Published for SISSA by Springer

Received: October 29, 2012
Accepted: December 3, 2012
Published: January 4, 2013

Sommerfeld effect in heavy quark chemical equilibration

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Abstract: The chemical equilibration of heavy quarks in a quark-gluon plasma proceeds via annihilation or pair creation. For temperatures T much below the heavy quark mass M, when kinetically equilibrated heavy quarks move very slowly, the annihilation in the colour singlet channel is enhanced because the quark and antiquark attract each other which increases their probability to meet, whereas the octet contribution is suppressed. This is the so-called Sommerfeld effect. It has not been taken into account in previous calculations of the chemical equilibration rate, which are therefore incomplete for $T \lesssim \alpha_{\rm s}^2 M$. We compute the leading-order equilibration rate in this regime; there is a large enhancement in the singlet channel, but the rate is dominated by the octet channel, and therefore the total effect is small. In the course of the computation we demonstrate how operators that represent the annihilation of heavy quarks in non-relativistic QCD can be incorporated into the imaginary-time formalism.

Keywords: Thermal Field Theory, Resummation, Quark-Gluon Plasma, Heavy Quark Physics

ArXiv ePrint: 1210.6153

doi:10.1007/JHEP01(2013)037

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Contents

1 Introduction

If heavy quarks of mass M in a quark-gluon plasma are initially out of thermal equilibrium, they quickly equilibrate kinetically by multiple scatterings with gluons and light quarks [1]–[5]. (In this paper we consider an ideal limit in which the plasma lives for a long time.) At temperatures $T \ll M$ chemical equilibration is much slower because it requires quark-antiquark annihilation or pair creation.[1] In fact, given that the probability to find a target is Boltzmann suppressed, the chemical equilibration rate $\Gamma_{\rm chem}$ is exponentially small at low temperatures, $\Gamma_{\rm chem} \sim e^{-M/T}$ [6–8].

Given that the heavy quarks are in kinetic equilibrium they move with non-relativistic velocity. When the reacting particles have small relative velocity v, their mutual interactions can have a large influence on the annihilation or production cross section [9, 10], a phenomenon known as the Sommerfeld effect. In perturbation theory this effect would first show up in the 1-loop correction to a tree-level cross section σ_0:

$$\sigma = \sigma_0 \left[1 + O\left(\frac{\alpha_{\rm s}}{v}\right) + O\big(\alpha_{\rm s} \ln v\big) + O\big(\alpha_{\rm s}\big)\right]. \tag{1.1}$$

For Coulomb-like interactions the Sommerfeld effect manifests itself as a non-vanishing contribution of $O(\alpha_s/v)$. When v becomes as small as $\alpha_{\rm s}$, the 1-loop correction can become larger than the tree-level result. The naive loop expansion then breaks down, and the enhanced terms need to be resummed. This effect has to be taken into account for any particle reactions close to threshold [11], and has been widely discussed, e.g., in connection with $t\bar{t}$ and gluino and squark pair production in hadronic collisions (see e.g. refs. [12–14] for recent work and references). It may also play an important role in the indirect detection of dark matter particles [15].

[1] Weak interactions are not considered here.

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At finite temperature the typical heavy quark velocity is of order $v \sim \sqrt{T/M} \ll 1$. Thus the naive perturbative expansion breaks down for $v \lesssim \alpha_{\rm s}$, i.e. $T \lesssim \alpha_{\rm s}^2 M$. Then a similar resummation is needed as in hadronic collisions at zero temperature. The Sommerfeld effect in thermal dark matter freeze-out has indeed been discussed in many recent works, such as refs. [16–18], and it may also play a role in certain leptogenesis scenarios [19]. However, to the best of our knowledge, the Sommerfeld effect has not been taken into account in previous calculations of the chemical equilibration rate of heavy quarks.

In ref. [20], a definition of the heavy quark chemical equilibration rate was given which is non-perturbative and thus goes beyond the usual formulation in terms of the Boltzmann equation. At leading order it gives the same rate as the Boltzmann equation. To include more terms of the perturbative expansion, it would be convenient to use non-relativistic QCD (NRQCD [21]) for computing the rate. The pair annihilation of heavy quarks is represented in NRQCD by an imaginary part in a coefficient of a 4-fermion operator [22]. Such a complex coefficient is related to the analytic structure of a corresponding Green's function. One purpose of the present paper is to give a formulation of such operators which can be used in the imaginary-time formalism. Subsequently, the NRQCD analysis allows us to disentangle the contributions from the colour singlet and octet operators to the heavy quark chemical equilibration rate, a necessary first step for discussing the Sommerfeld effect.

This paper is organized as follows. In section 2 we discuss how pair annihilation can be incorporated in the imaginary-time formulation of NRQCD, and use this to determine the contributions of singlet and octet operators to the chemical equilibration rate. In section 3 we compute the leading-order chemical equilibration rate taking into account the Sommerfeld effect. A brief summary is presented in section 4.

2 Non-relativistic QCD in the imaginary-time formalism

2.1 General formulation

NRQCD [21] describes non-relativistic heavy quarks, and gluons and light quarks with momenta much smaller than M. Therefore the annihilation of a heavy quark-antiquark pair cannot be described in terms of the fields of the theory. However, when integrating out the scale M, one obtains 4-fermion operators. When this is done in real time, their coefficients have an imaginary part which corresponds to the annihilation process [22].

The imaginary parts of these coefficients arise from a cut, or discontinuity, of 4-point functions, viewed as a function of some energy variable ω, across the real ω-axis. If manipulations are carried out in the complex ω-plane, as is necessary e.g. in thermal field theory, then the imaginary parts of these coefficients have to be represented in a way which reflects this analytic structure. This can be achieved by expressing them in a suitable spectral representation.

The spectral representation of a 2-point function Π can be written as

$$\Pi(\omega, \mathbf{k}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \left(\frac{1}{\omega - z} - \frac{1}{\omega + z} \right) \rho(z, \mathbf{k}) \,, \tag{2.1}$$

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where the spectral density ρ is a real and odd function of z. In this form Π can be evaluated both for real ω, which corresponds to real time, and for imaginary $\omega = i\omega_n$, where ω_n is a Matsubara frequency, which corresponds to imaginary time. By approaching the real ω-axis in different ways one obtains different operator orderings (for instance, setting $\omega = \mathrm{Re}\,\omega + i0^+$ yields a retarded correlator, which for $\omega \gg T$ is equivalent to the time-ordered one). The function $\Pi(i\omega_n, \mathbf{k})$ is purely real. Spectral representations are routinely used in finite temperature perturbation theory for Hard Thermal Loop [23, 24] resummed propagators.

The 4-fermion operators of ref. [22] are

$$\delta\mathcal{L}_M = \frac{f_1(^1S_0)}{M^2}\,\mathcal{O}_1(^1S_0) + \frac{f_1(^3S_1)}{M^2}\,\mathcal{O}_1(^3S_1) + \frac{f_8(^1S_0)}{M^2}\,\mathcal{O}_8(^1S_0) + \frac{f_8(^3S_1)}{M^2}\,\mathcal{O}_8(^1S_1)\,, \quad (2.2)$$

$$\begin{aligned} \mathcal{O}_1(^1S_0) &\equiv \psi^\dagger\chi\,\chi^\dagger\psi\,, & \mathcal{O}_1(^3S_1) &\equiv \psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\psi\,, \\ \mathcal{O}_8(^1S_0) &\equiv \psi^\dagger T^a\chi\,\chi^\dagger T^a\psi\,, & \mathcal{O}_8(^3S_1) &\equiv \psi^\dagger\vec{\sigma}T^a\chi\cdot\chi^\dagger\vec{\sigma}T^a\psi\,. \end{aligned} \quad (2.3)$$

Here ψ, χ are 2-component non-relativistic spinors, σ are the Pauli matrices, and T^a are generators of $\mathrm{SU}(N_\mathrm{c})$, normalized as $\mathrm{Tr}[T^aT^b] = \frac{\delta^{ab}}{2}$. The subscripts 1, 8 refer to singlet and octet channels, respectively. The absorptive parts of the coefficients read [22]

$$\begin{aligned} \mathrm{Im}\, f_1(^1S_0) &= \frac{C_\mathrm{F}}{2N_\mathrm{c}}\,\pi\alpha_\mathrm{s}^2 + O(\alpha_\mathrm{s}^3)\,, & \mathrm{Im}\, f_1(^3S_1) &= O(\alpha_\mathrm{s}^3)\,, \\ \mathrm{Im}\, f_8(^1S_0) &= \frac{N_\mathrm{c}^2-4}{4N_\mathrm{c}}\,\pi\alpha_\mathrm{s}^2 + O(\alpha_\mathrm{s}^3)\,, & \mathrm{Im}\, f_8(^3S_1) &= \frac{N_\mathrm{f}}{6}\,\pi\alpha_\mathrm{s}^2 + O(\alpha_\mathrm{s}^3)\,, \end{aligned} \quad (2.4)$$

where $C_\mathrm{F} \equiv (N_\mathrm{c}^2-1)/2N_\mathrm{c}$. The corrections of $O(\alpha_\mathrm{s}^3)$ are also known, but not needed here.

The spectral representation of the most general 4-point function involves three energy variables instead of a single one as in eq. (2.1). Fortunately, this complication can be avoided for the operators of eq. (2.3) at leading order: only the sum of the energies of the annihilating particles appears. This is obvious for the s-channel annihilation. It is also true for the t and u-channel annihilation, because the virtual heavy quark is far off-shell, and effectively leads to a point-like interaction of the annihilating pair and the produced two gluons. Consequently, the operators can be represented in a form similar to eq. (2.1),

$$\begin{aligned} \delta S_M^{(i)} = {} & \int_\mathcal{X}\int_{\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4} e^{i(\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4)\cdot\mathcal{X}}\;\psi_r^*(\mathcal{K}_1)\,\chi_s(\mathcal{K}_2)\,\chi_t^*(\mathcal{K}_3)\,\psi_u(\mathcal{K}_4) \\ & \times\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{2\pi}\left[\frac{\rho^{(i)}_{rstu}(z)}{k_1^0+k_2^0-z} - \frac{\rho^{(i)}_{utsr}(z)}{k_1^0+k_2^0+z}\right], \end{aligned} \quad (2.5)$$

where r, s, t, u contain both spin and colour indices, i enumerates the four cases in eq. (2.2), $\mathcal{X} \equiv (t, \mathbf{x})$ and $\mathcal{K}_i \equiv (k_i^0, \mathbf{k}_i)$. Setting e.g. $k_1^0, k_2^0 \to M + i0^+$, $\mathbf{k}_1, \mathbf{k}_2 \to \mathbf{0}$, the absorptive parts can be read off:

$$\frac{\mathrm{Im}\, f^{(i)}}{M^2} \Leftrightarrow -\frac{1}{2}\Big[\rho^{(i)}_{rstu}(2M) - \rho^{(i)}_{utsr}(-2M)\Big]\,, \quad (2.6)$$

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where a suitable choice of indices is understood. Subsequently, computations can be carried out also in the imaginary-time formalism, by including

$$\delta S_E^{(i)} = -\int_X \sum\!\!\!\!\!\!\int_{\{K_1,K_2,K_3,K_4\}} e^{i(K_1+K_2+K_3+K_4)\cdot X}\, \psi_r^*(K_1)\,\chi_s(K_2)\,\chi_t^*(K_3)\,\psi_u(K_4) \times \int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{2\pi}\left[\frac{\rho^{(i)}_{rstu}(z)}{ik_{n1}+ik_{n2}-z} - \frac{\rho^{(i)}_{utsr}(z)}{ik_{n1}+ik_{n2}+z}\right] \tag{2.7}$$

in the Euclidean effective action. Here $X \equiv (\tau, \mathbf{x})$, $K \equiv (k_n, \mathbf{k})$, $\int_X \equiv \int_0^{1/T}\mathrm{d}\tau\int_{\mathbf{x}}$, $\sum\!\!\!\!\!\!\int_{\{K\}} \equiv T\sum_{\{k_n\}}\int_{\mathbf{k}}$, and $\sum_{\{k_n\}}$ denotes a sum over fermionic Matsubara frequencies.

2.2 Definition of the chemical equilibration rate

Physically, heavy quark chemical equilibration corresponds to the fact that the energy carried by kinetically equilibrated heavy quarks is not conserved because of annihilation or pair creation; the chemical equilibration rate is a "transport coefficient" describing the average non-conservation. Concretely, the task is to compute the connected correlator [20]

$$\Delta(\tau) \equiv \int_{\mathbf{x}} \Big\langle H(\tau,\mathbf{x})H(0,\mathbf{0})\Big\rangle_{\mathrm{c}}\,, \quad 0<\tau<\frac{1}{T}\,, \tag{2.8}$$

where H denotes the heavy quark Hamiltonian. In terms of the fields in eq. (2.3), the Hamiltonian reads $H = M(\psi^\dagger\psi - \chi^\dagger\chi) + O(1/M)$. We expand to first order in the absorptive action, eq. (2.7), which is $1/M^2$-suppressed. After a Fourier transformation, $\tilde{\Delta}(\omega_n) = \int_0^{1/T}\mathrm{d}\tau\, e^{i\omega_n\tau}\Delta(\tau)$, and analytic continuation, $\rho_\Delta(\omega) = \mathrm{Im}\,\tilde{\Delta}(\omega_n\to -i[\omega+i0^+])$, a coefficient denoted by Ω_{chem} in ref. [20] can be extracted as

$$\Omega_{\mathrm{chem}} = \lim_{\Gamma_{\mathrm{chem}}\ll\omega\ll\omega_{\mathrm{UV}}} \omega^2\big[1+2f_{\mathrm{B}}(\omega)\big]\rho_\Delta(\omega)\,, \tag{2.9}$$

where $\Gamma_{\mathrm{chem}} \sim e^{-M/T}$, $\omega_{\mathrm{UV}} \sim T$, and f_{B} denotes the Bose distribution. The chemical equilibration rate then follows from $\Gamma_{\mathrm{chem}} = \Omega_{\mathrm{chem}}/(2\chi_{\mathrm{f}}\,M^2)$, where χ_{f} denotes the heavy quark-number susceptibility.

2.3 Perturbative evaluation of the chemical equilibration rate

Whereas the formulation of section 2.2 is in principle non-perturbative (apart from the fact that the matching coefficients in eq. (2.4), reflecting ultraviolet dynamics at the energy scale $\sim M$, need to be computed perturbatively), we now expand in the gauge coupling as well. The free heavy quark propagators have the forms

$$\big\langle\psi_r(K_1)\psi_s^*(K_2)\big\rangle_0 = \frac{\delta_{rs}\,\delta\,(K_1+K_2)}{ik_{n1}+E_{k1}}\,, \quad \big\langle\chi_r(K_1)\chi_s^*(K_2)\big\rangle_0 = \frac{\delta_{rs}\,\delta\,(K_1+K_2)}{ik_{n1}-E_{k1}}\,, \tag{2.10}$$

where $E_k \equiv M + k^2/2M + \cdots$, and $\sum\!\!\!\!\!\!\int_K \delta\,(K) \equiv 1$. Feynman diagrams are illustrated in figure 1.

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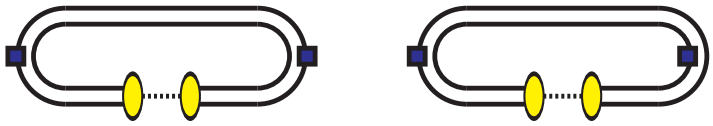

(a) (b)

Figure 1. Feynman diagrams for the computation of the heavy quark chemical equilibration rate within NRQCD. Two ovals connected by a dotted line represent the absorptive part of a 4-fermion operator; filled squares represent the heavy quark Hamiltonian; and solid lines represent heavy quark propagators. The heavy quarks propagate along the imaginary time direction.

Carrying out Wick contractions and Matsubara sums, performing the analytic continuation, and taking the cut, we obtain the spectral function

$$\rho_\Delta(\omega) = \frac{2M^2}{\omega^2}\sum_i \int_{\mathbf{k}_1,\mathbf{k}_2} \Big\{ \big[\rho^{(i)}_{rssr}(E_{k_1}+E_{k_2}+\omega) - \rho^{(i)}_{rssr}(-E_{k_1}-E_{k_2}-\omega)\big] \times \big[f_{\rm F}(E_{k_1}) + f_{\rm B}(E_{k_1}+E_{k_2}+\omega)\big]\big[f_{\rm F}(E_{k_2}+\omega) - f_{\rm F}(E_{k_2})\big] - (\omega\to-\omega)\Big\}\,, \quad (2.11)$$

where $f_{\rm F}$ denotes the Fermi distribution. Expanding in a small ω,

$$f_{\rm F}(E_{k_2}+\omega) - f_{\rm F}(E_{k_2}) \approx -\frac{\omega}{T}\, f_{\rm F}(E_{k_2})\big[1 - f_{\rm F}(E_{k_2})\big]\,, \quad (2.12)$$

and omitting exponentially small terms, the coefficient of eq. (2.9) is readily extracted:

$$\Omega_{\rm chem} = -8M^2 \sum_i \int_{\mathbf{k}_1,\mathbf{k}_2} \big[\rho^{(i)}_{rssr}(E_{k_1}+E_{k_2}) - \rho^{(i)}_{rssr}(-E_{k_1}-E_{k_2})\big] f_{\rm F}(E_{k_1}) f_{\rm F}(E_{k_2})\,. \quad (2.13)$$

Subsequently we may count the contractions for the operators in eq. (2.3):

$$\begin{aligned} &\rho_{rssr}[\mathcal{O}_1(^1S_0)] \to 2N_{\rm c}\,, && \rho_{rssr}[\mathcal{O}_1(^3S_1)] \to 6N_{\rm c}\,, \\ &\rho_{rssr}[\mathcal{O}_8(^1S_0)] \to 2N_{\rm c}C_{\rm F}\,, && \rho_{rssr}[\mathcal{O}_8(^3S_1)] \to 6N_{\rm c}C_{\rm F}\,. \end{aligned} \quad (2.14)$$

Identifying the absorptive coefficients from eq. (2.6) (noting that $E_{k_1}+E_{k_2} \approx 2M$ because of the exponential suppression factors), and inserting their values from eq. (2.4), then leads to

$$\Omega_{\rm chem} = 16M^2 \int_{\mathbf{k}_1,\mathbf{k}_2} f_{\rm F}(E_{k_1}) f_{\rm F}(E_{k_2}) \times \frac{\pi\alpha_{\rm s}^2 N_{\rm c} C_{\rm F}}{M^2} \Bigg(\underbrace{\frac{1}{N_{\rm c}}}_{\text{singlet},\mathcal{O}_1} + \underbrace{\frac{N_{\rm c}^2-4}{2N_{\rm c}} + N_{\rm f}}_{\text{octet},\mathcal{O}_8} \Bigg)\,. \quad (2.15)$$

This is the main information needed in the next section. (For completeness we note that a division by $2\chi_{\rm f}\, M^2 = 8N_{\rm c}M^2 \int_{\mathbf{k}_2} f_{\rm F}(E_{k_2})$ leads to $\Gamma_{\rm chem}$ of eq. (3.15) with $\bar{S}_1 = \bar{S}_8 = 1$.)

3 Sommerfeld effect in the chemical equilibration rate

Consider now the annihilation or pair creation of a heavy quark Q and antiquark $\bar{Q}$ with four-momenta $\mathcal{K}_1$ and $\mathcal{K}_2$. We define v as the velocity of Q in the $Q\bar{Q}$ rest frame:

$$v \equiv |\mathbf{v}|\,, \quad \mathbf{v} \equiv \frac{\mathbf{k}_1 - \mathbf{k}_2}{2M}\,. \quad (3.1)$$

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One has to resum the multiple exchange of gluons with typical momenta $\mathcal{Q} = (q^0, \mathbf{q})$, where

$$q^0 \sim Mv^2\,, \quad q \equiv |\mathbf{q}| \sim Mv\,. \tag{3.2}$$

In heavy-quark kinetic equilibrium eq. (3.2) corresponds to

$$q^0 \sim T\,, \quad q \sim \sqrt{MT}\,. \tag{3.3}$$

In particular, q is parametrically larger than the Debye scale which is of order gT, where g is the gauge coupling ($g \equiv \sqrt{4\pi\alpha_{\rm s}}$):

$$q \gg gT\,. \tag{3.4}$$

Therefore the Debye screening and Landau damping of the exchanged gluons by the hot plasma can be neglected.[2]

The heavy quarks interact with gluons in the plasma, constantly changing their colour charge. This could affect the Sommerfeld effect which depends on the colour charge of the pair. The scattering with the plasma is characterized by the thermal width γ, which for heavy quarks is of order $\alpha_{\rm s}T$ [26]. On the other hand, the virtuality $\Delta \equiv (\mathcal{K}-\mathcal{Q})^2 - M^2 = (k^0-q^0)^2 - (\mathbf{k}-\mathbf{q})^2 - M^2$ of the heavy quark lines is of the same order as the typical momentum transfer squared, i.e. $\Delta \sim MT$. Schematically, a thermal width would replace

$$(k^0-q^0)^2 \to (k^0-q^0+i\gamma)^2 \simeq (k^0-q^0)^2 + 2ik^0\gamma \tag{3.5}$$

in the propagator. Since $k^0\gamma \sim \alpha_{\rm s}MT \ll MT \sim \Delta$, the width and correspondingly the colour change due to scattering with the heat bath are small compared with virtuality, and can be neglected at leading order.

In ref. [20] it was shown that, ignoring the Sommerfeld effect, the leading order chemical equilibration rate can be obtained from a Boltzmann equation which contains the Born cross section. The resummation of the Sommerfeld-enhanced terms modifies the Born matrix elements as [9–11]

$$|\mathcal{M}_{\rm resummed}|^2 = S\,|\mathcal{M}_{\rm tree}|^2\,, \tag{3.6}$$

where $S = S(v)$ is the so-called Sommerfeld factor. When the $Q\bar{Q}$ pair is in a colour singlet state the Sommerfeld factor is $S = S_1$ with

$$S_1 = \frac{X_1}{1-e^{-X_1}}\,, \quad X_1 = C_{\rm F}\,\frac{g^2}{4v}\,, \tag{3.7}$$

whereas for the octet $S = S_8$ with

$$S_8 = \frac{X_8}{e^{X_8}-1}\,, \quad X_8 = \left(\frac{N_{\rm c}}{2} - C_{\rm F}\right)\frac{g^2}{4v}\,. \tag{3.8}$$

[2]This is true not only parametrically but also numerically: we have checked that, above threshold and for typical parameter values, eq. (3.7) is in excellent agreement with the ratio of resummed and tree-level singlet spectral densities [25], in which the effects of Debye screening and Landau damping are included.

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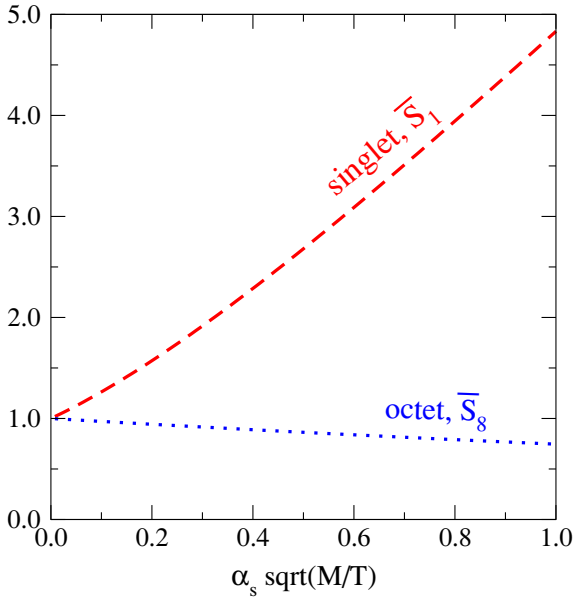

Figure 2. The averaged Sommerfeld factors, eq. (3.16), for the singlet and octet contributions.

At tree level the processes $gg \leftrightarrow Q\bar{Q}$ and $q\bar{q} \leftrightarrow Q\bar{Q}$ contribute to the chemical equilibration rate. The result of ref. [20] can be written as

$$\Gamma_{\rm chem} = \frac{M e^{-M/T}}{16\sqrt{2}\pi^3 N_{\rm c}} \int_0^\infty {\rm d}v\, v^2 e^{-Mv^2/T} \left\{ \frac{1}{2} \sum \left| \mathcal{M}^{\rm tree}_{gg\to Q\bar{Q}} \right|^2 + N_{\rm f} \sum \left| \mathcal{M}^{\rm tree}_{q\bar{q}\to Q\bar{Q}} \right|^2 \right\} , \quad (3.9)$$

where the sums are over all spin and colour degrees of freedom. In $q\bar{q} \leftrightarrow Q\bar{Q}$ the $Q\bar{Q}$ is in a colour octet state, whereas the process $gg \leftrightarrow Q\bar{Q}$ has both octet and singlet contributions. Denoting by r the ratio of octet to singlet contributions, eq. (2.15) implies that

$$r = \frac{N_{\rm c}^2 - 4}{2} = \frac{5}{2} . \quad (3.10)$$

According to eq. (3.6) one has to replace in eq. (3.9)

$$\left| \mathcal{M}^{\rm tree}_{gg\to Q\bar{Q}} \right|^2 \to \left| \mathcal{M}^{\rm tree}_{gg\to Q\bar{Q}} \right|^2 \left(\frac{1}{1+r}\, S_1 + \frac{r}{1+r}\, S_8 \right) , \quad (3.11)$$

$$\left| \mathcal{M}^{\rm tree}_{q\bar{q}\to Q\bar{Q}} \right|^2 \to \left| \mathcal{M}^{\rm tree}_{q\bar{q}\to Q\bar{Q}} \right|^2 S_8 . \quad (3.12)$$

The summed tree-level matrix elements in the non-relativistic limit are

$$\sum \left| \mathcal{M}^{\rm tree}_{gg\to Q\bar{Q}} \right|^2 = 4g^4 C_{\rm F} N_{\rm c} (4C_{\rm F} - N_{\rm c}) , \quad (3.13)$$

$$\sum \left| \mathcal{M}^{\rm tree}_{q\bar{q}\to Q\bar{Q}} \right|^2 = 4g^4 C_{\rm F} N_{\rm c} . \quad (3.14)$$

Thus we find

$$\Gamma_{\rm chem} = \frac{g^4 C_{\rm F}}{8\pi M^2} \left(\frac{MT}{2\pi} \right)^{3/2} e^{-M/T} \left[\left(2C_{\rm F} - \frac{N_{\rm c}}{2} \right) \left(\frac{1}{1+r} \bar{S}_1 + \frac{r}{1+r} \bar{S}_8 \right) + N_{\rm f}\, \bar{S}_8 \right] , \quad (3.15)$$

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with the thermally averaged Sommerfeld factors

$$\bar{S}_\alpha \equiv \frac{4}{\sqrt{\pi}} \left(\frac{M}{T}\right)^{3/2} \int_0^\infty \mathrm{d}v\, v^2 e^{-Mv^2/T}\, S_\alpha\,, \quad \alpha \in \{1,8\}\,. \tag{3.16}$$

After a rescaling of v, the Sommerfeld factors of eq. (3.16) are seen to be functions of $g^2\sqrt{M/T}$ only. A numerical evaluation is shown in figure 2. Analytically, for $T \ll \alpha_\mathrm{s}^2 M$ we get

$$\bar{S}_1 \approx \frac{g^2 C_\mathrm{F}}{2} \sqrt{\frac{M}{\pi T}}\,, \tag{3.17}$$

whereas $\bar{S}_8$ is exponentially small (although decreasing only slowly in figure 2). For $T \gg \alpha_\mathrm{s}^2 M$, on the other hand,

$$\bar{S}_1 \approx 1 + \frac{g^2 C_\mathrm{F}}{4} \sqrt{\frac{M}{\pi T}}\,, \quad \bar{S}_8 \approx 1 - \frac{g^2 (N_\mathrm{c} - 2C_\mathrm{F})}{8} \sqrt{\frac{M}{\pi T}}\,. \tag{3.18}$$

As an example, if we take $\alpha_\mathrm{s} \simeq 0.3$, $M \simeq 1.5\,\mathrm{GeV}$, and $T \simeq 300\,\mathrm{MeV}$, then $\bar{S}_1 \simeq 3.4$, $\bar{S}_8 \simeq 0.8$. For $N_\mathrm{f} = 3$, this implies that the square brackets in eq. (3.15) evaluate to 4.28 rather than the naive 4.17. In other words, the substantial Sommerfeld enhancement of the singlet channel is all but compensated for by the fact that most channels, in particular all associated with light quarks, are octets, and for octets there is a mild suppression.

4 Summary

In a heavy ion collision, the heavy quark chemical equilibration rate parametrizes the rate at which heavy quarks and antiquarks, produced in overabundance in an initial hard process, annihilate during the thermal stage of the fireball evolution. It can be viewed as a fundamental property of thermal QCD, whose systematic understanding may have interesting theoretical relations to cosmology, given that similar (co-)annihilation phenomena lie e.g. at the heart of computations determining the dark matter relic abundance (in some scenarios).

On general grounds, the perturbative expansion for the chemical equilibration rate has the same functional form as the cross section shown in eq. (1.1) with $v \sim \sqrt{T/M}$. In this paper, we have resummed the terms of $O(\alpha_\mathrm{s}^n/v^n)$, describing the Sommerfeld effect, to all orders. The result has the form shown in eq. (3.15), with numerical factors plotted in figure 2. Due to a fortuitous cancellation between a strongly enhanced but mildly weighted singlet contribution, and a mildly suppressed but strongly weighted octet contribution, the numerical results turn out to be largely insensitive to the resummation.

The cancellation is peculiar to $N_\mathrm{c} = 3$. For instance, for the fundamental representation of SU(2), possibly relevant for dark matter (co-)annihilation at temperatures above the electroweak scale, the repulsive non-singlet contribution is absent (cf. eq. (3.10)). There is only an attractive channel also for oppositely charged particles in U(1), and indeed the Sommerfeld effect is likely to play an important role in chemical equilibration in hot QED plasmas (see e.g. ref. [27] for a general discussion of the problem).

Even though the $O(\alpha_\mathrm{s}/v)$ contribution in eq. (1.1) is insignificant in practice for Γ_chem, the functions $O(\alpha_\mathrm{s} \ln v)$ and $O(\alpha_\mathrm{s})$ might well be large. Therefore their determination, as well as a fully non-perturbative study of the chemical equilibration rate remain, in our opinion, well-motivated challenges.

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Acknowledgments

We thank M. Beneke, X. Garcia i Tormo, and M. Garny for helpful discussions. The work of D.B. was supported in part by the DFG through the Graduate School GRK 881, and the work of M.L. by the Swiss National Science Foundation (SNF) under grant 200021-140234. D.B. thanks ITP/AEC Bern for kind hospitality and support.

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