The anisotropic $\lambda$-deformed $SU(2)$ model is integrable

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ARTICLE INFO

Article history:
Received 22 December 2014
Accepted 17 February 2015
Available online 20 February 2015
Editor: L. Alvarez-Gaumé

ABSTRACT

The all-loop anisotropic Thirring model interpolates between the WZW model and the non-Abelian T-dual of the anisotropic principal chiral model. We focus on the $SU(2)$ case and we prove that it is classically integrable by providing its Lax pair formulation. We derive its underlying symmetry current algebra and use it to show that the Poisson brackets of the spatial part of the Lax pair, assume the Maillet form. In this way we procure the corresponding $r$ and $s$ matrices which provide non-trivial solutions to the modified Yang–Baxter equation.

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1. Introduction and motivation

The general class of $\sigma$-models whose integrability properties will be investigated was constructed in [1]. The corresponding action is given by

$$S_{k,\lambda}(g) = S_{\text{WZW},k}(g) - \frac{k}{\pi} \int J_+^A M_{AB}^{-1} J_-. $$

$$M_{AB} = (\lambda^{-1} - D^T)_{AB}, \quad (1.1)$$

where the first term is the WZW model action for a semi-simple compact group $G$ and a group element $g \in G$ given by [2]

$$S_{\text{WZW},k}(g) = -\frac{k}{2\pi} \int \text{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g) + \frac{k}{6\pi} \int \text{Tr}(g^{-1} d g)^3. $$

(1.2)

This is a CFT with two commuting current algebras at level $k$. The second term in (1.1) represents the deformation from the conformal point. Our conventions are $^3$

$$f_+^A = \text{Tr}(\partial_+ g^{-1} g^{-1} \partial_- g), \quad f_-^A = \text{Tr}(\partial_- g^{-1} g^{-1} \partial_- g),$$

$$D_{AB} = \text{Tr}(\partial g t_{AB} g^{-1}). \quad (1.3)$$

The above action has a dim $G$ target space with coordinates the parameters in the group element $g \in G$. The $t_A$s are representation matrices obeying the Lie algebra $[t_A, t_B] = f_{ABC} t_C$ and normalized as $\text{Tr}(t_{AB}) = \delta_{AB}$. The deviation from the WZW model is parametrized by the coupling matrix elements $\lambda_{AB}$. For small such elements the Lagrangian density is proportional to the current bilinear $\lambda_{AB} f_+^A f_+^B$. Hence, the name $\lambda$-deformed models. The above action develops an extra local invariance under the vector action of a subgroup $H \subset G$ when $\lambda_{AB}$ assumes the block diagonal form $\lambda_{AB} = \text{diag}(t_{ab}, \lambda_{\alpha\beta})$, where the lower case Latin indices take values in the Lie algebra of $H$ and the Greek ones in the coset $G/H$. Due to this local invariance dim $H$ degrees of freedom become redundant. Hence, dim $H$ variables among those parameterizing $g$ should be gauged fixed. For vanishing $\lambda_{\alpha\beta}$ the $\sigma$-model corresponds to the coset $G/H$ CFT. In addition, the perturbation is driven by parafermion bilinears $\lambda_{\alpha\beta} \psi_\alpha^T \psi_\beta$, where the $\psi_\alpha^T$s are gauge invariant versions of the currents $f_+^A$. The renormalization group equations for $\lambda_{AB}$ in the action (1.1) have been computed for the isotropic case in [3] and in full generality in [4]. In addition, the (1.1) has been used as a building block to construct full solution of type-II supergravity in [5] which are likely also integrable at the string level.

In this paper we are interested in investigating integrability property of the above action. Integrability has been first proven for the isotropic case when $\lambda_{AB} = \lambda \delta_{AB}$ and a general semi-simple group $G$ in [1]. This was done by explicitly showing that certain algebraic conditions developed in [6] (based on earlier work in [7]) were satisfied. In addition, it has been proved that these models have an underlying Yangian symmetry [8]. In [1], integrability

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$^3$ The world-sheet coordinates ($\sigma^+$, $\sigma^-$) and ($\tau$, $\sigma$) are related by $\sigma^\pm = \tau \pm \sigma$, so that $\partial_0 = \partial_\tau = \partial_+ \partial_- \partial_\tau = \partial_\sigma - \partial_\tau ...$

http://dx.doi.org/10.1016/j.physletb.2015.02.040
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was also expected for the coset SU(2)/U(1) case by making contact with the work of [9] where a CFT approach was utilized. The most efficient way to prove integrability of (1.1) for specific choices of the matrix $\lambda$ is to employ its origin via a gauging procedure as much as possible. This was done for the aforementioned isotropic group case as well as for the general symmetric coset space, for isotropic coupling $\lambda_{\alpha\beta} = \lambda \delta_{\alpha\beta}$ in [10].

In the present paper we will generalize this approach for a general matrix $\lambda$ and we will prove integrability for the anisotropic SU(2) model for general diagonalizable matrix $\lambda_{AB}$. The computation amounts to showing integrability for the diagonal matrix $\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. In addition, we will compute the Poisson brackets of the spatial component of the corresponding monodromy matrix and we will provide non-trivial solutions to the modified Yang–Baxter equation.

2. Origin of integrability

In this section we review the construction of our models, derive the equations of motion and set them up in such a way that investigating the existence of a Lax pair formulation becomes immediate.

2.1. Review of the models

We review the construction of the models by following [1]. The starting point is the action

$$S(g, \bar{g}) = S_{WZW,k}(g) + S_{PCM,E}(\bar{g}).$$

(2.1)

where the first term is the WZW action (1.2) and the second term is the principal chiral model (PCM) action for $G$ using a group element $\bar{g} \in G$.

$$S_{PCM,E}(\bar{g}) = -\frac{1}{\pi} \int E_{AB} \text{Tr}(t^A \bar{g}^{-1} \partial_+ \bar{g}) \text{Tr}(t^B \bar{g}^{-1} \partial_- \bar{g}).$$

(2.2)

The action (2.1) is invariant under left–right current algebra symmetry of the WZW action and a global left symmetry of the PCM.

We will gauge the same global group

$$g \rightarrow \Delta^{-1} g A, \quad \bar{g} \rightarrow \Delta^{-1} \bar{g}, \quad \Delta \in G.$$

(2.3)

Hence we consider the action

$$S_{k,E}(g, \bar{g}) = S_{WZW,k}(g, A_{\pm}) + S_{PCM,E}(\bar{g}, A_{\pm}),$$

(2.4)

where

$$S_{WZW,k}(g, A_{\pm}) = S_{WZW,k}(g) + \frac{k}{\pi} \int \text{Tr} \left( A_- \partial_+ g g^{-1} \right.$$

$$- A_+ g^{-1} \partial_- g - A_- g A g^{-1} - A_- A_+ \left. \right)$$

(2.5)

and

$$S_{PCM,E}(\bar{g}, A_{\pm}) = -\frac{1}{\pi} \int E_{AB} \text{Tr}(t^A \bar{g}^{-1} \bar{D}_+ \bar{g}) \text{Tr}(t^B \bar{g}^{-1} \bar{D}_- \bar{g}),$$

(2.6)

with the covariant derivatives being $\bar{D}_\pm \bar{g} = \partial_\pm \bar{g} - A_{\pm} \bar{g}$. This action (2.5) is invariant under the local symmetry

$$\bar{g} \rightarrow \Delta^{-1} \bar{g}, \quad g \rightarrow \Delta^{-1} g A, \quad A_{\pm} \rightarrow \Delta^{-1} A_{\pm} \Lambda - \Lambda^{-1} \partial_\pm A.$$

(2.7)

We will use the coupling matrix $\lambda$ defined as $E = k(\lambda^{-1} - I)$. Finally we mention that the action (2.4) is invariant under the generalized parity symmetry

$$\sigma^+ \leftrightarrow \sigma^-, \quad g \mapsto g^{-1}, \quad \bar{g} \mapsto \bar{g}, \quad A_+ \leftrightarrow A_-, \quad \lambda \mapsto \lambda^T.$$
which obey two commuting copies of current algebras [7,11]
\[ S_{\pm}^A, S_{\pm}^B = f_{ABC} S_{\pm}^C \delta_{\sigma \sigma'} \pm \frac{k}{2} \delta_{AB} \delta_{\sigma \sigma'}, \quad \delta_{\sigma \sigma'} = \delta(\sigma - \sigma'), \]  
(2.17)
where we have dropped the time dependence at usual equal time Poisson brackets. Since the action does not depend on derivatives of \( A_\pm \), its equations-of-motion are second class constraints [10,12]
\[ S_+ = \frac{k}{2} (\lambda^{-1} A_+ - A_-), \quad S_- = \frac{k}{2} (\lambda^{-1} A_- - A_+), \]
(2.18)
and inversely
\[ A_+ = \frac{2}{k} g^{-1} l_\lambda T (S_+ + \lambda S_-), \quad A_- = \frac{2}{k} \bar{g}^{-1} \bar{\lambda} (S_- - \bar{\lambda} T S_+), \]
\[ g = 1 - \lambda T, \quad \bar{g} = 1 - \bar{\lambda} \lambda T, \]  
(2.19)
where we assume that \( g, \bar{g} \) are positive-definite matrices. It is just a matter of algebra to rewrite the current algebras for \( S_\pm \) in the base of \( A_\pm \), as we are going to present in the subsequent sections.

3. Known integrable cases

In this section we review the known (isotropic) integrable cases, semi-simple group and general symmetric coset spaces, using the previous formulation.

3.1. The isotropic group space

As a warmup, we review the integrability for the isotropic case for a semi-simple group \( G [1,10] \). Then the equations of motion for the gauge field read
\[ \partial_\pm A_\mp = \pm \frac{1}{1 + \lambda} [A_+, A_-], \]  
(3.1)
and a simple rescaling
\[ A_\pm = -\frac{1}{2} (1 + \lambda) I_\pm, \]  
(3.2)
proves the integrability. As for the Lax pair, this is given by
\[ L_\pm = \frac{2}{1 + \lambda} \frac{\mu}{T} A_\pm, \]  
(3.3)
where \( \mu \in \mathbb{C} \) is the spectral parameter.

3.1.1. Algebraic structure

Employing (2.17), (2.19) and (3.2), we find the Poisson brackets for \( l_\pm [6] \)
\[ \{ l_\pm^A, l_\pm^B \} = e^2 f_{ABC} \left( I_\mp^C + (1 + 2x) I_\pm^C \right) \delta_{12} \pm 2e^2 \delta_{AB} \delta_{12}, \]
\[ \{ I_\pm^A, I_\pm^B \} = -e^2 f_{ABC} \left( I_\mp^C + I_\pm^C \right) \delta_{12}, \]
(3.4)
where
\[ e = \frac{2 \lambda}{\sqrt{1 + \lambda^2}}, \quad x = \frac{1 + \lambda^2}{2 \lambda}, \quad x > 1, \]  
(3.5)
where the deformation parameter is a root of unity [10]. We note that the same underlying structure, but with \(-1 < x < 1\), corresponds to integrable deformations of the \( \sigma \)-model [13] constructed in [17,18], where the deformation parameter is real. The corresponding quantum properties at one-loop were studied in [14–16].

There are two interesting limits. Expanding \( \lambda \) near zero and rescaling \( I_\pm^A \mapsto -2e^2 x^2 I_\pm^A \) we find that
\[ \{ I_\pm^A, I_\pm^B \} = f_{ABC} I_\mp^C \delta_{\sigma \sigma'} \pm \frac{k}{2} \delta_{AB} \delta_{\sigma \sigma'}, \quad \{ I_\pm^A, I_\pm^B \} = 0. \]  
(3.6)
These are two commuting current algebras in accordance with the fact that in this limit the \( \sigma \)-model corresponds to a CFT.

Parameterizing \( \lambda \) as \( \lambda = k (1 + \nu)^{-1} \) and then letting \( k \to 1 \), we find the algebra of the non-Abelian T-dual of the PCM on \( G \)
\[ \{ I_\pm^A, I_\pm^B \} = \frac{1}{2e} f_{ABC} (I_+^C - 3I_-^C) \delta_{12} \pm \frac{1}{e} \delta_{AB} \delta_{12}, \]
\[ \{ I_\pm^A, I_\pm^B \} = -\frac{1}{2e} f_{ABC} (I_+^C + I_-^C) \delta_{12}, \]  
(3.7)
This is the same as the algebra for the PCM on \( G \), in accordance with the fact that the two cases are related by a canonical transformation.

3.2. The isotropic symmetric coset

Let us consider a semi-simple group \( G \) and its decomposition to a semi-simple subgroup \( H \) and a symmetric coset \( G/H \). Take the case where the matrix \( \lambda \) has elements \( \lambda_{ab} = \lambda_{h} \delta_{ab} \), \( \lambda_{ab} = \lambda_{h} \delta_{gh} \). The restriction to symmetric cosets translates to structure constants \( f_{abc} = 0 \), whereas \( f_{ABC}, f_{ABC} \neq 0 \). For \( \lambda_{h}, \lambda_{G/H} \neq 1 \) we have that (2.13) or (2.14) read
\[ \partial_\pm A_\pm = \pm (1 + \lambda_{H})^{-1} \left( [A_+, A_-] + \frac{\lambda_{H}}{\lambda_{G/H}} [B_+, B_-] \right), \]
\[ \partial_\pm B_\pm = \frac{1}{\lambda_{H}(1 - \lambda_{G/H})} \left( (\lambda_{G/H} H - \lambda_{H}) [B_+, A_-] \right) + \lambda_{G/H}(1 - \lambda_{H}) [B_+, A_+] \right), \]  
(3.8)
where \( A_\pm \) and \( B_\pm \) are Lie algebra valued one forms \( (A_\pm = A_\pm^{ab} t_a, B_\pm = B_\pm^a t_a) \), on the subgroup and coset respectively. The above consideration is drastically modified in the two cases we have excluded. The first special case is when \( \lambda_{H} = 1 \). In this singular limit we have to use (2.13) and the equations of motion simplify drastically
\[ \partial_+ A_- - \partial_- A_+ = [A_+, A_-] + \frac{1}{\lambda_{G/H}} [B_+, B_-], \]
\[ \partial_\pm B_\pm = -[B_\mp, A_\pm], \]  
(3.9)
and the two come for \( A_\pm \) in (3.8) are replaced by their difference. This case was shown to be integrable in [10] with Lax pair given by
\[ L_\pm = A_\pm + \frac{\mu^{\pm 1}}{\sqrt{\lambda_{G/H}}} B_\pm, \]  
(3.10)
where \( \mu \in \mathbb{C} \). It can be readily checked that then (2.15) is satisfied.

4. The anisotropic SU(2) case

In this section we consider the other special case in which the subgroup \( H \) is Abelian. In addition to demanding that the space \( G/H \) is symmetric, restrict our considerations to the group case

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4 In the conventions of Section 1 we denote subgroup indices by Latin letters and coset indices by Greek letters.

5 Note that for \( \lambda_{G/H} = 1, \lambda_{H} \) turns to be one for finiteness of the expressions. For general cosets \( G/H \) the equations for \( B_\pm \) contain the additional term \( (1 + \lambda_{G/H})^{-1} [B_+, B_-] \).

6 We have tried to construct a Lax pair for (3.8) of the form \( L_\pm = a_\pm A_\pm + b_\pm B_\pm \) where the coefficients are constants. For \( \lambda_{H} \neq 1 \) and for non-Abelian subgroup \( H \) one obtains a linear algebraic inhomogeneous system with has a unique solution. This implies that within this ansatz for the Lax pair one cannot prove integrability.
SU(2). We will consider the cases of a diagonalizable matrix \(\lambda_{AB}\), 
\(A, B = 1, 2, 3\). Then as explained before it is sufficient to consider the case with 
\[\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3).\]  
(4.1)

that is the fully anisotropic albeit diagonal case. In this case the generators are \(f_{AB} = -i\sigma_A / \sqrt{2}\), where \(\sigma_A\) are the Pauli matrices, so that \(f_{ABC} = \sqrt{2}\delta_{ABC}\). Then a straightforward computation shows that 
\[\hat{\sigma}_\pm A_\mp^1 = \frac{\sqrt{2}\lambda_1}{(1 - \lambda_1^2)\lambda_2\lambda_3} \left[ (\lambda_2 - \lambda_1\lambda_3)A_\mp^2 A_\pm^3 - (\lambda_3 - \lambda_1\lambda_2)A_\mp^3 A_\pm^2 \right],\]  
(4.2)

and cyclic in 1, 2 and 3.

This is in agreement with (3.8), for \(\lambda_H = \lambda_1, \lambda_{G/H} = \lambda_2 = \lambda_3\), where \(H = U(1)\) and SU(2)/U(1) symmetric coset. Moreover, along the results of Section 3.2 and [10], the coset limit \(\lambda_1 = 1\) is integrable and for compatibility \(\lambda_2 = \lambda_3\). As shown in [1] expanding the \(\lambda_i\)'s around one, we get the non-Abelian T-dual of the anisotropic PCM for SU(2), which is integrable due to the fact that the PCM is integrable [19,20] and non-Abelian T-duality preserves integrability [21]. All these are signals that the more general case we consider here with (4.1) is likely integrable as well.

Let's define a convenient set of fields given by
\[X_\pm^1 = \frac{A_\pm^1}{\lambda_1\sqrt{(1 - \lambda_2^2)(1 - \lambda_3^2)}}\]  
(4.3)

and cyclic in 1, 2 and 3. Then we assume the following expression for a Lax pair
\[L_{\pm}^A(\tau, \sigma; \mu) = v_+^A(\mu)X_{\pm}^A,\]  
where \(\mu \in \mathbb{C}\) and \(v_\pm^A\) satisfy six non-linear equations
\[c_2 v_\pm^1 v_\mp^1 + c_3 v_\pm^1 v_\pm^3 + c_1 v_\pm^3 = v_\mp^1 v_\pm^1,\]  
\[c_1 v_\pm^2 + c_2 v_\pm^3 = v_\mp^2 v_\pm^2,\]  
(4.5)

with \(c_1 = \lambda_1 - \lambda_2\lambda_3\) and cyclic in 1, 2 and 3. This system turns out to have a one parameter solution. To prove this, we solve for example the first and the fourth with respect to \(v_\pm^{1,2}\) and we do the same by solving the fifth and the sixth. By equating these alternative expressions for \(v_\pm^{1,2}\) we find that \((v_\pm^1)^2 - (v_\pm^2)^2 = c_1^2 - c_2^2\). Working analogously we find two more conditions following by cyclic permutation of 1, 2 and 3 and three analogue expressions for \(v_\pm^3\) through a parity transformation \(v_\pm^3 \leftrightarrow v_\pm^3\). Hence all together we have the conditions
\[(v_\pm^1)^2 - (v_\pm^2)^2 = c_1^2 - c_2^2,\]  
\[(v_\pm^2)^2 - (v_\pm^3)^2 = c_2^2 - c_3^2,\]  
\[(v_\pm^3)^2 - (v_\pm^1)^2 = c_3^2 - c_1^2,\]  
(4.6)

We proceed by solving them as
\[v_\pm^A = \sqrt{z_\mp + c_\mp},\]  
\[z_\pm \in \mathbb{C}, A = 1, 2, 3.\]  
(4.7)

Plugging the latter in (4.5) and after some algebraic manipulations, we find one more independent condition for \(z_\pm\)
\[z_\pm z_- = c_1^2 c_2^2 - c_2 c_3^2 - c_1 c_3^2\]  
\[= 4c_1^2 c_2^2 c_3^2 (z_+ + z_- + c_1^2 + c_2^2 + c_3^2).\]  
(4.8)

This condition determines \(z_+\) in terms of an arbitrary complex number \(z_-\) or vise versa and so we have proved that there is a spectral parameter (\(z_+\) or \(z_-\)). As a check, in the isotropic case, where \(\lambda_A = \lambda\), using (4.3) we find that the construction yields (3.3)
\[v_\pm = 2c \frac{\mu}{\mu \mp 1},\]  
\[z_\pm = c^2 (3\mu \mp 1)(\mu \pm 1),\]  
\[c = (1 - \lambda), \quad \mu \in \mathbb{C}.\]  
(4.9)

4.1. The Poisson algebra

Employing (2.17) and (2.19) we find the Poisson brackets for the currents
\[\{A_\pm^1, A_\pm^2\} = \frac{2\sqrt{2} \lambda_1 \lambda_2}{k(1 - \lambda_1^2)(1 - \lambda_2^2)\lambda_3} \left( c_2 A_\mp^2 + c_1 A_\pm^3 \right) \delta_{12},\]  
(4.10)

and with cyclic permutations in 1, 2 and 3 for the other pairs. For consistency we have checked that they satisfy the Jacobi identity.

Rescaling the gauge fields \(A_\pm^1 \mapsto \lambda_A A_\pm^1\), we can easily take the limit \(\lambda_A \to 0\)
\[\{A_\pm^1, A_\pm^2\} = \frac{2\sqrt{2}}{k} A_\mp^3 \delta_{12}, \quad \{A_\pm^1, A_{\pm}^1\} = \pm \frac{2}{k} \delta_{12},\]  
\[\{A_\pm^1, A_{\pm}^3\} = 0,\]  
(4.11)

and with cyclic permutations in 1, 2 and 3 for the other pairs. These expressions can be also obtained from (3.6) by an appropriate rescaling.

Expanding \(\lambda_A\) near the identity, we find the algebra of the non-Abelian T-dual of the PCM on SU(2)
\[\{A_\pm^1, A_\pm^2\} = \frac{1}{\sqrt{2} \epsilon_1 \epsilon_2} \left( A_\pm^1 (\epsilon_3 + \epsilon_1 - \epsilon_2) \right.\]  
\[\quad + A_\mp^3 (\epsilon_2 + \epsilon_3 - \epsilon_1) \left. \right) \delta_{12},\]  
(4.12)

\[\{A_\pm^1, A_\pm^1\} = \pm \frac{\delta_{12}}{\epsilon_1},\]  
\[\{A_\pm^1, A_{\pm}^3\} = - \frac{1}{\sqrt{2} \epsilon_1 \epsilon_2} \left( A_\mp^3 (\epsilon_1 + \epsilon_2 - \epsilon_3) \right.\]  
\[\quad - A_\pm^1 (\epsilon_1 + \epsilon_2 + \epsilon_3) \left. \right) \delta_{12},\]

where we have let \(\lambda_A = 1 - \frac{\epsilon_A}{k}\) for \(k \gg 1\). This algebra should be equivalent to the anisotropic PCM since they are related by a canonical transformation.

In the isotropic case, where \(\epsilon_A = \epsilon\), it is in accordance with (3.7) under the identification given in (3.2).

4.2. Maillet brackets

Following Sklyanin [22], we compute the equal time Poisson bracket of \(L_4^1\):
\[\{L_4^{(1)}(\sigma_1; \mu), L_4^{(2)}(\sigma_2; v)\} = \{L_4^1(\sigma_1; \mu), L_4^1(\sigma_2; v)\} \otimes t_C,\]  
(4.13)
where $L_1 = L_2^A t_A$ and the superscript in parenthesis denotes
the vector spaces on which the matrices act. These brackets
assume the Maillet form [23]
\begin{equation}
[ r_{-\mu\nu}, L_1^{(1)}(\sigma^i; \mu) ] + [ r_{+\mu\nu}, L_2^{(2)}(\sigma^i; v)] \delta_{12} - 2s_{\mu\nu} \delta_{12},
\end{equation}
where $r_{\pm\mu\nu} = r_{\pm\mu\nu} + s\delta_{\mu\nu}$ and $r_{\mu\nu} + s\delta_{\mu\nu}$
are matrices on the basis $\tau_B \otimes \tau_C$ depending on $(\mu, v)$.
This is guaranteed to give a consistent Poisson structure, provided
the Jacobi identities for these brackets are obeyed. This enforces
$r_{\pm\mu\nu}$ to satisfy the modified classical Yang–Baxter relation
\begin{equation}
[ r_{+(12)}^{(13)} + r_{-(1+2)}^{(23)} + r_{+(1+2)}^{(23)} + r_{+(+12)}^{(13)} ] = 0.
\end{equation}
The non-vanishing coefficient of the $\delta$ term in (4.14) is
responsible for the above modification, appearance of the
classical Yang–Baxter relation. In what follows within this section,
we shall rewrite (4.13) and (4.14) and retrieve $r_{\pm\mu\nu}$.

Expanding the Poisson bracket (4.13) we find that
\begin{align}
&v^B_{\mu} v^C_{\nu} X^B_{\mu} X^C_{\nu} + v^B_{-\mu} v^C_{-\nu} ( X^B_{\mu} X^C_{\nu} ) - v^B_{\mu} v^C_{-\nu} ( X^B_{-\mu} X^C_{\nu} ) \\
&- v^B_{-\mu} v^C_{\nu} ( X^B_{\mu} X^C_{-\nu} ),
\end{align}
As noted this will take the form of (4.14). To proceed we decompose
this in two terms corresponding to $\delta_{12}$ and $\delta_{13}$.
To compute the coefficient of $\delta_{12}$ we use (4.12) with (4.3)
and (4.16). We find that $s_{\mu\nu}$ has only diagonal elements
\begin{equation}
s_{\mu\nu}^{11} = -\frac{1}{k(1 - \lambda_1^2)(1 - \lambda_2^2)(1 - \lambda_3^2)} ( v^1_{\mu} v^1_{\nu} - v^1_{-\mu} v^1_{-\nu} )
\end{equation}
and with cyclic permutations in 1, 2 and 3 the other two. Note
that they are symmetric under the exchange of $\mu, \nu$ as expected
by the antisymmetry of the Poisson bracket [23].
To compute the coefficient of $\delta_{13}$ we expand in the $\tau_B \otimes \tau_C$ basis
and we obtain
\begin{equation}
\partial_3 r_{\mu\nu}^{BC} + \sqrt{\tilde{E}_{EABD}} \tilde{r}_{mu}^{DC} \tau^A_{1\mu} - \sqrt{\tilde{E}_{EADC}} \tilde{r}_{mu}^{BD} \tau^A_{1\nu}.
\end{equation}
Using (4.16) and (4.18), we find that $r_{\mu\nu}$ has only diagonal
elements. Analyzing the $t_1 \otimes t_2$ component and heavily using (4.5)
we find that
\begin{align}
&k \big( 1 - \lambda_1^2 \big) \big( 1 - \lambda_2^2 \big) \big( 1 - \lambda_3^2 \big) \\
&\big( v^3_{\mu} v^3_{\nu} - v^3_{-\mu} v^3_{-\nu} \big) r_{\mu\nu}^{11} \\
&= (c_3(1 - \lambda_1 \lambda_2 \lambda_3) - c_1 c_2) (v^1_{\mu} v^1_{\nu} + v^1_{-\mu} v^1_{-\nu}) \\
&+ (c_1(1 - \lambda_1 \lambda_2 \lambda_3) - c_2 c_3) (v^2_{\mu} v^2_{\nu} + v^2_{-\mu} v^2_{-\nu}) \\
&- c_1 (v^1_{\mu} v^2_{\nu} + v^2_{\mu} v^1_{\nu}) \\
&- c_3 (v^1_{\mu} v^2_{\nu} + v^2_{\mu} v^1_{\nu})
\end{align}
and
which expressions determine $r_{\mu\nu}^{11}$ and $r_{\mu\nu}^{22}$. The rest of the coefficients
are determined by a cyclic permutations in 1, 2 and 3.
Although cyclicity is not profound in the above expressions,
we can restore it by adding the corresponding equivalent expressions
evaluated by the other components.
Finally, as it was stated in (4.15), $r_{\pm\mu\nu}$ satisfy the modified
classical Yang–Baxter equation, which in our case reduces to six
equations given compactly by
\begin{align}
&\tilde{r}_{\mu\nu}^{AA} \tilde{r}_{\mu\nu}^{CC} = \tilde{r}_{\mu\nu}^{BB} \tilde{r}_{\mu\nu}^{CC} + \tilde{r}_{\mu\nu}^{AA} \tilde{r}_{\mu\nu}^{BB} + \tilde{r}_{\mu\nu}^{BC} + \tilde{r}_{\mu\nu}^{CD},
& A \neq B \neq C.
\end{align}
The explicit form of the equations can be extracted from the
coefficients of the combination $r_{ABC} \otimes \tau_B \otimes \tau_C$.
We have checked that this condition is indeed satisfied through a heavy use of (4.5).

5. Conclusion and outlook

In this paper we proved that the $\sigma$-model action (1.1) for
the group $SU(2)$ and for a diagonalizable coupling matrix $\lambda_{AB}$
is classically integrable. We achieved this by explicitly constructing
the spectral depending Lax pair (4.4) and thus giving rise to an
infinite number of conserved charges. We computed the Poisson bracket
of the spatial part $L_1$ of the Lax pair and demonstrated that it
assumes the Maillet-type form [23,24] from which we read off the
$r$ and $s$ matrices satisfying the modified Yang–Baxter equation,
from the Jacobi identity for these Poisson brackets.
Our result establish an integrable interpolation between the WZW
model (CFT) and the non-Abelian T-dual for the anisotropic PCM
for $SU(2)$.

In the context of $\lambda$-deformations, integrability has been proven
so far for three cases: The isotropic case, i.e. single coupling and
any group $G$, the symmetric coset case $G/H$ again for a single
coupling, and finally for the anisotropic SU(2) case with a
diagonalizable coupling matrix in the present paper. The latter case
is special as it possesses only Abelian subgroups which seems to
be at the root of the integrability proof we have achieved. One may
wonder if there exist other cases, based either on groups or on
(non) symmetric cosets for which specific choices of the matrix $\lambda$
may render the corresponding $\sigma$-model as classically integrable.
A starting point in this direction could be to examine if with the
right amount of torsion non-symmetric coset spaces may prove
integrable. In fact the $U(3)/U(1)^3$ non-symmetric coset was recently
shown to belong in this category [25], although the two-form takes
imaginary values.

Acknowledgements

The research of K. Sfetsos is implemented under the ARISTEIA
action (D.654 of GGET) of the operational programme education
and lifelong learning and is co-funded by the European Social Fund (ESF)
and National Resources (2007–2013). The research of K. Siampos
has been supported by the Swiss National Science Foundation.
The authors thank the University of Patras for kind hospitality where
this work was initiated.
Appendix A. Anisotropic PCM and its non-abelian T-dual

In this appendix we prove that the equations of motion and the Bianchi identities of the anisotropic PCM and its non-abelian T-dual are mapped to each other.

The anisotropic PCM action (2.2) can be reformulated as

$$ S = \frac{1}{2\pi} \int \text{Tr} (j \wedge \star Gj - j \wedge Bj), \quad j = g^{-1} dg, \quad E = G + B. $$

(A.1)

Varying with respect to $g$ we find the eom

$$ G d \star j = B d j - (G \star j - Bj) \wedge j - j \wedge (G \star j - Bj), $$

(A.2)

plus the flatness condition for $j$

$$ d j + j \wedge j = 0. $$

(A.3)

We would like to show that these follow from (2.14) by letting $k \gg 1$

$$ \lambda = E + O \left( \frac{1}{k^2} \right), $$

(A.4)

and keeping the leading term in the $\frac{1}{k}$ expansion. Indeed one easily obtains that

$$ \partial_+ A_- = \left( E + E^T \right)^{-1} \left( E^T [A_+, A_-] + [A_+, EA_-] - [E^T A_+, A_-] \right), $$

$$ \partial_- A_+ = \left( E + E^T \right)^{-1} \left( -E [A_+, A_-] + [A_+, EA_-] 

- [E^T A_+, A_-] \right). $$

(A.5)

These can be rewritten as

$$ \partial_+ A_- \partial_- A_+ = [A_+, A_-] \quad \text{or} \quad dA = A \wedge A $$

(A.6)

and

$$ E \partial_+ A_- + E^T \partial_- A_+ = [A_+, EA_-] - [E^T A_+, A_-]. $$

(A.7)

It is elementary to prove that these can be mapped to (A.2), (A.3) for $A = -j$.

References


