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Phys. Rev. D **86**, 114031 — Published 20 December 2012

DOI: [10.1103/PhysRevD.86.114031](https://doi.org/10.1103/PhysRevD.86.114031)

## Determination of $\alpha_s$ from the QCD static energy

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(Dated: August 31, 2012)

We compare lattice data for the short-distance part of the static energy in 2 + 1 flavor quantum chromodynamics (QCD) with perturbative calculations, up to next-to-next-to-next-to-leading-logarithmic accuracy. We show that perturbation theory describes very well the lattice data at short distances, and exploit this fact to obtain a determination of the product of the lattice scale  $r_0$  with the QCD scale  $\Lambda_{\overline{\text{MS}}}$ . With the input of the value of  $r_0$ , this provides a determination of the strong coupling  $\alpha_s$  at the typical distance scale of the lattice data. We obtain  $\alpha_s(1.5\text{GeV}) = 0.326 \pm 0.019$ , which provides a novel determination of  $\alpha_s$  with three-loop accuracy (including resummation of the leading ultrasoft logarithms), and constitutes one of the few low-energy determination of  $\alpha_s$  available. When this value is evolved to the  $Z$ -mass scale  $M_Z$ , it corresponds to  $\alpha_s(M_Z) = 0.1156^{+0.0021}_{-0.0022}$ .

PACS numbers: 12.38.Aw, 12.38.Bx, 12.38.Gc

The static energy in quantum chromodynamics (QCD), i.e. the energy between a static quark and a static antiquark separated by a distance  $r$ , is a basic object to understand the behavior of the theory [1] and constitutes a fundamental ingredient in the description of many physical processes [2]. The short-distance part of the static energy can be computed using perturbative techniques, and it is nowadays known at next-to-next-to-next-to leading-logarithmic (N<sup>3</sup>LL) accuracy, i.e. including terms up to order  $\alpha_s^{4+n} \ln^n \alpha_s$  with  $n \geq 0$  [3–9] ( $\ln \alpha_s$  terms appear due to virtual emissions of ultrasoft gluons, which can change the color state of the quark-antiquark pair [10, 11]). It can also be computed on the lattice, and the comparison of the two approaches tests our ability to describe the short-distance regime of QCD, besides providing information on the region of validity of the perturbative weak-coupling approach [12]. A comparison of the static energy at N<sup>3</sup>LL accuracy with quenched lattice data [13] was presented in Ref. [3]. Here we present lattice data for the short-distance part of the static energy in 2 + 1 flavor QCD and compare it with the perturbative calculation up to N<sup>3</sup>LL accuracy. This allows us to determine the strong coupling  $\alpha_s$  at three-loop accuracy (including resummation of the leading ultrasoft logarithms), in a way which is largely independent from the other determinations that currently enter in the world average [14]. The natural scale where our determination is performed corresponds to the inverse of the typical distance where we have lattice data, i.e. around 1.5 GeV. Therefore, our analysis provides a determination of  $\alpha_s$  at a scale smaller than those entering the current world average [14], and constitutes in this way an important ingredient to further test asymptotic freedom in QCD.

The static energy has been calculated on the lattice in 2 + 1 flavor QCD using a combination of tree-level improved gauge action and highly-improved staggered quark (HISQ) action [15] in Ref. [16]. The strange-quark mass  $m_s$  was fixed to its physical value, while the light-quark masses were chosen to be  $m_l = m_s/20$ . These correspond to the pion mass of about 160 MeV in the continuum limit, which is very close to the physical value. The calculation of the static energy was performed in a wide range of gauge couplings  $5.9 \leq \beta \equiv 10/g^2 \leq 7.28$ . At each value of the gauge coupling we calculate the scale parameters  $r_0$  and  $r_1$  defined in terms of the static energy  $E_0(r)$  as follows [17, 18]

$$r^2 \frac{dE_0(r)}{dr} \Big|_{r=r_0} = 1.65, \quad r^2 \frac{dE_0(r)}{dr} \Big|_{r=r_1} = 1. \quad (1)$$

The values of  $r_0$  and  $r_1$  were given in Ref. [16] for each  $\beta$ . The above range of the gauge couplings corresponds to lattice spacing  $1.909/r_0 \leq a^{-1} \leq 6.991/r_0$ . Using the most recent value  $r_0 = 0.468 \pm 0.004$  fm [16] we get  $0.805 \text{ GeV} < a^{-1} < 2.947 \text{ GeV}$ . Thus we can study the static energy down to distances  $r = 0.14r_0$  or  $r \simeq 0.065$  fm. For the comparison with perturbation theory the most relevant data set is the one corresponding to lattice gauge coupling  $\beta = 6.664, 6.740, 6.800, 6.880, 6.950, 7.030, 7.150, 7.280$ , which is what we will use here. The static energy can be calculated in units of  $r_0$  or  $r_1$ . Since the static energy has an additive ultraviolet renormalization we need to normalize the results calculated at different lattice spacings to a common value at a certain distance. We fix the static energy in units of  $r_0$  to 0.954 at  $r = r_0$  [16]. At distances comparable to the lattice spacing the static energy suffers from lattice artifacts. To correct for these artifacts we use tree level improvement. From the lattice Coulomb potential

$$C_L(r) = \int \frac{d^3k}{(2\pi)^3} D_{00}(k_0 = 0, \vec{k}) e^{i\vec{k}\vec{r}}, \quad (2)$$

we can define the improved distance  $r_I = (4\pi C_L(r))^{-1}$  for each separation  $r$ . Here  $D_{00}$  is the tree level gluon propagator for the  $a^2$  improved gauge action. The tree level improvement amounts to replacing  $r$  by  $r_I$  [13]. Alternatively following Ref. [18, 19] we fit the lattice data at short distances to the form  $const - a/r + \sigma r + a'(1/r - 1/r_I)$  and subtract the last term from the lattice data. We have found that both methods of correcting for lattice artifacts lead to the same results within errors of the calculations. Furthermore, the static energies calculated for different lattice spacings agree well with each other after the removal of lattice artifacts. The corrected lattice data obtained for several lattice spacings are shown in Fig. 1 as the points. All the lattice data seem to lie on a single curve even at short distances, indicating that the above procedure of removing the lattice artifacts works.

As mentioned before, the static energy is known at N<sup>3</sup>LL accuracy in perturbation theory. Detailed expressions for  $E_0$  were given in Ref. [3] (and references therein) and will not be reproduced here. For our present analysis, it is only important to recall that: (i) In order to obtain a well behaved perturbative series, it is necessary to implement a scheme that cancels the leading renormalon singularity [20]. This kind of schemes introduce dependence on a dimensional scale in the problem, which we denote as  $\rho$ . In particular, we implement the renormalon cancellation according to the scheme described in Ref. [21]. Then, the natural value for the scale  $\rho$  corresponds to the center of the range where we have lattice data (ii) At N<sup>3</sup>LL accuracy the perturbative expression depends on an additional constant (which was not present at lower levels of accuracy). This is due to the structure of the renormalization group equations at that order. We call this constant  $K_2$ . It should satisfy the power counting condition  $|K_2| \sim \Lambda_{\overline{\text{MS}}}$  (where  $\Lambda_{\overline{\text{MS}}}$  is the QCD scale in the  $\overline{\text{MS}}$  scheme), but apart from that it is unconstrained.

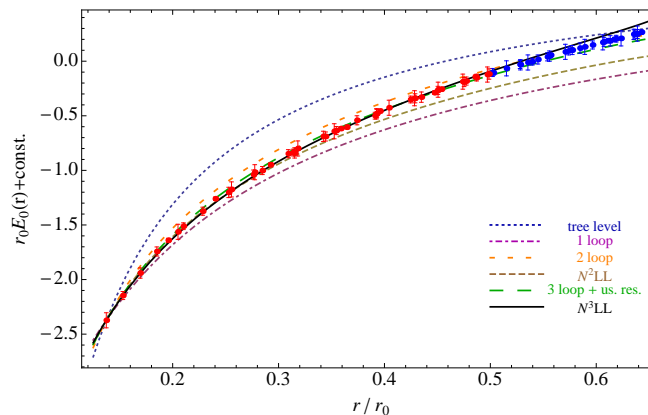


FIG. 1. Comparison of the singlet static energy with lattice data (red -lighter- points). [The comparison (and all the analysis in the text) is done for  $r < 0.5r_0 \simeq 0.234$  fm, which is the region where perturbation theory is reliable. The (blue -darker-) points and curves for  $r > 0.5r_0$  are shown just for illustration]. The dotted blue curve is at tree level, the dot-dashed magenta curve is at one loop, the long-spaced-dashed orange curve is at two loops, the dashed brown curve is at  $N^2LL$  accuracy, the long-dashed green curve is at three loops plus leading ultrasoft logarithmic resummation, and the solid black curve is at  $N^3LL$  accuracy.  $r_0\Lambda_{\overline{MS}} = 0.70$  was used in all the curves. The additive constant in the perturbative expression for the static energy is taken such that each curve coincides with the lattice data point at the shortest distance.

We can now compare the perturbative results for the static energy with the lattice data. This comparison goes along the same lines as the quenched case in Ref. [3], except that now we use  $n_f = 3$  everywhere ( $n_f$  is the number of light flavors); we also include finite strange mass effects at one loop, although they turn out to be negligible. We use the maximum known accuracy (four loop) for the running of  $\alpha_s$  everywhere (as opposed to changing the accuracy for the running depending on the order we are working at), since, for the  $n_f = 3$  case, the hierarchy of scales underlying the perturbative calculation would not be well satisfied with the running of  $\alpha_s$  at one loop. The perturbative expressions depend on the value of the quantity  $r_0\Lambda_{\overline{MS}}$ , and we will use the lattice data to determine it. That is, we search for the range of  $r_0\Lambda_{\overline{MS}}$  that is allowed by lattice data, taking into account all the uncertainties involved. Then, using the value for  $r_0$  determined in Ref. [16] we can obtain a determination of  $\alpha_s(M_Z)$  ( $M_Z$  is the  $Z$ -boson mass).

Under the assumption that perturbation theory by itself (after canceling the leading renormalon) is enough to accurately describe the lattice data in the range of distances we are considering (i.e.  $r < 0.5r_0$ ), a procedure to extract  $r_0\Lambda_{\overline{MS}}$  from the comparison of the perturbative expressions for the static energy with lattice data was devised in Ref. [3]. We will proceed in an analogous way here. The procedure exploits the fact that any value of  $\rho$  (around its natural value at the center of the range for which we have lattice data, i.e.  $\rho = 3.14r_0^{-1}$ ) cancels the renormalon and is therefore allowed. Following Ref. [3], we search for a set of  $\rho$  values which are optimal for the determination of  $r_0\Lambda_{\overline{MS}}$ . The procedure to do that consists of the following steps:

1. We vary  $\rho$  by  $\pm 25\%$  around its natural value  $\rho = 3.14r_0^{-1}$ , that is from  $\rho = 2.36r_0^{-1}$  to  $\rho = 3.93r_0^{-1}$ .
2. For each value of  $\rho$  and at each order in the perturbative expansion of the static energy, we perform a fit to the lattice data (that is we do fits at tree level, one loop, two loops, and three loops; in the last two cases with and without ultrasoft logarithmic resummation). The parameter in each of these fits is  $r_0\Lambda_{\overline{MS}}$ .
3. We select those  $\rho$  values for which the reduced  $\chi^2$  of the fits decreases when increasing the number of loops of the perturbative calculation.

For the analysis, we use the fits from tree level to three loop plus leading ultrasoft logarithmic resummation accuracy. We could also use the fits at  $N^3LL$  accuracy (i.e. three loops plus sub-leading ultrasoft logarithmic resummation), which would involve the additional constant  $K_2$  (that would also need to be fitted to the data, i.e. the fits involve one additional parameter at this order). We find that, with the present lattice data, the  $\chi^2$  as a function of  $r_0\Lambda_{\overline{MS}}$  is very flat in this case, and we cannot improve our extraction of  $r_0\Lambda_{\overline{MS}}$  by including the fits at  $N^3LL$  accuracy in the analysis. We interpret this as the unquenched lattice data not being accurate enough to be sensitive to subleading ultrasoft logarithms (unlike the quenched one used in [3]), a fact that leaves room for future improvements. Therefore, we take the numbers at three loop plus leading ultrasoft logarithmic resummation accuracy as our best result, and consider, at this order, the set of fitted values of  $r_0\Lambda_{\overline{MS}}$  for the  $\rho$  range obtained after step 3 above (we denote these values by  $x_i$ ). We assign a weight to each  $x_i$ , given by the inverse of the reduced  $\chi^2$  of the fit. We take the

Accuracy	$r_0\Lambda_{\overline{\text{MS}}}$	$\frac{r_0}{r_1} r_1\Lambda_{\overline{\text{MS}}}$
tree level	0.395	0.397
1 loop	0.848	0.862
2 loop	0.636	0.654
N <sup>2</sup> LL	0.756	0.783
3 loop	0.690	0.701
3 loop + us. res.	0.702	0.715

TABLE I. Values of  $r_0\Lambda_{\overline{\text{MS}}}$  obtained at different levels of accuracy. The second column shows the results obtained using the static energy normalized in units of the scale  $r_0$ . The third column shows the results obtained with the static energy normalized in units of the scale  $r_1$  and then (for easier comparison) transformed to  $r_0$  units, using the factor  $r_0/r_1 = 1.508 \pm 0.005$  [16]. “N<sup>2</sup>LL” stands for next-to-next-to leading-logarithmic (i.e. two loop plus leading ultrasoft logarithmic resummation) and “3 loop + us. res.” stands for three loop plus leading ultrasoft logarithmic resummation.

weighted average of the  $x_i$  as our central value for the determination of  $r_0\Lambda_{\overline{\text{MS}}}$ . To estimate the error that we should associate to this number, we consider the weighted standard deviation of this set of values, and the difference with the weighted average computed using the result at the previous perturbative order (with the corresponding  $\rho$  range that one obtains at that order; for illustration, we show the results for  $r_0\Lambda_{\overline{\text{MS}}}$  obtained at different levels of accuracy in Tab. I). We obtain  $r_0\Lambda_{\overline{\text{MS}}} = 0.7024 \pm 0.0011 \pm 0.0665 = 0.70 \pm 0.07$ , where the first error is due to the weighted standard deviation, the second to the difference with the two-loop result, and we summed the two errors linearly on the right-hand side of the equation. It is important to point out that the error assigned to the result must account for the uncertainties associated to the neglected higher-order terms in the perturbative expansion of the static energy; in that sense, assigning the difference with the result at the previous order as an error (as we do) is a quite conservative estimate. Note that, starting at the two-loop level, one can decide whether to perform resummation of the ultrasoft logarithms or not; when assigning the error, we take whichever difference is larger. We also mention that there is an error associated to each of the  $x_i$  coming from the fit to the lattice data, but the error that this induces in the average can be neglected. To further assess the systematic errors stemming from our procedure, we have redone the analysis using  $p$ -value weights, obtaining  $r_0\Lambda_{\overline{\text{MS}}} = 0.7022 \pm 0.0011 \pm 0.0628 = 0.70 \pm 0.06$ , and using constant weights, obtaining  $r_0\Lambda_{\overline{\text{MS}}} = 0.7022 \pm 0.0011 \pm 0.0666 = 0.70 \pm 0.07$ , which are both compatible with the previous analysis. In our final result we quote an error that covers the whole range spanned by the three analyses. As an additional cross-check of the result, we have redone the analysis with the static energy normalized in units of the scale  $r_1 = 0.3106 \pm 0.0020$  fm (rather than  $r_0$ ); these numbers are presented in the third column of Tab. I, and are consistent with our previous results. Finally, we point out that in some cases the  $\chi^2$  as a function of  $r_0\Lambda_{\overline{\text{MS}}}$  at next-to-next-to leading-logarithmic (N<sup>2</sup>LL) accuracy develops a second local minimum for larger values of  $r_0\Lambda_{\overline{\text{MS}}}$ . To discern which minimum should be taken as the physical result when this happens, we have redone the fits using smaller  $r$  ranges. We found that the position of the second minimum is not stable, while the position of the first one is. Furthermore the minima of the  $\chi^2$  from lower orders in perturbation theory come closer to the first of the two minima at N<sup>2</sup>LL, when decreasing the  $r$  range we use. In view of the above, when a second minimum develops, we keep the first one, which is stable and preferred by lower perturbative orders.

According to the discussion in the previous paragraph, our final result reads

$$r_0\Lambda_{\overline{\text{MS}}} = 0.70 \pm 0.07, \quad (3)$$

which using  $r_0 = 0.468 \pm 0.004$  fm [16] gives

$$\alpha_s(\rho = 1.5\text{GeV}, n_f = 3) = 0.326 \pm 0.019, \quad (4)$$

the uncertainty in  $r_0$  is negligible in the final error above. When we evolve Eq. (4) to the scale  $M_Z$  we obtain

$$\alpha_s(M_Z, n_f = 5) = 0.1156_{-0.0022}^{+0.0021}, \quad (5)$$

where we have used the `Mathematica` package `RunDec` [22] to obtain the above number (4 loop running, with the charm quark mass equal to 1.6 GeV and the bottom quark mass equal to 4.7 GeV). We mention that the final result employing the static energy normalized in units of  $r_1$  is  $\alpha_s(M_Z) = 0.1160_{-0.0022}^{+0.0021}$ , which is compatible with our result in Eq. (5) and further shows its robustness. Figure 1 shows a comparison of the perturbative expressions for the static energy with lattice data using our result  $r_0\Lambda_{\overline{\text{MS}}} = 0.70$  in Eq. (3) ( $\rho$  is set at the natural value,  $\rho = 3.14r_0^{-1}$ ). We can see that the perturbative series converges, approaches the lattice data, and reproduces it very well at N<sup>3</sup>LL accuracy

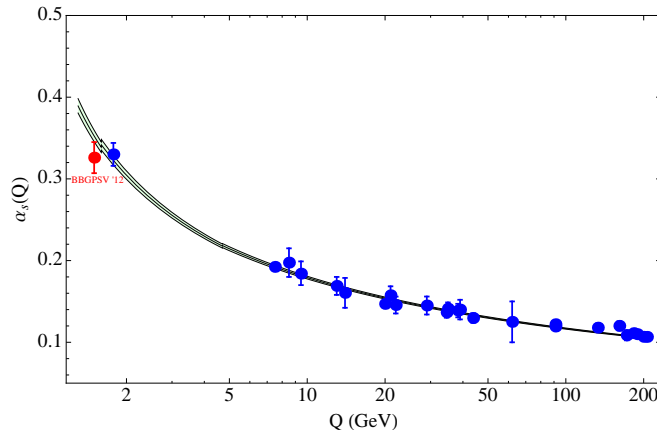


FIG. 2. Determinations of  $\alpha_s$  that enter in the world average [14] (blue -darker- points) compared with our result (most-left red -lighter- point), as a function of the energy scale  $Q$ . The band is the world-average value of  $\alpha_s(M_Z)$ , evolved with 4-loop accuracy.

(the constant  $K_2$  that appears at  $N^3\text{LL}$  is fixed by a fit to the lattice data, which gives  $r_0 K_2 = -2.3$ , fulfilling the power counting condition). Note also that our results for  $\alpha_s$  are not sensitive to the specific value of  $r$  that we consider as the upper limit where perturbation theory is reliable, as it is manifest from Fig. 1.

Our result in Eq. (5) constitutes a novel determination of  $\alpha_s$  (since it is largely independent of the other available determinations), that stems from a perturbative calculation of the QCD static energy at three loop plus leading ultrasoft logarithmic resummation accuracy. With respect to the other determinations currently entering the world average it represents the one at lowest energy. The lowest-energy determination so far was that from the  $\tau$  system, performed at  $m_\tau = 1.78$  GeV. Our result is therefore an important new ingredient to test the running of  $\alpha_s$ .

Other recent determinations of  $\alpha_s$ , that also employ comparisons with lattice data, include Refs. [23, 24] where several observables related to Wilson loops (but not the static energy) are used, Refs. [24, 25] which employ moments of heavy quark correlators, Ref. [26] that uses the vacuum polarization function, Ref. [27] which uses the so-called Schrödinger functional scheme (albeit employing rather high pion masses), and Ref. [28] that employs the ghost-gluon coupling; they deliver numbers that are mostly compatible with our result, although with central values a bit higher than ours. We also mention that comparisons of perturbative calculations for the static energy with lattice data in QCD with  $n_f = 2$  flavors have been presented recently in Refs. [29, 30].

Let us also point out that the comparison of the perturbative result with lattice data, shown in Fig. 1, is interesting in itself, since the static energy constitutes a basic ingredient in the description of many physical processes [2]. As an example, it is relevant for the study of quarkonium production in heavy-ion collisions: A direct lattice calculation of the quarkonium spectral functions is known to be difficult [31]. The only viable option could be calculating quarkonium spectral functions within an effective field theory framework, as potential Non-Relativistic QCD (pNRQCD) provides [32, 33]. To access the validity of a weak coupling pNRQCD approach at non-zero temperature, one eventually will have to compare weak-coupling calculations of the static quark-antiquark correlators with the corresponding lattice calculations. The comparison of the static energy at zero temperature to the perturbative results, in 2+1 flavor QCD with physical quark masses (as we have provided), is an important first step in this direction.

In summary, we have shown that perturbation theory (after canceling the leading renormalon singularity) can describe the short-distance part of the QCD static energy computed in the lattice, see Fig. 1; this is the first time that this is done for QCD with  $n_f = 2 + 1$  dynamical quarks. Exploiting this fact, we have obtained the range of  $r_0 \Lambda_{\overline{\text{MS}}}$  that is allowed by lattice data. Using the value of  $r_0$  as an additional input, this provides a determination of  $\alpha_s$ . We obtained  $\alpha_s(1.5\text{GeV}) = 0.326 \pm 0.019$ , which represents one of the few low-energy determinations of  $\alpha_s$  available and, when evolved to the scale  $M_Z$ , corresponds to  $\alpha_s(M_Z) = 0.1156^{+0.0021}_{-0.0022}$ . A comparison of our result with the determinations of  $\alpha_s$  that currently enter in the world average [14] is shown in Fig. 2.

We thank Philippe de Forcrand for pointing out Ref. [12]. This work was supported in part by U.S. Department of Energy under Contract No. DE-AC02-98CH10886. N.B. and A.V. acknowledge financial support from the DFG cluster of excellence “Origin and structure of the universe” ([www.universe-cluster.de](http://www.universe-cluster.de)). J.S. is supported by the CPAN CSD2007-00042 Consolider-Ingenio 2010 program (Spain), the 2009SGR502 CUR grant (Catalonia) and the FPA2010-16963 project (Spain).

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- [1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
- [2] N. Brambilla, S. Eidelman, B. K. Heltsley, R. Vogt, G. T. Bodwin, E. Eichten, A. D. Frawley and A. B. Meyer *et al.*, Eur. Phys. J. C **71**, 1534 (2011) [arXiv:1010.5827 [hep-ph]].
- [3] N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo, Phys. Rev. Lett. **105**, 212001 (2010) [arXiv:1006.2066 [hep-ph]].
- [4] A. Pineda and M. Stahlhofen, Phys. Rev. D **84**, 034016 (2011) [arXiv:1105.4356 [hep-ph]].
- [5] A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. **104**, 112002 (2010) [arXiv:0911.4742 [hep-ph]].
- [6] C. Anzai, Y. Kiyo and Y. Sumino, Phys. Rev. Lett. **104**, 112003 (2010) [arXiv:0911.4335 [hep-ph]].
- [7] N. Brambilla, A. Vairo, X. Garcia i Tormo and J. Soto, Phys. Rev. D **80**, 034016 (2009) [arXiv:0906.1390 [hep-ph]].
- [8] A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Lett. B **668**, 293 (2008) [arXiv:0809.1927 [hep-ph]].
- [9] N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo, Phys. Lett. B **647**, 185 (2007) [hep-ph/0610143].
- [10] T. Appelquist, M. Dine and I. J. Muzinich, Phys. Rev. D **17**, 2074 (1978).
- [11] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D **60**, 091502 (1999) [hep-ph/9903355].
- [12] A. Billoire, Phys. Lett. B **104**, 472 (1981).
- [13] S. Necco and R. Sommer, Nucl. Phys. B **622**, 328 (2002).
- [14] S. Bethke, Eur. Phys. J. C **64**, 689 (2009) [arXiv:0908.1135 [hep-ph]].
- [15] E. Follana *et al.* [HPQCD and UKQCD Collaborations], Phys. Rev. D **75**, 054502 (2007) [hep-lat/0610092].
- [16] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H. T. Ding, S. Gottlieb, R. Gupta and P. Hegde *et al.*, Phys. Rev. D **85**, 054503 (2012) [arXiv:1111.1710 [hep-lat]].
- [17] R. Sommer, Nucl. Phys. B **411**, 839 (1994) [hep-lat/9310022].
- [18] C. Aubin, C. Bernard, C. DeTar, J. Osborn, S. Gottlieb, E. B. Gregory, D. Toussaint and U. M. Heller *et al.*, Phys. Rev. D **70**, 094505 (2004) [hep-lat/0402030].
- [19] S. P. Booth *et al.* [UKQCD Collaboration], Nucl. Phys. B **394**, 509 (1993) [hep-lat/9209007].
- [20] M. Beneke, Phys. Lett. B **434**, 115 (1998) [hep-ph/9804241].
- [21] A. Pineda, JHEP **0106** (2001) 022 [hep-ph/0105008].
- [22] K. G. Chetyrkin, J. H. Kuhn and M. Steinhauser, Comput. Phys. Commun. **133**, 43 (2000) [hep-ph/0004189].
- [23] C. T. H. Davies *et al.* [HPQCD Collaboration], Phys. Rev. D **78**, 114507 (2008) [arXiv:0807.1687 [hep-lat]].
- [24] C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel and G. P. Lepage, Phys. Rev. D **82**, 034512 (2010) [arXiv:1004.4285 [hep-lat]].
- [25] I. Allison *et al.* [HPQCD Collaboration], Phys. Rev. D **78**, 054513 (2008) [arXiv:0805.2999 [hep-lat]].
- [26] E. Shintani, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, T. Onogi and N. Yamada, Phys. Rev. D **82**, 074505 (2010) [arXiv:1002.0371 [hep-lat]].
- [27] S. Aoki *et al.* [PACS-CS Collaboration], JHEP **0910**, 053 (2009) [arXiv:0906.3906 [hep-lat]].
- [28] B. Blossier, P. Boucaud, M. Brinet, F. De Soto, X. Du, V. Morenas, O. Pene and K. Petrov *et al.*, arXiv:1201.5770 [hep-ph].
- [29] K. Jansen *et al.* [ETM Collaboration], JHEP **1201**, 025 (2012) [arXiv:1110.6859 [hep-ph]].
- [30] F. Knechtli and B. Leder, arXiv:1112.1246 [hep-lat].
- [31] P. Petreczky, J. Phys. G G **37**, 094009 (2010) [arXiv:1001.5284 [hep-ph]].
- [32] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998) [hep-ph/9707481].
- [33] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B **566**, 275 (2000) [hep-ph/9907240].