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# Uniqueness of the embedding continuous convolution semigroup of a Gaussian probability measure on the affine group and an application in mathematical finance

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**Abstract** Let $\{\mu_t^{(i)}\}_{t\geq 0}$ $(i = 1, 2)$ be continuous convolution semigroups (c.c.s.) of probability measures on $\mathbf{Aff}(\mathbf{1})$ (the affine group on the real line). Suppose that $\mu_1^{(1)} = \mu_1^{(2)}$. Assume furthermore that $\{\mu_t^{(1)}\}_{t\geq 0}$ is a Gaussian c.c.s. (in the sense that its generating distribution is a sum of a primitive distribution and a second-order differential operator). Then $\mu_t^{(1)} = \mu_t^{(2)}$ for all $t \geq 0$. We end up with a possible application in mathematical finance.

**Keywords** Continuous convolution semigroups of probability measures · Affine group · Lévy processes · Brownian motion

**Mathematic Subject Classification** 60B15 · 22E25 · 60G51

## 1 Introduction

Let $G$ be a locally compact group, $e$ the neutral element, $G^* := G\backslash\{e\}$. The triple $(M^1(G), *, \stackrel{w}{\to})$ denotes the topological semigroup of (regular) probability measures on $G$, equipped with the operation of convolution and the weak topology (cf. [5], Theorem 1.2.2). A continuous convolution semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on $G$ (c.c.s. for short) is a continuous semigroup homomorphism

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$$([0, \infty[, +) \ni t \mapsto \mu_t \in (M^1(G), *, \xrightarrow{w}),$$
$$\mu_0 = \varepsilon_e$$

($\varepsilon_x$ denoting the Dirac probability measure at $x \in G$). Let $G$ be a Lie group, $C_b^\infty(G)$ the space of bounded complex-valued $C^\infty$-functions on $G$, $\mathcal{D}(G)$ the subspace of complex-valued $C^\infty$-functions with compact support.

The generating distribution $\mathcal{A}$ of a c.c.s. $\{\mu_t\}_{t\geq 0}$ is defined (for $f \in \mathcal{D}(G)$) as

$$\mathcal{A}(f) := \lim_{t\to 0+} \frac{1}{t} \int_G (f(x) - f(e))\mu_t(dx)$$
$$= \frac{d}{dt}|_{t=0+} \int_G f(x)\mu_t(dx) \quad (f \in \mathcal{D}(G)).$$

It exists on the whole of $C_b^\infty(G)$ (cf. [13], p.119). Consider the group $\mathbf{Aff}(\mathbf{1})$, i.e. the group of all affine transformations

$$\mathbb{R} \ni u \mapsto au + b \in \mathbb{R} \quad (b \in \mathbb{R}, a > 0) \tag{1}$$

of the real line. This is the identity component of the group of all transformations of the shape (1), but with arbitrary $a \neq 0$. It is no loss of generality to restrict ourselves to the identity component, since (as is well-known) Gaussian laws are concentrated on the identity component. (There seems to be some ambiguity in the literature; some authors use the term “affine group” for the larger one of the before-mentioned groups with two connected components.) We will always write

$$\mathbf{Aff}(\mathbf{1}) := \{z = (d, b) =: (p_1(z), p_2(z)) \in \mathbb{R}^2\}, \tag{2}$$

equipped with the product

$$(d_1, b_1) \cdot (d_2, b_2) := (d_1 + d_2, b_1 + e^{d_1} b_2) \tag{3}$$

(so $d = \log a$). Observe that $p_1 : \mathbf{Aff}(\mathbf{1}) \to \mathbb{R}$ is a homomorphism. Of course, the group $\mathbf{Aff}(\mathbf{1})$ is not commutative; for this reason in products of elements of $\mathbf{Aff}(\mathbf{1})$ the order of the factors matters. One sees that for $z_k := (d_k, b_k) \in \mathbf{Aff}(\mathbf{1})$, we have that

$$\prod_{k=1}^n z_k := z_1 \cdot z_2 \cdot \ldots \cdot z_n = \left( \sum_{k=1}^n d_k, \sum_{k=1}^n \left( \exp\left( \sum_{\ell=1}^{k-1} d_\ell \right) \right) \cdot b_k \right). \tag{4}$$

In what follows, we will - whenever convenient - identify $\mathbf{Aff}(\mathbf{1})$ and $\mathbb{R}^2$ as $C^\infty$-manifolds in the before-mentioned canonical way (2). Let now $G$ be one of the groups $\mathbf{Aff}(\mathbf{1})$ or $(\mathbb{R}^2, +)$. The generating distribution of a c.c.s. on $G$ assumes a very explicit form. For $z \in G$, the symbol $||z||$ will mean the euclidean norm of $z$ on the underlying vector space $(\mathbb{R}^2, +) \cong G$ (cf. (2)). For these groups $G \in \{\mathbf{Aff}(\mathbf{1}), (\mathbb{R}^2, +)\}$, the functional $\mathcal{A}$ on $C_b^\infty(G)$ is the generating distribution of a c.c.s. $\{\mu_t\}_{t\geq 0}$ on $G$ iff it has the form (Lévy-Hinčin formula)

$$\mathcal{A}(f) = \langle \xi, \nabla \rangle f(e) + \frac{1}{2} \langle \nabla, M \cdot \nabla \rangle f(e) + \int_{G^*} (f(z) - f(e) - \Psi(f, z)) \eta(dz), \quad (5)$$

where

$$\Psi(f, z) := \begin{cases} \langle z, \nabla \rangle f(e) & : \ ||z|| \leq 1, \\ \langle \frac{z}{||z||}, \nabla \rangle f(e) & : \ ||z|| > 1. \end{cases}$$

($f \in C_b^\infty(G)$), $\xi \in \mathbb{R}^2$, $M$ is a positive semidefinite $2 \times 2$-matrix, and $\eta$ is a Lévy measure on $G^*$, i.e. a non-negative (not necessarily finite) measure on $G^*$ satisfying

$$\int_{0<||z||\leq 1} ||z||^2 \eta(dz) + \eta(\{z \in G : ||z|| > 1\}) < \infty.$$

Of course, the analogue of this formula holds in particular on $\mathbb{R}$ mutatis mutandis. The first summand in the Lévy-Hinčin formula is called the primitive term, the second one the centered Gaussian term, and the third one (the integral expression) the generalized Poisson distribution (here - as always in this text - the word "distribution" is used in the sense of a linear functional rather than in the sense of a probability law). The data $\xi, M, \eta$ are uniquely determined by $\{\mu_t\}_{t\geq 0}$ (cf. [12], Satz 1). Observe that the Jacobian matrix of the exponential map from the underlying Lie algebra into the group **Aff(1)** at 0 is just the two-dimensional unit matrix. As a shorthand we will write $\mathcal{A} = [\xi, M, \eta]$. The distribution $\mathcal{A}$ on $C_b^\infty(G)$ uniquely determines the c.c.s. $\{\mu_t\}_{t\geq 0}$, for this reason we may write $\mu_t =: \operatorname{Exp} t\mathcal{A} \quad (t \geq 0)$. Furthermore, every triple $[\xi, M, \eta]$ of the above-mentioned type generates a c.c.s. on $G$. (Cf. [12], Satz 1.) The generating distribution $\mathcal{A}$ is called Gaussian if $\eta = 0$. A c.c.s. $\{\operatorname{Exp} t\mathcal{A}\}_{t\geq 0}$ is called Gaussian if $\mathcal{A}$ is Gaussian. A c.c.s. will be called non-Gaussian if its generating distribution is not a Gaussian one, i.e. iff its Lévy measure is non-zero. A probability measure $\mu \in M^1(G)$ is called Gaussian if it is embeddable into a Gaussian c.c.s., i.e. if $\mu = \mu_1$ for some Gaussian c.c.s. $\{\mu_t\}_{t\geq 0}$ on $G$. A $G$-valued random variable $Z$ is called Gaussian if its law $\mathcal{L}(Z) \in M^1(G)$ is Gaussian. A lot of research has been done on the question of embeddability of probability measures in c.c.s. on different types of groups (and other algebraic structures). But now also the question of uniqueness of the embedding c.c.s. (which is well-known to hold on finite-dimensional vector spaces) is of great importance: If on **Aff(1)** a sequence of ("approximating") c.c.s. converges for time $t = 1$ to a limit measure which is embeddable into a unique ("limit") c.c.s., then it follows (as on finite-dimensional vector spaces) that the sequence of approximating c.c.s. converges to the limit c.c.s. as a whole (i.e. for all $t \geq 0$; cf. [3], Theorem 2.2; [5], Theorem 3.1.23 and Example 3.1.25). This is equivalent to the convergence of the corresponding generating distributions (for all $f \in C_b^\infty(G)$ (cf. [3], Facts 2.2 and 2.3)). By [3], Theorems 2.1 and 2.2 it follows that if $\mu_1$ is embeddable into a unique c.c.s. $\{\mu_t\}_{t\geq 0}$ with generating distribution $\mathcal{A}$, then for a strictly increasing sequence $\{k(n)\}_{n\geq 1}$ of natural numbers and some sequence $\{\nu_n\} \subset M^1(G)$, the relation $\nu_n^{*k(n)} \xrightarrow{w} \mu_1 \quad (n \to \infty)$ implies $\nu_n^{*\lfloor k(n)t \rfloor} \xrightarrow{w} \mu_t \quad (n \to \infty) \quad (t \geq 0)$ and that again, the latter is equivalent to the convergence of the corresponding Poisson generators

$k(n)\int_G (f(x) - f(e))\nu_n(dx) \to \mathcal{A}(f)$ as $n \to \infty$ ($f \in C_b^\infty(G)$). (Cf. [3,4,6] and the literature cited in these works.) For an account of the history of the uniqueness problem for c.c.s. of probability measures on groups see e.g. [7]. It can e.g. be shown that Gaussian measures on simply connected nilpotent Lie groups have exactly one embedding c.c.s. (cf. [7]). In [1], it has been shown that Gaussian probability measures on $\mathbf{Aff(1)}$ can be embedded into exactly one Gaussian c.c.s. on $\mathbf{Aff(1)}$. But it was left open if Gaussian measures on $\mathbf{Aff(1)}$ may also be embedded into non-Gaussian c.c.s. on $\mathbf{Aff(1)}$. The main goal of our paper is to show that a Gaussian measure on $\mathbf{Aff(1)}$ can in fact not be embedded into a non-Gaussian c.c.s. on $\mathbf{Aff(1)}$. So the just mentioned result of [1] indeed yields the uniqueness of the embedding c.c.s for a Gaussian probability measure on $\mathbf{Aff(1)}$ among all c.c.s. on $\mathbf{Aff(1)}$. As a general method, our way of proof will be somewhat an adaption of the method of recursive calculation of moments as used in [7] for simply connected nilpotent Lie groups.

In general, c.c.s. $\{\mu_t\}_{t\geq 0}$ on Lie groups represent Lévy processes with values in these groups. These are group-valued processes $\{Z(t)\}_{t\geq 0}$ starting at the neutral element and with stationary and independent increments with respect to the group multiplication. So for time points $s < t$, the increment (more precisely: "right increment") from time $s$ to time $t$ is given by $Z(s)^{-1}Z(t)$. The measure $\mu_t$ then is the law of $Z(t)$.

If $\{(D(t), X(t))\}_{t\geq 0}$ is a Lévy process on $(\mathbb{R}^2, +)$, then

$$\{Z(t)\}_{t\geq 0} := \{(D(t), H(t))\}_{t\geq 0} := \{(D(t), \int_{[0,t[} \exp D(s)dX(s))\}_{t\geq 0}$$

(stemming from (4)) is a Lévy process on $\mathbf{Aff(1)}$ with the "same" (with respect to the always used identification (cf. (1), (2), (3)) of $\mathbf{Aff(1)}$ and $(\mathbb{R}^2, +)$ as $C^\infty$-manifolds) generating distribution as $\{(D(t), X(t))\}_{t\geq 0}$.

Our paper will end up with a possible application in mathematical finance.

For more background information on the whole theory see e.g. [3–6,12,13], and the literature cited in these works. For the classical case of the real line see [2].

## 2 Uniqueness of the embedding c.c.s. for Gaussian measures on the group Aff(1)

Now we will state our main result:

**Theorem 1** *Let* $\mathcal{S}^{(i)} = \{\mu_t^{(i)}\}_{t\geq 0}$ *(*$i = 1, 2$*) be c.c.s. on the affine group on the real line* $\mathbf{Aff(1)}$. *Suppose that* $\mathcal{S}^{(1)}$ *is a Gaussian c.c.s. and that* $\mu_1^{(1)} = \mu_1^{(2)}$. *Then* $\mu_t^{(1)} = \mu_t^{(2)}$ *for all* $t \geq 0$.

In general, the proof of Theorem 1 is, to a certain extent, related to the proof of the Gaussian part of Theorem 1 in [7]. The following property of recursive calculability of moments will be of crucial importance. The following lemma is related to Lemma 1 in [7] (there for simply connected nilpotent Lie groups). For a measurable map $f$ and a measure $\mu$ we define the measure $f(\mu)$ by $f(\mu)(B) := \mu(f^{-1}(B))$ for measurable sets $B$. If $\ell = (\ell_1, \ell_2) \in \mathbb{N}_0^2$ and $\mu$ is a probability measure on $\mathbf{Aff(1)}$, then the symbol $M_\ell(\mu)$ will stand for the "mixed exponential/power moment"

$$M_\ell(\mu) := \int_G \exp(d\ell_1) b^{\ell_2} \mu(dz) = \int_G \exp(p_1(z)\ell_1) p_2(z)^{\ell_2} \mu(dz).$$

Consider on $\mathbb{N}_0^2$ the "lexicographic ordering from behind" defined as follows: Put $(a_1, a_2) < (b_1, b_2)$ if either

$$a_2 < b_2$$

or

$$a_2 = b_2, \quad a_1 < b_1.$$

**Lemma 1** *Suppose $\mu$, $\nu$ are probability measures on $\mathbf{Aff(1)}$ satisfying $\mu = \nu * \nu$. Assume that the Laplace Transform of $p_1(\mu)$ (moment generating function), $\varphi_{p_1(\mu)}(u) := E(\exp(uR))$ where $R$ denotes a real-valued random variable with $\mathcal{L}(R) = p_1(\mu)$ exists for all real $u$ and that all power functions of $p_2(z)$ are absolutely $p_2(\mu)$-integrable. Then all functions $z \mapsto p_2(z)^{\ell_2}$ and $z \mapsto \exp(\ell_1 p_1(z))$ $(\ell_1, \ell_2 \in \mathbb{N}_0)$ are absolutely $\nu$-integrable and the moments $M_\ell(\nu)$ $(\ell \in \mathbb{N}_0^2)$ may be calculated out of the moments $M_\ell(\mu)$ $(\ell \in \mathbb{N}_0^2)$ recursively with respect to $\ell$.*

*Proof* Of course, the map $p_1 : \mathbf{Aff(1)} \to (\mathbb{R}, +)$ is a homomorphism, hence (as $M_{(l_1,0)}(\mu) = \varphi_{p_1(\mu)}(l_1)$) we obtain $M_{(l_1,0)}(\mu) = M_{(l_1,0)}(\nu)^2$. Thus the assertions are obvious for $l_2 = 0$, hence w.l.o.g. we may assume $l_2 \geq 1$. Let $Z$, $W$ be i.i.d. $\mathbf{Aff(1)}$-valued random variables with law $\mathcal{L}(Z) = \mathcal{L}(W) = \nu$. Then by the assumption of the Lemma it holds that

$$E(|p_2(W) + e^{p_1(W)} p_2(Z)|^{l_2}) = E(|p_2(W \cdot Z)|^{l_2}) < \infty.$$

If $E(|p_2(Z)|^{l_2}) = \infty$, then for any fixed $w \in \mathbf{Aff(1)}$ it holds that $E(|p_2(w) + e^{p_1(w)} p_2(Z)|^{l_2}) = \infty$. By the independence of $W$ and $Z$, and Fubini's Theorem, this implies $E(|p_2(W) + e^{p_1(W)} p_2(Z)|^{l_2}) = \infty$, which leads to a contradiction. Hence indeed $E(|p_2(Z)|^{l_2}) < \infty$. Now we have

$$\begin{aligned} M_\ell(\mu) &= M_\ell(\nu * \nu) \\ &= E(\exp(\ell_1(p_1(W) + p_1(Z))) \cdot (p_2(W) + \exp(p_1(W)) p_2(Z))^{\ell_2}). \end{aligned}$$

By multiplying out the term $(\ldots)^{\ell_2}$ by the Binomial Theorem we find

$$\begin{aligned} M_\ell(\mu) &= E(\exp(\ell_1 p_1(W)) + \exp((\ell_1 + \ell_2) p_1(W))) \cdot M_\ell(\nu) + S \\ &= (M_{(\ell_1,0)}(\nu) + M_{(\ell_1+\ell_2,0)}(\nu)) M_\ell(\nu) + S \end{aligned} \tag{6}$$

where

$$S = \sum_{k=1}^{\ell_2 - 1} \binom{\ell_2}{k} M_{(\ell_1+\ell_2-k,k)}(\nu) M_{(\ell_1,\ell_2-k)}(\nu)$$

is an expression where only those $M_{\ell'}(\nu)$ with $\ell' < \ell$ occur. The same is also true for the coefficient

$$M_{(\ell_1,0)}(\nu) + M_{(\ell_1+\ell_2,0)}(\nu) > 0,$$

so that the linear Eq. (6) has exactly one solution $M_\ell(\nu)$. □

The following lemma follows from [3], Theorem 2.1 (Approximating Poisson Laws Theorem) and [10], Theorem 5.1:

**Lemma 2** *Let $\{\mu_t\}_{t\geq 0}$ be a c.c.s. on* $\mathbf{Aff}(\mathbf{1})$ *with generating distribution $\mathcal{A} = [\xi, M, \eta]$. Write the matrix $M$ in the form $M =: (m_{i,j})_{1\leq i,j\leq 2}$. Then*

$$2^n \mu_{2^{-n}}(B) \to \eta(B) \quad (n \to \infty)$$

*(for every Borel subset $B$ of* $\mathbf{Aff}(\mathbf{1}) \cong \mathbb{R}^2$ *bounded away from $e$ and carrying $\eta$-measure zero on its boundary). Furthermore, we have*

$$\limsup_{n\to\infty} 2^n \int\limits_{\{z\in\mathbf{Aff}(\mathbf{1}):||z||\leq\varepsilon\}} p_i(z)p_j(z)\mu_{2^{-n}}(dz) \to m_{i,j} \quad (\varepsilon \to 0) \quad (1 \leq i, j \leq 2).$$

The term "Brownian motion" (on $(\mathbb{R}^d, +)$) will always be used in the general sense, i.e. the expectation parameter $\xi$ and the variance resp. covariance matrix $M$ can be arbitrary. A standard Brownian motion is one with zero expectation parameter and the unit matrix as its covariance matrix (resp. with variance one in the one-dimensional case).

*Proof of Theorem 1*
1. By the above-mentioned uniqueness result for the embedding Gaussian c.c.s. of a Gaussian measure on $\mathbf{Aff}(\mathbf{1})$ due to Barczy, Pap [1] it suffices to show that also $\mathcal{S}^{(2)}$ has to be Gaussian.
2. Define, for $t > 0$, the bounded non-negative measures $\eta_t^{(i)} := (1/t)\mu_t^{(i)}$ and the non-negative measures

$$\iota_t^{(i)}(B) := \int\limits_{B\{e\}} ||z||^4 \eta_t^{(i)}(dz)$$

for Borel subsets $B$ of $\mathbf{Aff}(\mathbf{1})$. (A priori, it is not excluded that $\iota_t^{(i)}(\mathbf{Aff}(\mathbf{1}))$ could be $\infty$, however our subsequent estimates will show that this is indeed not the case.) Let $\{G(t)\}_{t\geq 0} = \{(G_1(t), G_2(t))\}_{t\geq 0}$ be any Brownian motion on $(\mathbb{R}^2, +)$. Put

$$H(t) := \int\limits_0^t \exp(G_1(s))dG_2(s). \tag{7}$$

Hence $\{Z(t)\}_{t\geq 0} = \{(G_1(t), H(t))\}_{t\geq 0}$ is any Gaussian Lévy process on $\mathbf{Aff}(\mathbf{1})$ (hence also called Brownian motion on $\mathbf{Aff}(\mathbf{1})$). The c.c.s. belonging to the processes $\{G(t)\}_{t\geq 0}$ and to $\{Z(t)\}_{t\geq 0}$ have the "same" generating distributions via the always used identification of **Aff**(**1**) and $(\mathbb{R}^2, +)$ as $C^\infty$-manifolds, cf. (2). If $\{Z(t)\}_{t\geq 0}$ is the Brownian motion on **Aff**(**1**) belonging to the c.c.s. $\{\mu_t^{(1)}\}_{t\geq 0}$, then we will write

$$\{Z(t)\}_{t\geq 0} =: \{Z^{(1)}(t)\}_{t\geq 0} =: \{G_1^{(1)}(t), H^{(1)}(t)\}_{t\geq 0}.$$

We have to estimate the moments of $H^{(1)}(t)$ $(0 < t \leq 1)$. Since every centered Brownian motion on $(\mathbb{R}^2, +)$ is the image under a linear endomorphism of a standard Brownian motion $\{(B_1(t), B_2(t))\}_{t\geq 0}$ on $(\mathbb{R}^2, +)$, it follows that $H^{(1)}(t)$ can be written as a linear combination (with coefficients not depending on $t$) of stochastic integrals of the type

$$J_{(\alpha_1,\alpha_2,\alpha_3)}(t) := \int_0^t \exp(\alpha_1 B_1(s) + \alpha_2 B_2(s) + \alpha_3 s) dB_1(s)$$

and

$$I_{(\alpha_1,\alpha_2,\alpha_3)}(t) := \int_0^t \exp(\alpha_1 B_1(s) + \alpha_2 B_2(s) + \alpha_3 s) ds$$

for suitable parameters $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. We first want to estimate the behavior of the moments of the laws of $I_{(\alpha_1,\alpha_2,\alpha_3)}(t)$. Suppose $0 < t \leq 1$ and let $p > 1$. By the subadditivity of the $p$-norm we find

$$\begin{aligned} E^{1/p}(I_{(\alpha_1,\alpha_2,\alpha_3)}(t)^p) &= E^{1/p}\left(\left(\int_0^t e^{\alpha_1 B_1(s)+\alpha_2 B_2(s)+\alpha_3 s} ds\right)^p\right) \\ &\leq \int_0^t E^{1/p}(e^{p\alpha_1 B_1(s)+p\alpha_2 B_2(s)+p\alpha_3 s}) ds \\ &\leq C \cdot t \end{aligned}$$

for some suitable constant $C > 0$, since

$$E(e^{p\alpha_1 B_1(s)+p\alpha_2 B_2(s)+p\alpha_3 s}) = e^{(((p\alpha_1)^2/2)+((p\alpha_2)^2/2)+p\alpha_3)s}$$

is bounded for $0 \leq s \leq 1$. Hence

$$\frac{1}{t} E(I_{(\alpha_1,\alpha_2,\alpha_3)}(t)^p) = O(t^{p-1}) \to 0 \quad (t \to 0) \tag{8}$$

for all $p > 1$. Denote by $\{\langle Y, Y\rangle(t)\}_{t\geq 0}$ the quadratic variation process of the process $\{Y(t)\}_{t\geq 0}$. Now suppose $p > 2$. The Burkholder-Davis-Gundy Inequality (see e.g. [11] Theorem 4.1 in Chapter 4) yields

$$E(|J_{(\alpha_1,\alpha_2,\alpha_3)}(t)|^p) \leq C \cdot E(\langle J_{(\alpha_1,\alpha_2,\alpha_3)}, J_{(\alpha_1,\alpha_2,\alpha_3)}\rangle(t)^{p/2}).$$

However, we have (due to the relation $dB(t) \cdot dB(t) = dt$ for a real-valued standard Brownian motion $\{B(t)\}_{t\geq 0}$)

$$\langle J_{(\alpha_1,\alpha_2,\alpha_3)}, J_{(\alpha_1,\alpha_2,\alpha_3)}\rangle(t) = \int_0^t (e^{\alpha_1 B_1(s)+\alpha_2 B_2(s)+\alpha_3 s})^2 ds = I_{(2\alpha_1,2\alpha_2,2\alpha_3)}(t).$$

So for all fixed $p > 2$ we also have

$$\frac{1}{t} E(|J_{(\alpha_1,\alpha_2,\alpha_3)}(t)|^p) \to 0 \quad (t \to 0)$$

by (8).

Hence we have found that in particular

$$2^n E(I_{(\alpha_1,\alpha_2,\alpha_3)}(2^{-n})^4) \to 0 \quad (n \to \infty) \tag{9}$$

and

$$2^n E(J_{(\alpha_1,\alpha_2,\alpha_3)}(2^{-n})^4) \to 0 \quad (n \to \infty). \tag{10}$$

So from the conditions (9) and (10) we obtain that

$$2^n E(H^{(1)}(2^{-n})^4) \to 0 \quad (n \to \infty).$$

In other words,

$$2^n M_{(0,4)}(\mu^{(1)}_{2^{-n}}) = p_2(\iota^{(1)}_{2^{-n}})(\mathbf{Aff}(\mathbf{1})) \to 0 \quad (n \to \infty). \tag{11}$$

By Lemma 1 and the hypothesis of theorem 1 it follows that the measures $p_2(\mu^{(2)}_{2^{-n}})$ possess all absolute moments and moreover

$$M_{(0,4)}(\mu^{(1)}_{2^{-n}}) = M_{(0,4)}(\mu^{(2)}_{2^{-n}}) \quad (n \geq 0).$$

Thus we obtain that also

$$p_2(\iota^{(2)}_{2^{-n}})(\mathbf{Aff}(\mathbf{1})) \to 0 \quad (n \to \infty).$$

Hence by Lemma 2 (applied to $\mathcal{S}^{(2)}$) the projection of the restriction of the Lévy measure of $\mathcal{S}^{(2)}$ to $\mathbb{R}^* \times \mathbb{R} \subset \mathbf{Aff}(\mathbf{1})$ (with respect to the always used identification $\mathbf{Aff}(\mathbf{1}) \cong \mathbb{R}^2$ used up to now) onto the second coordinate (with respect to (2)) must be zero, too. The same is of course also true for the projection of the restriction of the Lévy measure to $\mathbb{R} \times \mathbb{R}^* \subset \mathbf{Aff}(\mathbf{1})$ onto the first coordinate by the fact that Theorem 1 is in particular true for $\mathbb{R}$ instead of $\mathbf{Aff}(\mathbf{1})$ mutatis mutandis (this follows by the afore mentioned uniqueness of embedding for the real line). Hence also the Lévy measure of $\mathcal{S}^{(2)}$ vanishes on $\mathbf{Aff}(\mathbf{1})^*$, which proves our claim that $\mathcal{S}^{(2)}$ has to be Gaussian, too. □

*Remark 1* In fact, the above proof of Theorem 1 shows that for a Gaussian c.c.s. $\mathcal{S} = \{\mu_t\}_{t\geq 0}$ with generating distribution $\mathcal{A} = [\xi, M, 0]$ on $\mathbf{Aff}(\mathbf{1})$ (where, as usual, we will write $M =: (m_{i,j})_{1\leq i,j\leq 2}$ and $\xi =: (\xi_1, \xi_2)$) it holds that e.g.

$$2^n \int\limits_{\{z\in\mathbf{Aff}(\mathbf{1}):||z||>\varepsilon\}} ||z||^2 \mu_{2^{-n}}(dz) \to 0 \quad (n\to\infty) \quad (\varepsilon > 0).$$

By Lemma 2 and standard estimations this implies that

$$2^n \int\limits_{\mathbf{Aff}(\mathbf{1})} p_i(z) p_j(z) \mu_{2^{-n}}(dz) \to m_{i,j} \quad (n\to\infty) \quad (1\leq i,j\leq 2).$$

We now suppose that, in addition to the hypotheses of Theorem 1, it holds furthermore that both $\mathcal{S}^{(i)}$ $(i = 1, 2)$ are Gaussian. Then by Lemma 1 and the just verified fact for Gaussian c.c.s. $\mathcal{S}$ on $\mathbf{Aff}(\mathbf{1})$ we obtain that

$$M^{(1)} = M^{(2)}.$$

The fact that $\xi_1^{(1)} = \xi_1^{(2)}$ follows from the corresponding property on the real line, whereas afterwards $\xi_2^{(1)} = \xi_2^{(2)}$ then follows from the property that for any non-degenerate real-valued random variable $Y$ the equality $\mathcal{L}(Y) = \mathcal{L}(Y + c)$ (for some $c\in\mathbb{R}$) can only hold iff $c = 0$.

So in fact our proof of Theorem 1 can be kept self-contained without referring to the result of [1]. In other words, we have also given another proof of the corresponding result in the latter paper.

## 3 An application to mathematical finance

This section is related to [8,9]. Lévy processes on $\mathbf{Aff}(\mathbf{1})$ have a natural interpretation in mathematical finance in the sense that the second (i.e. the non-trivial) coordinate can be interpreted as the present value of a random payment stream in a model with stochastic interest where the joint process consisting of the logarithm of the discount factors until some time $t$ and the book values of the random payment stream at this time is a Lévy process. In detail, let $e^{D(t)}$ be the discount factor from time 0 up to time $t$ and let $X(t)$ be the book value at time $t$ of some random payment stream. Assume $\{(D(t), X(t))\}_{t\geq 0}$ is a Lévy process on $(\mathbb{R}^2, +)$. Then $H(t)$ (defined as a stochastic integral on the interval $[0, t[$ in the analogous way as in (7)) can be interpreted as the present value of the payment stream from time 0 up to time $t$. Define (as above) $\{Z(t)\}_{t\geq 0} := \{(D(t), H(t))\}_{t\geq 0}$. Our result then tells that in case $\{(D(t), X(t))\}_{t\geq 0}$ is assumed to be a Brownian motion on $(\mathbb{R}^2, +)$, then for any process $\{\tilde{Z}(t)\}_{t\geq 0} = \{(\tilde{D}(t), \tilde{H}(t))\}_{t\geq 0}$ on $\mathbf{Aff}(\mathbf{1})$ with $\tilde{H}(t) = \int_{[0,t[} \exp(\tilde{D}(s))d\tilde{X}(s)$ and where $\{(\tilde{D}(t), \tilde{X}(t))\}_{t\geq 0}$ is supposed to be any Lévy process on $(\mathbb{R}^2, +)$, the hypothesis that $Z(1)$ and $\tilde{Z}(1)$ coincide in law implies that the processes $\{(D(t), X(t))\}_{t\geq 0}$ and $\{(\tilde{D}(t), \tilde{X}(t))\}_{t\geq 0}$ coincide in law. (By Theorem 1, it first follows that the processes $\{(D(t), H(t))\}_{t\geq 0}$ and $\{(\tilde{D}(t), \tilde{H}(t))\}_{t\geq 0}$ coincide in law; this implies equality of their generating distributions on $C_b^\infty(\mathbf{Aff}(\mathbf{1}))$ and hence of the counterparts of the latter generating distributions on $C_b^\infty((\mathbb{R}^2, +))$, which generate the c.c.s. belonging to the processes $\{(D(t), X(t))\}_{t\geq 0}$ and $\{(\tilde{D}(t), \tilde{X}(t))\}_{t\geq 0}$ on $(\mathbb{R}^2, +)$.) In statistical language, this means that the law $\mathcal{L}(Z(1))$ (thus the joint law of the interest rate process and the present value at time one) is in some sense a "sufficient statistic" for the law of the joint process of the interest rate and the book value (i.e. for the process $\{(D(t), X(t))\}_{t\geq 0}$) among all models where the latter joint process is assumed to be a Lévy process on $(\mathbb{R}^2, +)$. This joint law at time point one can e.g. be determined by sampling (always at time one) from several i.i.d models. Concretely, this can be used to reveal the investor's policy (depending possibly on the development of the interest rate in the economy). Of course, in practice, the variance parameter of $\{D(t)\}_{t\geq 0}$ will be assumed to be small in comparison with the absolute value of its (negative) expectation parameter, so that the probability of $D(t) - D(s)$ becoming positive for a fixed time interval $[s, t]$ $(0 \leq s < t < \infty)$ becomes small (in the obvious quantitative way), but remains still positive. This corresponds to a situation where negative interest is improbable, but not completely excluded. In many economic situations, this is a not unrealistic model. On the one hand, the assumption that the logarithm of the discount factor obeys a Lévy process (in particular a Brownian motion) does not correspond to many more sophisticated stochastic interest models. But we think that nevertheless in certain cases (in particular for rather short observation periods and if the shift parameter is relatively strongly negative in comparison with the variance parameter) it is quite suitable as an approximation. More specifically, one can e.g. take a Cox-Ross-Rubinstein binomial model for the interest rate as one of the simplest possible extensions of the model with constant force of interest and then interpret the Brownian motion as a continuous-time approximation of the binomial model. One can imagine that the interest itself is a traded asset. In this case, the accumulation factor (i.e. the reciprocal value of the discount factor) plays in fact the role of a numéraire, i.e. some reference asset with respect to which the payments are discounted. The probability that during some fixed interval $[0, t]$ the interest rate can once become negative is given by the finite-horizon ruin probability (with zero initial capital). Note that for a negative expectation parameter of the Brownian motion $\{D(t)\}_{t\geq 0}$ and some fixed $t > 0$ the finite-horizon ruin probability with zero initial capital is strictly smaller than one by the theory of the Lebesgue needle. It is well-known that in risk theory much work has been done about estimation of ruin probabilities. (Cf. e.g. [9] and the literature cited there.)

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