Quasi self-dual exponential Lévy processes *

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Abstract

The important application of semi-static hedging in financial markets naturally leads to the notion of quasi self-dual processes. The focus of our study is to give new characterizations of quasi self-duality for exponential Lévy processes such that the resulting market does not admit arbitrage opportunities. We derive a set of equivalent conditions for the stochastic logarithm of quasi self-dual martingale models and derive a further characterization of these models not depending on the Lévy-Khintchine parametrization. Since for non-vanishing order parameter two martingale properties have to be satisfied simultaneously, there is a non-trivial relation between the order and shift parameter representing carrying costs in financial applications. This leads to an equation containing an integral term which has to be inverted in applications. We first discuss several important properties of this equation and, for some well-known models, we derive a family of closed-form inversion formulae leading to parameterizations of sets of possible combinations in the corresponding parameter spaces of well-known Lévy driven models.

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1 Introduction

The duality principle in option pricing relates different financial products by a certain change of measure. It allows to transform complicated financial derivatives into simpler ones in a suitable dual market. For a comprehensive treatment, see [9, 10] and the literature cited therein.

Sometimes it is even possible to semi-statically hedge path-dependent barrier options with European ones. These are options which only depend on the asset price at maturity. The possibility of this hedge, however, requires a certain symmetry property of the asset price which has to remain invariant under the duality transformation. Non-vanishing carrying costs like interest rates, dividends etc. are handled in the previous literature by quasi self-dual processes which remain invariant under duality after a power transform, see [6] and more recently e.g. [7, 20]. Furthermore, this allows more modelling flexibility since quasi self-duality is a less restrictive requirement on the

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price process compared to classical self-duality. For references to the large literature of the special case of self-duality, often called put-call symmetry, see [7, 11, 14, 28].

Moreover, quasi-self duality in one period shows up in the study of probabilistic representations of the Riemann zeta function, see e.g. equation (11.4) in [19].

The focus of our study is to give new characterizations of quasi self-duality for exponential Lévy processes such that the corresponding market does not admit arbitrage opportunities. Indeed, it is discussed in [20] that ensuring quasi self-duality and the absence of arbitrage simultaneously is a delicate task in many Lévy settings. We derive a set of equivalent conditions for the stochastic logarithm of quasi self-dual martingale models and derive a further characterization of these models not depending on the Lévy-Khintchine parametrization. Furthermore, we complement the characterization of quasi self-dual martingale models obtained in [20] for our slightly different definition, where the order parameter of quasi self-duality is allowed to vanish. Since for non-vanishing order parameter two martingale properties have to be satisfied simultaneously there is a non-trivial relation between the order and the shift parameter representing carrying costs in financial applications. This leads to an equation containing an integral term which has to be inverted in applications. We first discuss several important properties of this equation and for some well-known models, we derive a family of closed-form inversion formulae leading to parameterizations of sets of possible parameter combinations in the corresponding parameter spaces of well-known Lévy driven models. Furthermore, we discuss an example where we do not end up with a unique inversion formula for possible carrying costs after fixing all parameters up to the shift (carrying costs) and the order parameter.

Since hedging portfolios in applications, i.e. the construction of semi-static hedging strategies, substantially depend on the order parameter, this study leads to new explicit semi-static hedging portfolios in markets without arbitrage. In real market applications often the assumption that the asset price follows certain exponential Lévy processes will typically not be completely fulfilled and the possibility of jumps will lead to certain hedging errors. However, several comparative studies, see e.g. [12, 22], have confirmed a relatively good performance of "symmetry based" semi-static hedges, even if the assumptions behind the semi-static hedges are not exactly satisfied.

2 Definitions and applications

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration satisfies the usual conditions with $\mathcal{F}_0$ being trivial apart from P-null sets, and fix a finite but arbitrary time horizon $T > 0$. All stochastic processes are RCLL and defined on $[0, T]$ unless otherwise stated. We understand positive and negative in the strict sense. As far as the definitions are concerned we follow [24].

Definition 1 *Let M be an adapted process. M is **conditionally symmetric** if for any stopping time $\tau \in [0, T]$ and any non-negative Borel function f*

$$E\left[\, f\left(M_T - M_\tau\right) \middle|\, \mathcal{F}_\tau\right] = E\left[\, f\left(M_\tau - M_T\right) \middle|\, \mathcal{F}_\tau\right]. \tag{1}$$

Here it is permissible that both sides of the equation are infinite. If M is an integrable conditionally symmetric process, then Condition (1) implies that M is a martingale by choosing $f(x) = x$ $(= x^+ - x^-)$.

Definition 2 *An adapted positive process S is* ***quasi self-dual*** *of order $\alpha \in \mathbb{R}$ if for any stopping time $\tau \le T$ and any non-negative Borel function f it holds that*

$$E_P\left[f\left(\frac{S_T}{S_\tau}\right) \middle| \mathcal{F}_\tau \right] = E_P\left[\left(\frac{S_T}{S_\tau}\right)^\alpha f\left(\frac{S_\tau}{S_T}\right) \middle| \mathcal{F}_\tau \right]. \tag{2}$$

In particular, for all $\tau \le T$

$$E_P\left[\left(\frac{S_T}{S_\tau}\right)^\alpha \middle| \mathcal{F}_\tau \right] = 1.$$

Note that even for $\alpha = 1$ these definitions differ slightly from the one used in [28] who uses bounded measurable f instead. Moreover, in view of applications we use bounded stopping times instead of deterministic times. However, all corresponding results in [28] applied in this paper can be easily adapted to our corresponding setting as is straightforward to check.

In the case when S is a martingale, we can define a probability measure Q, the so-called *dual measure*, via

$$\frac{dQ}{dP} = \frac{S_T}{S_0}. \tag{3}$$

Similarly, if $E\left[\sqrt{S_T}\right] < \infty$, or $E\left[S_T^w\right] < \infty$ for a $w \in [0,1]$, respectively, we define probability measures H, sometimes called 'half measure', respectively P^w, via

$$\frac{dH}{dP} = \frac{\sqrt{S_T}}{E\left[\sqrt{S_T}\right]}, \quad \frac{dP^w}{dP} = \frac{S_T^w}{E\left[S_T^w\right]}. \tag{4}$$

Note that the integrability of $S_T = S_0 \exp(X_T)$ under P implies the existence of the moment generating function of X_T under H for an open interval including the origin, i.e. X_T has all moments under H.

By Bayes' formula, the self-duality condition (2) for $\alpha = 1$ can be expressed for a martingale S in terms of the dual measure Q defined in (3) as

$$E_P\left[f\left(\frac{S_T}{S_\tau}\right) \middle| \mathcal{F}_\tau \right] = E_Q\left[f\left(\frac{S_\tau}{S_T}\right) \middle| \mathcal{F}_\tau \right]. \tag{5}$$

For the measure H, i.e. $w = \frac{1}{2}$, the following proposition has been stated in slightly different settings in [7, 20] and [28], and also for $w = 1$, i.e. for Q. Similar unconditional multivariate results are given in [21].

Proposition 3 *Let $S = \exp(X)$ be a martingale. Then S is self-dual if and only if for any stopping time $\tau \in [0,T]$ and any non-negative Borel function f*

$$E_{P^w}\left[f\left(X_T - X_\tau\right) \middle| \mathcal{F}_\tau \right] = E_{P^{1-w}}\left[f\left(X_\tau - X_T\right) \middle| \mathcal{F}_\tau \right] \tag{6}$$

holds for at least one (and then necessarily for all) $w \in [0,1]$.

Proof. See [24], Proposition 4; note that this proof does not need that S is continuous. ■

For the half measure we immediately obtain the known special case, which was formulated in slightly different settings in [7, 20, 24, 28].

Corollary 4 *Let $S = \exp(X)$ be a martingale. Then S is self-dual if and only if X is conditionally symmetric with respect to H.*

Note that this corollary shows that the self-duality definition employed here ($\alpha = 1$) is equivalent to the assumption that put-call symmetry at stopping times, as defined in [7], holds for all stopping times $\tau \in [0, T]$.

The following result is sometimes used as definition or as starting point in slightly different settings, see [7], who treat the case of vanishing parameter in concrete models separately, and [20]. The advantage of Definition 2 is that we do not need to exclude $\alpha = 0$ at the starting point or in the definition. Note furthermore, that our terminology also minimally differs from the one in [20] where the part without carrying costs is called to be self-dual and not the asset price process itself.

Proposition 5 *S is quasi self-dual of order $\alpha \neq 0$ if and only if S^α is self-dual.*

Proof. This follows by considering for each f the functions g defined by $g(x) = f(x^\alpha)$, respectively h given by $h(x) = f\left(x^{1/\alpha}\right)$, $x > 0$. ■

For S^α being a martingale for non-vanishing α and such that the discounted asset price process is also a martingale, a machinery of hedging strategies has been derived in [7, Section 6.2] and is concretely discussed in the geometric Brownian motion case, there also for vanishing α as in [6]. Certain extensions to geometric Brownian motion including jumps to zero and to one-dimensional diffusions are derived in [7, Section 7], while the structural results in [24] show that a reasonable calibration of continuous asset price models leading at the same time to quasi self-duality can usually not be expected due to symmetry reasons.

For illustration we repeat here one particular hedging strategy, other ones can be derived from [7] in an analogous way. Consider

$$X = f(S_T)1\!\!1_{\left\{\exists t\in[0,T],\ S_t \leq H\right\}},$$

along with

$$g(S_T) = f(S_T)1\!\!1_{\left\{S_T \leq H\right\}} + \left(\frac{S_T}{H}\right)^\alpha f\left(\frac{H^2}{S_T}\right)1\!\!1_{\left\{S_T < H\right\}}, \tag{7}$$

such that f and g are non-negative, integrable payoff functions ($S_0 > H$, $H > 0$). Furthermore, assume that S is positive quasi self-dual of non-vanishing order α under a chosen risk-neutral measure and that S cannot jump over the barrier H. Then we can hedge the path dependent claim X by the non-vanilla European claim defined by g. The hedge works as follows

- If X never knocks in, then the claim in (7) expires worthless.
- If and when the barrier is hit, we can exchange (7) for a claim on $f(S_T)$ at zero costs, as proved in [7].

If the asset process can jump over the barrier the hedging strategy may no longer exactly replicate the knock-in claim. A criterion for a superreplication in the self-dual case is given in [7, Remark 5.17]; for quasi self-dual cases we refer to [20, Remark 7.4]. In practice the claim $g(S_T)$ could be synthetically approximated by bonds, forwards and lots of vanilla options, cf. [7].

The hedge in (7) and also other hedges derived in [7, Section 6.2] heavily depend on the order parameter α. The usual choice for α is then $\alpha = 1 - 2\lambda/\sigma_{BS}^2$ where λ represents the carrying costs and σ_{BS} corresponds to the implied volatility in the Black–Scholes model. But with this approach, the empirically already quite well performing hedges (we again refer to [12, 22]) are *a priori* Black–Scholes semi-static hedges. In view of that it seems natural to discuss the derivation of semi-static hedging portfolios for exponential Lévy processes in models with non-trivial carrying costs. And exactly this will be the main application of the results presented in the rest of this paper.

Related to the above discussion we should mention that different asset price models can lead to the same hedge portfolios. Indeed, if we assume that there are no-carrying costs then all assets price models of the form $S = S_0\mathcal{E}(M)$, for M being a continuous Ocone martingale, would lead to the hedging portfolio (7) with $\alpha = 1$. This can directly be derived with the help of analogous conditioning arguments as presented in [24]. A well-known example of an Ocone martingale is given by the solution of the stochastic differential equation

$$
\begin{aligned}
dM_t &= V_t\,dB_t,\\
dV_t &= -\mu V_t\,dt + \sqrt{V_t}\,dW_t,
\end{aligned}
$$

where $\mu > 0$ and B, W are two independent Brownian motions, see e.g. [24]. Further examples of Ocone martingales can e.g. be found in [29].

3 Exponential Lévy processes: general results

In this and the following section we consider a process S which is the exponential of a Lévy process X, $X_0 = 0$, characterized by the Lévy-Khintchine formula for the characteristic function $E(e^{iuX_t}) = e^{t\psi(u)}$ for $u \in \mathbb{R}$ with

$$
\psi(u) = \kappa(iu)\,, \quad \kappa(z) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int (e^{zx} - 1 - zxc(x))\nu(dx)\,, \tag{8}
$$

with $c(x) = \mathbb{I}_{|x|\le 1}$, for $z \in \mathbb{C}$, such that $\Re z = c$ satisfies $\int_{|x|>1} e^{cx}\nu(dx) < \infty$ (the latter is the case if and only if $Ee^{cX_t} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$), where ν is the Lévy measure, i.e. $\nu(\{0\}) = 0$ and $\int \min(x^2, 1)\,\nu(dx) < \infty$, see e.g. [26], in particular Theorem 25.17.

There are no problems with strict local martingales here since a local martingale which is a Lévy process is a martingale and since a local martingale of the form e^X, with X being a Lévy process, is a martingale, see e.g. [18, Lemma 4.4]. Moreover, if Y is a Lévy martingale with $\Delta Y > -1$, then $\mathcal{E}(Y)$ is a martingale, see [23, Corollary 7].

Furthermore, as in [20], we can use the Lévy property in order to reduce the analysis of quasi self-duality to the analysis of infinitely divisible distributions.

Proposition 6 *Let L be a Lévy process with $L_0 = 0$.*

(A) *L with $L_0 = 0$ is conditionally symmetric if and only if for any non-negative Borel function f*

$$
E_P[f(L_T)] = E_P[f(-L_T)]\,.
$$

(B) *A process $S = S_0 \exp(\lambda t + L)$ with existing α-moment for all $t \in [0, T]$ (or equivalently for one $t > 0$) is quasi self-dual of order α if and only if for any non-negative Borel function f it holds that*

$$
E_P\left[f\left(\frac{S_T}{S_0}\right)\right] = E_P\left[\left(\frac{S_T}{S_0}\right)^{\alpha} f\left(\frac{S_0}{S_T}\right)\right].
$$

An immediate consequence of (A) is that an integrable symmetric Lévy process is a martingale and in fact also a conditionally symmetric one. Both observations are not true in general cases, where one has to distinguish carefully between different notions of symmetry, see e.g. [24] and the literature cited therein. The fact that an integrable symmetric Lévy process is a martingale is also a direct consequence of [26, Exercise 18.1, Example 25.12] combined with [8, Propositions 3.17].

Proof of Proposition 6. (A) Symmetry of the distribution of L_t is not a time-dependent distributional property of Lévy processes, see e.g. [26, Section 23]. Furthermore, by [5, Chapter 1, Propoposition 6] we have that for a stopping time $\tau \in [0,T]$ the process $\tilde{L}$ defined by $\tilde{L}_s = L_{\tau+s} - L_\tau$ is a copy of L independent of $\mathcal{F}_\tau$ with $\tilde{L}_{T-\tau} = L_T - L_\tau$. Hence,

$$E[f(L_T - L_\tau)|\mathcal{F}_\tau] = E[f(\tilde{L}_{T-\tau})|\mathcal{F}_\tau] = E[f(-\tilde{L}_{T-\tau})|\mathcal{F}_\tau] = E[f(L_\tau - L_T)|\mathcal{F}_\tau]\,.$$

(B) For non-vanishing α we can e.g. directly combine the well-known results about Lévy processes in [26, Theorems 7.10, 11.5, Corollary 8.3,] with the result presented in [20, Theorem 5.3] in order to see that (B) for $T > 0$ already implies (B) for all other t. Furthermore, as a consequence of (A), the same is true for vanishing α. The rest of the proof uses the above argument. ■

In the light of Proposition 6 for $\alpha = 1$, i.e. in the put-call symmetry or self-dual case, the well-known result in [13] reads as follows, see also [7] for the conditional statement.

Theorem 7 ([13, 7]) *Let $S = \exp(X)$ be a martingale for a Lévy process X with triplet $\left(\gamma, \sigma^2, \nu\right)$ where ν is the Lévy measure. Then S is self-dual iff*

$$\nu\,(dx) = e^{-x}\,\nu\,(-dx)\,. \tag{9}$$

Remark 8 *We stress that for self-duality the martingale assumption is important in the above statement, since for integrable $S = \exp(X)$ (9) does not imply self-duality. However, an integrable S is self-dual if and only if (9) holds along with the "drift" restriction which forces an integrable S to be a martingale, see [20].*

In view of Proposition 5 and Remark 8 it is obvious that in the integrable quasi self-dual case of order $\alpha \neq 0$, S^α needs to be a martingale due to symmetry reasons where $e^{-\lambda t}S$ is assumed to be a martingale in order to obtain a risk-neutral setting. Hence, we have only one "drift" which has to satisfy two (different) martingale assumptions simultaneously, one for symmetry reasons and the other one due to risk-neutrality. Roughly speaking we can say that this intuitively describes the origin of (12) in Theorem 11. Note that these two conditions coincide in the self-dual case with vanishing carrying costs.

The Conditions (i), (ii)′, and (iii)′ as well as, (a) and (b), respectively, will be unified in Theorem 11 to the Conditions (i), (ii), and (iii).

Proposition 9 *Let $S = e^{\lambda t}\exp(X)$ for $\lambda \in \mathbb{R}$ and a Lévy-process X such that S_t and $(S_t)^\alpha$ are integrable for some $t > 0$. Then S is quasi self-dual of order α with $\exp(X)$ being a martingale if and only if the following conditions hold.*

For $\alpha \neq 0$:

(i) *The Lévy measure satisfies*

$$\nu\,(dx) = e^{-\alpha x}\,\nu\,(-dx)\;, \tag{10}$$

i.e. $\nu(B) = \int_{-B} e^{\alpha x}\,d\nu(x)$ for a Borel set B in $\mathbb{R}\setminus\{0\}$.

(ii)′ *The process* S^α *is a martingale.*

(iii)′ *The process* $\exp(X)$ *is a martingale.*

For $\alpha = 0$*:*

(a) *The process* $X = \log(S)$ *is a (conditionally) symmetric Lévy process.*

(b) *The process* $S = \exp(X)$ *is a martingale.*

Remark 10 *Hence, if for* $\alpha \neq 0$ S^α *and* $\exp(X)$ *are martingales (then the imposed integrability assumptions are automatically satisfied) and if* (i) *holds, then we are in a risk-neutral setting where the quasi self-duality property* (2) *holds.*

Note that as a consequence of [26, Proposition 11.10], the Lévy triplet $(\tilde{\gamma}, \tilde{\sigma}^2, \tilde{\nu})$ *of* $Z = \alpha\lambda t + \alpha X$*,* $\alpha \neq 0$*, in terms of the triplet* (γ, σ^2, ν) *of* X *is given by*

$$(\tilde{\gamma}, \alpha^2\sigma^2, (\nu\alpha^{-1}))\,, \quad \text{where} \quad \tilde{\gamma} = \alpha((\lambda+\gamma) + \int x(1\!\!1_{|\alpha x|\leq 1} - 1\!\!1_{|x|\leq 1})\nu(dx))$$

and $(\nu\alpha^{-1})(B) = \nu(\{x \in \mathbb{R} : \alpha x \in B\})$*.*

Proof of Proposition 9. For $\alpha \neq 0$ we only need to prove the equivalence of the Definition 2 with (i) and (ii)′, since e^X is a martingale in either case. If S is quasi self-dual of order α then we can apply Definition 2 for f being identically one and $\tau = 0$ in order to see that given the imposed integrability assumptions $E[e^{Z_T}] = 1$, i.e. S^α is a martingale, since Z is a Lévy process, see [8, Proposition 3.17]. Furthermore, Proposition 5 then implies that e^Z is self-dual so that by Theorem 7 the Lévy measure $\tilde{\nu}$ satisfies (9). Hence, for any Borel set $B \subset \mathbb{R} \setminus \{0\}$

$$\begin{aligned}\int_B \nu(dx) &= \int 1\!\!1_B(\alpha^{-1}y)\tilde{\nu}(dy) \\ &= \int 1\!\!1_B(\alpha^{-1}y)e^{-y}\tilde{\nu}(-dy) = \int 1\!\!1_B(-\alpha^{-1}y)e^{y}\tilde{\nu}(dy) = \int_{-B} e^{\alpha x}\nu(dx)\,,\end{aligned}$$

i.e. (10) holds so that (i) and (ii)′ hold.

Conversely, given (ii)′ from Proposition 9 we have for every Borel set $B \subset \mathbb{R} \setminus \{0\}$

$$\int_B \tilde{\nu}(dy) = \int 1\!\!1_B(\alpha x)\nu(dx) = \int 1\!\!1_B(\alpha x)e^{-\alpha x}\nu(-dx) = \int 1\!\!1_B(-\alpha x)e^{\alpha x}\nu(dx) = \int_{-B} e^y\tilde{\nu}(dy)\,,$$

so that (9) follows. Since furthermore e^Z is now assumed to be a martingale, we obtain its self-duality property by Theorem 7. By Proposition 5 we end up with the quasi self-duality of S.

For $\alpha = 0$ we can use again the strong Markov as well as the independent and stationarity property of the increments of Lévy processes in order to see that (2) is equivalent to the property that $Z_t = (\lambda t + X_t)$ has an even distribution for some, or equivalently, for all $t \in (0, T]$ (recall that symmetry is not a time-dependent distributional property of a Lévy process) since the condition is reflected in the Lévy triplet characterizing the distribution of the whole process. ■

For the case $\alpha \neq 0$ the following theorem is a summary of various results presented in [20], adapted to the present setting, with an alternative proof based on Proposition 9. In order to exclude any confusion related to carrying costs let us stress that consistently with (8) the meaning of the Lévy triplet of a Lévy process X is that $P_{X_1} = \mu$ for an infinitely divisible distribution μ and $P_{X_t} = \mu^t$, where the latter infinitely divisible *distributions* have generating triplet $(t\gamma, tA, t\nu)$ (we assume w.l.o.g. that the process is defined on an interval including 1).

Theorem 11 *Let $S = e^{\lambda t}\exp(X)$ for $\lambda \in \mathbb{R}$ and a Lévy-process X with triplet $\left(\gamma, \sigma^2, \nu\right)$, such that S_t and $(S_t)^\alpha$ are integrable for some $t > 0$. Then S is quasi self-dual of order α with $\exp(X)$ being a martingale if and only if the following conditions hold.*

(i) *The Lévy measure satisfies Condition* (i) *from Proposition* 9*.*

(ii) *The entries of the triplet satisfy*

$$\gamma = \int_{|x|\leq 1} x\left(1 - e^{\frac{1}{2}\alpha x}\right)\nu\left(dx\right) - \frac{1}{2}\alpha\sigma^2 - \lambda. \tag{11}$$

(iii) *The parameters λ and α are related by*

$$\lambda = (1-\alpha)\frac{\sigma^2}{2} + \int\left(e^x - xe^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\leq 1} - 1\right)\nu\left(dx\right). \tag{12}$$

Proof. Given the Conditions (i), (ii), and (iii) and the imposed integrability assumptions we can substitute (12) in (11) in order to obtain

$$\gamma = -\frac{\sigma^2}{2} + \int\left(x\mathbb{1}_{|x|\leq 1} + 1 - e^x\right)\nu\left(dx\right), \tag{13}$$

so that in view of the imposed integrability assumptions (iii)$'$ follows, see e.g. [8, Proposition 3.18]. Furthermore, by multiplying (11) with $\alpha \neq 0$, adding $\alpha\int x(\mathbb{1}_{|\alpha x|\leq 1} - \mathbb{1}_{|x|\leq 1})\nu(dx)$, Condition (ii) can be written as

$$\alpha(\gamma + \lambda + \int x(\mathbb{1}_{|\alpha x|\leq 1} - \mathbb{1}_{|x|\leq 1})\nu(dx)) = -\frac{1}{2}(\alpha\sigma)^2 - \alpha\int_{\mathbb{R}} x(e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\leq 1} - \mathbb{1}_{|\alpha x|\leq 1})\nu(dx) = \tilde{\gamma}\,. \tag{14}$$

On the other hand, by substituting, we can rewrite the integral expression in the (given the integrability) martingale condition of S^α, i.e.

$$\tilde{\gamma} = -\frac{1}{2}(\alpha\sigma)^2 + \int(x\mathbb{1}_{|x|\leq 1} + 1 - e^x)(\nu\alpha^{-1})(dx)\,, \tag{15}$$

as

$$\int(e^y - 1 - y\mathbb{1}_{|y|\leq 1})(\nu\alpha^{-1})(dy) = \int(e^{\alpha x} - 1 - \alpha x\mathbb{1}_{|\alpha x|\leq 1})\,\nu(dx)\,.$$

Hence, (14) coincides with (15) since $\int(e^{\alpha x} - 1 - \alpha xe^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\leq 1})\nu(dx)$ vanishes. The latter is a consequence of (i), concretely

$$
\begin{aligned}
&\int (e^{\alpha x}-1-\alpha x e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1})\nu(dx) \\
&\quad = \int_{-\infty}^{0} (e^{\alpha x}-1-\alpha x e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1}) e^{-\alpha x}\nu(-dx) + \int_{0}^{\infty} (e^{\alpha x}-1-\alpha x e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1})\nu(dx) \\
&\quad = \int_{0}^{\infty} (1-e^{\alpha x}+\alpha x e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1})\,\nu(dx) \\
&\qquad + \int_{0}^{\infty} (e^{\alpha x}-1-\alpha x e^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1})\nu(dx) = 0\,. \qquad (16)
\end{aligned}
$$

Conversely, if we start with (i), (ii)$'$, and (iii)$'$ we can use (i) to see that (16) holds again so that (ii)$'$ can be rewritten as (ii) (respectively, (15) as (11)) by the converse calculations from above (for $\alpha \neq 0$). By equating the martingale condition $\gamma = -\frac{\sigma^2}{2} + \int \left(x\mathbb{1}_{|x|\le 1} + 1 - e^{x}\right)\,\nu\,(dx)$ with (11) we arrive at (12), i.e. Condition (iii) is implied. Essentially the same proof results if we use that the martingale condition of $e^{\alpha(\lambda t + X)}$ is equivalent to $E[e^{\alpha(\lambda t + X_t)}] = 1$ and then apply [26, Theorem 27.15] instead of [26, Proposition 11.10].

For $\alpha = 0$ recall that we can use the strong Markov as well as the independent and stationarity property of the increments of Lévy processes in order to see that (2) is equivalent to the property that $Z_t = (\lambda t + X_t)$ has an even distribution for some, or equivalently, for all $t \in (0,T]$ (where symmetry is not a time-dependent distributional property of a Lévy process), since the latter is the case if and only if the triplet $(\gamma+\lambda, \sigma^2, \nu)$ of Z satisfies that $\gamma+\lambda$ vanishes and ν is even, see again [26, Exercise 18.1], i.e. if and only if (i) and (ii) are satisfied for $\alpha = 0$. In view of the existence of the first exponential moment we have that the martingale property of $\exp(X)$ is equivalent to (13). Hence, quasi self-duality of order $\alpha = 0$ implies (i) and (ii) for vanishing α and since furthermore $\exp(X)$ is a martingale we can plug (ii) for vanishing α in (13) in order to end up with (iii) for vanishing α. Conversely, (i) and (ii) for vanishing α imply quasi self-duality of order $\alpha = 0$ and by plugging (ii) in (iii) for vanishing α we end up with (13) implying the martingale property of $\exp(X)$, given the assumed integrability of $\exp(X)$. ■

Remark 12 *Both formulations, the one from Proposition* 9 *and the one from Theorem* 11 *have their merits. An advantage of the first formulation is that this statement does not depend anymore on the choice of the function* $c(\cdot)$ *in* ψ*, while* (12) *will be particularly useful for uniqueness discussions of the parameter* α*. Furthermore, the formulation in Theorem* 11 *needs no distinction of the case of vanishing* α*. For the concrete derivation of* α*, both characterizations can be helpful.*

Now we discuss an analogue of Theorem 11 for the case when the price process is represented as a stochastic rather than an ordinary exponential. As preparation we recall the following well-known result which can be found in [17], Theorem II.8.10.

Proposition 13 ***Equivalence between two different representations of exponential Lévy processes.*** *Let* X*,* Y *be two Lévy processes such that* $X_0 = 0$ *and* $\Delta Y > -1$*. Then we have*

$\exp(X) = \mathcal{E}\,(Y)$ *if and only if*

$$X = Y - Y_0 - \frac{1}{2}\,[Y^c] + (\log{(1+y)} - y) * \mu^Y, \tag{17}$$
$$Y = Y_0 + X + \frac{1}{2}\,[X^c] + (e^x - 1 - x) * \mu^X,$$
$$x * \mu^X = \log{(1+y)} * \mu^Y, \qquad y * \mu^Y = (e^x - 1) * \mu^X.$$

Here μ^X, μ^Y *are the jump measures of* X*, respectively* Y*.*

In the following we denote by $\left(\gamma^X, \sigma^2, \nu^X\right)$, $\left(\gamma^Y, \sigma^2, \nu^Y\right)$ the triplets of X, respectively Y and we again put $X_0 = 0$, $Y_0 = 0$. The relations between the Lévy triplets are e.g. given in [4, Corollary 4.1]. In the sequel we apply the self-inverse function $\chi : (-1,\infty) \to (-1,\infty)$, defined by

$$\chi(y) = -\frac{y}{1+y}\,.$$

Note that by setting $\alpha = 1$ and $\lambda = 0$, we obtain an analogon to the characterization of self-dual continuous processes obtained by Tehranchi in [28, Theorem 3.1]. However, the characterization in terms of the conditional symmetry of the stochastic logarithm does not hold in this Lévy setting, as in particular we here have to invoke the (restricted) Möbius transform χ.

Theorem 14 *In the setting of Theorem* 11*, for* $\alpha \neq 0$ *let* Y, $\tilde{Y}$ *be Lévy processes with* $\Delta Y, \Delta\tilde{Y} > -1$ *related to* X *and* $Z = \alpha(\lambda t + X)$ *by* (17)*, respectively (i.e.* $\mathcal{E}(Y) = \exp(X)$ *and* $\mathcal{E}(\tilde{Y}) = S^\alpha$*). Then* S *is quasi self-dual of order* α *with* $\mathcal{E}\,(Y)$ *being a martingale if and only if the following conditions hold.*

(a) *For all Borel sets* B *the Lévy measure satisfies*

$$\nu^Y(B) = \int_{\chi(B)} (1+y)^\alpha \nu^Y(dy)\,,$$

(b) *The process* $\tilde{Y}$ *is a martingale.*

(c) *The process* Y *is a martingale.*

Note that Condition (a) can equivalently be written as

$$\nu^Y(B) = \int_B (1+y)^{-\alpha} \left(\nu^Y \chi^{-1}\right)(dy)\,. \tag{18}$$

Proof. Since $\mathcal{E}(Y) = \exp(X)$ we have to show the equivalence of Conditions (a), (b), and (c) with the Condition (i) from Theorem 11 and the Conditions (ii)$'$ as well as (iii)$'$ from Proposition 9.

If (10) holds then

$$\begin{aligned}
\nu^Y(B) &= \int \mathbb{1}_B(y)\nu^Y(dy) = \int \mathbb{1}_B(e^x - 1)\nu^X(dx) \\
&= \int \mathbb{1}_B(e^x - 1)e^{-\alpha x}\nu^X(-dx) = \int \mathbb{1}_B(e^{-x} - 1)e^{\alpha x}\nu^X(dx) \\
&= \int \mathbb{1}_B\left(\frac{-y}{1+y}\right)(1+y)^\alpha \nu^Y(dy) = \int_{\chi(B)} (1+y)^\alpha \nu^Y(dy)\,,
\end{aligned}$$

where on the r.h.s. of the last equation we can equivalently write $\int_B (1+y)^{-\alpha}\left(\nu^Y \chi^{-1}\right)(dy)$, i.e. Condition (a) follows. If (a) holds then

$$\begin{aligned}\nu^X(B) &= \int \mathbb{I}_B(x)\nu^X(dx) = \int \mathbb{I}_B(\log(1+y))\nu^Y(dy)\\ &= \int \mathbb{I}_B(\log(1+y))(1+y)^{-\alpha}\left(\nu^Y\chi^{-1}\right)(dy) = \int \mathbb{I}_B\left(\log\left(\frac{1}{1+y}\right)\right)(1+y)^{\alpha}\nu^Y(dy)\\ &= \int \mathbb{I}_B(-x)\,e^{\alpha x}\nu^X(dx) = \int_{-B} e^{\alpha x}\nu^X(dx)\ ,\end{aligned}$$

i.e. we end up with (10).

The remaining two equivalences are a consequence of the Lévy property, since the stochastic exponential $\mathcal{E}(Z)$ for a Lévy process Z is a (local) martingale iff so is Z. ■

Remark 15 *If we want to include the special case of vanishing α in the context of Theorem* 14 *then we can rewrite the conditions of Theorem* 11*. As in the proof of Theorem* 14 *we then end up with Condition* (a) *for the Lévy measure. The other two conditions are*

(b′) $\gamma^Y = \frac{1}{2}\sigma^2(1-\alpha) - \lambda + \int (y\mathbb{I}_{|y|\le 1} - \log(1+y)(1+y)^{\frac{1}{2}\alpha}\mathbb{I}_{|\log(1+y)|\le 1})\nu^Y(dy)$,

(c′) $\lambda = \frac{1}{2}\sigma^2(1-\alpha) + \int (y - \log(1+y)(1+y)^{\frac{1}{2}\alpha}\mathbb{I}_{|\log(1+y)|\le 1})\nu^Y(dy)$.

Condition (b′) *is obtained by changing variables and applying* [4, *Corollary* 4.1] *while* (c′) *is simply obtained by changing variables. Note that given that Y is integrable plugging in* (c′) *into* (b′) *yields the martingale condition for Y.*

The next result excludes certain combinations of parameters.

Proposition 16 *Assume that* $\exp(X)$ *is a martingale,* $S = e^{\lambda t}\exp(X)$ *has finite α-moments, and either*

- $\lambda > 0$ *along with* $\alpha > 1$ *,*
- $\lambda < 0$ *along with* $\alpha < 1$*,* $\alpha \neq 0$*,*

Then S cannot be quasi self-dual of order α.

Note that the omitted case $\alpha = 1$ corresponds to the self-dual case where λ needs to vanish (related to a self-dual Lévy measure) and $\alpha = 0$ where $\gamma = -\lambda = -\frac{1}{2}\sigma^2 - \int (e^x - 1 - x\mathbb{I}_{|x|\le 1})\nu_0(dx)$ is needed in order to end up with the corresponding quasi self-duality along with risk neutrality.
Proof. For $\alpha > 1$, $f : (0,\infty) \to (0,\infty)$, $f(x) = x^\alpha$ is strictly convex increasing so that by the Jensen inequality, the martingale property of $\exp(X)$ and the assumption $\lambda > 0$ we obtain for $t > s$ that

$$E\big[\left(e^{\lambda t + X_t}\right)^\alpha | \mathcal{F}_s\big] \ge \left(E\big[e^{\lambda t + X_t}|\mathcal{F}_s\big]\right)^\alpha = \left(e^{\lambda t}e^{X_s}\right)^\alpha > \left(e^{\lambda s + X_s}\right)^\alpha\ ,$$

i.e. S^α cannot be a martingale so that Condition (ii)′ of Proposition 9 does not hold.

For $\alpha \in (0,1)$ the function $-f$ is strictly convex and increasing so that the Jensen inequality, the martingale property of $\exp(X)$ and $\lambda < 0$ imply that S^α is a supermartingale, but not a martingale, so that the quasi self-duality of order α can again not hold.

The last case follows again by the fact that for $\alpha < 0$ the function f is strictly convex but now decreasing. ■

In view of Proposition 16 we now turn our attention to the integral in Equation (12), and discuss first an important monotonicity property.

Proposition 17 *Assume that the Lévy measure can be written in the form*

$$\nu(dx) = e^{-\frac{1}{2}\alpha x}\nu_0(dx)\,, \tag{19}$$

with an even measure ν_0, $\nu_0(dx) = \nu_0(-dx)$, being independent of α. Furthermore, take $\alpha_1 < \alpha_2$ and assume that $E\left[S_t^{\alpha_i}\right] < \infty$ for $\alpha_i = \alpha_1\,, \alpha_2\,, 1$, and some $t > 0$. Then the integral in (12) *satisfies*

$$\int \left(e^x - xe^{\frac{\alpha_1}{2}x}\mathbb{1}_{|x|\leq 1} - 1\right)\,\nu\,(dx) \geq \int \left(e^x - xe^{\frac{\alpha_2}{2}x}\mathbb{1}_{|x|\leq 1} - 1\right)\,\nu\,(dx)\ . \tag{20}$$

If furthermore the measure ν_0 satisfies that $\nu_0(B) > 0$ for some Borel set in $\mathbb{R}\setminus\{0\}$ then the inequality in (20) *is strict.*

As a consequence of Hölder's inequality the set of existing α-moments is convex, see [26, Theorem 25.17].

Proof. Assume that $\alpha_1 < \alpha_2$. By assumption the integrals in (20) can be written as

$$\int \left(e^{(1-\frac{1}{2}\alpha_i)x} - x\mathbb{1}_{|x|\leq 1} - e^{-\frac{\alpha_i}{2}x}\right)\,\nu_0\,(dx)\ ,\quad i = 1, 2\,.$$

If we consider for every *fixed* $x \in \mathbb{R}\setminus\{0\}$ the integrand as a function of α, i.e.

$$g_x(\alpha) = e^{(1-\frac{1}{2}\alpha)x} - x\mathbb{1}_{|x|\leq 1} - e^{-\frac{1}{2}\alpha x}\,,$$

then we obtain a family of differentiable functions with

$$g_x'(\alpha) = \frac{1}{2}xe^{-\frac{1}{2}\alpha x}(1-e^x)\,.$$

Since for every $x > 0$ and any $\alpha \in \mathbb{R}$ (note that we only consider the integrand, the integral does not need to converge for any $\alpha \in \mathbb{R}$) $\frac{1}{2}xe^{-\frac{1}{2}\alpha x} > 0$ while $1 - e^x < 0$, and because for every $x < 0$ and any $\alpha \in \mathbb{R}$ we have that $\frac{1}{2}xe^{-\frac{1}{2}\alpha x} < 0$ along with $1-e^x > 0$, we end up with $g_x'(\alpha) < 0$, i.e. the functions $g_x(\alpha)$ are strictly monotonically decreasing. Hence, for every $x \in \mathbb{R}\setminus\{0\}$ we have that

$$e^{(1-\frac{1}{2}\alpha_1)x} - x\mathbb{1}_{|x|\leq 1} - e^{-\frac{\alpha_1}{2}x} > e^{(1-\frac{1}{2}\alpha_2)x} - x\mathbb{1}_{|x|\leq 1} - e^{-\frac{\alpha_2}{2}x}\,.$$

Since the integral is order preserving we end up with the claim. ■

We will see in Section 4.3 that the monotonicity (20) does not always hold. However, as an immediate consequence of Proposition 17, the monotonicity holds for the well-known family of Generalized Hyperbolic (GH) models and also for the CGMY model.

Remark 18 (Monotonicity in Generalized Hyperbolic models) *The infinitely divisible Generalized Hyperbolic distribution was introduced by Barndorff-Nielsen in [1]. The corresponding processes became quite popular in the recent financial literature, see e.g. [4, 25] and the literature cited* *therein. GH processes have no centered Gaussian term and their Lévy measures have a density given by*

$$\nu(x) = \frac{e^{bx}}{|x|}\left(\int_0^\infty \frac{\exp(-|x|\sqrt{2y+a^2})}{\pi^2 y(J^2_{|v|}(d\sqrt{2y}) + N^2_{|v|}(d\sqrt{2y}))}\,dy + \max(0,v)e^{-a|x|}\right), \tag{21}$$

where the functions J_v *and* N_v *are the Bessel functions of the first and second kind, see e.g.* [25, Appendix A], *and where we consider the parameters restricted to* $d > 0$, $a > \frac{1}{2}$, $-a < b < a-1$, *in order to ensure that the first exponential moment always exists. If we rewrite the density by* $b = -\frac{1}{2}\alpha$ *for* $\alpha \in (-2(a-1), 2a)$, *then the Lévy measure is of the form* (19) *where the* α*th exponential moments also exist. Hence, with the imposed parameter restrictions and with respect to the Lévy measure with density given in* (21) *with* $b = -\frac{1}{2}\alpha$, *the integral in* (12) *strictly decreases as a function of* $\alpha \in (-2(a-1), 2a)$.

Remark 19 (Monotonicity in the CGMY model) *In the classical four-parameter CGMY model we again have no centered Gaussian term and the Lévy measure has a density given by*

$$\nu(x) = \frac{C}{|x|^{1+Y}}e^{-G|x|}\mathbb{1}_{x<0} + \frac{C}{|x|^{1+Y}}e^{-M|x|}\mathbb{1}_{x>0}\,, \tag{22}$$

where we consider the parameters $C > 0$, $G > 0$, $M > 1$, $Y < 2$ *in order to ensure that the first exponential moments exist. For* $Y = 0$ *we obtain VG processes, which are considered in detail in Section 4.2. For negative* Y *we obtain compound Poisson models, for* $0 < Y < 1$ *"infinite activity" models (i.e. on any time interval* $(0, t]$, $t > 0$, *the process* X_t, $t \geq 0$, *has, with probability one, an infinite number of jumps) exhibiting trajectories of finite variation. If* $Y \geq 1$ *the variation is infinite. When* Y *is close to 2 then the process behaves much like a Brownian motion, for more details we refer e.g. to* [8, Section 4.5].

By choosing

$$M = \beta + \frac{1}{2}\alpha > 1\,, \quad \text{and} \quad G = \beta - \frac{1}{2}\alpha > 0\,, \tag{23}$$

we ensure that the Lévy measure is given by

$$\nu(x) = e^{-\frac{1}{2}\alpha x}\nu_0(x)\,, \quad \text{with} \quad \nu_0(x) = \frac{Ce^{-\beta|x|}}{|x|^{1+Y}}\mathbb{1}_{x\neq 0}\,,$$

i.e. we have that Condition (19) *is satisfied so that the integral expression in* (12), *respectively, in* [20, Remark 5.8], *is strictly monotonically decreasing in* $\alpha \in (-2(\beta-1), 2\beta)$, *where* $\beta > \frac{1}{2}$ *is needed in order to ensure that this interval is not empty. The integral expression in* [20, Remark 5.8] *is related to the choice of* $c(x) = 1$ *in* ψ *being possible and quite popular in the CGMY model. The proof of the monotonicity property in this case is easily obtained by changing* $c(x) = \mathbb{1}_{|x|\leq 1}$ *to* $c(x) = 1$ *in the proof of Proposition* 17.

In order to efficiently derive a suitable α for given carrying costs λ, the following result is often useful. We stress that in the following proposition, $\psi.$, $\kappa.$ are defined as in (8) with $c(x) = \mathbb{1}_{|x|\leq 1}$.

Proposition 20 *Let* $S = e^{\lambda t}\exp(X)$ *for* $\lambda \in \mathbb{R}$ *and a Lévy-process* X *with triplet* (γ, σ^2, ν) *(with corresponding* ψ*), such that* S_t *and* $(S_t)^\alpha$ *are integrable for some* $t > 0$. *Furthermore, assume that the Lévy measure is of the form*

$$\nu(dx) = e^{-\frac{1}{2}\alpha x}\nu_0^\alpha(dx) \tag{24}$$

with $\nu_0^\alpha(dx) = \nu_0^\alpha(-dx)$, i.e. even but possibly depending on α. Then Condition (12) *can be written as*

$$\lambda = (1-\alpha)\frac{\sigma^2}{2} + \int \left(e^x - xe^{\frac{1}{2}\alpha x}1\!\!1_{|x|\le 1} - 1\right)\nu(dx)$$
$$= \psi_0^{(\alpha)}\big(-i(1-\frac{1}{2}\alpha)\big) - \psi_0^{(\alpha)}\big(i\frac{1}{2}\alpha\big) = \kappa_0^{(\alpha)}\big(1-\frac{1}{2}\alpha\big) - \kappa_0^{(\alpha)}\big(-\frac{1}{2}\alpha\big) \quad (25)$$

where ψ_0, κ_0, correspond to the triplet $\left(0, \sigma^2, \nu_0^\alpha\right)$.

Hence, with the above observation the task of finding α for given λ is similar to the task of finding the parameter for the Esscher martingale transform for exponential processes, see [16]. Note however, that $\psi_0^{(\alpha)}$, $\kappa_0^{(\alpha)}$ may depend on α, which can be a source of non-uniqueness of α in inverting (25). For some restricted special cases where the Lévy process has vanishing centered Gaussian part and the Lévy measure ν_0^α has finite Laplace transform on the real line, a corresponding result in terms of the Laplace transform of the Lévy measure has been derived in [20, Remark 5.8]
Proof. By [26, Theorem 27.15], the imposed integrability assumptions, Condition (24), and the assumption that ν_0^α is even combined with a substitution, we have that

$$\int_{|x|>1} e^{\alpha x}\nu(dx) = \int_{|x|>1} e^{\frac{1}{2}\alpha x}\nu_0^\alpha(dx) = \int_{|x|>1} e^{-\frac{1}{2}\alpha x}\nu_0^\alpha(dx) < \infty\,,$$
$$\int_{|x|>1} e^{x}\nu(dx) = \int_{|x|>1} e^{(1-\frac{1}{2}\alpha)x}\nu_0^\alpha(dx) < \infty\,,$$

so that, again by [26, Theorem 27.15], $\psi_0\big(-i(1-\frac{1}{2}\alpha)\big) - \psi_0\big(i\frac{1}{2}\alpha\big)$ is definable (and given by the above triplets). It follows that

$$\psi_0\big(-i(1-\frac{1}{2}\alpha)\big) - \psi_0\big(i\frac{1}{2}\alpha\big) = \kappa_0\big(1-\frac{1}{2}\alpha\big) - \kappa_0\big(-\frac{1}{2}\alpha\big)$$
$$= (1-\alpha)\frac{1}{2}\sigma^2 + \int (e^{(1-\frac{1}{2}\alpha)x} - e^{-\frac{1}{2}\alpha x} - x1\!\!1_{|x|\le 1})\nu_0^\alpha(dx)$$
$$= (1-\alpha)\frac{\sigma^2}{2} + \int \left(e^x - xe^{\frac{1}{2}\alpha x}1\!\!1_{|x|\le 1} - 1\right)e^{-\frac{1}{2}\alpha x}\nu_0^\alpha(dx)\,,$$

so that by (24) we arrive at (25). ■

4 Exponential Lévy processes: specific models

4.1 Quasi self-dual Normal Inverse Gaussian models

Define $S = e^{\lambda t + X}$, $\lambda \in \mathbb{R}$, with X a Lévy process with characteristic function

$$\varphi_{X_t}(u) = \exp\left(t\big(ium + d\big(\sqrt{a^2-b^2} - \sqrt{a^2-(b+iu)^2}\big)\big)\right), \quad (26)$$

with $a > 0$, $-a < b < a$, $d > 0$, $m \in \mathbb{R}$, as e.g. in [25, Sections 5.3.8, 5.4]. Note that here we do not use standard Greek letters for the parameters since we intend to reparameterize the model in order to obtain the quasi self-dual parameter as a model parameter. The process X, introduced by Barndorff-Nielsen in [2, 3], and frequently used in the financial literature, see e.g. [4, 8, 25] and the literature cited therein, is called Normal Inverse Gaussian (NIG) process. It can be constructed by time changing a Brownian motion with drift, is an "infinite activity" process, and the trajectories of a NIG-process are of unbounded variation. Furthermore, for the above parameter restrictions the NIG distribution, i.e. the distribution of X_1, has semi-heavy tails. The NIG-processes belong to the class of the Generalized Hyperbolic Lévy processes. However, unlike several other processes in this class, the NIG-processes (along with the Variance Gamma processes, which will be analyzed in the next section) exhibit the important property that for any $t \neq 1$ the distribution of X_t is of the same type as the distribution of X_1. This makes the NIG processes preferable when one works with empirical data, see [4].

While the centered Gaussian term vanishes, the Lévy measure of NIG-processes and the γ is given by

$$\nu(dx) = \frac{da}{\pi}\frac{e^{bx}}{|x|}K_1(a|x|)dx\,, \quad \gamma = m + \frac{2da}{\pi}\int_0^1 \sinh(bx)K_1(ax)dx\,, \tag{27}$$

see e.g. [3], [25, Sections 5.3.8, 5.4], where K_1 denotes the modified Bessel function of the third kind with index 1, see e.g. [25, p. 148].

By further restricting the parameter range for b to the interval $(-a, a-1)$, cf. e.g. [4], where at the same time we assume that $a > 1/2$ in order to avoid that this interval is empty, we obtain an existing first exponential moment (for one or equivalently for all $t > 0$, being equivalent to $\int_{|x|>1} e^x\nu(dx) < \infty$, see e.g. [26, Theorem 25.17]).

If we rewrite the parameter $b = -\frac{1}{2}\alpha$ then (27) reads

$$\nu(dx) = e^{-\frac{1}{2}\alpha x}\nu_0(x)dx\,, \quad \text{with} \quad \nu_0(x) = \frac{da}{\pi|x|}K_1(a|x|)\,, \tag{28}$$

where $\alpha \in (-2(a-1), 2a)$. Since ν_0 is an even function, ν is of the form (24) so that it easily follows that ν satisfies (19).

Proposition 21 *Assume that $S = e^{t\lambda+X}$, $\lambda \in \mathbb{R}$, where X is characterized by* (26) *with $b = -\frac{1}{2}\alpha$, $a > \frac{1}{2}$, $\alpha \in (-2(a-1), 2a)$, $d > 0$. Then*

(i) *For α such that*

$$\lambda = -d\Big(\sqrt{a^2 - \frac{1}{4}(2-\alpha)^2} - \sqrt{a^2 - \frac{1}{4}\alpha^2}\Big)\,,$$

and by subsequently choosing γ (via $m = -\lambda$) as in (11)*, the asset price model is quasi self-dual of order α with respect to λ and e^X is a martingale.*

(ii) *The functions*

$$f_{a,d} : (-2(a-1), 2a) \to (-d\sqrt{2a-1}, d\sqrt{2a-1})$$

defined by

$$f_{a,d}(\alpha) = -d\Big(\sqrt{a^2 - \frac{1}{4}(2-\alpha)^2} - \sqrt{a^2 - \frac{1}{4}\alpha^2}\Big) \tag{29}$$

are vanishing if and only if $\alpha = 1$, strictly monotonically decreasing, and bijective with inverse mapping

$$\alpha_{a,d} : (-d\sqrt{2a-1}, d\sqrt{2a-1}) \to (-2(a-1), 2a)$$

defined by

$$\alpha_{a,d}(\lambda) = 1 - \lambda \frac{\sqrt{4a^2d^2 - d^2 - \lambda^2}}{d\sqrt{\lambda^2 + d^2}} \,. \tag{30}$$

Hence, if for given λ, the parameters a, d, are chosen such that $|\lambda| < d\sqrt{2a-1}$ (along with the above conditions) then we can find the corresponding α. Furthermore, note that (ii) implies that $f_{a,d}$ is non-negative on $(-2(a-1), 1]$ and strictly negative on $(1, 2a)$, consistent with Proposition 16.

Proof. Under the imposed parameter restrictions we have that S_t and $(S_t)^\alpha$ are integrable for some $t > 0$ so that we can apply Proposition 20, i.e.

$$\lambda = \int \left(e^x - x e^{\frac{1}{2}\alpha x} \mathbb{I}_{|x| \le 1} - 1 \right) \nu\,(dx)$$
$$= \psi_0\big(-i(1 - \frac{1}{2}\alpha)\big) - \psi_0\big(i\frac{1}{2}\alpha\big) = d\Big(\sqrt{a^2 - \frac{1}{4}\alpha^2} - \sqrt{a^2 - \frac{1}{4}(2-\alpha)^2}\,\Big)\,,$$

where the last equality is justified by the fact that $\phi_0(-iv) = e^{\psi_0(-iv)}$, for $v \in (-a, a)$, is the (real valued) moment generating function which is known in the literature, see e.g. [16], or which can be derived from (26) by the standard arguments as in the proofs of [26, Theorems 25.17; 24.11] and by setting $b = 0$, $m = 0$ and using that $\sinh(0) = 0$ so that $\gamma = 0$ also holds (while $\alpha \in (-2(a-1), 2a)$ implies that $-\frac{1}{2}\alpha$ and $1 - \frac{1}{2} \in (-a, a)$). It remains to show (ii). By $\alpha \in (-2(a-1), 2a)$, $a > \frac{1}{2}$, we have that $a^2 - \frac{1}{4}(2-\alpha)^2 > 0$ as well as $a^2 - \frac{1}{4}\alpha^2 > 0$, i.e. $f_{a,d}$ is differentiable on this interval (with continuous extension to the closure). Hence,

$$\lim_{\alpha \to (-2(a-1))+} f_{a,d}(\alpha) = d\sqrt{2a-1}\,, \qquad \lim_{\alpha \to (2a)-} f_{a,d}(\alpha) = -d\sqrt{2a-1}\,.$$

Since ν satisfies (19), the property that the function $f_{a,d}$ is strictly monotonically decreasing is a direct consequence of Proposition 17. Alternatively this can also be seen by analyzing the derivative. Hence, $f_{a,d}$ is a bijective mapping from $(-2(a-1), 2a)$ to $(-d\sqrt{2a-1}, d\sqrt{2a-1})$. Furthermore, for given a, d, $f_{a,d}$ vanishes if and only if $\alpha = 1$ (note that due to $a > \frac{1}{2}$ we always have $1 \in (-2(a-1), 2a)$).

The calculation for deriving (30) is essentially the same as needed for the results in [16], we give details in an appendix.

We additionally remark that the equation $m = -\lambda$ for non-vanishing α is a consequence of the fact that given (10) the condition (11) translates into the martingale property of $e^{\alpha(\lambda t + X_t)}$, so that the corresponding expectations are identically one. For vanishing α it is a direct consequence of (11). ■

4.2 Quasi self-dual Variance Gamma models

As already mentioned in the last section, the Variance Gamma (VG) processes also belong to the class of Generalized Hyperbolic processes and, as Normal Inverse Gaussian processes, exhibit the property that for any $t \neq 1$ the distribution of X_t is of the same type as the distribution of X_1. The characteristic function can be parameterized in different ways, see e.g. [25, Sections 5.3, 5.4]. We use here a parametrization which shows that these processes also belong to the family of (extended) CGMY processes. Hence, we define $S = e^{\lambda t + X}$, $\lambda \in \mathbb{R}$, with X being a Lévy process with characteristic function

$$\varphi_{X_t}(u) = \exp\Big(t\big(ium - C(\log(1-\frac{iu}{M}) + \log(1+\frac{iu}{G}))\big)\Big)$$
$$= \exp\Big(t\big(iu\gamma + \int_{\mathbb{R}}(e^{iux} - 1 - iux\mathbb{1}_{|x|\le 1})\nu(x)\,dx\big)\Big)\,, \tag{31}$$

where

$$\nu(x) = \frac{C}{|x|}e^{-G|x|}\mathbb{1}_{x<0} + \frac{C}{|x|}e^{-M|x|}\mathbb{1}_{x>0}\,, \tag{32}$$

with $C > 0$, $G > 0$, $M > 0$, along with

$$\gamma = m - \frac{C(G(e^{-M}-1) - M(e^{-G}-1))}{MG}\,. \tag{33}$$

These well-known processes X also appear frequently in the financial literature, see again e.g. [4, 8, 25] and the literature cited therein. VG processes possess the "infinite-activity" property but have paths of bounded variation.

By choosing

$$M = \beta + \frac{1}{2}\alpha > 1\,, \quad \text{and} \quad G = \beta - \frac{1}{2}\alpha > 0\,, \tag{34}$$

we ensure that the Lévy measure is again of the form (19), more concretely

$$\nu(x) = e^{-\frac{1}{2}\alpha x}\nu_0(x)\,, \quad \text{with} \quad \nu_0(x) = \frac{Ce^{-\beta|x|}}{|x|}\mathbb{1}_{x\neq 0}\,,$$

i.e. we have that Condition (10) is satisfied, and in view of Proposition 17 that the integral expression in (12) is monotone in $\alpha \in (-2(\beta-1), 2\beta)$, where $\beta > \frac{1}{2}$ is needed in order to ensure that this interval is not empty. As in Remark 18 note also that we have chosen $M > 1$, not only $M > 0$, in order to ensure that the first exponential moment along with the characteristic function (including the Lévy-Khintchine representation) at the corresponding point in the complex plane exists, cf. e.g. [26, Theorem 25.17].

Proposition 22 *Assume that* $S = e^{t\lambda + X}$, $\lambda \in \mathbb{R}$, *where* X *is characterized by* (31) *with* $\alpha \in (-2(\beta-1), 2\beta)$, $\beta > \frac{1}{2}$. *Then*

(i) *For* α *such that*

$$\lambda = -C\Big(\log\big(1 - \frac{1}{\beta + \frac{1}{2}\alpha}\big) + \log\big(1 + \frac{1}{\beta - \frac{1}{2}\alpha}\big)\Big) \tag{35}$$

and by subsequently choosing γ *(via* $m = -\lambda$*) as in* (11) *the asset price model is quasi self-dual of order* α *with respect to* λ *and* e^X *is a martingale.*

(ii) *The functions*

$$f_{C,\beta} : (-2(\beta-1), 2\beta) \to (-\infty, \infty)$$

defined by

$$f_{C,\beta}(\alpha) = -C\Big(\log\big(1 - \frac{1}{\beta + \frac{1}{2}\alpha}\big) + \log\big(1 + \frac{1}{\beta - \frac{1}{2}\alpha}\big)\Big) \tag{36}$$

are vanishing if and only if $\alpha = 1$*, are strictly monotonically decreasing, and bijective with inverse mapping*

$$\alpha_{C,\beta} : (-\infty, \infty) \to (-2(\beta - 1), 2\beta)$$

defined by

$$\alpha_{C,\beta}(\lambda) = \begin{cases} \frac{-2+2\sqrt{e^{-\frac{\lambda}{C}}+\beta^2(e^{-\frac{\lambda}{C}}-1)^2}}{e^{-\frac{\lambda}{C}}-1} & \text{for } \lambda \neq 0 \\ 1 & \text{for } \lambda = 0\,. \end{cases} \tag{37}$$

Proof. Under the imposed parameter restrictions we have that S_t and $(S_t)^\alpha$ are integrable for a $t > 0$ so that we can apply Proposition 20, i.e.

$$\begin{aligned} \lambda &= \int \Big(e^x - xe^{\frac{1}{2}\alpha x}\mathbb{1}_{|x|\le 1} - 1\Big)\,\nu\,(dx) = \psi_0\big(-i(1-\frac{1}{2}\alpha)\big) - \psi_0\big(i\frac{1}{2}\alpha\big) \\ &= -C(\log(1 - \frac{1-\frac{1}{2}\alpha}{\beta}) + \log(1 + \frac{1-\frac{1}{2}\alpha}{\beta}) - \log(1+\frac{\alpha}{2\beta}) - \log(1-\frac{\alpha}{2\beta}))\,, \end{aligned}$$

where the last equality can be obtained by a direct calculation or with the help of the arguments in the Proof of Proposition 21, e.g. based on (31) by choosing $\alpha = 0$ and $m = 0$ so that also $\gamma = 0$ (and $M = G = \beta$). Hence, (35) follows after collecting the first and the third as well as the second and the fourth summand.
It remains to show (ii). Obviously, for $\alpha = 1$ we have that $f_{C,\beta}$ vanishes. By $\alpha \in (-2(\beta-1), 2\beta)$, we obtain that $f_{C,\beta}$ is differentiable on this interval. Furthermore, note that

$$\lim_{\alpha\to -2(\beta-1)+} f_{C,\beta}(\alpha) = \infty\,, \quad \lim_{\alpha\to 2\beta-} f_{C,\beta}(\alpha) = -\infty\,.$$

Since ν satisfies (19), Proposition 17 implies that $f_{C,\beta}$ is strictly monotonically decreasing. Alternatively this can also be seen by the (non-vanishing) derivative on $(-2(\beta-1), 2\beta)$ given by

$$f'_{C,\beta}(\alpha) = -\frac{C}{2}\left(\frac{1}{(\beta+\frac{1}{2}\alpha-1)(\beta+\frac{1}{2}\alpha)} + \frac{1}{(\beta-\frac{1}{2}\alpha)(\beta-\frac{1}{2}\alpha+1)}\right)\,,$$

which is, in view of (34), $C > 0$, obviously negative on the considered interval.

It remains to show that (37) is the corresponding inverse mapping. First note that since the negative continuous derivative of $f_{C,\beta}$ never vanishes on $(-2(\beta-1), 2\beta)$ we have that the inverse mapping is continuously differentiable and strictly monotonically decreasing. Finally, with $C > 0$, $M = \beta + \frac{1}{2}\alpha > 1$, and $G = \beta - \frac{1}{2}\alpha > 0$, (36) can be rewritten as

$$-\frac{\lambda}{C} = \log\big(1 - \frac{1}{\beta+\frac{1}{2}\alpha}\big) + \log\big(1 + \frac{1}{\beta-\frac{1}{2}\alpha}\big)\,,$$

or equivalently

$$(e^{-\frac{\lambda}{C}} - 1)\alpha^2 + 4\alpha - 4(1 + \beta^2(e^{-\frac{\lambda}{C}} - 1)) = 0$$

so that

$$\alpha = \frac{-2 \pm 2\sqrt{e^{-\frac{\lambda}{C}} + \beta^2(e^{-\frac{\lambda}{C}} - 1)^2}}{e^{-\frac{\lambda}{C}} - 1}\,, \quad \text{for } \lambda \neq 0\,.$$

In order to ensure

$$\lim_{\lambda\to-\infty}\alpha_{C,\beta}(\lambda)=2\beta\,,\quad \lim_{\lambda\to\infty}\alpha_{C,\beta}(\lambda)=-2(\beta-1)\,,\quad \lim_{\lambda\to 0}\alpha_{C,\beta}(\lambda)=1\,,$$

we arrive at (37). Furthermore, note that the equation $m=-\lambda$ follows by the reasons given in the proof of Proposition 21. ■

4.3 Quasi self-dual Meixner models

In this section we consider share prices modelled by $S=e^{t\lambda+X}$, $\lambda\in\mathbb{R}$, with X, being a Lévy process with characteristic function

$$\varphi_{X_t}(u)=e^{i(tm)u}\left(\frac{\cos(\frac{b}{2})}{\cosh((au-ib)/2)}\right)^{2dt}\,,$$

where

$$\nu(dx)=d\frac{e^{\frac{b}{a}x}}{x\sinh\frac{\pi}{a}x}\,dx$$

with $a>0$, $b\in(-\pi,\pi)$, $d>0$, $m\in\mathbb{R}$, see e.g. [15] or [25] and the literature cited therein. Again, this process has no centered Gaussian term. The trajectories of the process $(X_t)_{t\in[0,T]}$ have unbounded variation and the Meixner distribution of X_1 has semi-heavy tails, see again [15]. In order to ensure $E(e^{X_1})<\infty$ (and that the characteristic function is correspondingly extendable) we additionally restrict b to $(-\pi,\pi-a)$, see e.g. [15, Proof of Theorem 1], so that we also have to restrict a to values strictly below 2π in order to avoid that this interval is empty. By keeping the parameters b, d, and m but writing $\frac{b}{a}=-\frac{1}{2}\alpha$ ($b\neq 0$ in order to avoid division by zero), the Lévy measure reads

$$\nu(dx)=e^{-\frac{1}{2}\alpha x}\nu_0^\alpha(x)dx\,,\quad\text{with}\quad \nu_0^\alpha(x)=\frac{d}{x\sinh((-\frac{\alpha\pi}{2b})x)}\,,$$

with ν_0^α being again even in x, however, still depending on α, so that (24) holds, but the conditions in Proposition 17 are not satisfied. Note that the assumption $b\neq 0$ immediately implies that α cannot vanish, an assumption, which is *a priori* not needed. For brevity we focus in the sequel on the cases $\alpha\neq 0$. However, on the basis of Proposition 9 it can be derived that the quasi self-duality of order $\alpha=0$ enforces $b=0$ (so that $a\in(0,\pi)$ is assumed in order to ensure the existence of the first exponential moment) and that $m=-\lambda$. Furthermore, the additional martingale property of $\exp(X)$ for positive carrying costs can be obtained by ensuring $\lambda=-2d\log(\cos(a/2))$, for suitably chosen $a\in(0,\pi)$, $d>0$.

Now we discuss the case $\alpha\neq 0$ in more detail. In order to ensure that $a=-\frac{2b}{\alpha}>0$, and that the first exponential moment exists, we consider the two cases

(M1) $b\in(0,\pi)$ and $\alpha\in(-\infty,-\frac{2b}{\pi-b})\subset(-\infty,0)$, ($d>0$, $m\in\mathbb{R}$);

(M2) $b\in(-\pi,0)$ and $\alpha\in(-\frac{2b}{\pi-b},\infty)\subset(0,\infty)$ ($d>0$, $m\in\mathbb{R}$).

Since $\alpha=0$ is not an admissible parameter for the chosen parametrization, we take this opportunity to derive here the risk-neutral self-duality based on Remark 10. In our notation and with the parameter restrictions the characteristic function reads

$$\varphi_{X_t}(u) = e^{i(tm)u} \left(\frac{\cos(\frac{b}{2})}{\cosh(b\,\frac{2u+i\alpha}{2\alpha})} \right)^{2(dt)} . \tag{38}$$

In view of the imposed integrability assumptions and the fact that X is a Lévy process, it suffices to ensure that $\varphi_{X_1}(-i) = 1$, i.e.

$$m = -2d \log \left(\frac{\cos(\frac{b}{2})}{\cos(\frac{b}{2} - \frac{b}{\alpha})} \right) , \tag{39}$$

where we refer to the proof of Theorem 1 in [15] for the fact that the moment generating function at one is of the form of the r.h.s. of (38) for $u = -i$ (for our parameter restrictions). Now ensure the martingale property of S^α. Note that as a consequence of [26, Proposition 11.10] or directly with the help of (38) we obtain that $Y = \alpha\lambda t + \alpha X$ is again a Meixner process with characteristic function

$$\varphi_{Y_t}(u) = e^{it(\alpha\lambda + \alpha m)u} \left(\frac{\cos(\frac{\tilde{b}}{2})}{\cosh(\tilde{b}\,\frac{2u+i}{2})} \right)^{2(dt)} \tag{40}$$

with new parameter $\tilde{m} = \alpha(\lambda + m)$, unchanged parameter $d > 0$, and new α now being identically one, as it should be in the self-dual case. Furthermore, the parameter b turns the sign in the first case (M1) and remains unchanged in the second case (M2). Note that in both cases the first exponential moment of Y_t exists. Hence, by using that Y is a Lévy process we get the martingale property of S^α by ensuring that $\varphi_{Y_1}(-i) = 1$, i.e. in both cases

$$e^{\alpha\lambda + \alpha m} \left(\frac{\cos(\frac{b}{2})}{\cos(\frac{b}{2})} \right)^{2d} = 1 ,$$

i.e.

$$\lambda = -m , \tag{41}$$

where we again refer to the proof of Theorem 1 in [15] as far as the form of the moment generating function at one is concerned (note that condition (41) can also be derived with the help of the moment generating function of X_1). Hence, in particular in view of (39, 41) we obtain the first part of the following result. In the following we always restrict the arccos to its principal branch with range $[0, \pi]$, in particular we then have for $x \in [-\pi, 0]$ that $\arccos(\cos(x)) = \arccos(\cos(-x)) = -x$. Furthermore, note that the case (M1) and the second inversion formula in the case (M2) do not include the special case $\alpha = 1$.

Proposition 23 *Assume that $S = e^{t\lambda + X}$, $\lambda \in \mathbb{R}$, where X is characterized by* (38) *with parameter restrictions given in* (M1) *or in* (M2)*. Then*

(i) *For α such that*

$$\lambda = 2d \log \left(\frac{\cos(\frac{b}{2})}{\cos(\frac{b}{2} - \frac{b}{\alpha})} \right) ,$$

and by subsequently choosing $\lambda = -m$, Condition (39) *is satisfied so that e^X is a martingale and S is quasi self-dual of order α with respect to λ.*

(ii) *The behavior of the corresponding function depends on the following cases.*

(M1) *The functions*

$$f_{b,d} : (-\infty, -\frac{2b}{\pi - b}) \to (0, \infty)$$

defined by

$$f_{b,d}(\alpha) = 2d \log \left(\frac{\cos(\frac{b}{2})}{\cos(\frac{b}{2} - \frac{b}{\alpha})} \right) , \tag{42}$$

are strictly monotonically increasing, and bijective with inverse mapping

$$\alpha_{b,d} : (0, \infty) \to \big(-\infty, -\frac{2b}{\pi - b}\big)$$

defined by

$$\alpha_{b,d}(\lambda) = \frac{2b}{b - 2\arccos(\cos(\frac{b}{2})e^{-\frac{\lambda}{2d}})} . \tag{43}$$

(M2) *The functions*

$$f_{b,d} : (-\frac{2b}{\pi - b}, \infty) \to [2d \log(\cos(b/2)), \infty)$$

defined by (42) *are vanishing if and only if* $\alpha = 1$, *are piecewise injective respectively on* $(-\frac{2b}{\pi-b}, 2]$ *where they are strictly monotonically decreasing, and on* $(2, \infty)$ *where they are strictly increasing. The corresponding inverse mappings are*

$$\alpha_{b,d} : [2d \log(\cos(b/2)), \infty) \to (-\frac{2b}{\pi - b}, 2]$$

again defined by (43) *as well as*

$$\bar{\alpha}_{b,d} : (2d \log(\cos(b/2)), 0) \to (2, \infty)$$

here defined by

$$\bar{\alpha}_{b,d}(\lambda) = \frac{2b}{b + 2\arccos(\cos(b/2)e^{-\frac{\lambda}{2d}})} . \tag{44}$$

Proof. It remains to show (ii). We start with the case (M1). For $\alpha \in (-\infty, \frac{-2b}{\pi-b})$ (note that 0 is not contained in this interval) we have that $\frac{b}{2} - \frac{b}{\alpha} \in (\frac{b}{2}, \frac{\pi}{2})$ where the cosine does not vanish. Hence, $f_{b,d}$ is differentiable on this interval. Furthermore, note that

$$\lim_{\alpha \to (-\frac{2b}{\pi-b})-} f_{b,d}(\alpha) = \infty , \quad \lim_{\alpha \to -\infty} f_{b,d}(\alpha) = 0 ,$$

and that the derivative can be written as

$$f'_{b,d}(\alpha) = 2d \frac{\sin(\frac{b}{2} - \frac{b}{\alpha})b}{\cos(\frac{b}{2} - \frac{b}{\alpha})\alpha^2} , \tag{45}$$

where for $b \in (0, \pi)$ and since $\frac{b}{2} - \frac{b}{\alpha} \in (\frac{b}{2}, \frac{\pi}{2})$ all expressions in this quotient are positive so that the derivative is positive for all $\alpha \in (-\infty, -\frac{2b}{\pi - b})$, i.e. the functions $f_{b,d}$ are strictly monotonically *increasing*. Now we derive the corresponding inverse mappings. Since $d > 0$ we can rewrite (42) as

$$\frac{\lambda}{2d} = \log\left(\frac{\cos(\frac{b}{2})}{\cos(\frac{b}{2} - \frac{b}{\alpha})}\right)$$

where we note that none of the expressions vanishes for the imposed parameter restrictions. This is equivalent to

$$\cos(\frac{b}{2} - \frac{b}{\alpha}) = \cos(\frac{b}{2})e^{-\frac{\lambda}{2d}}$$

where we note that $e^{-\frac{\lambda}{2d}} < 1$ for the present parameter restrictions and $\lambda > 0$. Since $\frac{b}{2} - \frac{b}{\alpha} \in (\frac{b}{2}, \frac{\pi}{2})$ this yields

$$\frac{b}{2} - \frac{b}{\alpha} = \arccos(\cos(\frac{b}{2})e^{-\frac{\lambda}{2d}})\,. \tag{46}$$

By the monotonicity property of arccos we have that $\arccos(\cos(\frac{b}{2})e^{-\frac{\lambda}{2d}}) \neq \frac{b}{2}$ for the present parameter restrictions and $\lambda > 0$ so that we end up with (43) and we finally note that

$$\lim_{\lambda \to 0+} f_{b,d}(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} f_{b,d}(\lambda) = -\frac{2b}{\pi - b}\,.$$

For the cases (M2) we first observe that for the imposed parameter restrictions the functions $f_{b,d}$ are differentiable on $(-\frac{2b}{\pi-b}, \infty)$ with derivative given in (45). However, here we have that for $b < 0$ with $\alpha \in (-\frac{2b}{\pi-b}, 2)$, $\sin(\frac{b}{2} - \frac{b}{\alpha})$ along with the other remaining expressions is still positive so that the derivatives are negative and the functions $f_{b,d}$ monotonically decreasing. However, where for $\alpha \in (2, \infty)$ the sine turns its sign so that the derivatives become positive, i.e. the functions are monotonically increasing there, the derivative vanishes at the minimum of $f_{b,d}$ in $\alpha = 2$. As far as the inversions are concerned, we remark that for $\lambda \geq 2d\log(\cos(\frac{b}{2}))$ we get $\cos(\frac{b}{2})e^{-\frac{\lambda}{2d}} \leq 1$. Since for $\alpha \in (-\frac{2b}{\pi-b}, 2]$ we have that $\frac{b}{2} - \frac{b}{\alpha}$ is non-negative, the inversion follows by the same steps as in the case (M1) so that (43) again follows. However, for $\alpha > 2$ we have that $\frac{b}{2} - \frac{b}{\alpha} < 0$ so that we obtain

$$\frac{b}{\alpha} - \frac{b}{2} = \arccos(\cos(\frac{b}{2})e^{-\frac{\lambda}{2d}})$$

instead of (46) so that we finally end up with (44) (where we stress that $\lambda \in (2d\log(\cos(b/2)), 0)$). To conclude, we remark that

$$\lim_{\lambda \to \infty} \alpha_{b,d}(\lambda) = -\frac{2b}{\pi - b}\,, \quad \lim_{\lambda \to 2d\log(\cos(b/2))+} \bar{\alpha}(\lambda) = 2\,, \quad \lim_{\lambda \to 0-} \bar{\alpha}(\lambda) = \infty\,,$$

while $\alpha_{b,d}(2d\log(\cos(b/2))) = 2$. ■

Appendix: Inverse in the Normal Inverse Gaussian case

First note that due to $|\lambda| < d\sqrt{2a - 1}$ (and $d > 0$) as well as because of $a > \frac{1}{2}$ we have

$$\lambda^2 < d^2(2a-1) < 4a^2d^2 - d^2\,, \quad \text{equivalently} \quad 4a^2d^2 - d^2 - \lambda^2 > 0\,.$$

Furthermore, $d>0$ implies that $d\sqrt{\lambda^2+d^2}$ is non-vanishing and that

$$g_{a,d}(\lambda)=\frac{\sqrt{4a^2d^2-d^2-\lambda^2}}{d\sqrt{\lambda^2+d^2}}>0 \quad \text{for all} \quad \lambda\in(-d\sqrt{2a-1},d\sqrt{2a-1})\,,$$

i.e.

$$\alpha_{a,d}(\lambda)=1-\lambda g_{a,d}(\lambda)\,, \quad \text{with} \quad g_{a,d}>0 \ \text{ for all } \lambda\in(-d\sqrt{2a-1},d\sqrt{2a-1})\,. \tag{47}$$

The function $\alpha_{a,d}$ is differentiable on $(-d\sqrt{2a-1},d\sqrt{2a-1})$ (this could also be seen by noticing that the derivative of $f_{a,d}$ does not vanish and by the inversion below), with derivative

$$\alpha'_{a,d}(\lambda)=-\frac{4a^2d^4-(d^2+\lambda^2)^2}{d(\lambda^2+d^2)^{(3/2)}\sqrt{4a^2d^2-d^2-\lambda^2}}\,,$$

where, again by $|\lambda|<d\sqrt{2a-1}$, $a>\frac{1}{2}$, we obtain $(\lambda^2+d^2)^2<(2ad)^2=4a^2d^4$, i.e. $\alpha_{a,d}$ is strictly monotonically decreasing on $(-d\sqrt{2a-1},d\sqrt{2a-1})$ with

$$\lim_{\lambda\to(-d\sqrt{2a-1})+}\alpha_{a,d}(\lambda)=2a\,, \qquad \lim_{\lambda\to(d\sqrt{2a-1})-}\alpha_{a,d}(\lambda)=-2(a-1)\,.$$

Furthermore, by $d>0$, (29) can be rewritten as

$$\sqrt{a^2-\frac{1}{4}\alpha^2}-\frac{\lambda}{d}=\sqrt{a^2-\frac{1}{4}(2-\alpha)^2}\,,$$

where we recall that $a^2-\frac{1}{4}\alpha^2>0$, and $a^2-(1-\frac{1}{2}\alpha)^2>0$ for $\alpha\in(-2(a-1),a)$, by squaring and rearranging, this implies

$$(1+(\frac{\lambda}{d})^2)-\alpha=2\frac{\lambda}{d}\sqrt{a^2-\frac{1}{4}\alpha^2}\,.$$

Squaring again, multiplying by d^4, and rearranging yields

$$(d^2(d^2+\lambda^2))\alpha^2-2d^2(d^2+\lambda^2)\alpha+((d^2+\lambda^2)^2-4a^2\lambda^2d^2)=0\,.$$

Again by noticing that $d>0$, $d^2(d^2+\lambda^2)>0$, and $4a^2d^2-d^2-\lambda^2>0$ we see that the solution needs to be of the form

$$\begin{aligned}\alpha_{1,2}(\lambda)&=\frac{2(d^2+\lambda^2)d^2\pm\sqrt{4d^4(d^2+\lambda^2)^2-4d^2(d^2+\lambda^2)((\lambda^2+d^2)^2-4a^2\lambda^2d^2)}}{2d^2(d^2+\lambda^2)}\\&=1\pm\frac{\sqrt{4d^2(d^2+\lambda^2)\lambda^2(4a^2d^2-\lambda^2-d^2)}}{2d^2(\lambda^2+d^2)}\\&=1\pm|\lambda|\frac{\sqrt{4a^2d^2-d^2-\lambda^2}}{d\sqrt{d^2+\lambda^2}}\,.\end{aligned}$$

Since $g_{a,d}(\lambda)>0$ for all $|\lambda|<2\sqrt{2a-1}$ (where $a>\frac{1}{2}$) and because the function is decreasing in λ we end up with (30).

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