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Dispersive Approach to Hadronic Light-by-Light

Gilberto Colangelo[1], Martin Hoferichter[1,2,3], Massimiliano Procura[1], and Peter Stoffer[1]

[1] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH–3012 Bern, Switzerland*
[2] *Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*
[3] *ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für Schwerionenforschung GmbH, D-64291 Darmstadt, Germany*

Abstract. A recently proposed dispersive approach to hadronic light-by-light is described.

In this talk I have presented a dispersive approach to hadronic light-by-light (HLbL) which has been recently proposed in [1]. This approach aims to take into account only the cuts in the hadronic tensor which are due to single- or double-pion intermediate states – this approximation is justified by the fact that in explicit calculations higher-lying singularities (like the one due to two kaons) give small contributions [2]. Further, we split the hadronic tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots \,, \tag{1}$$

where the first term takes into account the one-pion pole, the second one two-pion intermediate states with simultaneous cuts in the s and t channel (and all possible cyclic permutations including u), and the third one is the one for which we write down a dispersion relation.

We briefly discuss the three contributions.

1 Pion pole

The dominant contribution to HLbL scattering at low energy is given by the π^0-poles. Their residues are determined by the on-shell, doubly-virtual pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$, which is defined as the current matrix element

$$i\int d^4x\, e^{iq\cdot x}\langle 0\big|T\{j_\mu(x)j_\nu(0)\}\big|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q^2,(p-q)^2). \tag{2}$$

In these conventions, the π^0-pole HLbL amplitude reads

$$\begin{aligned}\Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} &= \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,0)}{s-M_{\pi^0}^2}\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \epsilon_{\lambda\sigma\gamma\delta}q_3^\gamma k^\delta \\ &+ \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{t-M_{\pi^0}^2}\epsilon_{\mu\lambda\alpha\beta}q_1^\alpha q_3^\beta \epsilon_{\nu\sigma\gamma\delta}q_2^\gamma k^\delta \\ &+ \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,0)}{u-M_{\pi^0}^2}\epsilon_{\nu\lambda\alpha\beta}q_2^\alpha q_3^\beta \epsilon_{\mu\sigma\gamma\delta}q_1^\gamma k^\delta .\end{aligned} \tag{3}$$

Its contribution to a_μ can be expressed as [3]

$$a_\mu^{\pi^0\text{-pole}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)} \tag{4}$$
$$\times \left\{ \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(s,0)}{s - M_{\pi^0}^2} T_1(q_1, q_2; p) + \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(s, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, 0)}{q_1^2 - M_{\pi^0}^2} T_2(q_1, q_2; p) \right\},$$

with

$$T_1 = \frac{8}{3}\left\{2p\cdot q_1\, p\cdot q_2\, q_1\cdot q_2 + p\cdot q_1\, q_2^2\Big(q_1\cdot q_2 + q_1^2 - 2p\cdot q_1\Big) - \frac{m^2\lambda_{12}}{4}\right\}, \tag{5}$$
$$T_2 = \frac{16}{3}\left\{p\cdot q_1\Big(p\cdot q_2\, q_1\cdot q_2 - p\cdot q_1\, q_2^2 + (q_1\cdot q_2)^2\Big) - \frac{q_1^2}{2}\Big(3p\cdot q_1\, q_2^2 - p\cdot q_2\, q_1\cdot q_2\Big) - \frac{m^2\lambda_{12}}{4}\right\}.$$

Due to the $q_1 \leftrightarrow -q_2$ symmetry of the integrand, the t- and u-channel terms give the same contribution.

2 FsQED contribution

The precise meaning of the FsQED contribution can be explained as follows: $\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}}$ includes the contribution due to simultaneous two-pion cuts in two of the channels (by crossing symmetry it contains three contributions with simultaneous singularities in the (s,t), (s,u), and (t,u) channels, respectively). One first takes the two-pion cut in the s-channel, which gives the discontinuity as the product of two $\gamma^*\gamma^* \to \pi\pi$ amplitudes, and then selects the Born term (the pure pole term) in each of the two amplitudes. The singularity of this diagram is therefore given by four $\pi^+\pi^-\gamma^*$ vertices with on-shell pions—which implies that these vertices are nothing but the full pion vector form factors. On the other hand, the singularity structure of this contribution is identical to that of a Feynman box diagram with four pion propagators: since the four vertices depend only on the momentum squared of the external photons and on none of the internal momenta, this contribution is given by the box-diagram multiplied by three pion vector form factors (since one of the photons is on-shell). In sQED the box diagram is not gauge invariant on its own, however. The photon–scalar–scalar vertex comes together with the seagull term (two-photon–two-scalar vertex), with couplings strictly related to each other: in any amplitude with two or more photons both vertices have to be taken into account to form a subset of gauge-invariant diagrams. Therefore, in sQED the box diagram has to be accompanied by a triangle and a bulb diagram in order to respect gauge invariance. We do the same here and define our gauge-invariant box diagram as the charged pion loop calculated within sQED multiplied by the pion vector form factors.

We find the representation

$$a_\mu^{\text{FsQED}} = \frac{2e^6}{3\pi^2} \int \frac{\mathrm{d}^4q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4q_2}{(2\pi)^4} \frac{F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(s)(I_s + 2I_u + J_1 + J_2)}{q_1^2 q_2^2 s((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)}, \tag{6}$$

where $F_\pi^V(s)$ is the pion vector form factor and the expression for $I_{s,u}$ and $J_{1,2}$ can be found in [1].

3 Two-pion cuts

A central result of our analysis is that after separating $\Pi_{\mu\nu\lambda\sigma}^{\pi^0-\text{pole}}$ and $\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}}$ from the rest, we have been able to derive explicit unitarity relations for the remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ and relate the imaginary parts to the helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$.

In general, the HLbL tensor with one of the four photons on-shell contains 29 independent scalar amplitudes. We have explicitly constructed 29 independent gauge-invariant Lorentz tensors, but doing so in a way that makes crossing symmetry manifest, or even easy to express, is more difficult. For our purposes we find it more convenient to use a redundant basis, in which however crossing symmetry is evident. Therefore, we exploit the crucial property of the $A_{i,s}^{\mu\nu\lambda\sigma}$ that if we project the s-channel HLbL tensor on helicity amplitudes, only a single function $\Pi_i^s \equiv \Pi_i(s,t,u)$ contributes for each helicity amplitude, and write

$$\bar{\Pi}^{\mu\nu\lambda\sigma}(s,t,u) = \sum_{i=1}^{15}\Big(A_{i,s}^{\mu\nu\lambda\sigma}\Pi_i(s,t,u) + A_{i,t}^{\mu\nu\lambda\sigma}\Pi_i(t,s,u) + A_{i,u}^{\mu\nu\lambda\sigma}\Pi_i(u,t,s)\Big). \tag{7}$$

The 45 tensors in (7) form a complete, though redundant, basis. In fact, already the 30 tensors $A_{i,s}^{\mu\nu\lambda\sigma}$ and $A_{i,t}^{\mu\nu\lambda\sigma}$ are sufficient to saturate the number of linearly independent structures.

The construction of dispersion relations for the Π_i functions becomes greatly simplified if we consider that here we are only interested in the HLbL contribution to a_μ. This involves the derivative of the HLbL tensor with respect to k evaluated at $k=0$. We therefore construct dispersion relations only for this very special limit and omit from the start any contribution to the HLbL tensor of $O(k^2)$. The dispersive representation of the Π_i^s amplitudes which we have provided has the following properties

1. For each Π_i^s we only take into account the discontinuity due to the lowest partial wave.

2. We fix the discontinuity to what unitarity prescribes.

3. The Π_i^s amplitudes have the required soft-photon zeros.

4. The exact form of the soft-photon zeros follows from $\gamma^*\gamma^* \to \pi\pi$ by means of factorization.

5. The number of subtractions is chosen according to what the implementation of the soft-photon zeros naturally generates (which is sufficient to ensure convergence).

These arguments uniquely lead to the following dispersive integrals for the Π_i^s amplitudes[1]

$$\Pi_1^s = \bar{h}^0_{++,++}(s) = \frac{s-q_3^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\,\bar{h}^0_{++,++}(s'), \tag{8}$$

$$-\frac{q_1^2q_2^2}{\xi_1\xi_2}\Pi_2^s = \bar{h}^0_{00,++}(s) = \frac{s-q_3^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\,\bar{h}^0_{00,++}(s'),$$

$$-\frac{2\sqrt{6}}{75}\Pi_3^s = \bar{h}^2_{++,+-}(s) = \frac{(s-q_3^2)\lambda_{12}}{s\,\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'\,s'}{(s'-q_3^2)\lambda'_{12}}\left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\,\bar{h}^2_{++,+-}(s'),$$

[1] We omit here non-diagonal terms, which are discussed in [1].

and similarly for the remaining ones not given explicitly here. The imaginary parts read

$$
\begin{aligned}
\mathrm{Im}\,\bar{h}^0_{++,++}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{0,++}(s;q_1^2,q_2^2)h^*_{0,++}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^0_{00,++}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{0,00}(s;q_1^2,q_2^2)h^*_{0,++}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^2_{++,+-}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{2,++}(s;q_1^2,q_2^2)h^*_{2,+-}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^2_{+-,+-}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{2,+-}(s;q_1^2,q_2^2)h^*_{2,+-}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^2_{+0,+-}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{2,+0}(s;q_1^2,q_2^2)h^*_{2,+-}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^2_{0+,+-}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{2,0+}(s;q_1^2,q_2^2)h^*_{2,+-}(s;q_3^2,0)\Big],\\
\mathrm{Im}\,\bar{h}^2_{00,+-}(s) &= \frac{\sigma_s}{16\pi}\theta(s-4M_\pi^2)\,\mathcal{S}\Big[h_{2,00}(s;q_1^2,q_2^2)h^*_{2,+-}(s;q_3^2,0)\Big].
\end{aligned}
\tag{9}
$$

The relations (9) without the bars on the left-hand side and the $\mathcal{S}[\ldots]$ operators, defined in (10), on the right-hand side simply express unitarity for partial-wave helicity amplitudes. Since we have subtracted the FsQED contributions and are dealing with subtracted partial-wave helicity amplitudes, we have to correspondingly adapt the unitarity relations. This is taken care of by the operator $\mathcal{S}[\ldots]$, which either subtracts the FsQED contribution for charged (c) pions, or restores the symmetry factor for neutral (n) pions

$$
\begin{aligned}
\mathcal{S}\Big[h^{\mathrm{c}}_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)\Big(h^{\mathrm{c}}_{J,\lambda_3\lambda_4}(s;q_3^2,0)\Big)^*\Big] &\equiv h^{\mathrm{c}}_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)\Big(h^{\mathrm{c}}_{J,\lambda_3\lambda_4}(s;q_3^2,0)\Big)^*\\
&\quad - N_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)N_{J,\lambda_3\lambda_4}(s;q_3^2,0),\\
\mathcal{S}\Big[h^{\mathrm{n}}_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)\Big(h^{\mathrm{n}}_{J,\lambda_3\lambda_4}(s;q_3^2,0)\Big)^*\Big] &\equiv \frac{1}{2}h^{\mathrm{n}}_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)\Big(h^{\mathrm{n}}_{J,\lambda_3\lambda_4}(s;q_3^2,0)\Big)^*.
\end{aligned}
\tag{10}
$$

Our representation for $\bar{\Pi}_{\mu\nu\lambda\sigma}$ can be viewed as a generalization of the reconstruction theorem [4] originally derived for the $\pi\pi$ scattering amplitude to the hadronic light-by-light tensor.

4 Master formula

When evaluating the HLbL contribution to a_μ one has to take the limit $k \to 0$. In this limit the D-wave contributions involve terms which are ambiguously defined. To overcome this technical difficulty we have followed an approach that relies on an angular average over the spatial directions of k, wherein the limit $k \to 0$ and the loop integrations may be interchanged. After doing that we obtain

$$
\begin{aligned}
a_\mu^{\pi\pi} &= e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 q_2}{(2\pi)^4}\frac{I^{\pi\pi}}{q_1^2 q_2^2 s((p+q_1)^2-m^2)((p-q_2)^2-m^2)},\\
I^{\pi\pi} &= \sum_{i\in\{1,2,3,6,14\}}\Big(T_{i,s}I_{i,s}+2T_{i,u}I_{i,u}\Big)+2T_{9,s}I_{9,s}+2T_{9,u}I_{9,u}+2T_{12,u}I_{12,u},
\end{aligned}
\tag{11}
$$

with dispersive integrals $I_{i,(s,u)}$ and integration kernels $T_{i,(s,u)}$ to be found in [1]. Throughout, we used the symmetry of the integrand under $q_1 \leftrightarrow -q_2$ to map the t-channel contributions onto the u-channel and simplify the s-channel kernels. Moreover, this symmetry transforms the amplitudes corresponding to $h^2_{+0,+-}$ and $h^2_{0+,+-}$ into each other, with the t-channel of one equaling the u-channel of the other, and makes the s-channel contribution of $h^2_{0+,+-}$ coincide with the one generated by $h^2_{+0,+-}$. More details about the meaning and interpretation of the master formula (11) can be found in [1].

5 Concluding remarks

The final goal of the approach discussed here is a calculation of HLbL scattering consistent with the general principles of analyticity, unitarity, crossing symmetry, and gauge invariance and backed by data as closely as possible. Ultimately, this approach should allow for a more reliable estimate of uncertainties in the HLbL contribution to the anomalous magnetic moment of the muon. An overview of the theoretical foundations for a data-driven evaluation of the HLbL and experimental information useful to this goal can be found in [5].

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