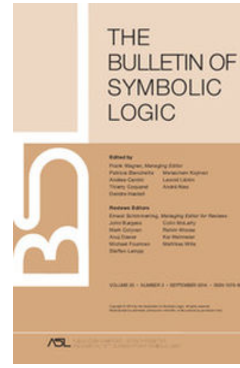


# EXPLICIT MATHEMATICS AND OPERATIONAL SET THEORY: SOME ONTOLOGICAL COMPARISONS

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# EXPLICIT MATHEMATICS AND OPERATIONAL SET THEORY: SOME ONTOLOGICAL COMPARISONS

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**Abstract.** We discuss several ontological properties of explicit mathematics and operational set theory: global choice, decidable classes, totality and extensionality of operations, function spaces, class and set formation via formulas that contain the definedness predicate and applications.

**§1. Introduction.** The purpose of this article is to discuss several ontological properties of explicit mathematics and operational set theory, not least of all for the sake of pointing out some principal differences between explicit mathematics and operational set theory. Very often, operational set theory is regarded as the set-theoretic counterpart of explicit mathematics, and this point of view is certainly justified – but only to a certain extent.

Both explicit mathematics and operational set theory give operations a prominent role, self-application is possible though not necessarily defined. And in both cases the universe of discourse is a partial combinatory algebra. However, differences occur, for example, with respect to global choice, the possibility of asking for totality and extensionality of operations, and set or class formation by means of formulas that contain the definedness predicate and applications.

In the following section we briefly introduce the formalism of explicit mathematics, review several of its known ontological properties and turn to some additional ones that have not yet been published in this form. Interesting observations tell us that choice is problematic and decidability of classes can only be permitted for "small" classes.

Afterwards we turn to operational set theory and show that it is not consistent to claim that all operations are total or extensional and that the collection of all operations from a set $a$ to a set $b$ do not form a set in all relevant cases. We also analyze the situation of set formation via formulas that permit the definedness predicate and application terms and point out a significant difference between uniform and nonuniform such set formations.

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**§2. Explicit mathematics.** Explicit mathematics and in particular the axiomatic system $\mathsf{T}_0$ – then formulated in intuitionistic logic – were introduced by Feferman in the nineteen seventies and originally designed as a framework for formalizing Bishop-style constructive mathematics. But soon it became evident that systems of explicit mathematics (based on intuitionistic or classical logic) play an independent important role in proof theory. The three articles Feferman [7, 8, 9] provide an excellent introduction into explicit mathematics and put it into a general context.

Here we do not work with Feferman's original formalization of systems of explicit mathematics; instead we treat them as theories of types and names as developed in Jäger [14] and used in, for example, Jäger and Strahm [20, 21] and Jäger and Studer [22]. The following description of the relevant systems of explicit mathematics is more or less as in [21].

The applicative theory with elementary typing ($\mathsf{AET}$) that we will consider is formulated in the second order language $\mathbb{L}$ for individuals and types. It comprises individual variables $a, b, c, f, g, h, u, v, w, x, y, z, \dots$ as well as type variables $U, V, W, X, Y, Z, \dots$ (both possibly with subscripts). $\mathbb{L}$ also includes the individual constants $\mathsf{k}, \mathsf{s}$ (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projections), $0$ (zero), $\mathsf{s_N}$ (successor), $\mathsf{p_N}$ (predecessor), $\mathsf{d_N}$ (definition by numerical cases), and additional individual constants that will be used for the uniform naming of types, namely $\mathsf{nat}$ (natural numbers), $\mathsf{id}$ (identity), $\mathsf{co}$ (complement), $\mathsf{un}$ (union), $\mathsf{dom}$ (domain), and $\mathsf{inv}$ (inverse image). There is one binary function symbol $\cdot$ for (partial) application of individuals to individuals. Further, $\mathbb{L}$ has unary relation symbols $\downarrow$ (defined), $\mathsf{N}$ (natural numbers), as well as three binary relation symbols $\in$ (membership), $=$ (equality), and $\Re$ (naming, representation).

The *individual terms* $(r, s, t, r_1, s_1, t_1, \dots)$ of $\mathbb{L}$ are built up from individual variables and individual constants by means of our function symbol $\cdot$ for forming applications $(s \cdot t)$. In the following we often abbreviate $(s \cdot t)$ as $(st)$ or – if no confusion arises – simply as $st$. We further adopt the convention of association to the left so that $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 \cdot s_2) \dots s_n)$, and we often also write $s(t_1, \dots, t_n)$ for $st_1 \dots t_n$. Further notations:

$$<s, t> \;:=\; \mathsf{p} s_1 s_2, \qquad t' \;:=\; \mathsf{s_N} t, \qquad \text{and} \qquad 1 \;:=\; 0'.$$

The *atomic formulas* of $\mathbb{L}$ are the expressions $\mathsf{N}(s)$, $s\downarrow$, $(s = t)$, $(U = V)$, $(s \in U)$, and $\Re(s, U)$; the *formulas* $(A, B, C, A_1, B_1, C_1, \dots)$ of $\mathbb{L}$ are generated from the atomic formulas by closing under negation, disjunction, conjunction, implication, equivalence, as well as existential and universal quantification for individuals and types. The *free variables* of $t$ and $A$ are defined in the standard way; the closed $\mathbb{L}$ terms and closed $\mathbb{L}$ formulas, also called $\mathbb{L}$ sentences, are those that do not contain free variables.

Since we work with a logic of partial terms, it is not guaranteed that all terms have values, and $s\downarrow$ is read as $s$ *is defined* or $s$ *has a value*. Moreover, $\mathsf{N}(s)$ says that $s$ is a natural number, and the formula $\Re(s, U)$ is used to express that the individual $s$ *represents* the type $U$ or is a *name* of $U$.

We often omit parentheses and brackets whenever there is no danger of confusion. Moreover, we frequently make use of the vector notation

$\vec{U}$ and $\vec{s}$ for finite strings of type variables $U_1, \dots, U_m$ and individual terms $s_1, \dots, s_n$, respectively, whose length is not important or given by the context.

Suppose now that $\vec{a} = a_1, \dots, a_n$ and $\vec{s} = s_1, \dots, s_n$. Then $A[\vec{s}/\vec{a}]$ is the $\mathbb{L}$ formula that is obtained from the $\mathbb{L}$ formula $A$ by simultaneously replacing all free occurrences of the variables $\vec{a}$ by the terms $\vec{s}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. If the $\mathbb{L}$ formula $A$ is written as $B[\vec{a}]$, then we often simply write $B[\vec{s}]$ instead of $A[\vec{s}/\vec{a}]$. Further variants of this notation below will be obvious. The substitution of $\mathbb{L}$ terms for variables in $\mathbb{L}$ terms is treated accordingly. The following table contains a list of useful abbreviations:

$$\begin{aligned}
(s \neq t) &:= \neg(s = t), \\
(s \simeq t) &:= s{\downarrow} \vee t{\downarrow} \;\rightarrow\; s = t, \\
(s \in \mathsf{N}) &:= \mathsf{N}(s), \\
(V \subseteq W) &:= \forall x(x \in V \;\rightarrow\; x \in W), \\
(s \mathrel{\dot{\in}} t) &:= \exists X(\Re(t, X) \;\wedge\; s \in X), \\
\Re(s) &:= \exists X\, \Re(s, X), \\
\Re(\vec{r}, \vec{U}) &:= \Re(r_1, U_1) \wedge \ldots \wedge \Re(r_n, U_n),
\end{aligned}$$

where the vector $\vec{r}$ consists of the individual terms $r_1, \dots, r_n$ and the vector $\vec{U}$ of the type variables $U_1, \dots, U_n$.

The underlying logic of $\mathsf{AET}$ is given by Beeson's classical *logic of partial terms* (cf. Beeson [2] or Troelstra and van Dalen [25]) for the individuals and classical logic with equality for the types. We also include the usual strictness axioms. The nonlogical axioms of $\mathsf{AET}$ can be divided into the following groups:

**I. Applicative axioms.** These axioms formalize that the individuals form a partial combinatory algebra, that we have pairing and projection and the usual closure conditions on the natural numbers plus definition by numerical cases.

(1) $\mathsf{k}ab = a$,
(2) $\mathsf{s}ab{\downarrow} \wedge \mathsf{s}abc \simeq (ac)(bc)$,
(3) $\mathsf{p}_0(<a, b>) = a \wedge \mathsf{p}_1(<a, b>) = b$,
(4) $0 \in \mathsf{N}$,
(5) $a \in \mathsf{N} \;\rightarrow\; a' \in \mathsf{N}$,
(6) $a \in \mathsf{N} \;\rightarrow\; a' \neq 0 \wedge \mathsf{p_N}(a') = a$,
(7) $a \in \mathsf{N} \wedge a \neq 0 \;\rightarrow\; \mathsf{p_N}a \in \mathsf{N} \wedge (\mathsf{p_N}a)' = a$,
(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \;\rightarrow\; \mathsf{d_N}(x, y, a, b) = x$,
(9) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \;\rightarrow\; \mathsf{d_N}(x, y, a, b) = y$.

**II. Explicit representation and extensionality.** The following axioms state that each type has a name, that there are no homonyms and that equality of types is extensional.

(1) $\exists x\, \Re(x, U)$,
(2) $\Re(a, U) \wedge \Re(a, V) \;\rightarrow\; U = V$,
(3) $\forall x(x \in U \leftrightarrow x \in V) \;\rightarrow\; U = V$.

**III. Basic type existence axioms.** In the following we provide a finite axiomatization of uniform elementary comprehension.

*Natural numbers*

(1) $\Re(\mathsf{nat}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{nat} \leftrightarrow \mathsf{N}(x))$.

*Identity*

(2) $\Re(\mathsf{id}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{id} \leftrightarrow \exists y(x = {<}y, y{>}))$.

*Complements*

(3) $\Re(a) \;\rightarrow\; \Re(\mathsf{co}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{co}(a) \;\leftrightarrow\; x \mathrel{\dot\notin} a)$.

*Unions*

(4) $\Re(a) \wedge \Re(b) \;\rightarrow\; \Re(\mathsf{un}(a, b)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{un}(a, b) \;\leftrightarrow\; x \mathrel{\dot\in} a \vee x \mathrel{\dot\in} b)$.

*Domains*

(5) $\Re(a) \;\rightarrow\; \Re(\mathsf{dom}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{dom}(a) \;\leftrightarrow\; \exists y({<}x, y{>} \mathrel{\dot\in} a))$.

*Inverse images*

(6) $\Re(a) \;\rightarrow\; \Re(\mathsf{inv}(a, f)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{inv}(a, f) \;\leftrightarrow\; f x \mathrel{\dot\in} a)$.

As usual from the axioms of a partial combinatory algebra, i.e., from the applicative axioms (1) and (2) above, we can introduce for each $\mathbb{L}$ term $t$ an $\mathbb{L}$ term $(\lambda x.t)$ whose variables are those of $t$ other than $x$ such that

$$(\lambda x.t){\downarrow} \;\wedge\; (\lambda x.t)y \;\simeq\; t[y/x].$$

Of course, we can generalize $\lambda$ abstraction to several arguments by simply iterating abstraction for one argument. Accordingly, we set for all $\mathbb{L}$ terms $t$ and all variables $x_1, \ldots, x_n$,

$$(\lambda x_1 \ldots x_n.t) \;:=\; (\lambda x_1.(\ldots(\lambda x_n.t)\ldots)).$$

Often the term $(\lambda x_1 \ldots x_n.t)$ is simply written as $\lambda x_1 \ldots x_n.t$. If $\vec{x}$ is the sequence $x_1, \ldots, x_n$, then $\lambda\vec{x}.t$ stands for $\lambda x_1 \ldots x_n.t$.

The applicative axioms (1) and (2) also provide us with a closed $\mathbb{L}$ term $\mathsf{fix}$ – a so-called fixed point operator – such that

$$\mathsf{fix}(f){\downarrow} \;\wedge\; \mathsf{fix}(f, x) \;\simeq\; f(\mathsf{fix}(f), x).$$

If an $\mathbb{L}$ formula $A$ is called *elementary* provided that it contains neither the relation symbol $\Re$ nor bound type variables, then we have the following result; see Feferman and Jäger [12].

Theorem 2.1. *For every elementary formula $A[u, \vec{v}, \vec{W}]$ with at most the indicated free variables there exists a closed term $t_A$ such that* AET *proves*:

1. $\Re(\vec{w}, \vec{W}) \;\rightarrow\; \Re(t_A(\vec{v}, \vec{w}))$,
2. $\Re(\vec{w}, \vec{W}) \;\rightarrow\; \forall x(x \mathrel{\dot\in} t_A(\vec{v}, \vec{w}) \;\leftrightarrow\; A[x, \vec{v}, \vec{W}])$.

A trivial consequence of this theorem is that there exist the type $\mathbb{V}$ of all objects (the universal type) and the type of the natural numbers, denoted by $\mathsf{N}$ as the corresponding relation symbol.

Our theory $\mathsf{AET}$ is a subsystem of the theory $\mathsf{EET}$, which has been considered in, for example, Feferman and Jäger [12]. $\mathsf{EET}$ comprises additional axioms for primitive recursion, but they are of no relevance for the following ontological considerations. However, before turning to some new results, let me recall some well-known inconsistencies. By Feferman [7] and some straightforward considerations we know that $\mathsf{AET}$ is inconsistent with:

- the totality statement $\forall x \forall y (xy\downarrow)$ plus full definition by cases.
- the totality statement $\forall x \forall y (xy\downarrow)$ plus $\forall x \mathsf{N}(x)$.
- extensionality of operations plus full definition by cases.
- extensionality of operations plus $\forall x \mathsf{N}(x)$.

Furthermore, in Feferman [7], it is also shown that $\mathsf{AET}$ is inconsistent with the schema of comprehension for arbitrary $\mathbb{L}$ formulas, and Jäger [15] tells us that the names of a type never form a type.

**2.1. Global and weak choice.** The axioms of operational set theory ($\mathsf{OST}$) comprise an axiom for global choice, and we will show in this section that the corresponding statement is inconsistent with explicit mathematics. In order to prove this, we first turn to (names of) types that represent graphs of functions in the set-theoretic sense and ask the question whether such graphs can be represented in an operational sense.

Definition 2.2.

1. $\mathcal{G}[a] \;:=\; \begin{cases} \Re(a) \;\wedge\; (\forall x \,\dot{\in}\, a)(x = < \mathsf{p}_0 x, \mathsf{p}_1 x >) \;\wedge \\ (\forall x, y \,\dot{\in}\, a)(\mathsf{p}_0 x = \mathsf{p}_0 y \;\rightarrow\; \mathsf{p}_1 x = \mathsf{p}_1 y). \end{cases}$
2. $\mathcal{O}[a, f] \;:=\; \mathcal{G}[a] \;\wedge\; (\forall x \,\dot{\in}\, a)(f(\mathsf{p}_0 x) = \mathsf{p}_1 x)$.

The formula $\mathcal{G}[a]$ says that $a$ is the name of a type that represents the graph of a function. On the other hand, $\mathcal{O}[a, f]$ means that $a$ represents the graph of a function and $f$ is an operation that yields the same values as this function.

Theorem 2.3. *In* $\mathsf{AET}$ *not every graph of a set-theoretic function can be simulated by an operation*; *i.e.*,

$$\mathsf{AET} \;\vdash\; \exists a(\mathcal{G}[a] \;\wedge\; \forall f \neg \mathcal{O}[a, f]).$$

Proof. We work in $\mathsf{AET}$ and let $A[u]$ be the elementary $\mathbb{L}$ formula

$$(u = < \mathsf{p}_0 u, 0 > \;\wedge\; (\mathsf{p}_0 u)(\mathsf{p}_0 u) = 1) \;\vee\; (u = < \mathsf{p}_0 u, 1 > \;\wedge\; (\mathsf{p}_0 u)(\mathsf{p}_0 u) \neq 1).$$

Thus Theorem 2.1 implies that there exists the name $a$ of a type such that $(\forall x \,\dot{\in}\, a)(x = < \mathsf{p}_0 x, \mathsf{p}_1 x >)$ and for all individuals $u$ and $v$,

$$< u, v > \,\dot{\in}\, a \;\leftrightarrow\; (uu = 1 \;\wedge\; v = 0) \;\vee\; (uu \neq 1 \;\wedge\; v = 1).$$

Clearly, we have $\mathcal{G}[a]$. Now assume that there exists an $f$ such that $\mathcal{O}[a, f]$. Then we have

$$\forall x \forall y (< x, y > \,\dot{\in}\, a \;\rightarrow\; fx = y).$$

Altogether we thus have $(ff = 1 \;\leftrightarrow\; ff \neq 1)$, a contradiction. Hence there is no $f$ with $\mathcal{O}[a, f]$, and our theorem is proved. $\dashv$

Making use of this theorem we can easily derive that explicit mathematics does not permit a form of global choice as in operational set theory. Actually, even a very weak form of global choice will be seen to be inconsistent with $\mathsf{AET}$.

Definition 2.4.

1. $\mathcal{C}[f] \;:=\; \forall x(\Re(x) \wedge \exists y(y \dot{\in} x) \;\rightarrow\; fx \dot{\in} x)$.
2. $\mathcal{C}_1[f] \;:=\; \forall x(\exists y \forall z(z \dot{\in} x \leftrightarrow z = y) \;\rightarrow\; fx \dot{\in} x)$.

Hence $\mathcal{C}[f]$ formalizes that $f$ is a global operation picking from any nonempty type an element; $\mathcal{C}_1[f]$ is a weak version of global choice claiming only that $f$ selects the uniquely determined element of every type that contains exactly one element. It is obvious that $\mathcal{C}[f]$ implies $\mathcal{C}_1[f]$.

Theorem 2.5. $\mathsf{AET}$ *is inconsistent with the statement that there exists a weak global choice operation, i.e.,*

$$\mathsf{AET} \;\vdash\; \neg \exists f\, \mathcal{C}_1[f].$$

Proof. We work within $\mathsf{AET}$, pick the formula $A[u]$ introduced in the proof of the previous theorem, let $a$ be the name of the type defined by $A[u]$, and recall from the proof of the previous theorem that

$$\forall f \neg \mathcal{O}[a, f]. \tag{*}$$

We also set $B[u, v, W] := (<v, u> \in W)$ and assume $\mathcal{C}_1[g]$ for some individual $g$. First observe that Theorem 2.1 provides us with a closed $\mathbb{L}$ term $t_B$ such that

$$\Re(b) \;\rightarrow\; (\Re(t_B(x, b)) \;\wedge\; \forall y(y \dot{\in} t_B(x, b) \;\leftrightarrow\; <x, y> \dot{\in} b)).$$

Since $a$ is a name, this implies that

$$y \dot{\in} t_B(x, a) \;\leftrightarrow\; ((y = 0 \;\wedge\; xx = 1) \;\vee\; (y = 1 \;\wedge\; xx \neq 1)).$$

It only remains to define $s := \lambda ux.g(t_B(x, u))$. Then $sa{\downarrow}$ and, because of $\mathcal{C}_1[g]$, we also have

$$(xx = 1 \;\rightarrow\; s(a, x) = 0) \;\wedge\; (xx \neq 1 \;\rightarrow\; s(a, x) = 1),$$

meaning that $\mathcal{O}[a, sa]$. This is a contradiction to (*), and thus there cannot exist a $g$ with $\mathcal{C}_1[g]$. ⊣

Corollary 2.6. $\mathsf{AET}$ *is inconsistent with the existence of a global choice operation, i.e.,*

$$\mathsf{AET} \;\vdash\; \neg \exists f\, \mathcal{C}[f].$$

Please keep in mind that these forms of weak global choice and global choice must not be confused with other forms of choice such as

$$\forall x \exists y A[x, y] \;\rightarrow\; \exists f \forall x A[x, fx], \tag{AC}$$

where $A[u, v]$ may be any $\mathbb{L}$ formula. However, by taking up the argument in Feferman [9], one can easily see that $\mathsf{AET} + (\mathsf{AC})$ is inconsistent as well.

**2.2. Decidable and semidecidable types.** In explicit mathematics we call a subtype $W$ of a type $V$ *decidable on* $V$ if and only if there exists an operation that is total on V and yields 0 for all elements of $W$ and 1 for all elements of $V$ not in $W$. Accordingly, a subtype $W$ of a type $V$ is denoted as *semidecidable on* $V$ if and only if there exists an operation $f$ such that for all elements $x$ of $V$ we have $fx = 0$ exactly for the elements of $W$.

DEFINITION 2.7.

1. $\mathcal{T}[V, f] := (\forall x \in V)(fx = 0 \vee fx = 1)$.
2. $\mathcal{D}[V, W] := W \subseteq V \wedge \exists f(\mathcal{T}[V, f] \wedge (\forall x \in V)(fx = 0 \leftrightarrow x \in W))$.
3. $\mathcal{SD}[V, W] := W \subseteq V \wedge \exists f(\forall x \in V)(fx = 0 \leftrightarrow x \in W)$.

In addition, we write $\mathcal{D}[W]$ and $\mathcal{SD}[W]$ for $\mathcal{D}[\mathbb{V}, W]$ and $\mathcal{SD}[\mathbb{V}, W]$, respectively.

In $\mathsf{AET}$ or stronger systems like $\mathsf{T}_0$ we cannot prove that a type $U$ is decidable if and only if $U$ and the complement of $U$ are semidecidable. On the other hand, it seems that we can consistently add such a statement in all relevant cases. What is not allowed is to assume that all types are semidecidable.

THEOREM 2.8. $\mathsf{AET}$ *is inconsistent with the statement that all types are semidecidable, i.e.*,

$$\mathsf{AET} \vdash \neg\forall X \mathcal{SD}[X].$$

PROOF. Working informally in $\mathsf{AET}$, use Theorem 2.1 to introduce the type $U$ for which

$$\forall x(x \in U \leftrightarrow xx \neq 0).$$

Hence if $\forall X \mathcal{SD}[X]$ is assumed we have an operation $f$ such that

$$\forall x(fx = 0 \leftrightarrow x \in U).$$

This implies $(ff = 0 \leftrightarrow ff \neq 0)$, a contradiction. ⊣

While it is inconsistent to assume that every type is semidecidable on the universe, we may consistently claim that every subtype of the natural numbers is even decidable on the naturals. To see why, consider the level $V_{\omega+\omega}$ in the cumulative hierarchy and construct the full set-theoretic model as described in Feferman [9]. Then every set-theoretic function belonging to $V_{\omega+\omega}$ – and thus every set of natural numbers – is represented by an operation.

THEOREM 2.9. *The theory* $\mathsf{AET} + (\forall X \subseteq \mathsf{N})\mathcal{D}[\mathsf{N}, X]$ *is consistent*.

It is an easy exercise to extend this theorem to $\mathsf{AET} + (\forall X \subseteq U)\mathcal{D}[U, X]$ for all types $U$ that are bounded in the sense of Feferman [7, 9].

A first approach to dealing with "ordinary" set theory in explicit mathematics is to interpret sets as (names of) types. However, as shown in Jäger [15], it is inconsistent with $\mathsf{AET}$ that the names of the empty type form a type and, consequently, that the strong form of the power types axiom

$$\forall X \exists Y \forall z(z \in Y \leftrightarrow (\exists Z \subseteq X)\Re(z, Z)) \qquad \text{(S-Pow)}$$

is inconsistent with AET as well. The situation is different if we only require that for every type $X$ there exists a type $Y$ that consists of names of all subtypes of $X$,

$$\forall X \exists Y((\forall z \in Y)(\exists Z \subseteq X)\Re(z, Z) \wedge (\forall Z \subseteq X)(\exists z \in Y)\Re(z, Z)). \tag{W-Pow}$$

According to Feferman [9], this weak power type axiom is consistent with AET. From the proof there we can even conclude that the uniform version of (W-Pow) is consistent with AET (and many extensions of AET), but recall from [9] that inconsistencies arise as soon as the join axiom, which allows the formation of disjoint unions of families of types, is added. Also, the weak power type axiom is not really in the spirit of explicit mathematics since the selection of the names of the subtypes that go into the power type is not made explicit.

Because of these complications in dealing with power types, the interpretation of sets as (names of) types does not lead to a satisfactory treatment of set theory within explicit mathematics. Of course, such complications vanish in operational set theory.

**§3. Operational set theory.** Feferman's original motivation for operational set theory was to provide a setting for the operational formulation of large cardinal statements directly over set theory in a way that seemed to him to be more natural mathematically than the metamathematical formulations using reflection and indescribability principles, etc. He saw operational set theory as a natural extension of the von Neumann approach to axiomatizing set theory.[†]

The system OST has been introduced in Feferman [10] and further discussed in Feferman [11] and Jäger [16, 17, 18, 19]. For a first discussion of operational set theory and some general motivation we refer to these articles, in particular to [11].

Besson [3] presents rule based extensions of set theory and in this sense there is some similarity to operational set theory, though starting off from a different motivation. Also, his main system ZFR is conservative over Zermelo–Fraenkel set theory and thus significantly stronger than OST. Cantini and Crosilla [5, 6] and Cantini [4] are about the interplay between some constructive variants of operational set theory and constructive set theory.

In the next paragraphs we present the syntax of operational set theory, though not in its original form (as in the articles mentioned above) but in a slightly modified and essentially equivalent way similar to Zumbrunnen [27].

Let $\mathcal{L}$ be a typical language of first order set theory with the binary symbols $\in$ and $=$ as its only relation symbols and countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \ldots$ (possibly with subscripts). We further assume that $\mathcal{L}$ has a constant $\omega$ for the collection of all finite von Neumann ordinals. The formulas of $\mathcal{L}$ are defined as usual.

[†]Another principal motivation of Feferman [10, 11] was to relate formulations of classical large cardinal statements to their analogues in admissible set theory. However, in view of Jäger and Zumbrunnen [23] this aim of OST has to be analyzed further.

The language $\mathcal{L}^\circ$ of operational set theory extends $\mathcal{L}$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ for definedness and a series of constants: (i) the combinators $\mathsf{k}$ and $\mathsf{s}$, (ii) $\top$, $\bot$, **el**, **non**, **dis**, and **e** for logical operations, (iii) $\mathbb{D}$, $\mathbb{U}$, $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$ for set-theoretic operations. The meaning of these constants will be specified by the axioms below.

As in explicit mathematics, the *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\circ$ are built up from the variables and constants, now by means of our function symbol $\circ$ for application to form expressions $(s \circ t)$. Taking up the conventions of explicit mathematics, $(s \circ t)$ is often abbreviated as $st$ or simply as $st$, again association to the left is made use of so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 \circ s_2)\ldots s_n)$, and frequently we write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$.

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. All expressions of the form $(s \in t)$, $(s = t)$, and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$, the so-called *atomic* formulas.
2. If $A$ and $B$ are formulas of $\mathcal{L}^\circ$ , then so are $\neg A$, $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.
3. If $A$ is a formula of $\mathcal{L}^\circ$ and if $t$ is a term of $\mathcal{L}^\circ$ which does not contain $x$, then $(\exists x \in t)A$, $(\forall x \in t)A$, $\exists x A$, and $\forall x A$ are formulas of $\mathcal{L}^\circ$.

The notions of free variables, $A[\vec{s}/\vec{a}]$, and $B[\vec{s}]$ are as in $\mathbb{L}$, and we often omit parentheses and brackets whenever there is no danger of confusion. The negation $(s \neq t)$ of $(s = t)$ and the partial equality $(s \simeq t)$ are defined as above.

To increase readability, we freely use standard set-theoretic terminology. For example, if $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ denotes the collection of all sets satisfying $A$; it may be (extensionally equal to) a set, but this is not necessarily the case. Special cases are

$$\mathbb{V} := \{x : x\downarrow\}, \quad \emptyset := \{x : x \neq x\}, \quad \text{and} \quad \mathbb{B} := \{x : x = \top \vee x = \bot\}$$

so that $\mathbb{V}$, as in explicit mathematics, denotes the collection of all sets (it is not a set itself), $\emptyset$ stands for the empty collection, and $\mathbb{B}$ for the unordered pair consisting of the truth values $\top$ and $\bot$ (it will turn out that $\emptyset$ and $\mathbb{B}$ are sets in $\mathsf{OST}$). The following shorthand notation, for $n$ an arbitrary natural number greater than 0,

$$(f : a^n \rightarrow b) := (\forall x_1, \ldots, x_n \in a)(f(x_1, \ldots, x_n) \in b)$$

expresses that $f$, in the operational sense, is an $n$-ary mapping from $a$ to $b$. It does not say, however, that $f$ is an $n$-ary function in the set-theoretic sense. In this definition the set variables $a$ and $b$ may be replaced by $\mathbb{V}$ and $\mathbb{B}$. So, for example, $(f : a \rightarrow \mathbb{V})$ means that $f$ is total on $a$, and $(f : \mathbb{V} \rightarrow b)$ means that $f$ maps all sets into $b$.

As in the case of explicit mathematics, also the logic of operational set theory is Beeson's classical logic of partial terms with strictness, including the common equality axioms. The nonlogical axioms of $\mathsf{OST}$ comprise axioms about the applicative structure of the universe, some basic set-theoretic properties, the representation of elementary logical connectives as operations, and operational set existence axioms.

**I. Applicative axioms.**

(A1) $\mathsf{k}xy = x$,

(A2) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq (xz)(yz)$.

**II. Basic set-theoretic axioms.** They comprise: (i) the usual extensionality axiom; (ii) assertions that give the appropriate meaning to the constant $\omega$; (iii) $\in$-induction for arbitrary formulas $A[u]$ of $\mathcal{L}^\circ$,

$$\forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x].$$

**III. Logical operations axioms.**

(L1) $\top \neq \bot$,

(L2) $(\mathbf{el} : \mathbb{V}^2 \to \mathbb{B}) \wedge \forall x \forall y(\mathbf{el}(x, y) = \top \leftrightarrow x \in y)$,

(L3) $(\mathbf{non} : \mathbb{B} \to \mathbb{B}) \wedge (\forall x \in \mathbb{B})(\mathbf{non}(x) = \top \leftrightarrow x = \bot)$,

(L4) $(\mathbf{dis} : \mathbb{B}^2 \to \mathbb{B}) \wedge (\forall x, y \in \mathbb{B})(\mathbf{dis}(x, y) = \top \leftrightarrow (x = \top \vee y = \top))$,

(L5) $(f : a \to \mathbb{B}) \to (\mathbf{e}(f, a) \in \mathbb{B} \wedge (\mathbf{e}(f, a) = \top \leftrightarrow (\exists x \in a)(fx = \top)))$.

**IV. Set-theoretic operations axioms.**

(S1) Unordered pair:

$$\mathbb{D}(a, b)\downarrow \wedge \forall x(x \in \mathbb{D}(a, b) \leftrightarrow x = a \vee x = b).$$

(S2) Union:

$$\mathbb{U}(a)\downarrow \wedge \forall x(x \in \mathbb{U}(a) \leftrightarrow (\exists y \in a)(x \in y)).$$

(S3) Separation for definite operations:

$$(f : a \to \mathbb{B}) \to (\mathbb{S}(f, a)\downarrow \wedge \forall x(x \in \mathbb{S}(f, a) \leftrightarrow (x \in a \wedge fx = \top))).$$

(S4) Replacement:

$$(f : a \to \mathbb{V}) \to (\mathbb{R}(f, a)\downarrow \wedge \forall x(x \in \mathbb{R}(f, a) \leftrightarrow (\exists y \in a)(x = fy))).$$

(S5) Choice:

$$\exists x(fx = \top) \to (\mathbb{C}f\downarrow \wedge f(\mathbb{C}f) = \top).$$

This finishes our description of the system $\mathsf{OST}$. It is known from Feferman [10, 11] and Jäger [16] that $\mathsf{OST}$ is proof-theoretically equivalent to Kripke–Platek set theory with infinity. A recent result of Sato and Zumbrunnen even shows that $\mathsf{OST}$ without the choice axiom (S5) is of the same proof-theoretic strength as Kripke–Platek set theory with infinity; cf. also Zumbrunnen [27].

According to the applicative axioms the universe is a partial combinatory algebra, and thus we have $\lambda$ abstraction and a fixed point operator $\mathsf{fix}$ exactly as in explicit mathematics.

Although $\mathsf{OST}$ itself does not include an axiom for power sets, operational set theory – in contrast to explicit mathematics – provides an ideal framework for introducing them. We simply select a new constant $\mathbb{P}$ and let $\mathsf{OST}(\mathbb{P})$ be the extension of $\mathsf{OST}$ obtained by adding

$$(\mathbb{P} : \mathbb{V} \to \mathbb{V}) \wedge \forall x \forall y(y \in \mathbb{P}x \leftrightarrow y \subseteq x)$$

and formulating all axioms of OST for the new language. We will not consider this system further in the following.

**3.1. Totality.** A first significant difference between explicit mathematics and operational set theory has to do with totality: AET and many extensions such as $\mathsf{T}_0$ are consistent with the totality assumption $\forall x \forall y(xy\downarrow)$. In operational set theory this is not the case.

THEOREM 3.1.

1. *There exists a closed $\mathcal{L}^\circ$ term $t$ such that* OST *proves $t\downarrow$ and $\forall x \neg(tx\downarrow)$.*
2. OST *proves $\neg\forall x \forall y(xy\downarrow)$.*

PROOF. Let $s$ be the term $\lambda xy.\mathbb{D}(xy, xy)$ and set $t := \mathsf{fix}(s)$. Then we have $t\downarrow$ and for any set $u$,

$$tu \simeq s(t, u) \simeq \mathbb{D}(tu, tu) \simeq \{tu\}.$$

Because of the wellfoundedness of the $\in$ relation this is only possible if $tu$ is not defined. Therefore, we have the first assertion, and the second is an immediate consequence. ⊣

A next interesting distinction between explicit mathematics and operational set theory has to do with totality checking. To show this, we make use of the well-known term representation of $\Delta_0$ formulas and an extended form of definition by $\Delta_0$ cases.

The $\Delta_0$ *formulas* of $\mathcal{L}^\circ$ are defined to be those $\mathcal{L}^\circ$ formulas which do not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers. Hence they are the $\Delta_0$ formulas of traditional set theory, possibly containing additional constants. The logical operations make it possible to represent all $\Delta_0$ formulas by constant $\mathcal{L}^\circ$ terms. For a proof of the following lemma see Feferman [10, 11].

LEMMA 3.2. *Let $\vec{u}$ be the sequence of variables $u_1, \dots, u_n$. For every $\Delta_0$ formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term $t_A$ such that* OST *proves*

$$t_A\downarrow \;\wedge\; (t_A : \mathbb{V}^n \to \mathbb{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

In combination with the axiom (S3) about separation for definite operations, this lemma provides us with a uniform version of $\Delta_0$ separation.

THEOREM 3.3. *Let $\vec{u}$ be the sequence of variables $u_1, \dots, u_n$. For every $\Delta_0$ formula $A[\vec{u}, v]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}, v$ free, there exists a closed $\mathcal{L}^\circ$ term $\mathsf{rp}_A$ such that* OST *proves*

$$\mathsf{rp}_A\downarrow \;\wedge\; (\mathsf{rp}_A : \mathbb{V}^{n+1} \to \mathbb{V}) \;\wedge\; \forall \vec{x} \forall y(\mathsf{rp}_A(\vec{x}, y) = \{z \in y : A[\vec{x}, z]\}).$$

Please observe that the relation symbol $\downarrow$ and applications are not permitted in the formulas $A$ of the previous lemma and theorem. We will see later that this restriction is crucial.

As shown in Zumbrunnen [26], the previous lemma can be extended to definition by cases with respect to $\Delta_0$ formulas of $\mathcal{L}^\circ$.

LEMMA 3.4. *Let $\vec{u}$ be the sequence of variables $u_1, \dots, u_n$. For every $\Delta_0$ formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term $s_A$ such that* OST *proves*:

1. $s_A(u,v)\downarrow \ \wedge \ (s_A(u,v) : \mathbb{V}^n \to \mathbb{D}(u,v))$.
2. $(A[\vec{w}] \ \to \ s_A(u,v,\vec{w}) = u) \ \wedge \ (\neg A[\vec{w}] \ \to \ s_A(u,v,\vec{w}) = v)$.

We may ask the question whether testing all operations for totality is consistent with explicit mathematics and operational set theory.

Definition 3.5. We call an operation $f$ a *totality checker* if and only if it has the property $\mathcal{TC}[f]$, where

$$\mathcal{TC}[f] \ := \ (f : \mathbb{V} \to \mathbb{B}) \wedge \forall x(fx = \top \leftrightarrow \forall y(xy\downarrow)).$$

The consistency of explicit mathematics with the existence of a totality checker is a trivial consequence of the fact that explicit mathematics is consistent with the assumption that all operations are total. The situation in operational set theory is different.

Theorem 3.6. $\mathsf{OST}$ *is inconsistent with the existence of a totality checker, i.e.,*

$$\mathsf{OST} \ \vdash \ \neg\exists f \mathcal{TC}[f].$$

Proof. We work within $\mathsf{OST}$ and assume that there exists an $f$ such that $\mathcal{TC}[f]$. Then consider the $\Delta_0$ formula $A[u] := (u = \top)$ and select the term $s_A$ according to Lemma 3.4. Now set

$$r_0 \ := \ \lambda xy.(s_A(\lambda uv.\mathbb{D}(uv,uv), \lambda uv.\bot, fx)xy)$$

and obtain $\forall x \forall y(r_0(x,y)\downarrow)$. More precisely, we have for all $x$ and $y$ that

$$r_0(x,y) \ = \ \begin{cases} \mathbb{D}(xy,xy) & \text{if } fx = \top, \\ \bot & \text{if } fx = \bot. \end{cases} \tag{1}$$

Finally, for $r_1 := \mathsf{fix}(r_0)$ and any $a$ the properties of fix yield $r_1(a) \simeq r_0(r_1,a)$. Since $\forall x \forall y(r_0(x,y)\downarrow)$ this even implies

$$r_1(a) = r_0(r_1,a) \quad \text{and} \quad r_1(a)\downarrow \tag{2}$$

for all $a$. Hence $f(r_1) = \top$, and (1) and (2) give us $r_1(a) = \mathbb{D}(r_1(a), r_1(a))$ for any $a$. As in the proof of Theorem 3.1 this is a contradiction. Hence a totality checker cannot exist in $\mathsf{OST}$. ⊣

**3.2. Function spaces and extensionality.** Let $U$ and $V$ be types in explicit mathematics, for example, in the theory $\mathsf{AET}$. Then elementary comprehension implies the existence of types $W_1$ and $W_2$ with the following properties:

(i) $f \in W_1 \ \leftrightarrow \ \forall x(x \in U \ \to \ fx \in V)$;
(ii) $f \in W_2 \ \leftrightarrow \ \forall x(x \in U \ \to \ fx \in V) \wedge \forall x(x \notin U \ \to \ \neg(fx\downarrow))$.

This means that $\mathsf{AET}$ and explicit mathematics in general allow the formation of (i) the type of all operations from a given $U$ to a given $V$ as well as the formation of (ii) the type of all operations from a given $U$ to a given $V$ that are undefined outside $U$. Our next theorem tells us that a corresponding result is false in operational set theory.

Theorem 3.7. *The following three assertions are provable in* $\mathsf{OST}$:

1. *If set $a$ contains at least one element and set $b$ contains at least two elements, then $\{f : (f : a \to b)\}$ is not a set.*

2. *If set $a$ contains at least one element and set $b$ contains at least two elements, then the collection*

$$\{f : (f : a \to b) \wedge \forall x(x \notin a \to \neg(fx\downarrow))\}$$

*is not a set.*

3. *If set $a$ contains at least one element and set $b$ contains at least two elements, then, for any set $w$, the collection*

$$\{f : (f : a \to b) \wedge \forall x(x \notin a \to (fx = w))\}$$

*is not a set.*

Proof. We confine ourselves to proving the first assertion; the proofs of the second and third are obtained by suitable modifications and given in Zumbrunnen [26]. So let $a$ and $b$ be sets in $\mathsf{OST}$ with an element $a_0 \in a$ and two different elements $b_1, b_2 \in b$. Also, assume that $\{f : (f : a \to b)\}$ is a set $c$. For the $\Delta_0$ formula $A[u, v] := (u = v)$ we first pick the closed $\mathcal{L}^\circ$ term $s_A$ according to Lemma 3.4 and then define

$$r_0 \ := \ \lambda f x.s_A(b_1, b_2, fx, b_2).$$

For all $f \in c$ and $x \in a$ we thus have $r_0(f, x) \in b$ and

$$r_0(f, x) \ = \ \begin{cases} b_1 & \text{if } fx = b_2, \\ b_2 & \text{if } fx \neq b_2. \end{cases} \tag{1}$$

Now we take the $\Delta_0$ formula $B[u, v] := (u \in v)$, select the closed $\mathcal{L}^\circ$ term $s_B$ according to Lemma 3.4 and define

$$r_1 \ := \ \lambda gy.(s_B(r_0, \lambda fx.b_1, g, c)gy).$$

In view of (1) it is easy to see that for any $g$ and all $y \in a$,

$$r_1(g, y) \ = \ \begin{cases} r_0(g, y) & \text{if } g \in c, \\ b_1 & \text{if } g \notin c. \end{cases} \tag{2}$$

It only remains to set $r_2 := \mathsf{fix}(r_1)$. Consequently, $r_2(z) \simeq r_1(r_2, z)$ for all $z$. In particular, for $z \in a$ the equations (1) and (2) yield $r_1(r_2, z) \in b$, thus $r_2(z) \in b$ as well. This means that $r_2 \in c$.

Finally, take the element $a_0$ of $a$. Making use of (1) and (2) once more, we derive the following sequence of equations,

$$r_2(a_0) \ = \ r_1(r_2, a_0) \ = \ r_0(r_2, a_0) \ = \ \begin{cases} b_1 & \text{if } r_2(a_0) = b_2, \\ b_2 & \text{if } r_2(a_0) \neq b_2. \end{cases}$$

Since $b_1$ and $b_2$ are different, this is a contradiction. Hence $\{f : (f : a \to b)\}$ cannot be a set in $\mathsf{OST}$. $\dashv$

A similar problem has been discussed in Cantini and Crosilla [5, 6]. There Cantini and Crosilla study systems $\mathsf{COST}$ and $\mathsf{EST}$ of constructive operational set theory and mention that Minari observed the inconsistency of $\mathsf{EST}$ plus the assertion

$$\forall x \forall y \exists z(z = \{f : (f : x \to y)\}).$$

Their strategy of proof is different. In these two articles, Cantini and Crosilla also address the question of operational extensionality,

$$\forall f \forall g(\forall x(fx \simeq gx) \rightarrow f = g) \tag{$\mathsf{EXT}$}$$

and show that $(\mathsf{EXT})$ is inconsistent with their $\mathsf{COST}$ and $\mathsf{EST}$. They argue that for every total operation $f$ in an extensional partial combinatory algebra there exists an $x$ such that $fx = x$ and proved that this is not the case in models of $\mathsf{COST}$ and $\mathsf{EST}$. We obtain this as an immediate consequence of our previous theorem.

Corollary 3.8. $\mathsf{OST}$ *is inconsistent with* $(\mathsf{EXT})$, *i.e.*,

$$\mathsf{OST} \vdash \neg(\mathsf{EXT}).$$

Proof. In $\mathsf{OST}$ we have the one-element set $a := \mathbb{D}(\bot, \bot) = \{\bot\}$ and the two-element set $b := \mathbb{D}(\bot, \top) = \{\bot, \top\}$; obviously, $\bot$ is provably different from $\top$. Consider the $\mathcal{L}^\circ$ terms

$$r_0 := \lambda x.\bot \qquad \text{and} \qquad r_1 := \lambda x.s_A(\top, \bot, x, \bot),$$

with $A[u, v] := (u = v)$ and $s_A$ chosen according to Lemma 3.4. For any $f$ satisfying

$$(f : a \rightarrow b) \wedge \forall x(x \notin a \rightarrow fx = \bot)$$

we thus conclude

$$\forall x(fx = r_0(x)) \vee \forall x(fx = r_1(x)). \tag{*}$$

Now assume $(\mathsf{EXT})$. Then $(*)$ yields

$$\{f : (f : a \rightarrow b) \wedge \forall x(x \notin a \rightarrow fx = \bot)\} = \mathbb{D}(r_0, r_1).$$

Since $\mathbb{D}(r_0, r_1)$ is a set, this contradicts Theorem 3.7. Hence $(\mathsf{EXT})$ has to be false in $\mathsf{OST}$. $\dashv$

**3.3. Separation with definedness and application.** In view of Theorem 2.1 we know that in explicit mathematics all elementary $\mathbb{L}$ formulas $A[u]$ can be used to form the type $U$ of all elements $x$ satisfying $A[x]$,

$$\forall x(x \in U \leftrightarrow A[x]),$$

and elementary $\mathbb{L}$ formulas may contain the definedness relation $\downarrow$ as well as application terms, i.e., terms of the form $st$. Hence in explicit mathematics types can be formed with reference to definedness assertions and applications of terms to each other.

In this section we will show that the situation is more intricate in the case of operational set theory. In a nutshell: (i) From Theorem 3.3 we know that uniform $\Delta_0$ separation can be proved in $\mathsf{OST}$, but $\Delta_0$ formulas of $\mathcal{L}^\circ$ must not contain the definedness predicate and applications. (ii) Even the simplest forms of *uniform* separation with definedness and application lead to inconsistencies. (iii) The *nonuniform* versions of these separations are consistent with $\mathsf{OST}$, but adding them to $\mathsf{OST}$ as further axioms increases the proof-theoretic strength from that of Kripke–Platek set theory with infinity to that of Kripke–Platek set theory with infinity and $\Sigma_1$ separation.

THEOREM 3.9. $\mathsf{OST}$ *is inconsistent with uniform comprehension (separation) for formulas involving definedness and application; in particular, we have:*

1. $\mathsf{OST} \vdash \neg\exists f \forall x(fx\downarrow \wedge fx = \{y \in x : yy\downarrow\})$.
2. $\mathsf{OST} \vdash \neg\exists f \forall x \forall g(f(x,g)\downarrow \wedge f(x,g) = \{y \in x : gy\downarrow\})$.

PROOF. Working in $\mathsf{OST}$, we proceed indirectly for establishing the first assertion and assume that $f$ is an operation that satisfies

$$\forall x(fx\downarrow \wedge fx = \{y \in x : yy\downarrow\}).$$

For the $\mathcal{L}^\circ$ term $r_0 := \lambda x.f(\mathbb{D}(x,x))$ and any $a$ we thus have

$$r_0(a)\downarrow \ \wedge \ (r_0(a) \neq \emptyset \leftrightarrow aa\downarrow). \tag{*}$$

Now let $A[u]$ be the $\Delta_0$ formula $(u \neq \emptyset)$ and $B[u]$ the $\Delta_0$ formula $(u \neq \top)$. For $A[u]$ we choose an $\mathcal{L}^\circ$ term $s_A$ according to Lemma 3.4 and for $B[u]$ an $\mathcal{L}^\circ$ term $t_B$ according to Lemma 3.2. Then we define

$$r_1 \ := \ \lambda x.(s_A(\lambda y.t_B(yy), \lambda y.\bot, r_0(x))x).$$

Given an arbitrary $a$, statement (*) yields $r_0(a)\downarrow$ and we conclude

$$r_1(a) \ = \ \begin{cases} (\lambda y.t_B(yy))a & \text{if } r_0(a) \neq \emptyset, \\ (\lambda y.\bot)a & \text{if } r_0(a) = \emptyset. \end{cases}$$

Together with (*), we thus obtain

$$r_1(a) \ = \ \begin{cases} t_B(aa) & \text{if } aa\downarrow, \\ \bot & \text{if } \neg(aa\downarrow), \end{cases}$$

and because of the properties of $t_B$ this implies

$$r_1(a) \ = \ \begin{cases} \top & \text{if } aa\downarrow \wedge aa \neq \top, \\ \bot & \text{if } aa\downarrow \wedge aa = \top, \\ \bot & \text{if } \neg(aa\downarrow). \end{cases}$$

Hence $r_1(a)\downarrow$ for all $a$ and $(r_1(r_1) = \top \leftrightarrow r_1(r_1) \neq \top)$, a contradiction. This settles the first assertion of our theorem.

For the proof of the second assertion, assume that there is an $f$ with

$$\forall x \forall g(f(x,g)\downarrow \wedge f(x,g) = \{y \in x : gy\downarrow\}).$$

Now consider the $\mathcal{L}^\circ$ term $t := \lambda x.f(x, \lambda y.yy)$ and observe that

$$\forall x(tx\downarrow \wedge tx = \{y \in x : yy\downarrow\}).$$

This reduces the second assertion to the first, and the proof of our theorem is completed. ⊣

What about nonuniform separations with definedness and applications? Let us define the $\Delta_0^+$ formulas of $\mathcal{L}^\circ$ to be the $\mathcal{L}^\circ$ formulas without unbounded quantifiers. So, in contrast to the $\Delta_0$ formulas of $\mathcal{L}^\circ$, the $\Delta_0^+$ formulas may contain the definedness relation and application terms. Below we show that $\mathsf{OST}$ plus nonuniform $\Delta_0^+$ separation, i.e.,

$$\forall \vec{a} \forall b \exists c(c = \{z \in b : A[\vec{a}, z]\}) \tag{$\Delta_0^+$-$\mathsf{Sep}$}$$

where $A[\vec{u}, v]$ is a $\Delta_0^+$ formula of $\mathcal{L}^\circ$, is consistent. To calibrate the exact consistency strength of $\mathsf{OST} + (\Delta_0^+\text{-}\mathsf{Sep})$ we refer to a well-known extension of Kripke–Platek set theory.

The theory KP is the standard system of Kripke–Platek set theory with infinity as presented, for example, in Barwise [1], Jäger [13], or Rathjen [24]. It is formulated in the language $\mathcal{L}$, and $\Sigma_1$ separation is the schema

$$\forall \vec{a} \forall b \exists c (c = \{z \in b : \exists x A[\vec{a}, x, z]\}) \tag{$\Sigma_1$-Sep}$$

for $A[\vec{u}, v, w]$ ranging over all $\Delta_0$ formulas of $\mathcal{L}$. As usual we write $(\mathbf{V} = \mathbf{L})$ for the axiom of constructibility.

THEOREM 3.10. *The theory* $\mathsf{OST} + (\Delta_0^+\text{-}\mathsf{Sep})$ *can be interpreted in the theory* $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep}) + (\mathbf{V} = \mathbf{L})$.

PROOF. In Jäger and Zumbrunnen [23] a natural translation is introduced that maps an $\mathcal{L}^\circ$ formula $A$ to a formula $A^\star$ such that

$$\mathsf{OST} \vdash A \quad \Longrightarrow \quad \mathsf{KP} + (\mathbf{V} = \mathbf{L}) \vdash A^\star.$$

The crucial point of this interpretation is a $\Sigma_1$ formula $App[u, v, w]$, for application, taking care of the $\mathcal{L}^\circ$ formula $(uv = w)$. Based on that, it can be easily shown that every $\Delta_0^+$ formula $A$ of $\mathcal{L}^\circ$ translates into a formula $A^\star$ of $\mathcal{L}$ that is $\Delta_0$ in $\Sigma_1$. Since separation for this class of formulas is provable in $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep}) + (\mathbf{V} = \mathbf{L})$ we have our result. ⊣

For the converse direction we will now see that adding to OST the very special instance

$$\forall x \forall f \exists y (y = \{z \in x : fz\downarrow\}) \tag{DEF}$$

of $(\Delta_0^+\text{-}\mathsf{Sep})$, testing only for definedness, is sufficient for establishing $\Sigma_1$ separation.

THEOREM 3.11. *Every instance of* $(\Sigma_1\text{-}\mathsf{Sep})$ *is provable in* $\mathsf{OST} + (\mathsf{DEF})$.

PROOF. Let $A[\vec{u}, v, w]$ be a $\Delta_0$ formula of $\mathcal{L}$ with at most the variables $\vec{u}, v, w$ free and select a closed $\mathcal{L}^\circ$ term $t_A$ according to Lemma 3.2. Depending on this $t_A$ we now define

$$s \ := \ \lambda \vec{x} z.\mathbb{C}(\lambda y.t_A(\vec{x}, y, z)).$$

The axiom (S5) of OST about the choice operator $\mathbb{C}$ therefore yields for all $\vec{a}$ and $z$ that

$$\exists x A[\vec{a}, x, z] \ \leftrightarrow \ (s(\vec{a}, z)\downarrow \ \wedge \ A[\vec{a}, s(\vec{a}, z), z]). \tag{*}$$

Hence for any $b$ our additional axiom (DEF) implies the existence of a set $c$ such that

$$c \ = \ \{z \in b : s(\vec{a}, z)\downarrow\}.$$

For the $\mathcal{L}^\circ$ term $r := \lambda \vec{x} \lambda z.t_A(\vec{x}, s(\vec{x}, z), z)$ and all $\vec{a}$ we can easily verify that

$$r(\vec{a}) : c \to \mathbb{B}.$$

Thus we are ready to make use of axiom (S3) about separation for definite operations and obtain the set $\mathbb{S}(r(\vec{a}), c)$ for which

$$z \in \mathbb{S}(r(\vec{a}), c) \ \leftrightarrow \ z \in c \ \wedge \ r(\vec{a}, z) = \top,$$

consequently also

$$z \in \mathbb{S}(r(\vec{a}), c) \;\leftrightarrow\; z \in b \,\wedge\, s(\vec{a}, z)\!\downarrow \,\wedge\, t_A(\vec{a}, s(\vec{a}, z), z) = \top.$$

Due to the properties of $t_A$ and (*) this yields that

$$\mathbb{S}(r(\vec{a}), c) \;=\; \{z \in b : \exists x A[\vec{a}, x, z]\}.$$

So we have shown separation for the $\Sigma_1$ formula $\exists x A[\vec{u}, x, v]$ of $\mathcal{L}^\circ$ and arbitrary parameters $\vec{a}$ and $b$. ⊣

COROLLARY 3.12. *$\mathsf{OST} + (\Delta_0^+\text{-}\mathsf{Sep})$ and $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ are equiconsistent.*

This corollary is immediate from Theorem 3.10 and Theorem 3.11 simply by recalling that $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep}) + (\mathbf{V} = \mathbf{L})$ is conservative over $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ for absolute formulas and since $\mathsf{OST}$ contains $\mathsf{KP}$ according to Feferman [10, 11] and Jäger [16].

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