

# $\alpha_s$ from the static energy in QCD

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## Abstract

Comparing perturbative calculations with a lattice computation of the static energy in quantum chromodynamics at short distances, we obtain a determination of the strong coupling  $\alpha_s$ . Our determination is performed at a scale of around 1.5 GeV (the typical distance scale of the lattice data) and, when evolved to the Z-boson mass scale  $M_Z$ , it corresponds to  $\alpha_s(M_Z) = 0.1156^{+0.0021}_{-0.0022}$ .

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This talk is based on Ref. [1], to which we refer for additional details.

The energy between a static quark and a static antiquark that are separated a distance  $r$ , i.e. the quantum chromodynamics (QCD) static energy,  $E_0(r)$ , is a good object to study in order to understand the behavior of the theory. One can identify a long-distance part and a short-distance part of the static energy, both of which can be computed with lattice simulations. Here we will focus only on the short-distance part, i.e. on distances  $r \lesssim 0.234$  fm, where weak-coupling calculations are also reliable. The comparison of the lattice computation with the perturbative calculation tests our ability to describe the short-distance regime of QCD, and provides information on the region of validity of the weak-coupling approach. This comparison also allow us to determine the strong coupling  $\alpha_s$ , which is the subject of this talk. There has been a lot of recent activity regarding both, the lattice computations and the perturbative calculations of the static energy, which has allowed for a precise determination of  $\alpha_s$  from it to be possible [1].

On the perturbative side, the static energy is known, at present, including terms up to order  $\alpha_s^{4+n} \ln^n \alpha_s$  with  $n \geq 0$  [2–8]. That is, three-loop with resummation at sub-leading accuracy of the  $\ln \alpha_s$  terms that appear at short distances. We refer to this level of accuracy as next-to-next-to-next-to leading-logarithmic (N<sup>3</sup>LL).

On the lattice side, the static energy has recently been calculated in 2 + 1 flavor QCD [9], using a combination of tree-level improved gauge action and highly-improved staggered quark action [10]. This computation employed the physical value for the strange-quark mass  $m_s$  and light quark masses equal to  $m_s/20$ , which

correspond to a pion mass of about 160 MeV in the continuum limit, very close to the physical value. The computation was performed for a wide range of gauge couplings, and was corrected for lattice artifacts. It allows to study the static energy down to distances  $r \approx 0.065$  fm.

The perturbative expressions for the static energy depend on the value of the QCD scale  $\Lambda_{\overline{\text{MS}}}$  (in the  $\overline{\text{MS}}$  scheme), and we can use the comparison with lattice data to determine it. For that, we assume that perturbation theory (after implementing a cancellation of the leading renormalon singularity) is enough to describe lattice data in the range of distances we are considering. Then, the general idea is that we can search for the values of  $\Lambda_{\overline{\text{MS}}}$  for which the agreement with lattice improves when the perturbative order of the calculation is increased; and in that way find the values of  $\Lambda_{\overline{\text{MS}}}$  that are allowed by lattice data. This same program was already performed for the quenched case in Ref. [2]; the unquenched computation of the static energy in Ref. [9] allows us to do the same here in the unquenched case, and therefore obtain a value for  $\alpha_s$ .

The static energy on the lattice is calculated in units of the scales  $r_0$  or  $r_1$ , defined as [11, 12]

$$r^2 \frac{dE_0(r)}{dr} \Big|_{r=r_0} = 1.65, \quad r^2 \frac{dE_0(r)}{dr} \Big|_{r=r_1} = 1; \quad (1)$$

we use the values of  $r_0$  or  $r_1$  in Ref. [9] to obtain  $\Lambda_{\overline{\text{MS}}}$  in physical units. In the perturbative calculation one needs to implement a scheme that cancels the leading renormalon singularity [13]<sup>1</sup>. This kind of schemes introduce

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<sup>1</sup>In the lattice computation the results calculated at different lattice spacings are normalized to a common value at a certain distance.

an additional dimensional scale in the problem (that we denote as  $\rho$ ). We implement the renormalon cancellation according to the scheme described in Ref. [14]; then, the natural value of the scale  $\rho$  is at the center of the range for which we have lattice data. But since any value of  $\rho$  around this natural value cancels the renormalon, we can exploit this freedom to search for a set of  $\rho$  values that allow for an optimal determination of  $r_0\Lambda_{\overline{\text{MS}}}$ . To obtain our central value for  $r_0\Lambda_{\overline{\text{MS}}}$  we let  $\rho$  vary around its natural value; then, for each value of  $\rho$  and at each order in the perturbative expansion, we perform a fit to the lattice data ( $r_0\Lambda_{\overline{\text{MS}}}$  is the parameter of the fits); and finally select the  $\rho$  values for which the reduced  $\chi^2$  of the fit decreases when increasing the perturbative order. Our central value for  $r_0\Lambda_{\overline{\text{MS}}}$  is then given by the average (weighted by the inverse  $\chi^2$ ) of those fit values<sup>2</sup>. We can perform the above analysis at different orders of accuracy; at N<sup>3</sup>LL accuracy the perturbative expression depends on an additional constant (due to the structure of the renormalization group equations at this order), which would also need to be fitted to the lattice data. When we do the fits at N<sup>3</sup>LL accuracy, we find that the  $\chi^2$  as a function of  $r_0\Lambda_{\overline{\text{MS}}}$  is very flat; we interpret this fact as the data not being sensitive to the sub-leading ultrasoft logarithms. Therefore we take the result at three loops with resummation of the leading ultrasoft logarithms as our best result. For illustration, we present here the expressions for the static energy at this level of accuracy. The static energy at next-to-next-to-leading order (N<sup>3</sup>LO) is given by

$$\begin{aligned}
E_0^{\text{N}^3\text{LO}}(r) = & -\frac{C_F\alpha_s(1/r)}{r} \left\{ 1 + \tilde{a}_1 \frac{\alpha_s(1/r)}{4\pi} \right. \\
& + \tilde{a}_2 \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 + \left[ \frac{16\pi^2}{3} C_A^3 \ln \frac{C_A\alpha_s(1/r)}{2} \right. \\
& \left. \left. + \tilde{a}_3 \right] \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \right\} + K_1, \quad (2)
\end{aligned}$$

<sup>2</sup>Note that the absolute value of the  $\chi^2$  (from the fits of the perturbative expressions to the lattice data), at a given order in perturbation theory, does not have any particular significance. The reason for that is that, in principle, one does not know exactly how large the terms at the next perturbative order will be. Therefore, one does not know how accurately the perturbative expression should describe the lattice data at a given order. For that reason, our procedure to determine  $r_0\Lambda_{\overline{\text{MS}}}$ , described above, does not use absolute values of the  $\chi^2$ s, but rather comparisons between  $\chi^2$  values at different perturbative orders. Otherwise there would be the danger that one artificially reduces the  $\chi^2$  at a low perturbative order, by using a “wrong” value of  $r_0\Lambda_{\overline{\text{MS}}}$  that is not suitable for a more precise expression at a higher perturbative order.

with (numerically, for  $N_c = 3$ )

$$\tilde{a}_1 = 23.032 - 1.8807n_f, \quad (3)$$

$$\tilde{a}_2 = 1396.3 - 192.90n_f + 4.9993n_f^2, \quad (4)$$

$$\begin{aligned} \tilde{a}_3 = & 108654. - 21905.2n_f + 1284.69n_f^2 \\ & - 20.6009n_f^3, \end{aligned} \quad (5)$$

where  $C_F = (N_c^2 - 1)/(2N_c)$ ,  $C_A = N_c$ ,  $n_f$  is the number of light flavors (i.e.  $n_f = 3$  in our case), and  $K_1$  is a constant that, in the comparison with data, gets absorbed in the constant used to make the static energy coincide with the lattice point at the shortest distance available. If we include the resummation of the leading ultrasoft logarithms we have

$$\begin{aligned}
E_0^{\text{N}^3\text{LO}+\text{us.res.}}(r) = & \left\{ \text{Eq. (2)} + \frac{C_F\alpha_s^4(1/r)}{r} \frac{1}{12\pi} C_A^3 \right. \\
& \times \ln \frac{C_A\alpha_s(1/r)}{2} \left. \right\} + \frac{2C_FC_A^3}{12\beta_0 r} \alpha_s^3(1/r) \ln \frac{\alpha_s(\mu)}{\alpha_s(1/r)} \\
& - \frac{C_FC_A^3}{12\pi r} \alpha_s^4(1/r) \ln \frac{C_A\alpha_s(1/r)}{2r\mu}, \quad (6)
\end{aligned}$$

where  $\mu$  is the ultrasoft scale (of order  $\alpha_s/r$ ), and  $\beta_0 = (11/3)C_A - (4/3)T_F n_f$ , with  $T_F = 1/2$ . To implement the required renormalon cancellation we use the so-called RS scheme [14]. That is, if we calculate the static energy at  $m$ -loop order in perturbation theory we add the following term to it [6]

$$\begin{aligned}
\text{RSsubtr.} = & R_s \rho \sum_{n=1}^m \left( \frac{\beta_0}{2\pi} \right)^n \alpha_s(\rho)^{n+1} \\
& \times \sum_{k=0}^2 d_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}, \quad (7)
\end{aligned}$$

with  $R_s = -1.123$  the normalization of the  $u = 1/2$  renormalon singularity (which we computed approximately according to the procedure in Ref. [15]), and

$$\begin{aligned}
d_0 = & 1, \\
d_1 = & \frac{\beta_1^2 - \beta_2\beta_0}{4b\beta_0^4}, \\
d_2 = & \frac{-2\beta_0^4\beta_3 + 4\beta_0^3\beta_1\beta_2}{32(b-1)b\beta_0^8} \\
& + \frac{\beta_0^2(\beta_2^2 - 2\beta_1^3) - 2\beta_0\beta_1^2\beta_2 + \beta_1^4}{32(b-1)b\beta_0^8}, \quad (8)
\end{aligned}$$

with

$$b = \frac{\beta_1}{2\beta_0^2}, \quad (9)$$

(the higher order coefficients of the beta function,  $\beta_{1,2,3}$ , can be found, for instance, in Refs. [16, 17]).

Having determined our central value for  $r_0\Lambda_{\overline{\text{MS}}}$  we now need to assign an error to it. The error must reflect the uncertainties associated to the neglected higher-order terms in the perturbative expansion. To account for that, we consider the weighted standard deviation in the set of  $\rho$  values we found before, and the difference with the weighted average computed at the previous perturbative order. The latter term turns out to be the dominant error; we then add the two errors linearly. Additionally, we also redo the analysis with alternative weight assignments ( $p$ -value, and constant weights); we obtain compatible results, and quote an error that covers the whole range spanned by the three analyses. As a further cross-check, we can compare the analysis performed with the static energy normalized in units of  $r_0$  (our default choice) and the one with the static energy normalized in units of  $r_1$ ; we find that the two analyses give consistent results<sup>3</sup>.

Our final result reads

$$r_0\Lambda_{\overline{\text{MS}}} = 0.70 \pm 0.07, \quad (10)$$

which using the value of  $r_0$  from Ref. [9] gives

$$\alpha_s(\rho = 1.5\text{GeV}, n_f = 3) = 0.326 \pm 0.019. \quad (11)$$

When we evolve Eq. (11) to the  $Z$ -mass scale,  $M_Z$ , we obtain

$$\alpha_s(M_Z, n_f = 5) = 0.1156_{-0.0022}^{+0.0021}, \quad (12)$$

where we have used the `Mathematica` package `RunDec` [18] to obtain the above number (4 loop running, with the charm quark mass equal to 1.6 GeV and the bottom quark mass equal to 4.7 GeV).

We compare our result with other recent lattice determinations of  $\alpha_s$  in Fig. 1. Our central value is a bit lower than those of the other lattice determinations.

Our determination is performed at a scale of around 1.5 GeV. This scale corresponds to (the inverse of) the typical distance where: (i) we have lattice data, and (ii) the weak-coupling calculation is reliable. This means that our analysis represents the lowest-energy determination of  $\alpha_s$  available, and can therefore be an important new ingredient to further test the running of  $\alpha_s$ . Previously, the lowest-energy determination was that coming from hadronic  $\tau$  decays (performed at  $m_\tau = 1.78$  GeV). For comparison, the value of the pre-average of

<sup>3</sup>This is a cross-check and not just a trivial rescaling, because the errors and systematics entering the lattice analyses normalized in units of  $r_0$  or  $r_1$  are different.

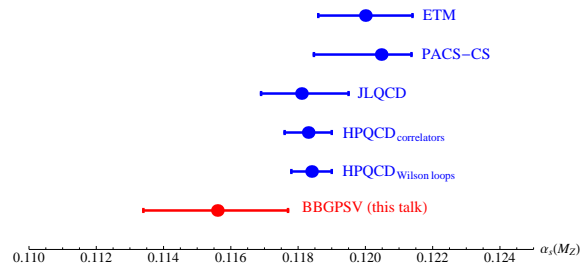


Figure 1: Comparison of our result (red -lighter- point) with other recent lattice determinations of  $\alpha_s$  (blue -darker- points). The references are: HPQCD [19], JLQCD [20], PACS-CS [21], ETM [22].

$\alpha_s$  determinations from  $\tau$  decays that is currently used by the Particle Data Group (PDG) [23] is  $\alpha_s(M_Z) = 0.1197 \pm 0.0016$ .

To summarize, we have obtained a determination of  $\alpha_s$  by comparing perturbative calculations with a lattice computation of the short-distance part of the QCD static energy. Our determination is at three-loop accuracy (including resummation of the leading ultrasoft logarithms), and is performed at a scale of 1.5 GeV (and therefore it constitutes the lowest-energy determination of  $\alpha_s$  available). When evolved to the scale  $M_Z$ , it corresponds to  $\alpha_s(M_Z) = 0.1156_{-0.0022}^{+0.0021}$ . A very important outcome of our work is also that our analysis shows, for the first time in QCD with  $n_f = 2 + 1$  flavors, that perturbation theory (after cancellation of the leading renormalon singularity) can describe the short-distance part of the static energy. This is illustrated in Fig. 2.

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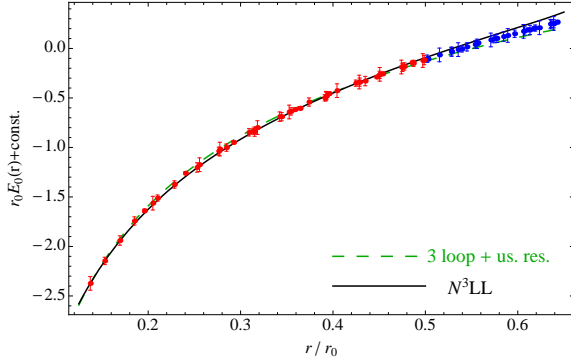


Figure 2: Comparison of the singlet static energy with lattice data (red -lighter- points). [The comparison (and all the analysis in the text) is done for  $r < 0.5r_0 \approx 0.234$  fm, which is the region where perturbation theory is reliable. The (blue -darker-) points and curves for  $r > 0.5r_0$  are shown just for illustration]. The long-dashed green curve is at three loops plus leading ultrasoft logarithmic resummation, and the solid black curve also includes resummation of the sub-leading ultrasoft logarithms (i.e. it is at  $N^3$ LL accuracy).  $r_0 \Lambda_{\overline{\text{MS}}} = 0.70$  was used in all the curves. The additive constant in the perturbative expression for the static energy is taken such that each curve coincides with the lattice data point at the shortest distance.

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