Starting from hyperbolic dispersion relations for the invariant amplitudes of pion–nucleon scattering together with crossing symmetry and unitarity, one can derive a closed system of integral equations for the partial waves of both the $s$-channel ($\pi N \rightarrow \pi N$) and the $t$-channel ($\pi \pi \rightarrow NN$) reaction, called Roy–Steiner equations. After giving a brief overview of the Roy–Steiner system for $\pi N$ scattering, we demonstrate that the solution of the $t$-channel subsystem, which represents the first step in solving the full system, can be achieved by means of Muskhelishvili–Omnès techniques. In particular, we present results for the $P$-waves featuring in the dispersive analysis of the electromagnetic form factors of the nucleon.
1. Introducing Roy–Steiner equations for $\pi N$ scattering

Partial-wave dispersion relations (PWDRs) together with unitarity and crossing symmetry as well as isospin and chiral symmetry (i.e. all available symmetry constraints) have repeatedly proven to be a powerful tool for studying processes at low energies with high precision [1-4]. For $\pi N$ scattering the (unsubtracted) hyperbolic dispersion relations (HDRs) for the usual Lorentz-invariant amplitudes read [5] (using the notation of [6], see [7] for more details)

$$A^+(s,t) = \frac{1}{\pi} \int_{s_\pm}^{\infty} \frac{ds'}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a} \right] \text{Im} A^+(s',t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{dt}{t'-t} \right],$$

$$B^+(s,t) = N^+(s,t) + \frac{1}{\pi} \int_{s_\pm}^{\infty} \frac{ds'}{s'-s} - \frac{1}{s'-u} \right] \text{Im} B^+(s',t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{dt}{t'-t} \right],$$

$$N^+(s,t) = g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right], \quad (s-a)(u-a) = b = (s'-a)(u'-a), \quad (1.1)$$

and similarly for $A^-$, $B^-$, and $N^-$, where $N^\pm$ are the nucleon pole terms and the “external” (unprimed) and “internal” (primed) kinematics are related by real hyperbola parameters $a$ and $b$ (as well as via $s + t + u = 2m^2 + M_\pi^2 = s' + t' + u'$), so that HDRs allow for the combination of all physical regions, which is known to be crucial for a reliable continuation into the subthreshold region and hence for an accurate determination of the $\pi N \sigma$-term. Furthermore, the imaginary parts are only needed in regions where the corresponding partial-wave decompositions converge and the range of convergence can be maximized by tuning the free hyperbola parameter $a$. While the $s$-channel integrals start at the threshold $s_\pm = W_\pm^2 = (m + M_\pi)^2$, the $t$-channel contributes already above the pseudothreshold $t_\pi = 4M_\pi^2$ far below the threshold $t_N = 4m^2$. Depending on the asymptotic behavior of the imaginary parts, in principle it could be necessary to subtract the HDRs to ensure the convergence of the integrals, thereby parameterizing high-energy information with polynomials containing a priori unknown subtraction constants. However, (additional) subtractions may also be introduced to lessen the dependence of the low-energy solution on high-energy input; the corresponding subtraction parameters then obey respective sum rules. For $\pi N$ scattering it proves particularly useful to subtract at the subthreshold point $(s = u, t = 0)$, as this preserves the $s \leftrightarrow u$ crossing symmetry (which can be made explicit in terms of the crossing variable $v = (s-u)/(4m)$ via $D^\pm(\nu, t) = A^\pm + vB^\pm = \pm D^\pm(-\nu, t)$). This is especially favorable for the $t$-channel subproblem and facilitates matching to chiral perturbation theory [8, 9] to determine the subtraction constants, which thus can be identified with the subthreshold expansion parameters.\(^1\)

In addition to the presentation in [6], we also introduce a (partial) third subtraction, which is related to the parameters $a_{10}^+$ and $a_{01}^+$ of the subthreshold expansions (with $d_{0n}^+ = a_{0n}^+$ for all $n \geq 0$)

$$A^+(v, t) = g^2 \frac{m}{m} + d_{10}^+ + d_{01}^+ t + a_{10}^+ v^2 + \mathcal{O}(v^4, v^2 t^2), \quad A^-(v, t) = a_{00}^- v + a_{01}^- vt + \mathcal{O}(v^3, v^2 t^2). \quad (1.2)$$

\(^1\)For the PWDRs of $\pi \pi$ scattering, called Roy equations [10], an analogous matching procedure for the $\pi \pi$ scattering lengths as pertinent subtraction parameters has been conducted in [11]. In contrast to $\pi \pi$ scattering, the $\pi N$ scattering lengths can be extracted with high accuracy from hadronic-atom data [12, 13] and may thus serve as additional constraints on the subtraction constants in the Roy–Steiner system.
In order to derive the partial-wave HDRs, called Roy–Steiner (RS) equations, one needs to expand the $s$- and $t$-channel imaginary parts in \( (1.1) \) into the respective partial waves and subsequently project the full expanded equations onto either $s$- or $t$-channel partial waves; the resulting sets of integral equations together with the respective partial-wave unitarity relations then form the $s$- and $t$-channel RS subsystems. According to \([5]\), the ( unsubtracted) $s$-channel RS equations read (based on the MacDowell symmetry $f^I_\ell(t+1)(W) = -f^I_\ell(-W)$ for all $\ell \geq 0$) \( (1.3) \)

\[
f^I_\ell(t+1)(W) = N^I_\ell(t+1)(W) + \frac{1}{\pi} \int_{t_\pi}^\infty dt' \sum_{J} \left( G_{IJ}(W,t') \text{ Im } f^I_\ell(t')(t') + H_{IJ}(W,t') \text{ Im } f^I_\ell(t') \right) \\
+ \frac{1}{\pi} \int_{W_+}^\infty \! dt' \sum_{J} \left( K^I_{IJ}(W,W') \text{ Im } f^I_\ell(t',W') + K^I_{IJ}(W,-W') \text{ Im } f^I_\ell(t+1,-W') \right),
\]

where due to $G$-parity only even/odd $J$ contribute for isospin $I = +/-$, respectively, and the partial-wave projections of the pole terms as well as the (lowest) kernels are analytically known, the latter including in particular the Cauchy kernel: $K^I_{IJ}(W,W') = \delta_{IJ}/(W'-W) + \ldots$. The $s$-channel $I = \pm$ partial waves are intertwined by the usual unitarity relations, which are diagonal in the $s$-channel isospin basis $I_5 \in \{1/2, 3/2\}$ only. Once the $t$-channel partial waves are known, the structure of the $s$-channel RS subsystem is therefore similar to the $\pi\pi$ Roy equations, cf. \([1]\). As shown in \([7]\), the corresponding ( unsubtracted) $t$-channel RS equations are given by

\[
f^I_\ell(t) = N^I_\ell(t) + \frac{1}{\pi} \int_{W_+}^\infty \! dt' \sum_{J} \left( \hat{G}_{IJ}(t',W') \text{ Im } f^I_\ell(t',W') + \hat{G}_{IJ}(t,-W') \text{ Im } f^I_\ell(t+1,-W') \right) \\
+ \frac{1}{\pi} \int_{t_\pi}^\infty dt' \sum_{J} \left( \hat{K}^I_{IJ}(t',t') \text{ Im } f^I_\ell(t',t') + \hat{K}^I_{IJ}(t,t') \text{ Im } f^I_\ell(t') \right) + \text{ Im } f^I_\ell(t') + \text{ Im } f^I_\ell(t').
\]

(1.4)

and similarly for the $f^I_\ell$ except for the fact that these do not receive contributions from the $f^I_\ell$. Here, only even or odd $J'$ couple to even or odd $J$ (corresponding to $t$-channel isospin $I_5 = 0$ or $I_5 = 1$), respectively, and $K^I_{IJ}$ (as well as the analogous $\hat{K}^I_{IJ}$ for the $f^I_\ell$) contains the Cauchy kernel. Moreover, it turns out that only higher $t$-channel partial waves contribute to lower ones. Assuming Mandelstam analyticity, the equations \((1.4)\) are valid for $\sqrt{t} \in [2M_\pi, 2.00\text{GeV}]$ using $a = -2.71 M_\pi^2$, whereas \((1.3)\) holds for $W \in [m + M_\pi, 1.38\text{GeV}]$ using $a = -23.19 M_\pi^2$. The $t$-channel unitarity relations are diagonal in $I_5$ and only linear in the $f^I_\pm$ (below the first inelastic threshold $t_{\text{inel}}$)

\[
\text{ Im } f^I_\pm(t) = \sigma^\pi_\pm (t^I_\pm(t))^* f^I_\pm(t) \vartheta(t-t_\pi), \quad \sigma^\pi_\pm(t^I_\pm(t)) = \sin \delta^I_\pm(t) e^{i \delta^I_\pm(t)}, \quad \sigma^\pi_\pm(t) = \sqrt{1-t_\pi/t},
\]

from which one can infer Watson’s final state interaction theorem \([5]\) stating that ( in the “elastic” region) the phase of $f^I_\pm$ is given by the phase $\delta^I_\pm$ of the respective $\pi\pi$ scattering partial wave $t^I_\pm$.

Due to the simpler recoupling scheme for the $f^I_\pm$, the $t$-channel RS subsystem can be recast as a (single-channel) Muskhelishvili–Omnès (MO) problem \([16]\) with a finite matching point $t_m$ \([1]\) for $f^0_+, f^0_-$, and the linear combinations $\Gamma^I(t) = m \sqrt{J/(J+1)} f^I_+(t) - f^I_-(t)$ with $\Gamma^I(t_{\text{inel}}) = 0$ for all $J \geq 1$ of the generic form (the details are given in \([1]\))

\[
f(t) = \Delta(t) + \frac{1}{\pi} \int_{t_\pi}^{t_m} dt \sin \delta(t') e^{-i \delta(t')} f^{I_\pm}(t') + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \text{ Im } f(t') \equiv |f(t)| e^{i \delta(t)} \quad \text{ for } t \leq t_m < t_{\text{inel}},
\]
Roy–Steiner equations for πN scattering

Christoph Ditsche

Figure 1: Flowchart of the solution strategy for the Roy–Steiner system for πN scattering. The third step consist in the self-consistent iteration (denoted by thick arrows) of the preceding steps until convergence.

where the inhomogeneities \( \Delta(t) \) subsume the nucleon pole terms, all \( s \)-channel integrals, and the higher \( t \)-channel partial waves. For \( t_\pi \leq t \leq t_m \), solving for \( |f(t)| \) only according to Watson’s theorem requires \( \delta(t) \) for \( t_\pi \leq t \leq t_m \) and \( \text{Im} f(t) \) for \( t \geq t_m \). Introducing \( n \geq 1 \) subtractions does not change the general structure of the RS/MO system, e.g. the \( P \)-waves are given by

\[
\Gamma^1(t) = \Delta^1(t) + \int_{t_\pi}^{t_m} \frac{dt'}{\pi} \frac{\text{Im} \Gamma^1(t')}{{t'}^{n-1}(t'-t_N)(t'-t)}, \quad f^1_l(t) = \Delta^1_l(t) + \int_{t_\pi}^{t_m} \frac{dt'}{\pi} \frac{\text{Im} f^1_l(t')}{{t'}^{n}(t'-t)},
\]

demonstrating that \( \Gamma^1 \) and hence \( f^1_l \) is effectively subtracted by one power less than \( f^1_l \), which motivates the additional (partial) third subtraction in \( A^\pm \), cf. (1.2), that affects solely the \( f^1_l \).

The solution strategy for the full RS system in the low-energy (or even subthreshold/pseudo-physical) regions, where only the lowest partial waves are relevant and inelastic contributions may be (approximately) neglected, is shown in Fig. 1; see [7] for more details.

2. The \( t \)-channel Muskhelishvili–Omnès problem: \( P \)-wave solutions

As the first step in the numerical solution of the full RS system, we check the consistency of our \( t \)-channel MO solutions with the results of the KH80 analysis [17], which are still used nowadays although no thorough error estimates are given (and despite the availability of more modern experimental data). Here, we present results for the \( P \)-waves in the (elastic) single-channel approximation of the MO problem, which is well justified for the \( P \)- and higher partial waves, whereas the \( S \)-wave requires a two-channel description including \( \bar{K}K \) intermediate states as described in [7]. To produce the results (that will also serve as input for the solution of the \( s \)-channel RS subsystem, cf. Fig. 1) partly shown in Fig. 2, we have used as input \( \pi\pi \) phase shifts from [18], \( s \)-channel partial waves \( (l \leq 4) \) from SAID [19] for \( W \leq 2.5 \text{ GeV} \), and above the Regge model of [20]. To facilitate
Figure 2: $n$-subtracted MO solutions for the $P$-wave moduli.

Figure 3: Two-pion-continuum contribution to $\text{Im}G^p_\pi$ and $\text{Im}G^p_M$.

comparison with the results of KH80, we use the respective subthreshold parameter values and a $\pi N$ coupling of $g^2/(4\pi) = 14.28$ [4, 7] (as starting point, the final values will result from the iteration procedure, cf. Fig. [8]). Moreover, KH80 uses different types of dispersion relations, in particular so-called fixed-$t$ ones, which can be emulated (up to the $t$-channel contributions that are not present at all in the fixed-$t$ case) by taking the “fixed-$t$ limit” $|a| \to \infty$. As argued in [8], all $t$-channel input above $\sqrt{t_m} = 0.98\text{ GeV}$ is set to zero, which forces the MO solutions to match zero at $t = t_m$. While Fig. [8] displays the results for $|a| \to \infty$, investigating the effect of using a different (i.e. higher) matching point leads to the same conclusion: with increasing number of subtractions, thus lowering the dependence on the high-energy input by introducing more subthreshold parameter contributions as subtraction polynomials, the solutions show a nice convergence pattern both in general (proving the internal consistency and numerical stability of our RS/MO framework) and in particular towards the KH80 results (being consistent with relying on KH80 values for $g$ and the subtraction parameters). The $P$-waves feature prominently in the dispersive analysis of the nucleon electromagnetic form factors, see e.g. [7, 8] and references therein, and in Fig. [8] we illustrate the effects on the spectral functions (by approximating the vector pion form factor $F^V_\pi$ via a

\footnote{Modern analyses yield significant smaller values for the $\pi N$ coupling, cf. e.g. $g^2/(4\pi) = 13.7 \pm 0.2$ of [4].}
twice-subtracted Omnès representation, cf. [3])

\[ \text{Im} G^E(t) = \frac{t(\sigma^\pi)^3}{8m} \left( F^V_\pi(t) \right)^* f^1_+(t) \theta(t - t_\pi), \]

\[ \text{Im} G^M(t) = \frac{t(\sigma^\pi)^3}{8\sqrt{2}} \left( F^V_\pi(t) \right)^* f^1_-(t) \theta(t - t_\pi). \]

We are confident that a self-consistent iteration procedure between the solutions for the \( s \)- and \( t \)-channel eventually will yield a consistent and precise description (including error estimates) of the low-energy \( \pi N \) scattering amplitude in all kinematical channels.

References