

# Roy–Steiner equations for $\pi N$ scattering

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**Abstract.** In this talk, we present a coupled system of integral equations for the  $\pi N \rightarrow \pi N$  ( $s$ -channel) and  $\pi\pi \rightarrow \bar{N}N$  ( $t$ -channel) lowest partial waves, derived from Roy–Steiner equations for pion–nucleon scattering. After giving a brief overview of this system of equations, we present the solution of the  $t$ -channel sub-problem by means of Muskhelishvili–Omnès techniques, and solve the  $s$ -channel sub-problem after finding a set of phase shifts and subthreshold parameters which satisfy the Roy–Steiner equations.

## 1. Introduction

A precise determination of the pion–nucleon ( $\pi N$ ) scattering amplitude is relevant for different aspects of nuclear and particle physics. In particular at low energies, it allows for the test of dynamical constraints imposed by the chiral symmetry of QCD.

However, despite numerous investigations performed, the  $\pi N$  scattering amplitude is still not known to sufficient precision in the low-energy region. This is particularly striking in the scalar-isoscalar sector, where the determination of the pion–nucleon  $\sigma$ -term is still far from satisfactory.

Dispersion relations have repeatedly proven to be a powerful tool for studying processes at low energies with high precision [1–4]. They are built upon very general principles such as Lorentz invariance, unitarity, crossing symmetry, and analyticity. Moreover, the dispersive formalism is model independent and relates amplitudes at a given energy with an integral over the whole energy range, increasing the precision and providing information on the amplitude either at energies where data are poor or even in the unphysical region.

In particular, for  $\pi\pi$  scattering, Roy equations [7] are obtained from a twice-subtracted fixed- $t$  dispersion relation, where the  $t$ -dependent subtraction constants are determined by means of  $s \leftrightarrow t$  crossing symmetry, and performing a partial-wave expansion. This leads to a coupled system of partial-wave dispersion relations (PWDRs) for the  $\pi\pi$  partial waves  $t_J^I(s)$  with isospin  $I$  and angular

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momentum  $J$

$$t_J^I(s) = S_J^I(s) + \sum_{I'=0}^2 \sum_{J'=0}^{\infty} \int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s, s') \text{Im } t_{J'}^{I'}(s'), \quad (1)$$

where  $K_{JJ'}^{II'}$  are known kinematical kernel functions and the scattering lengths—the only free parameters—appear in the subtraction terms  $S_J^I(s)$ . In addition, assuming elastic unitarity

$$\text{Im } t_J^I(s) = \sigma(s) |t_J^I(s)|^2, \quad t_J^I(s) = \frac{e^{2i\delta_J^I(s)} - 1}{2i\sigma(s)}, \quad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}, \quad (2)$$

Eq. (1) translates into a coupled integral equation for the phase shifts  $\delta_J^I$  themselves.

Unfortunately, in the case of  $\pi N$  scattering, a full system of PWDRs has to include dispersion relations for two distinct physical processes,  $\pi N \rightarrow \pi N$  ( $s$ -channel) and  $\pi\pi \rightarrow \bar{N}N$  ( $t$ -channel), and the use of  $s \leftrightarrow t$  crossing symmetry will intertwine  $s$ - and  $t$ -channel equations.

## 2. Roy–Steiner equations

Roy–Steiner (RS) equations [5] solve this problem by combining the  $s$ - and  $t$ -channel physical region by means of hyperbolic dispersion relations (HDRs). They are obtained by expanding the absorptive part of the HDRs into  $s$ - and  $t$ -channel partial waves, respectively, and subsequently by projecting the full HDRs onto  $s$ -channel partial waves, conventionally denoted by  $f_{l\pm}^I$  with isospin index  $I \in \{+, -\}$  and total angular momentum  $j = l \pm 1/2 = l\pm$ , or  $t$ -channel partial waves  $f_{\pm}^J$ , labeled by the parallel (+)/anti-parallel (−) antinucleon–nucleon helicities and the total  $t$ -channel angular momentum.

Therefore, the  $s$ -channel RS equations read [5]

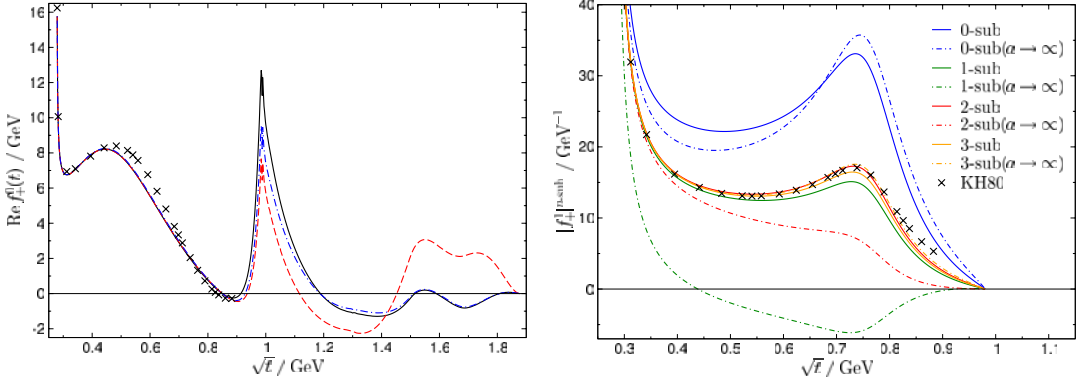
$$\begin{aligned} f_{l+}^I(W) = & N_{l+}^I(W) + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{IJ}(W, t') \text{Im } f_{+}^J(t') + H_{IJ}(W, t') \text{Im } f_{-}^J(t') \right\} \\ & + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{ll'}^I(W, W') \text{Im } f_{l'+}^I(W') + K_{ll'}^I(W, -W') \text{Im } f_{(l'+1)-}^I(W') \right\}, \end{aligned} \quad (3)$$

where due to  $G$ -parity only even/odd  $J$  contribute for isospin  $I = +/−$ , respectively. The kernels  $K_{ll'}^I(W, W)$ ,  $G_{IJ}(W, t)$ , and  $H_{IJ}(W, t)$  are known analytically, and  $N_{l+}^I(W)$  denotes the partial-wave projections of the pole terms.

The  $s$ -channel partial waves are intertwined by unitarity relations similar to Eq. (2), which are diagonal in the  $s$ -channel isospin basis  $I_s \in \{1/2, 3/2\}$ . In particular, it means that once the  $t$ -channel RS subproblem is solved, the structure of the  $s$ -channel RS subsystem is similar to  $\pi\pi$  Roy equations.

For the  $t$ -channel partial-wave projection, the corresponding  $t$ -channel RS equations are [6]

$$\begin{aligned} f_{+}^J(t) = & \tilde{N}_{+}^J(t) + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \left\{ \tilde{K}_{JJ'}^1(t, t') \text{Im } f_{+}^{J'}(t') + \tilde{K}_{JJ'}^2(t, t') \text{Im } f_{-}^{J'}(t') \right\} \\ & + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{\ell=0}^{\infty} \left\{ \tilde{G}_{J\ell}(t, W') \text{Im } f_{\ell+}^I(W') + \tilde{G}_{J\ell}(t, -W') \text{Im } f_{(\ell+1)-}^I(W') \right\}, \end{aligned} \quad (4)$$



**Figure 1.** Left: result for  $\text{Re } f_+^J(t)$ . The solid, dashed, and dot-dashed lines refer to three different variants of the input above the matching point [6]. The black crosses indicate the KH80 results [10]. Right: solution for  $|f_+^J(t)|$ . The different lines correspond to the number of subtractions considered [6].

and similarly for  $f_-^J$  except for the fact that these do not receive contributions from  $f_+^J$ . In addition, only even or odd  $J'$  couple to even or odd  $J$  (corresponding to  $t$ -channel isospin  $I_t = 0$  or  $I_t = 1$ ), respectively, and only higher  $t$ -channel partial waves contribute to lower ones.

Contrary to the  $s$ -channel, below the first inelastic threshold, the  $t$ -channel unitarity relations are linear in  $f_\pm^J$

$$\text{Im } f_\pm^J(t) = \sigma_t^\pi (t_j^I(t))^* f_\pm^J(t), \quad (5)$$

from which one can infer Watson's final-state interaction theorem [8], stating that (in the elastic region) the phase of  $f_\pm^J$  is given by the phase  $\delta_j^I$  of the respective  $\pi\pi$  scattering partial wave  $t_j^I$ .

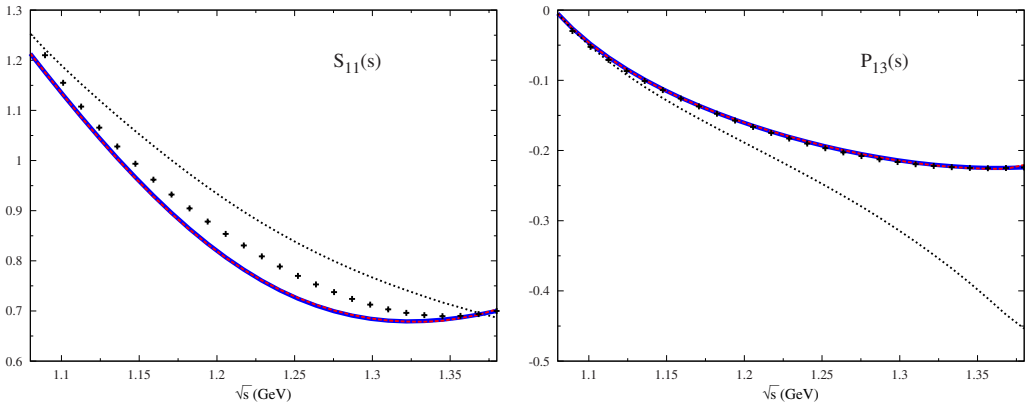
### 3. Solutions of the $t$ -channel and $s$ -channel subproblems

Due to its simpler recoupling scheme, the  $t$ -channel RS subsystem can be recast as a Muskhelishvili–Omnès (MO) problem [9] with a finite matching point [3], where the inhomogeneities subsume the nucleon pole terms, all  $s$ -channel integrals, and the higher  $t$ -channel partial waves [6].

The numerical solutions for the lowest partial waves of this MO problem were studied in Ref. [6] using the KH80 analysis [10] as input. For the  $P$ -waves, a single-channel approximation was considered, whereas for the  $S$ -waves a two-channel description including  $\bar{K}K$  intermediate states was used. As an example, we plot the solution for the real part of  $f_+^J(t)$  and the absolute value of  $f_+^J(t)$  in Fig. 1, which show a nice consistency with the KH80 results.

Once the  $t$ -channel equations are solved, the structure of the  $s$ -channel problem resembles the form of  $\pi\pi$  Roy equations, and should be amenable to similar solution techniques. As a first step, in order to investigate to what extent these equations are fulfilled for the SAID  $s$ -channel amplitudes [11], we compare the left- and right-hand side of Eq. (3). In Fig. 2, we present the results for the  $S_{11}$  and  $P_{13}$  partial waves, which show that the equation are fulfilled in the threshold region, while deviations emerge at higher energies in both waves.

In order to improve these results, we impose the modern  $\pi N$  coupling and  $S$ -wave scattering lengths determinations of Ref. [12], obtained from hadronic-atom data, as constraints, and minimize the difference between the left- and right-hand side of Eq. (3) by fitting the  $s$ -channel phase shifts together with the subtraction constants. The fitted results are also illustrated in Fig. 2, showing that the agreement between left- and right-hand side is now very good. For the  $P_{13}$  wave, this agreement is mainly due to the change of the subthreshold parameters, since the difference between the left-hand side of the RS



**Figure 2.** Left- and right-hand side of the RS equations for the real part of the  $S_{11}$  and  $P_{13}$  partial waves. The black crosses indicate the SAID results [11], i.e. the left-hand side of Eq. (3) before the fit, whereas the solid line corresponds to the left-hand side of Eq. (3) after the fit. The dotted and dashed lines denote the right-hand side of Eq. (3) before and after the fit, respectively.

equations before and after the fit is negligible, while for the  $S_{11}$  wave also the partial wave changes significantly.

The next step of this project will be a self-consistent iteration procedure between the solutions for the  $s$ - and  $t$ -channel, leading to a consistent and precise description of the low-energy  $\pi N$  scattering amplitude.

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## References

- [1] B. Ananthanarayan et al., Phys. Rept. **353**, 207 (2001)
- [2] R. García-Martín et al., Phys. Rev. D **83**, 074004 (2011)
- [3] P. Büttiker et al., Eur. Phys. J. C **33**, 409 (2004)
- [4] M. Hoferichter, D.R. Phillips and C. Schat, Eur. Phys. J. C **71**, 1743 (2011)
- [5] G.E. Hite and F. Steiner, Nuovo Cim. A **18**, 237 (1973)
- [6] C. Ditsche et al., JHEP **1206** (2012) 043; JHEP **1206**, 063 (2012)
- [7] S.M. Roy, Phys. Lett. B **36**, 353 (1971)
- [8] K.M. Watson, Phys. Rev. **95**, 228 (1954)
- [9] N.I. Muskhelishvili, *Singular Integral Equations*, Wolters-Noordhoff Publishing, Groningen, 1953; R. Omnès, Nuovo Cim. **8**, 316 (1958)
- [10] R. Koch and E. Pietarinen, Nucl. Phys. A **336**, 331 (1980)
- [11] R.A. Arndt et al., Eur. Phys. J. A **35**, 311 (2008)
- [12] V. Baru et al., Phys. Lett. B **694**, 473 (2011); Nucl. Phys. A **872**, 69 (2011)