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# A NEW PROOF FOR THE LÉVY CONSTRUCTION OF SECOND KIND FOR STABLE LAWS

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We give a direct proof for the "Lévy construction of second kind" for stable laws on the real line without referring to the construction of "first kind."

## 1. Introduction

Let $X$ be a real-valued non-gaussian $\alpha$-stable random variable. It is well known that this is the case iff the Fourier transform (characteristic function) of $X$ has the form

$$\begin{aligned}\varphi_X(u) &= \exp\left(iu\gamma + c_- \int_{-\infty}^{0} (e^{iux} - 1 - \frac{iux}{1+x^2}\right)|x|^{-(1+\alpha)}dx + \\ &\quad + c_+ \int_0^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) x^{-(1+\alpha)}dx), \qquad u \in \mathbb{R},\end{aligned}$$

with $0 < \alpha < 2$, $\gamma \in \mathbb{R}$, $c_-, c_+ \geqslant 0$, $c_- + c_+ > 0$.

Possible "constructions" of $X$ are the so-called Lévy constructions of "first" and "second kind." These are the following.

Assume $0 < \alpha < 2$. Let $\{N_t\}_{t \geqslant 0}$ be a Poisson process with parameter $\lambda > 0$ and suppose $\Gamma_j$ is the time of the $j$th jump of $\{N_t\}_{t\geqslant 0}$. Suppose $\{Y_j\}_{j \in \mathbb{N}}$ is a sequence of i.i.d. $\{-1, 1\}$-valued random variables that is independent of the process $\{N_t\}_{t\geqslant 0}$ and such that $P(Y_j = 1) = p$. Put $a_j := 0$ for $j \leqslant 1/\alpha$ and $a_j := E(Y_j)E(\Gamma_j^{-1/\alpha})$ for $j > 1/\alpha$. Set

$$S_n(\alpha, \lambda, p, \gamma) := \gamma + \sum_{j=1}^{n} (\Gamma_j^{-1/\alpha} Y_j - a_j).$$

**Theorem 1** (Lévy construction of second kind)**.** *The sum $S_n(\alpha, \lambda, p, \gamma)$ converges to some $S(\alpha, \lambda, p, \gamma)$ a.s. as $n \to \infty$, and $S(\alpha, \lambda, p, \gamma)$ exhausts all (nondegenerate) $\alpha$-stable laws as $(\lambda, p, \gamma) \in ]0, \infty[ \times [0, 1] \times \mathbb{R}$.*

For the "Lévy construction of first kind," one just uses that

$$\mathcal{L}((\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \mid \Gamma_{n+1} = t) = \mathcal{L}((U_{[n:1]}, U_{[n:2]}, \ldots, U_{[n:n]})), \tag{1}$$

where $U_{[n:1]} < U_{[n:2]} < \ldots < U_{[n:n]}$ denotes the increasing order statistics of independent random variables $U_1, U_2, \ldots, U_n$ distributed uniformly on $[0, t]$. Write

$$F_t(\alpha, \lambda, p, \gamma) := \gamma + \sum_{j=1}^{N_t} (U_{[N_t:j]}^{-1/\alpha} Y_j - a_j).$$

Then the "Lévy construction of first kind" is the following:

**Theorem 2** (Lévy construction of first kind)**.** *The sum $F_t(\alpha, \lambda, p, \gamma)$ converges weakly to some $F(\alpha, \lambda, p, \gamma)$ as $t \to \infty$, and $F(\alpha, \lambda, p, \gamma)$ exhausts all (nondegenerate) $\alpha$-stable laws as $(\lambda, p, \gamma) \in ]0, \infty[ \times [0, 1] \times \mathbb{R}$.*

The classical proof of Theorem 1 proceeds in the manner that one first verifies Theorem 2 by calculating the Fourier transform of $F_t(\alpha, \lambda, p, \gamma)$, then uses (1), and at the end takes the limit as $t \to \infty$. In other words, the Lévy construction of second kind is deduced from that of the first kind by the equivalence (1). From the pedagogical

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point of vue, this approach has one disadvantage: Although (1) seems to be quite intuitive, a formally absolute correct proof is quite cumbersome to write down. Often, textbooks just give a "proof" using manipulations with differentials (as, e.g., [1]). That is why in this note, we would like to show how a direct approach to the Lévy construction of second kind is possible without using (1). See, e.g., [3–7] for further information and generalizations of the Lévy construction.

## 2. Alternative proof of Theorem 1

The convergence result $S_n(\alpha,\lambda,p,\gamma) \stackrel{a.s.}{\to} S(\alpha,\lambda,p,\gamma)$ follows from the Three Series Theorem by observing that the sequence $\{\Gamma_j^{-1/\alpha}\}_{j\geqslant 1}$ behaves as $\{j/\lambda\}_{j\geqslant 1}$ and the conditional variance of $\Gamma_j^{-1/\alpha}Y_j$ given $Y_j$ is of the form $m\Gamma_j^{-2/\alpha} \sim m_1 j^{-2/\alpha}$ as $j\to\infty$ (cf. [2]). In order to verify the stability of $S(\alpha,\lambda,p,\gamma)$, observe that the addition of $n$ independent copies of $S(\alpha,\lambda,p,\gamma)$ corresponds to a superposition of $n$ independent sequences $\{\Gamma_j\}_{j\geqslant 1}$, i.e., to the addition of $n$ independent copies of the Poisson process $\{N_t\}_{t\geqslant 0}$, which is equivalent to the multiplication of the intensitiy parameter $\lambda$ by $n$. In the sequence of jump times $\{\Gamma_j\}_{j\geqslant 1}$ this corresponds to a division by $n$, hence in the sequence $\{\Gamma_j^{-1/\alpha}\}_{j\geqslant 1}$ to a multiplication with $n^{1/\alpha}$. More precisely: Let, for $1\leqslant k\leqslant n$, processes $\{N_t^{(k)}\}_{t\geqslant 0}$ and $\{Y_j^{(k)}\}_{j\geqslant 1}$ be given as above such that the processes $D^{(k)} := \{(N_t^{(k)}, Y_j^{(k)})\}_{t\geqslant 0, j\geqslant 1}$ are i.i.d., $\gamma^{(k)}=\gamma\in\mathbb{R}$, $S^{(k)}(\alpha,\lambda,p,\gamma)$ as above. Then

$$\begin{aligned}\mathcal{L}\left(\sum_{k=1}^{n} S^{(k)}(\alpha,\lambda,p,\gamma)\right) &= \mathcal{L}(n\gamma+\sum_{k=1}^{n}\sum_{j=1}^{\infty}((\Gamma_j^{(k)})^{-1/\alpha}Y_j^{(k)}-a_j)) = \\ &= \mathcal{L}\left(\tilde\gamma+\sum_{j=1}^{\infty}\tilde\Gamma_j^{-1/\alpha}(\tilde Y_j-\tilde a_j)\right),\end{aligned}$$

where $\{\tilde\Gamma_j\}_{j\geqslant 0}$ ($\tilde\Gamma_0 := 0$) is defined as a process with independent increments and

$$\mathcal{L}(\tilde\Gamma_{j+1}-\tilde\Gamma_j) = \mathcal{L}(\tilde\Gamma_1) = \mathcal{L}(\min_{1\leqslant k\leqslant n}\Gamma_1^{(k)}), \tag{2}$$

$\tilde Y_j$, $\tilde a_j$ by analogy as above ($\tilde\Gamma_j$ is the time of the $j$th jump of the superposition of the processes $\{N_t^{(k)}\}_{t\geqslant 0}$, $k=1,2,\ldots,n$; the property that the increments are i.i.d. follows from the fact that the processes $\{N_t^{(k)}\}_{t\geqslant 0}$ are themselves independent processes with i.i.d. increments). Now

$$\begin{aligned}P(\min_{1\leqslant k\leqslant n}\Gamma_1^{(k)}\geqslant x) &= \prod_{k=1}^{n} P(\Gamma_1^{(k)}\geqslant x) = \prod_{k=1}^{n} e^{-\lambda x} = e^{-n\lambda x} = \\ &= P(\Gamma_1^{(1)}\geqslant nx) = P(\Gamma_1^{(1)}/n\geqslant x),\end{aligned}$$

i.e., $\mathcal{L}(\min_{1\leqslant k\leqslant n}\Gamma_1^{(k)}) = \mathcal{L}(\Gamma_1^{(1)}/n)$, hence $\mathcal{L}(\tilde\Gamma_1^{-1/\alpha}) = \mathcal{L}(n^{1/\alpha}(\Gamma_1^{(1)})^{-1/\alpha})$. Thus

$$\begin{aligned}\mathcal{L}\left(\sum_{k=1}^{n} S^{(k)}(\alpha,\lambda,p,\gamma)\right) &= \mathcal{L}(\tilde\gamma+n^{1/\alpha}S^{(1)}(\alpha,\lambda,p,0)) = \\ &= \mathcal{L}(\tilde\gamma+n^{1/\alpha}S(\alpha,\lambda,p,0)).\end{aligned}$$

Since this is true for all $n\geqslant 1$, this means that $S(\alpha,\lambda,p,\gamma)$ obeys an $\alpha$-stable law.

It remains to show that every $\alpha$-stable law is of the form $\mathcal{L}(S(\alpha,\lambda,p,\gamma))$. It holds that

$$S(\alpha,1,1,0) \stackrel{a.s.}{=} \lim_{t\to\infty} S_t(\alpha,1),$$

where

$$S_t(\alpha,\lambda) := \sum_{j=1}^{N_t}\Gamma_j^{-1/\alpha} - \sum_{j=1}^{\lambda t} a_j.$$

Here $\{\Gamma_j\}_{j\geqslant 1}$ as above with parameter $\lambda$. Observe that $N_t-\lambda t \stackrel{a.s.}{=} o(t^{1/\alpha})$, $t\to\infty$, by the Law of the Iterated Logarithm and thus $\sum_{j=1}^{N_t} a_j - \sum_{j=1}^{\lambda t} a_j \stackrel{a.s.}{\to} 0$, $t\to\infty$ (cf. [2]). For every $n\geqslant 1$ we have that

$$\mathcal{L}(S_t(\alpha,1)) = \mathcal{L}(S_t(\alpha,1/n)+b_n)$$

for suitable $b_n \in \mathbb{R}$, i.e. $S_t(\alpha, 1)$ is infinitely divisible. Since for all $t \geqslant 0$ the Lévy measure in the Lévy-Hinčin formula of $\mathcal{L}(S_t(\alpha,1))$ is concentrated on $[0,\infty[$, the same must hold for the limit $\mathcal{L}(S(\alpha,1,1,0))$ (see e.g. [1, Theorem 9.22]), hence the Fourier transform of $\mu^{(0)} := \mathcal{L}(S(\alpha,1,1,0))$ is of the form

$$\hat{\mu}^{(0)}(u) = \exp\left( iu\gamma^{(0)} + c_+^{(0)} \int_0^\infty \left( e^{iux} - 1 - \frac{iux}{1+x^2} \right) x^{-1+\alpha} dx \right)$$

for some $c_+^{(0)} > 0$. Now take any (nondegenerate) $\alpha$-stable law $\mu$ given by the Fourier transform

$$\hat{\mu}(u) = \exp\left( iu\gamma^{(0)} + c_- \int_{-\infty}^0 \left( e^{iux} - 1 - \frac{iux}{1+x^2} \right) |x|^{-1+\alpha} dx + c_+ \int_0^\infty \left( e^{iux} - 1 - \frac{iux}{1+x^2} \right) x^{-1+\alpha} dx \right)$$

($c_- + c_+ > 0$). Then we have

$$\mu = \mathcal{L}(c'_+ S'(\alpha,1,1,0) - c'_- S''(\alpha,1,1,0) + \gamma')$$

with $c'_+ := (c_+/c_+^{(0)})^{1/\alpha}$ and $c'_- := (c_-/c_+^{(0)})^{1/\alpha}$, where $S'(\alpha,1,1,0)$ and $S''(\alpha,1,1,0)$ are i.i.d. random variables obeying the law $\mathcal{L}(S(\alpha,1,1,0))$. However,

$$\mathcal{L}(c'_+ S'(\alpha,1,1,0) - c'_- S''(\alpha,1,1,0) + \gamma') = \mathcal{L}(S(\alpha, (c_+ + c_-)/c_+^{(0)}, c_+/(c_+ + c_-), \gamma')),$$

i.e., $\mu$ has indeed a Lévy construction of the second kind.

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