-n_{j+1}}$. Note that the total length of all of these intervals is equal to 2^{-n_j}, which is the side length of P. Select the subcollection of these intervals, centered around the middle of P, of total length 2^{-bn_j}. Observe that this subcollection consists of $2^{n_{j+1}-bn_j}$ intervals each of length $2^{-n_{j+1}}$. In the inductive step, we consider only squares in some vertical column corresponding to one of these intervals and repeat the construction.

For each j, let
$$W_j = \{1, \dots, 2^{n_j - bn_{j-1}}\}$$
and denote by I_w, $w \in W_1 \times \cdots \times W_j$, the intervals at the jth level in the construction in the previous paragraph. The Cantor set in question is
$$E = \bigcap_{j \geq 1} \bigcup_{w \in W_1 \times \cdots \times W_j} I_w.$$
Using Proposition 5.4 with $M_j = 2^{n_j - bn_{j-1}}$ and $\epsilon_j \simeq 2^{-n_j}$ we find
$$\dim E \geq \lim_{j \to \infty} \frac{n_j - (b-1)\sum_{i=1}^{j-1} n_i}{bn_j} = \frac{1}{b} > (\frac{2}{\alpha} - 1) - \epsilon.$$
Now suppose that $a \in E$ and fix an integer $j \geq 1$. Then a is contained in a unique interval I_w with $w \in W_1 \times \cdots \times W_{j+1}$ which in turn is contained in a unique interval $I_{\hat{w}}$ with $\hat{w} \in W_1 \times \cdots \times W_j$. Let T be any square from $\mathcal{W}$ lying above the interval $I_{\hat{w}}$. Then the distance from γ_a to M_T is bounded above by $\frac{1}{2}2^{-bn_j}$ which is smaller than $\frac{1}{8}s_T = \frac{1}{8}2^{-n_j}$ provided that j is chosen sufficiently large. Note that there are
$$2^{n_j - n_{j-1}} - 1$$
such squares T. We define a sequence of scales (δ_j) depending on the point a; the desired estimate for the upper Minkowski dimension of $f(\gamma_a)$ will come from analyzing the covering number on this sequence of scales by an application of Lemma 5.3.

Let
$$\delta_j = c \operatorname{dist}(\gamma_a, M_T)^{1/D} s_T^{1-1/D} = c|a - m_j|^{1/D} 2^{-n_j(1-1/D)},$$
where m_j denotes the x-coordinate of the midline M_T. By Lemma 5.3(iv), we have
$$\mathbf{N}(g_T(\gamma_a \cap T), \delta_j) \geq \frac{s_T}{\operatorname{dist}(\gamma_a, M_T)} = \frac{2^{-n_j}}{|a - m_j|}.$$
Summing this over all of the relevant squares gives
$$\mathbf{N}(f(\gamma_a \cap Q), \delta_j) \geq (2^{n_j - n_{j-1}} - 1)\frac{2^{-n_j}}{|a - m_j|} \geq \frac{2^{-n_{j-1}}}{2|a - m_j|}.$$
We conclude that
$$\overline{\dim}_M f(\gamma_a \cap Q) \geq \limsup_{j \to \infty} \frac{-\log_2 |a - m_j| - n_{j-1} - 1}{-\frac{1}{D}\log_2 |a - m_j| + (1 - \frac{1}{D})n_j + C}. \tag{5.4}$$
Observing that $|a - m_j| \leq 2^{-bn_j - 1}$ and that the expression inside the limit on the right hand side of (5.4) is nondecreasing in the variable $-\log_2 |a - m_j|$, we conclude that
$$\overline{\dim}_M f(\gamma_a \cap Q) \geq D \cdot \limsup_{j \to \infty} \frac{bn_j - n_{j-1}}{(b + D - 1)n_j + DC + D} = \frac{bD}{b + D - 1} = \alpha$$
by the choice of D. This completes the proof.

Remark 5.5. For general n (still assuming $m = 1$) the proof is similar. We require the existence of D-dimensional von Koch snowflake curves in $\mathbb{R}^n$ for each $1 < D < n$. More precisely, we require a curve $\Gamma \subset \mathbb{R}^n$ such that $\Gamma = g(\mathbb{R})$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal map so that $|g(x)-g(y)| \simeq |x-y|^{1/D}$ for all $x, y \in \mathbb{R}$ with $|x-y| \le 1$. For a construction of such curves in $\mathbb{R}^3$, see Bonk and Heinonen [8]. A similar construction has been given by Ghamsari and Herron [17]. Using this construction, the proof of Theorem 5.2 for $m = 1$ and general n proceeds in a similar fashion.

Remark 5.6. The case $m \ge 2$ in Theorem 5.2 is more challenging. We require the existence of D-dimensional quasiconformal submanifolds of $\mathbb{R}^n$ of von Koch type. More precisely, we require a topological m-manifold $\Sigma \subset \mathbb{R}^n$ so that $\Sigma = g(\mathbb{R}^m)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal map so that

$$|g(x) - g(y)| \simeq |x - y|^{m/D}, \qquad \forall x, y \in \mathbb{R}^m, |x - y| \le 1. \tag{5.5}$$

Such *snowflaked quasiconformal submanifolds* were constructed by David and Toro [10] for a small range of values $D \in [m, m+\epsilon_{n,m})$. Using such submanifolds, one can establish an analog for Lemma 5.3 and thereby establish Theorem 5.2 for general m satisfying (1.1). The value of $\delta_{n,m}$ in Theorem 5.2 depends on the size of the interval $[m, m + \epsilon_{n,m})$ of dimensions of such snowflaked quasiconformal submanifolds. We leave to the interested reader the computation of a precise relationship between $\delta_{n,m}$ and $\epsilon_{n,m}$.

Snowflaked quasiconformal submanifolds were previously used in [8] and [30] to study the effect of smoothness on branching phenomena for quasiregular mappings.

Remark 5.7. Bishop [7] previously constructed a quasiconformal map g of $\mathbb{R}^3$ so that $g(W)$ contains no rectifiable curves, where $W \in G(3, 2)$ is a fixed plane. In particular, choosing $V \in G(3, 1)$ with $V \subset W$ and expressing $\mathbb{R}^3$ as an orthogonal sum

$$V \oplus (V^\perp \cap W) \oplus W^\perp \tag{5.6}$$

exhibits a one-dimensional family of parallel lines V_a, $a \in V^\perp \cap W$, all of whose images under g have no nontrivial rectifiable subcurves. The construction in [7], however, did not guarantee any dimension increase for the sets $g(V_a)$.

Using the aforementioned result of David and Toro and expressing $\mathbb{R}^n$ as an orthogonal sum of the form (5.6) for some $V \in G(n, k)$, $k < m$, $V \subset W$, we can exhibit an $(m - k)$-dimensional family of parallel lines V_a, $a \in V^\perp \cap W$, all of whose images under g have Hausdorff dimension at least a fixed value $D > m$.

Remark 5.8. Kovalev and Onninen [32, Corollary 1.6] have recently shown that, to every countable family of parallel lines $\{V_a\}$ in the plane, there corresponds a *reduced* quasiconformal map f of $\mathbb{R}^2$ with the property that each curve $f(V_a)$ has no nontrivial rectifiable subcurve. (See Definition 1.4 in [32] for the definition of reduced planar quasiconformal map.) It is not clear how to extend their construction to higher dimensions. Reduced quasiconformality implies that the image curves $f(V_a)$ necessarily have Hausdorff dimension equal to one [32, Theorem 1.7]. In Theorem 1.6, the curves $f(V_a)$ are nonrectifiable but locally rectifiable and also have Hausdorff dimension equal to one. However, the size of the family of lines allowed in Theorem 1.6 is substantially larger than that in [32].

5.2. Space-filling maps in subcritical Sobolev classes

We continue with a discussion of the critical and subcritical cases, i.e., the case

$$p \leq n.$$

We are interested in understanding the frequency of Hausdorff dimension distortion by a map f in $W^{1,p}(\Omega, Y)$. The first point to emphasize is that the problem is not precisely defined in this setting. Indeed, Sobolev maps in the critical class $W^{1,n}$ need not have continuous representatives. Varying the representative of f can affect the dimension distortion properties.

It is a standard fact of Sobolev space theory [49, Corollary 3.3.4] that $W^{1,p}$ maps admit p-quasicontinuous representatives, i.e. representatives which are continuously defined on the complement of sets of zero Bessel capacity $B_{1,p}$. We omit the definition of the Bessel capacity $B_{1,p}$ but we recall that $B_{1,p}(E) = 0$ whenever $\mathcal{H}^{n-p}(E) < \infty$, and $B_{1,p}(E) = 0$ implies that $\mathcal{H}^{n-p+\epsilon}(E) = 0$ for any $\epsilon > 0$; see [49, Theorem 2.6.16]. It is natural to restrict our attention to p-quasicontinuous representatives. Recently, Hencl and Honzík [28] proved the following extension of Theorem 1.3.

Theorem 5.9 (Hencl–Honzík)**.** *Let $m < \alpha < p \leq n$ and define $\beta = \beta(\alpha, p)$ as in* (1.5)*. Let f be the p-quasicontinuous representative of a mapping in $W^{1,p}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$. Then for each $V \in G(n,m)$ we have*

$$\dim\{a \in V^\perp : \dim f(V_a \cap \Omega) \geq \alpha\} \leq \beta.$$

In other words, the conclusion of our main Theorem 1.3 holds on the level of Hausdorff dimensions, even for (certain) integrability exponents p below the critical value n. The restriction on p stems from the fact that the dimension of the exceptional set for the p-quasicontinuous representative is at most $n - p$, which is strictly smaller than $\beta(\alpha, p)$ provided $p > \alpha$.

The situation for smaller values of p, and in particular for $p \leq m$, is more intriguing. For such a representative f we have no information whatsoever about the behavior of f on the $(n-p)$-dimensional exceptional set. Examples 5.11 and 5.12 below indicates the extent to which our results on dimension bounds can fail. The constructions in these examples are based on the following result taken from [24, Theorem 1.3].

Example 5.10 (Hajłasz–Tyson). Let $n \geq 2$. There exists a continuous map $g \in W^{1,n}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ and a set $F \subset [0,1]^n$ of Hausdorff dimension zero so that $\dim g(F) = \infty$. In particular, $\dim g([0,1]^n) = \infty$.

In other words, there exists a continuous $W^{1,n}$ map on the unit cube in $\mathbb{R}^n$ with infinite-dimensional target.

Next, we use Example 5.10 to illustrate what type of dimension distortion behavior can occur for maps in $W^{1,m}$. Note that here, in contrast with the rest of this paper, we require $m \geq 2$, since we appeal to Example 5.10. It is easy to see that Example 5.10 cannot extend to the case $n = 1$. Indeed, every $W^{1,1}$ map from $\mathbb{R}$ is absolutely continuous and the target has dimension at most one.

Example 5.11. Let $n \geq 3$ and $2 \leq m \leq n-1$ be integers. Then there exists a continuous map $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ with the property that $\dim f(\{a\} \times [0,1]^m) = \infty$ for all $a \in [0,1]^{n-m}$.

Proof. Let $g : [0,1]^m \to \ell^2$ be a continuous map in the class $W^{1,m}$ which is constant on the boundary of $[0,1]^m$ and for which $\dim g([0,1]^m) = \infty$. Define $f : [0,1]^n \to \ell^2$ by

$$f(a,x) = g(x), \qquad a \in \mathbb{R}^{n-m}, x \in \mathbb{R}^m.$$

Extend f to be constant on the complement of $[0,1]^n$. Then $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ and f is continuous. Moreover, for each $a \in [0,1]^{n-m}$, the set $f(\{a\} \times [0,1]^m) = g([0,1]^m)$ is infinite-dimensional.

We next modify the preceding example to illustrate what can happen for maps in $W^{1,p}$, $m < p < n$, with regard to almost sure dimension distortion of parallel subspaces. To accomplish this, we will need to modify the details of the construction of Example 5.10.

Example 5.12. Fix integers $1 \le m < n$ and let $m < p < n$. Then there exists a continuous map $f \in W^{1,p}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ and there exist compact sets $F \subset [0,1]^m$ and $E \subset [0,1]^{n-m}$ so that

1. the Hausdorff dimension of F is strictly less than $\frac{m}{p+1}$,
2. the Hausdorff dimension of E is in the interval $\left(n-p-\frac{m}{p+1}, n-p\right]$,
3. $\dim E \times F = \dim E + \dim F = n-p$, and
4. $\dim f(\{a\} \times F) = \infty$ for all $a \in E$.

The proof will show that when p is an integer, we may choose $\dim F = 0$ and $\dim E = n-p$.

We begin with some remarks.

The construction in Example 5.10 uses the fact that the n-capacity of a point in $\mathbb{R}^n$ is equal to zero. This allows us to build a $W^{1,n}$ map from a domain in $\mathbb{R}^n$ whose image is large with very small n-energy. In fact, the map is constructed first on the zero dimensional Cantor set F and then is extended to all of $[0,1]^n$ while preserving the finiteness of the n-energy.

The corresponding construction in Example 5.12 will use the p-capacity. The details are more technical, however, since we must work explicitly with the precise value of this capacity and relate it to the cardinality of various prefractals associated to the Cantor set F.

Let us recall the definition of capacity.

Definition 5.13. Let $F \subset \mathbb{R}^n$ be an open set and let E be a compact subset of F. The *p-capacity*, $p \ge 1$, of the pair (E,F) is the value

$$\mathrm{Cap}_p(E,F) = \inf \int_{\mathbb{R}^n} |\nabla\varphi|^p,$$

where the infimum is taken over all functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that $\varphi|_E = 1$ and $\varphi|_{\mathbb{R}^n\setminus F} = 0$.

We require knowledge of the behavior of the p-capacity of a ring domain. The following lemma is standard. Denote by $Q^n(r) = \{x \in \mathbb{R}^n : |x_i| \le r \ \forall i = 1,\dots,n\}$ the closed cube of side length $2r$ centered at the origin, and denote by $Q^n(r)^o$ its interior.

Lemma 5.14. *Let $0 < r < R < \infty$ and $1 < p < \infty$. Then*

$$\mathrm{Cap}_p(Q^n(r), Q^n(R)^o) = \begin{cases} c(n,p)\left|R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right|^{1-p}, & \text{if } p \neq n, \\ c(n)(\log R/r)^{1-n}, & \text{if } p = n. \end{cases}$$

In particular, if $1 < p < n$ and $2r < R$, then

$$C^{-1} r^{n-p} \le \mathrm{Cap}_p(Q^n(r), Q^n(R)^o) \le C r^{n-p} \tag{5.7}$$

for some constant $C = C(n,p)$.

Let $\varphi_{r,R;n,p} \in C_0^\infty(\mathbb{R}^n)$ be quasiextremal for the p-capacity of the ring domain $(Q^n(r), Q^n(R)^o)$, i.e.,

$$\varphi_{r,R;n,p}|_{Q^n(r)} = 1, \tag{5.8}$$

$$\varphi_{r,R;n,p}|_{\mathbb{R}^n \setminus Q^n(R)} = 0, \tag{5.9}$$

and

$$\int_{\mathbb{R}^n} |\nabla \varphi_{r,R;n,p}|^p \le C r^{n-p} \tag{5.10}$$

for some constant $C = C(n,p)$.

We now begin the construction of the mapping described in Example 5.12. The target will be the (compact) Hilbert cube

$$Y = \{y = (y_i) \in \ell^2 : |y_i| \le \frac{1}{i}\}.$$

In fact, any compact infinite-dimensional subset of ℓ^2 would work for our purposes.

There exists an increasing sequence of positive integers (N_j) and an increasing sequence of finite sets $Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y$ with the following properties:

- Y_j is 2^{-j}-dense in Y, i.e., every point of Y lies within distance 2^{-j} from a point of Y_j, and
- we can assign to each element y of Y_j a *parent* in Y_{j-1} which lies at distance 2^{-j} from y, so that each point in Y_{j-1} has at most 2^{N_j} children. The parent of any $y \in Y_1$ is $0 \in \ell^2$.

From the second condition, it follows that the cardinality of Y_j is at most $2^{\tilde{N}_j}$, where

$$\tilde{N}_j = N_1 + N_2 + \cdots + N_j.$$

We may assume that each of the integers N_j is a multiple of m.

For each point $y \in Y_j$, denote by γ_y the line segment in ℓ^2 joining y to its parent $\hat{y}$. The length of γ_y is at most 2^{-j}. We parameterize γ_y at constant speed by the interval $[0, 2^{-j}]$, in such a way that $\gamma_y(2^{-j}) = y$ and $\gamma_y(0) = \hat{y}$. As a map from $[0, 2^{-j}]$ to ℓ^2, γ_y is 1-Lipschitz.

We now return to the source space. Let k be the smallest integer greater than or equal to $p-m$ and write

$$\mathbb{R}^n = \mathbb{R}^{n-m-k} \times \mathbb{R}^k \times \mathbb{R}^m.$$

We will write points of $\mathbb{R}^n$ according to this splitting in the form $(a_1, a_2, x) = (a, x)$, where $a \in \mathbb{R}^{n-m}$ and $x \in \mathbb{R}^m$.

First, we construct a Cantor set in $\mathbb{R}^{k+m}$. Let $Q = [0,1]^{k+m}$ be the unit cube in $\mathbb{R}^{k+m}$. We partition Q into $2^{(\frac{k}{m}+1)N_1}$ nonoverlapping subcubes of side length $2^{-N_1/m}$. We denote these subcubes by P_w, where w is an index ranging over

$$W_1 = \{1, \ldots, 2^{kN_1/m}\} \times \{1, \ldots, 2^{N_1}\}.$$

Next, fix $\lambda < 1$. Inside each of the above subcubes, consider two further subcubes $Q_w \subset Q'_w \subset P_w$ so that

1. Q'_w has side length $R_1 = \beta_1 = \lambda \cdot 2^{-\frac{N_1}{m}}$,
2. Q_w has side length $r_1 = \alpha_1 = \lambda \cdot 2^{-\frac{m+k}{m+k-p} \cdot \frac{N_1}{m}}$, and
3. the distance between any two distinct cubes in $\{Q'_w\}_{w \in W_1}$ is comparable to $2^{-N_1/m}$.

For instance, we may choose Q_w and Q'_w to be concentric with each other and with the original cube P_w.

We now describe the inductive step. Assume that we are given a collection of disjoint cubes $\{Q_w\}$ indexed by the elements w in $W_1 \times \cdots \times W_j$, where

$$W_i = \{1, \ldots, 2^{kN_i/m}\} \times \{1, \ldots, 2^{N_i}\}.$$

We further assume that each of the cubes Q_w has side length $r_j = \alpha_1 \cdots \alpha_j$ where

$$\alpha_i = \lambda \cdot 2^{-\frac{m+k}{m+k-p} \cdot \frac{N_i}{m}}.$$

Let $R_j = \alpha_1 \cdots \alpha_{j-1} \cdot \beta_j$, where

$$\beta_i = \lambda \cdot 2^{-\frac{N_i}{m}}.$$

We partition each of the cubes Q_w into $2^{(\frac{k}{m}+1)N_{j+1}}$ nonoverlapping subcubes $P_{ww_{j+1}}$ of side length $2^{-N_{j+1}/m}$, which we index by a parameter w_{j+1} ranging over W_{j+1}.

Inside each of these subcubes, consider two further subcubes $Q_{ww_{j+1}} \subset Q'_{ww_{j+1}} \subset P_{ww_{j+1}}$ so that

1. $Q'_{ww_{j+1}}$ has side length $R_{j+1} = r_j \beta_{j+1}$,
2. $Q_{ww_{j+1}}$ has side length $r_{j+1} = r_j \alpha_{j+1}$, and
3. the distance between any two distinct cubes in $\{Q'_w\}_{w \in W_1 \times \cdots \times W_{j+1}}$ is comparable to $2^{-N_{j+1}/m} r_j$.

The Cantor set in question is

$$C = \bigcap_{j \geq 1} \bigcup_{w \in W_1 \times \cdots \times W_j} Q_w.$$

For each j, map W_j to the set $V_j := \{1, \ldots, 2^{N_j}\}$ by projecting to the second factor. This induces a map from $W_1 \times \cdots \times W_j$ to $V_1 \times \cdots \times V_j$.

By the choice of the sets Y_j, we can choose a surjective map from $V_1 \times \cdots \times V_j$ to Y_j for all j so that the following diagram commutes:

$$\begin{array}{ccccc} W_1 \times \cdots \times W_{j+1} & \to & V_1 \times \cdots \times V_{j+1} & \to & Y_{j+1} \\ \downarrow & & \downarrow & & \downarrow \\ W_1 \times \cdots \times W_j & \to & V_1 \times \cdots \times V_j & \to & Y_j \end{array}.$$

Here the left hand and central vertical maps are the natural projections, while the right hand map is the one which assigns to each point $y \in Y_{j+1}$ its parent $\hat{y} \in Y_j$. We denote by y_w the point in Y_j which corresponds to a given $w \in W_1 \times \cdots \times W_j$.

We now define a map $g : \mathbb{R}^{k+m} \to \ell^2$. If $w \in W_1 \times \cdots \times W_j$ and $(a_2, x) \in Q'_w \setminus Q_w$, then

$$g(a_2, x) = \gamma_{y_w}\left(2^{-j}\varphi_{r_j,R_j;m+k,p}((a_2, x) - c_w)\right),$$

where c_w denotes the center of the square Q_w. Observe that $g|_{\partial Q'_w} = \gamma_{y_w}(0) = \widehat{y_w}$ and $g|_{\partial Q_w} = \gamma_{y_w}(2^{-j}) = y_w$ by (5.8) and (5.9), respectively. Thus we may extend g to the sets $Q_w \setminus \bigcup_{w_{j+1}} Q'_{ww_{j+1}}$ for each w, and also to the set $\mathbb{R}^{k+m} \setminus Q$ in a continuous fashion, by setting g to an appropriate constant value in each of those sets. This defines g on the complement of C; we extend g by continuity to all of $\mathbb{R}^{k+m}$. Observe that for each $a_2 \in P_{\mathbb{R}^k}(C)$, the closed set $g(\{a_2\} \times P_{\mathbb{R}^m}(C))$ contains each of the sets Y_j, and hence contains all of Y.

We now define a map $f : \mathbb{R}^n \to \ell^2$ by setting $f(a, x) = f(a_1, a_2, x) = g(a_2, x)$ for all $a_1 \in [0,1]^{n-m-k}$ and extending by a suitable constant value for other values of a_1.

We claim that f is in the Sobolev space $W^{1,p}$. Since f is bounded, it suffices to verify that it has an upper gradient in L^p. For any w and for all (a_1, a_2, x) in the set $[0,1]^{n-m-k} \times (Q'_w \setminus Q_w)$,

$$|\nabla f(a_1, a_2, x)| = |\nabla g(a_2, x)| \le 2^{-j}|\nabla\varphi_{r_j,R_j;m+k,p}((a_2, x) - c_w)|.$$

At other points, ∇f vanishes. Thus we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f|^p &= \sum_w \int_{[0,1]^{n-m-k}\times(Q'_w\setminus Q_w)} |\nabla f|^p \\ &\le \sum_{j=1}^{\infty} 2^{-jp} \sum_{w\in W_1\times\cdots\times W_j} ||\nabla\varphi_{r_j,R_j;m+k,p}||^p_{L^p(\mathbb{R}^{k+m})} \\ &\le C\sum_{j=1}^{\infty} 2^{-jp} r_j^{m+k-p} \#(W_1 \times \cdots \times W_j)\end{aligned}$$

by (5.10)

$$\le C\sum_{j=1}^{\infty} 2^{-jp} r_j^{m+k-p} 2^{(\frac{k}{m}+1)\tilde{N}_j} = C\sum_{j=1}^{\infty} 2^{-jp} \prod_{i=1}^{j}\left(\alpha_i^{m+k-p} 2^{(\frac{k}{m}+1)N_i}\right).$$

By the choice of α_i, we easily see that

$$\alpha_i^{m+k-p} 2^{(\frac{k}{m}+1)N_i} = \lambda^{m+k-p} \le 1,$$

so the above product is bounded above by one and the sum converges. This shows that f is an element of the Sobolev space $W^{1,p}$.

Let F be the projection of C into the $\mathbb{R}^m$ factor, let E_2 be the projection of C into the $\mathbb{R}^k$ factor, and let $E = [0,1]^{n-m-k} \times E_2$. Using again the estimate in [13, Example 4.6], we find

$$\dim F = \lim_{j\to\infty} \frac{\log 2^{\tilde{N}_j}}{\log(1/r_j)} = m - \frac{pm}{m+k} < \frac{m}{p+1}$$

and

$$\dim E = n - m - k + \lim_{j\to\infty} \frac{\log 2^{\frac{k}{m}\tilde{N}_j}}{\log(1/r_j)} = n - m - \frac{pk}{m+k}.$$

Recalling that k is the smallest integer greater than or equal to $p-m$, we leave the details of the remaining claims to the reader. Note that $f(\{a\} \times F) \supset Y$ whenever $a \in E$.

6. Open problems and questions

Problem 6.1. Our main theorem estimates the size of the collection of parallel affine subspaces whose image under a fixed supercritical Sobolev mapping f exhibits a prespecified dimension jump. Do similar results hold for other parameterized families of subspaces?

As a sample of the type of problems which could be posed, we present the following variation on our main theme.

The Grassmanian manifold $G(n,m)$ is a smooth manifold of dimension $m(n-m)$. How many subspaces $V \in G(n,m)$ can have the property that their image under f exhibits a prespecified dimension jump? To be more precise, fix $p > n$ and α satisfying $m < \alpha < \frac{pm}{p-n+m}$. We ask for an estimate from above for the dimension of the set of subspaces $V \in G(n,m)$ for which $\dim f(V) \ge \alpha$. In fact, we seek an estimate of the form

$$\dim\{V \in G(n,m) : \dim f(V) \ge \alpha\} \le m(n-m) - \delta,$$

where $\delta = \delta(n,m,\alpha,p) > 0$.

The Grassmanian $G(n,1)$ coincides with the real projective space $P^{n-1}_{\mathbb{R}}$, which has dimension $n-1$. Using local triviality of the tautological line bundle over $G(n,1)$, one can recast the above problem into the framework of the product decomposition considered in our main theorem. The eventual conclusion matches that from Theorem 1.3, in the case $m = 1$. We omit the details, reserving discussion of this question for a later paper.

Problem 6.2. We anticipate that (3.9) is not sharp. Indeed, the dimension bounds in (3.8) can be improved in the case when E is a line. Smirnov [42] has shown that

$$\dim f(E) \le 1 + \left(\frac{K-1}{K+1}\right)^2 \tag{6.1}$$

whenever $E \subset \mathbb{R}^2$ is a line segment and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a K-quasiconformal map. We expect that (3.9) can be improved in the planar case to an estimate which recovers (6.1) at the borderline, when the exceptional set has zero dimension.

Problem 6.3. Does Theorem 1.6 hold with Minkowski dimension replaced by Hausdorff dimension?

Problem 6.3 asks about the existence of a planar quasiconformal map which sends a family of parallel lines, parameterized by a set of positive Hausdorff dimension, onto curves all of which have a specified lower bound on their Hausdorff dimension. Even a much weaker problem remains unsolved.

Problem 6.4. Does there exist a planar quasiconformal map f and an uncountable family of parallel lines $\{\ell_i\}$ so that $f(\ell_i)$ contains no nontrivial rectifiable arc for any i?

Problem 6.5. What can be said for other source spaces? The notion of Sobolev space defined on a metric measure space is by now well understood, see for instance [41], [18], [26], [21]. Even in the potentially simplest non-Euclidean setting, when the source is the sub-Riemannian Heisenberg group, it is unclear whether results analogous to those of this paper hold. We make substantial use of several purely Euclidean features, such as the Besicovitch covering theorem and the fact that the projection mappings $P_V : \mathbb{R}^n \to V$ are Lipschitz. In the Heisenberg group, the Besicovitch covering theorem is false and retractions along the fibers of a horizontal foliation are never Lipschitz. See [31] or [40] for details. At present, it appears that these complications preclude the development of a theory similar to that presented in this paper, in more general, non-Riemannian, contexts.

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