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The anisotropic λ-deformed $SU(2)$ model is integrable

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ABSTRACT

The all-loop anisotropic Thirring model interpolates between the WZW model and the non-Abelian T-dual of the anisotropic principal chiral model. We focus on the $SU(2)$ case and we prove that it is classically integrable by providing its Lax pair formulation. We derive its underlying symmetry current algebra and use it to show that the Poisson brackets of the spatial part of the Lax pair, assume the Maillet form. In this way we procure the corresponding r and s matrices which provide non-trivial solutions to the modified Yang–Baxter equation.

1. Introduction and motivation

The general class of σ-models whose integrability properties will be investigated was constructed in [1]. The corresponding action is given by

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) - \frac{k}{\pi}\int J_+^A M^{-1}_{AB} J_-^B\,,$$
$$M_{AB} = (\lambda^{-1} - D^T)_{AB}\,, \tag{1.1}$$

where the first term is the WZW model action for a semi-simple compact group G and a group element $g \in G$ given by [2]

$$S_{\mathrm{WZW},k}(g) = -\frac{k}{2\pi}\int \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \frac{k}{6\pi}\int_B \mathrm{Tr}(g^{-1}\mathrm{d}g)^3\,. \tag{1.2}$$

This is a CFT with two commuting current algebras at level k. The second term in (1.1) represents the deformation from the conformal point. Our conventions are[1]

$$J_+^A = \mathrm{Tr}(t^A\partial_+ g g^{-1})\,,\quad J_-^A = \mathrm{Tr}(t^A g^{-1}\partial_- g)\,,$$
$$D_{AB} = \mathrm{Tr}(t_A g t_B g^{-1})\,. \tag{1.3}$$

The above action has a $\dim G$ target space with coordinates the parameters in the group element $g \in G$. The t_A's are representation matrices obeying the Lie algebra $[t_A, t_B] = f_{ABC} t_C$ and normalized as $\mathrm{Tr}(t_A t_B) = \delta_{AB}$. The deviation from the WZW model is parametrized by the coupling matrix elements λ_{AB}. For small such elements the Lagrangian density is proportional to the current bilinear $\lambda_{AB} J_+^A J_-^B$. Hence the name λ-deformed models. The above action develops an extra local invariance under the vector action of a subgroup $H \subset G$ when λ_{AB} assumes the block diagonal form $\lambda_{AB} = \mathrm{diag}(\mathbb{I}_{ab}, \lambda_{\alpha\beta})$, where the lower case Latin indices take values in the Lie algebra of H and the Greek ones in the coset G/H. Due to this local invariance $\dim H$ degrees of freedom become redundant. Hence, $\dim H$ variables among those parameterizing g should be gauged fixed. For vanishing $\lambda_{\alpha\beta}$ the σ-model corresponds to the coset G/H CFT. In addition, the perturbation is driven by parafermion bilinears $\lambda_{\alpha\beta}\Psi_+^\alpha\Psi_-^\beta$, where the $\Psi_\pm^\alpha$'s are gauge invariant versions of the currents $J_\pm^\alpha$. The renormalization group equations for λ_{AB} in the action (1.1) have been computed for the isotropic case in [3] and in full generality in [4]. In addition, the (1.1) has been used as a building block to construct full solution of type-II supergravity in [5] which are likely also integrable at the string level.

In this paper we are interested in investigating integrability property of the above action. Integrability has been first proven for the isotropic case when $\lambda_{AB} = \lambda\delta_{AB}$ and a general semi-simple group G in [1]. This was done by explicitly showing that certain algebraic conditions developed in [6] (based on earlier work in [7]) were satisfied.[2] In addition, it has been proved that these models have an underlying Yangian symmetry [8]. In [1], integrability

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[1] The world-sheet coordinates (σ^+, σ^-) and (τ, σ) are related by $\sigma^\pm = \tau \pm \sigma$, so that $\partial_0 = \partial_\tau = \partial_+ + \partial_-$, $\partial_1 = \partial_\sigma = \partial_+ - \partial_-$.

[2] In this work the σ-model fields corresponding to (1.1) for the isotropic case and when $G = SU(2)$ were also constructed by a brute force computation which is not generalizable in practice for larger groups.

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was also expected for the coset $SU(2)/U(1)$ case by making contact with the work of [9] where a CFT approach was utilized. The most efficient way to prove integrability of (1.1) for specific choices of the matrix λ is to employ its origin via a gauging procedure as much as possible. This was done for the aforementioned isotropic group case as well as for the general symmetric coset space, for isotropic coupling $\lambda_{\alpha\beta}=\lambda\delta_{\alpha\beta}$ in [10].

In the present paper we will generalize this approach for a general matrix λ and we will prove integrability for the anisotropic $SU(2)$ model for general diagonalizable matrix λ_{AB}. The computation amounts to showing integrability for the diagonal matrix $\lambda=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$. In addition, we will compute the Poisson brackets of the spatial component of the corresponding monodromy matrix and we will provide non-trivial solutions to the modified Yang–Baxter equation.

2. Origin of integrability

In this section we review the construction of our models, derive the equations of motion and set them up in such a way that investigating the existence of a Lax pair formulation becomes immediate.

2.1. Review of the models

We review the construction of the models by following [1]. The starting point is the action

$$S(g,\tilde g)=S_{\mathrm{WZW},k}(g)+S_{\mathrm{PCM},E}(\tilde g), \tag{2.1}$$

where the first term is the WZW action (1.2) and the second term is the principal chiral model (PCM) action for G using a group element $\tilde g\in G$,

$$S_{\mathrm{PCM},E}(\tilde g)=-\frac{1}{\pi}\int E_{AB}\mathrm{Tr}(t^A\tilde g^{-1}\partial_+\tilde g)\mathrm{Tr}(t^B\tilde g^{-1}\partial_-\tilde g). \tag{2.2}$$

The action (2.1) is invariant under left–right current algebra symmetry of the WZW action and a global left symmetry of the PCM. We will gauge the same global group

$$g\to\Lambda^{-1}g\Lambda,\qquad \tilde g\to\Lambda^{-1}\tilde g,\qquad \Lambda\in G. \tag{2.3}$$

Hence we consider the action

$$S_{k,E}(g,\tilde g)=S_{\mathrm{gWZW},k}(g,A_\pm)+S_{\mathrm{gPCM},E}(\tilde g,A_\pm), \tag{2.4}$$

where

$$S_{\mathrm{gWZW},k}(g,A_\pm)=S_{\mathrm{WZW},k}(g)+\frac{k}{\pi}\int\mathrm{Tr}\Big(A_-\partial_+gg^{-1}-A_+g^{-1}\partial_-g+A_-gA_+g^{-1}-A_-A_+\Big), \tag{2.5}$$

and

$$S_{\mathrm{gPCM},E}(\tilde g,A_\pm)=-\frac{1}{\pi}\int E_{AB}\mathrm{Tr}(t^A\tilde g^{-1}\widetilde D_+\tilde g)\mathrm{Tr}(t^B\tilde g^{-1}\widetilde D_-\tilde g), \tag{2.6}$$

with the covariant derivatives being $\widetilde D_\pm\tilde g=\partial_\pm\tilde g-A_\pm\tilde g$. This action (2.5) is invariant under the local symmetry

$$\tilde g\to\Lambda^{-1}\tilde g,\qquad g\to\Lambda^{-1}g\Lambda,\qquad A_\pm\to\Lambda^{-1}A_\pm\Lambda-\Lambda^{-1}\partial_\pm\Lambda. \tag{2.7}$$

We will use the coupling matrix λ defined as $E=k(\lambda^{-1}-\mathbb{I})$. Finally we mention that the action (2.4) is invariant under the generalized parity symmetry

$$\sigma^+\leftrightarrow\sigma^-,\qquad g\mapsto g^{-1},\qquad \tilde g\mapsto\tilde g,\qquad A_+\leftrightarrow A_-,\qquad \lambda\mapsto\lambda^T. \tag{2.8}$$

2.2. Gauge fixing and equations of motion

We may choose the gauge $\tilde g=\mathbb{I}$. It is easily seen that the equation of motion followed by varying $\tilde g$ is automatically satisfied. Varying the action with respect to $A_\pm$ we find the constraints

$$D_+gg^{-1}=(\lambda^{-T}-\mathbb{I})A_+,\qquad g^{-1}D_-g=-(\lambda^{-1}-\mathbb{I})A_-, \tag{2.9}$$

where $D_\pm g=\partial_\pm g-[A_\pm,g]$, or equivalently

$$A_+=(\lambda^{-T}-D)^{-1}J_+,\qquad A_-=-(\lambda^{-1}-D^T)^{-1}J_-. \tag{2.10}$$

Varying with respect to the group element g we obtain that

$$D_-(D_+gg^{-1})=F_{+-},\qquad D_+(g^{-1}D_-g)=F_{+-},$$
$$F_{+-}=\partial_+A_--\partial_-A_+-[A_+,A_-], \tag{2.11}$$

which due to $[D_+,D_-]g=[g,F_{+-}]$, turn out to be equivalent. Substituting (2.9) into (2.11) we obtain that

$$D_-\Big((\lambda^{-T}-\mathbb{I})A_+\Big)=F_{+-},\qquad -D_+\Big((\lambda^{-1}-\mathbb{I})A_-\Big)=F_{+-}, \tag{2.12}$$

which can be cast as[3]

$$\partial_+A_--\partial_-(\lambda^{-T}A_+)=[\lambda^{-T}A_+,A_-],$$
$$\partial_+(\lambda^{-1}A_-)-\partial_-A_+=[A_+,\lambda^{-1}A_-]. \tag{2.13}$$

Unless $\lambda=\mathbb{I}$, the $A_\pm$ are not pure gauges. Solving for $\lambda\neq\mathbb{I}$ we obtain that

$$\partial_+A_-=(\mathbb{I}-\lambda\lambda^T)^{-1}\Big(-\lambda\lambda^T[\lambda^{-T}A_+,A_-]+\lambda[A_+,\lambda^{-1}A_-]\Big),$$
$$\partial_-A_+=(\mathbb{I}-\lambda^T\lambda)^{-1}\Big(\lambda^T\lambda[A_+,\lambda^{-1}A_-]-\lambda^T[\lambda^{-T}A_+,A_-]\Big). \tag{2.14}$$

Note that for any $\dim G\times\dim G$ diagonalizable matrix λ, with $\dim G$ linearly independent eigenvectors, it is sufficient to prove integrability using its diagonal form. To show this, we note that (2.13) (or (2.14)) are covariant under the similarity transformation $\lambda\mapsto S\lambda S^{-1}$, with $A_\pm\mapsto SA_\pm S^{-1}$. Specializing to symmetric matrices, this is always possible with S being an orthogonal matrix.

Our goal/effort would be to rewrite, if possible, the equations of motion (2.14) as a Lax equation

$$\mathrm{d}L=L\wedge L\quad\text{or}\quad \partial_+L_--\partial_-L_+=[L_+,L_-], \tag{2.15}$$

where $L_\pm=L_\pm(\tau,\sigma,\mu)$ depend on a spectral parameter $\mu\in\mathbb{C}$.

2.3. The current algebra

For a gauged WZW we can define

$$S_+=D_+gg^{-1}+A_+-A_-,\qquad S_-=-g^{-1}D_-g+A_--A_+, \tag{2.16}$$

[3] In components

$$PA_+=P_{BC}A_+^Ct_B,\qquad [PA_+,A_-]=f_{BCD}P_{CE}A_+^EA_-^Dt_B,$$

where P is an arbitrary square matrix.

which obey two commuting copies of current algebras [7,11]

$$\{S_\pm^A, S_\pm^B\} = f_{ABC} S_\pm^C \delta_{\sigma\sigma'} \pm \frac{k}{2}\delta_{AB}\delta'_{\sigma\sigma'}, \qquad \delta_{\sigma\sigma'} = \delta(\sigma-\sigma'), \tag{2.17}$$

where we have dropped the time dependence at usual equal time Poisson brackets. Since the action does not depend on derivatives of $A_\pm$, its equations-of-motion are second class constraints [10,12]

$$S_+ = \frac{k}{2}\left(\lambda^{-T}A_+ - A_-\right), \qquad S_- = \frac{k}{2}\left(\lambda^{-1}A_- - A_+\right) \tag{2.18}$$

and inversely

$$A_+ = \frac{2}{k} g^{-1}\lambda^T(S_+ + \lambda S_-), \qquad A_- = \frac{2}{k}\tilde{g}^{-1}\lambda(S_- + \lambda^T S_+),$$
$$g = \mathbb{I} - \lambda^T\lambda, \qquad \tilde{g} = \mathbb{I} - \lambda\lambda^T, \tag{2.19}$$

where we assume that $g, \tilde{g}$ are positive-definite matrices. It is just a matter of algebra to rewrite the current algebras for $S_\pm$ in the base of $A_\pm$, as we are going to present in the subsequent sections.

3. Known integrable cases

In this section we review the known (isotropic) integrable cases, semi-simple group and general symmetric coset spaces, using the previous formulation.

3.1. The isotropic group space

As a warmup, we review the integrability for the isotropic case for a semi-simple group G [1,10]. Then the equations of motion for the gauge field read

$$\partial_\pm A_\mp = \pm\frac{1}{1+\lambda}[A_+, A_-], \tag{3.1}$$

and a simple rescaling

$$A_\pm = -\frac{1}{2}(1+\lambda)I_\pm, \tag{3.2}$$

proves the integrability. As for the Lax pair, this is given by

$$L_\pm = \frac{2}{1+\lambda}\frac{\mu}{\mu\mp 1}A_\pm, \tag{3.3}$$

where $\mu \in \mathbb{C}$ is the spectral parameter.

3.1.1. Algebraic structure

Employing (2.17), (2.19) and (3.2), we find the Poisson brackets for $I_\pm$ [6]

$$\{I_\pm^A, I_\pm^B\} = e^2 f_{ABC}\left(I_\mp^C - (1+2x)I_\pm^C\right)\delta_{12} \pm 2e^2\delta_{AB}\delta'_{12},$$
$$\{I_\pm^A, I_\mp^B\} = -e^2 f_{ABC}\left(I_+^C + I_-^C\right)\delta_{12}, \tag{3.4}$$

where

$$e = \frac{2\lambda}{\sqrt{k(1-\lambda^2)}(1+\lambda)}, \qquad x = \frac{1+\lambda^2}{2\lambda} > 1, \tag{3.5}$$

where the deformation parameter is a root of unity [10]. We note that the same underlying structure, but with $-1 < x < 1$, corresponds to integrable deformations of the σ-model [13] constructed in [17,18], where the deformation parameter is real. The corresponding quantum properties at one-loop were studied in [14–16].

There are two interesting limits. Expanding λ near zero and rescaling $I_\pm^A \mapsto -2e^2 x I_\pm^A$ we find that

$$\{I_\pm^A, I_\pm^B\} = f_{ABC} I_\pm^C \delta_{\sigma\sigma'} \pm \frac{k}{2}\delta_{AB}\delta'_{\sigma\sigma'}, \quad \{I_+^A, I_-^B\} = 0. \tag{3.6}$$

These are two commuting current algebras in accordance with the fact that in this limit the σ-model corresponds to a CFT.

Parameterizing λ as $\lambda = k(k+\varepsilon)^{-1}$ and then letting $k \gg 1$, we find the algebra of the non-Abelian T-dual of the PCM on G

$$\{I_\pm^A, I_\pm^B\} = \frac{1}{2\varepsilon} f_{ABC}(I_\mp^C - 3I_\pm^C)\delta_{12} \pm \frac{1}{\varepsilon}\delta_{AB}\delta'_{12},$$
$$\{I_\pm^A, I_\mp^B\} = -\frac{1}{2\varepsilon} f_{ABC}(I_+^C + I_-^C)\delta_{12}. \tag{3.7}$$

This is the same as the algebra for the PCM for G, in accordance with the fact that the two cases are related by a canonical transformation.

3.2. The isotropic symmetric coset

Let us consider a semi-simple group G and its decomposition to a semi-simple subgroup H and a symmetric coset G/H.[4] Take the case where the matrix λ has elements $\lambda_{ab} = \lambda_H\delta_{ab}$, $\lambda_{\alpha\beta} = \lambda_{G/H}\delta_{\alpha\beta}$. The restriction to symmetric cosets translates to structure constants $f_{\alpha\beta\gamma} = 0$, whereas $f_{abc}, f_{\alpha\beta c} \neq 0$. For $\lambda_H, \lambda_{G/H} \neq 1$ we have that (2.13) or (2.14) read[5]

$$\partial_\pm A_\mp = \pm(1+\lambda_H)^{-1}\left([A_+, A_-] + \frac{\lambda_H}{\lambda_{G/H}}[B_+, B_-]\right),$$
$$\partial_\pm B_\mp = \frac{1}{\lambda_H(1-\lambda_{G/H}^2)}\Big((\lambda_{G/H}^2 - \lambda_H)[B_\mp, A_\pm]$$
$$+ \lambda_{G/H}(1-\lambda_H)[B_\pm, A_\mp]\Big), \tag{3.8}$$

where $A_\pm$ and $B_\pm$ are Lie algebra valued one forms ($A_\pm = A_\pm^a t_a$, $B_\pm = B_\pm^\alpha t_\alpha$), on the subgroup and coset respectively.[6] The above consideration is drastically modified in the two cases we have excluded. The first special case is when $\lambda_H = 1$. In this singular limit we have to use (2.13) and the equations of motion simplify drastically

$$\partial_+ A_- - \partial_- A_+ = [A_+, A_-] + \frac{1}{\lambda_{G/H}}[B_+, B_-],$$
$$\partial_\pm B_\mp = -[B_\mp, A_\pm], \tag{3.9}$$

and the two eom for $A_\pm$ in (3.8) are replaced by their difference. This case was shown to be integrable in [10] with Lax pair given by

$$L_\pm = A_\pm + \frac{\mu^{\pm 1}}{\sqrt{\lambda_{G/H}}}B_\pm, \tag{3.10}$$

where $\mu \in \mathbb{C}$. It can be readily checked that then (2.15) is satisfied.

4. The anisotropic $SU(2)$ case

In this section we consider the other special case in which the subgroup H is Abelian. In addition to demanding that the space G/H is symmetric, restrict our considerations to the group case

[4] In the conventions of Section 1 we denote subgroup indices by Latin letters and coset indices by Greek letters.

[5] Note that for $\lambda_{G/H} = 1$, λ_H turns to be one for finiteness of the expressions. For general cosets G/H the equations for $B_\pm$ contain the additional term $\pm(1+\lambda_{G/H})^{-1}[B_+, B_-]$.

[6] We have tried to construct a Lax pair for (3.8) of the form $L_\pm = a_\pm A_\pm + b_\pm B_\pm$ where the coefficients are constants. For $\lambda_H \neq 1$ and for non-Abelian subgroup H one obtains a linear algebraic inhomogeneous system with has a unique solution. This implies that within this ansatz for the Lax pair one cannot prove integrability.

$SU(2)$. We will consider the cases of a diagonizable matrix λ_{AB}, $A, B = 1, 2, 3$. Then as explained before it is sufficient to consider the case with

$$\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \tag{4.1}$$

that is the fully anisotropic albeit diagonal case. In this case the generators are $t_A = -i\sigma_A/\sqrt{2}$, where σ_A are the Pauli matrices, so that $f_{ABC} = \sqrt{2}\varepsilon_{ABC}$. Then a straightforward computation shows that

$$\begin{aligned}\partial_\pm A^1_\mp = \frac{\sqrt{2}\lambda_1}{(1-\lambda_1^2)\lambda_2\lambda_3}\Big[&(\lambda_2 - \lambda_1\lambda_3)A^2_\pm A^3_\mp \\ &- (\lambda_3 - \lambda_1\lambda_2)A^3_\pm A^2_\mp\Big],\end{aligned} \tag{4.2}$$

and cyclic in 1, 2 and 3.

This is in agreement with (3.8), for $\lambda_H = \lambda_1, \lambda_{G/H} = \lambda_2 = \lambda_3$, where $H = U(1)$ and $SU(2)/U(1)$ symmetric coset. Moreover, along the results of Section 3.2 and [10], the coset limit $\lambda_1 = 1$ is integrable and for compatibility $\lambda_2 = \lambda_3$. As shown in [1] expanding the λ_i's around one, we get the non-Abelian T-dual of the anisotropic PCM for $SU(2)$, which is integrable due to the fact that the PCM is integrable [19,20] and non-Abelian T-duality preserves integrability [21].[7] All these are signals that the more general case we consider here with (4.1) is likely integrable as well.

Let's define a convenient set of fields given by

$$X^1_\pm = \frac{A^1_\pm}{\lambda_1\sqrt{(1-\lambda_2^2)(1-\lambda_3^2)}} \tag{4.3}$$

and cyclic in 1, 2 and 3. Then we assume the following expression for a Lax pair

$$L^A_\pm(\tau, \sigma; \mu) = v^A_\pm(\mu) X^A_\pm, \tag{4.4}$$

where $\mu \in \mathbb{C}$ and $v^A_\pm$ satisfy six non-linear equations

$$\begin{aligned}&c_2 v^1_\mp + c_3 v^1_\pm = v^2_\pm v^3_\mp, \quad c_3 v^2_\mp + c_1 v^2_\pm = v^3_\pm v^1_\mp, \\ &c_1 v^3_\mp + c_2 v^3_\pm = v^1_\pm v^2_\mp,\end{aligned} \tag{4.5}$$

with $c_1 = \lambda_1 - \lambda_2\lambda_3$ and cyclic in 1, 2 and 3. This system turns out to have a one parameter solution. To prove this, we solve for example the first and the fourth with respect to $v^{1,2}_-$ and we do the same by solving the fifth and the sixth. By equating these alternative expressions for $v^{1,2}_-$ we find that $(v^1_+)^2 - (v^2_+)^2 = c_1^2 - c_2^2$. Working analogously we find two more conditions following by cyclic permutation of 1, 2 and 3 and three analogue expressions for v^A_- through a parity transformation $v^A_+ \mapsto v^A_-$. Hence all together we have the conditions

$$\begin{aligned}&(v^1_\pm)^2 - (v^2_\pm)^2 = c_1^2 - c_2^2, \quad (v^2_\pm)^2 - (v^3_\pm)^2 = c_2^2 - c_3^2, \\ &(v^3_\pm)^2 - (v^1_\pm)^2 = c_3^2 - c_1^2.\end{aligned} \tag{4.6}$$

We proceed by solving them as

$$v^A_\pm = \sqrt{z_\pm + c_A^2}, \quad z_\pm \in \mathbb{C}, \ A = 1, 2, 3. \tag{4.7}$$

Plugging the latter in (4.5) and after some algebraic manipulations, we find one more independent condition for $z_\pm$

$$\begin{aligned}&(z_+ z_- - c_1^2c_2^2 - c_2^2c_3^2 - c_3^2c_1^2)^2 \\ &\quad = 4c_1^2c_2^2c_3^2(z_+ + z_- + c_1^2 + c_2^2 + c_3^2).\end{aligned} \tag{4.8}$$

[7] We provide a detailed self-contained proof for the general case in Appendix A.

This condition determines z_+ in terms of an arbitrary complex number z_- or vise versa and so we have proved that there is a spectral parameter (z_+ or z_-). As a check, in the isotropic case, where $\lambda_A = \lambda$, using (4.3) we find that the construction yields (3.3)

$$\begin{aligned}&v_\pm = 2c\frac{\mu}{\mu \mp 1}, \qquad z_\pm = c^2\frac{(3\mu \mp 1)(\mu \pm 1)}{(\mu \mp 1)^2}, \\ &c = \lambda(1-\lambda), \ \mu \in \mathbb{C}.\end{aligned} \tag{4.9}$$

4.1. The Poisson algebra

Employing (2.17) and (2.19) we find the Poisson brackets for the currents

$$\begin{aligned}\{A^1_\pm, A^2_\mp\} &= \frac{2\sqrt{2}\lambda_1\lambda_2}{k(1-\lambda_1^2)(1-\lambda_2^2)\lambda_3}\Big(c_2 A^3_\pm + c_1 A^3_\mp\Big)\delta_{12}, \\ \{A^1_\pm, A^1_\pm\} &= \pm\frac{2\lambda_1^2}{k(1-\lambda_1^2)}\delta'_{12}, \\ \{A^1_\pm, A^2_\pm\} &= -\frac{2\sqrt{2}\lambda_1\lambda_2}{k(1-\lambda_1^2)(1-\lambda_2^2)\lambda_3} \\ &\qquad \times \Big(c_3 A^3_\mp - (1-\lambda_1\lambda_2\lambda_3)A^3_\pm\Big)\delta_{12},\end{aligned} \tag{4.10}$$

and with cyclic permutations in 1, 2 and 3 for the other pairs. For consistency we have checked that they satisfy the Jacobi identity.

Rescaling the gauge fields $A^A_\pm \mapsto \lambda_A A^A_\pm$, we can easily take the limit $\lambda_A \to 0$

$$\begin{aligned}&\{A^1_\pm, A^2_\mp\} = \frac{2\sqrt{2}}{k}A^3_\pm\delta_{12}, \qquad \{A^1_\pm, A^1_\pm\} = \pm\frac{2}{k}\delta'_{12}, \\ &\{A^1_\pm, A^2_\pm\} = 0,\end{aligned} \tag{4.11}$$

and with cyclic permutations in 1, 2 and 3 for the other pairs. These expressions can be also obtained from (3.6) by an appropriate rescaling.

Expanding λ_A near the identity, we find the algebra of the non-Abelian T-dual of the PCM on $SU(2)$

$$\begin{aligned}\{A^1_\pm, A^2_\mp\} &= \frac{1}{\sqrt{2}\varepsilon_1\varepsilon_2}\Big(A^3_\pm(\varepsilon_3 + \varepsilon_1 - \varepsilon_2) \\ &\qquad + A^3_\mp(\varepsilon_2 + \varepsilon_3 - \varepsilon_1)\Big)\delta_{12}, \\ \{A^1_\pm, A^1_\pm\} &= \pm\frac{\delta'_{12}}{\varepsilon_1}, \\ \{A^1_\pm, A^2_\pm\} &= -\frac{1}{\sqrt{2}\varepsilon_1\varepsilon_2}\Big(A^3_\mp(\varepsilon_1 + \varepsilon_2 - \varepsilon_3) \\ &\qquad - A^3_\pm(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\Big)\delta_{12},\end{aligned} \tag{4.12}$$

where we have let $\lambda_A = 1 - \frac{\varepsilon_A}{k}$, for $k \gg 1$. This algebra should be equivalent to the anisotropic PCM since they are related by a canonical transformation.

In the isotropic case, where $\varepsilon_A = \varepsilon$, it is in accordance with (3.7) under the identification given in (3.2).

4.2. Maillet brackets

Following Sklyanin [22], we compute the equal time Poisson bracket of L_1:

$$\{L_1^{(1)}(\sigma_1; \mu), L_1^{(2)}(\sigma_2; \nu)\} = \{L_1^B(\sigma_1; \mu), L_1^C(\sigma_2; \nu)\}t_B \otimes t_C, \tag{4.13}$$

where $L_1 = L_1^A t_A$ and the superscript in parenthesis denotes the vector spaces on which the matrices act.[8] These brackets assume the *Maillet* form [23]

$$\left([r_{-\mu\nu}, L_1^{(1)}(\sigma_1;\mu)] + [r_{+\mu\nu}, L_1^{(2)}(\sigma_1;\nu)]\right)\delta_{12} - 2s_{\mu\nu}\delta'_{12}, \quad (4.14)$$

where $r_{\pm\mu\nu} = r_{\mu\nu} \pm s_{\mu\nu}$ and $r_{\mu\nu}, s_{\mu\nu}$ are matrices on the basis $t_B \otimes t_C$ depending on (μ,ν). This is guaranteed to give a consistent Poisson structure, provided the Jacobi identities for these brackets are obeyed. This enforces $r_{\pm\mu\nu}$ to satisfy the modified classical Yang–Baxter relation

$$[r^{(13)}_{+\nu_1\nu_3}, r^{(12)}_{-\nu_1\nu_2}] + [r^{(23)}_{+\nu_2\nu_3}, r^{(12)}_{+\nu_1\nu_2}] + [r^{(23)}_{+\nu_2\nu_3}, r^{(13)}_{+\nu_1\nu_3}] = 0. \quad (4.15)$$

The non-vanishing coefficient of the δ' term in (4.14) is responsible for the above modification, appearance of $s_{\mu\nu}$, of the classical Yang–Baxter relation. In what follows within this section, we shall rewrite (4.13) and (4.14) and retrieve $r_{\pm\mu\nu}, s_{\mu\nu}$.

Expanding the Poisson bracket (4.13) we find that

$$\begin{aligned}&v^B_{+\mu}v^C_{+\nu}\{X^B_+, X^C_+\} + v^B_{-\mu}v^C_{-\nu}\{X^B_-, X^C_-\} - v^B_{+\mu}v^C_{-\nu}\{X^B_+, X^C_-\}\\ &\quad - v^B_{-\mu}v^C_{+\nu}\{X^B_-, X^C_+\}.\end{aligned} \quad (4.16)$$

As noted this will take the form of (4.14). To proceed we decompose this in two terms corresponding to δ'_{12} and δ_{12}.

To compute the coefficient of δ'_{12} we use (4.12) with (4.3) and (4.16). We find that $s_{\mu\nu}$ has only diagonal elements

$$s^{11}_{\mu\nu} = -\frac{1}{k(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_3^2)}\left(v^1_{+\mu}v^1_{+\nu} - v^1_{-\mu}v^1_{-\nu}\right), \quad (4.17)$$

and with cyclic permutations in 1, 2 and 3 the other two. Note that they are symmetric under the exchange of μ,ν as expected by the antisymmetry of the Poisson bracket [23].

To compute the coefficient of δ_{12} we expand in the $t_B \otimes t_C$ basis and we obtain

$$\partial_1 r^{BC}_{-\mu\nu} + \sqrt{2}\varepsilon_{ABD} r^{DC}_{-\mu\nu} L^A_{1\mu} - \sqrt{2}\varepsilon_{ADC} r^{BD}_{+\mu\nu} L^A_{1\nu}. \quad (4.18)$$

Using (4.16) and (4.18), we find that $r_{\mu\nu}$ has only diagonal elements. Analyzing the $t_1 \otimes t_2$ component and heavily using (4.5) we find that

$$\begin{aligned}&\frac{k}{2}(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_3^2)\left(v^3_{+\mu}v^3_{-\nu} - v^3_{+\nu}v^3_{-\mu}\right) r^{11}_{+\mu\nu}\\ &\quad = (c_3(1-\lambda_1\lambda_2\lambda_3) - c_1c_2)(v^2_{+\mu}v^2_{+\nu} + v^2_{-\mu}v^2_{-\nu})\\ &\qquad + (c_1(1-\lambda_1\lambda_2\lambda_3) - c_2c_3)(v^2_{+\mu}v^2_{-\nu} + v^2_{-\mu}v^2_{+\nu})\\ &\qquad - c_1\left(v^1_{+\mu}v^2_{-\nu}v^3_{+\mu} + v^1_{-\mu}v^2_{+\nu}v^3_{-\mu}\right)\\ &\qquad - c_3\left(v^1_{+\mu}v^2_{+\nu}v^3_{+\mu} + v^1_{-\mu}v^2_{-\nu}v^3_{-\mu}\right),\end{aligned} \quad (4.19)$$

and

$$\begin{aligned}&\frac{k}{2}(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_3^2)\left(v^3_{+\mu}v^3_{-\nu} - v^3_{+\nu}v^3_{-\mu}\right) r^{22}_{-\mu\nu}\\ &\quad = (c_3(1-\lambda_1\lambda_2\lambda_3) - c_1c_2)(v^1_{+\mu}v^1_{+\nu} + v^1_{-\mu}v^1_{-\nu})\\ &\qquad + (c_2(1-\lambda_1\lambda_2\lambda_3) - c_1c_3)(v^1_{+\mu}v^1_{-\nu} + v^1_{-\mu}v^1_{+\nu})\\ &\qquad - c_2\left(v^1_{+\mu}v^2_{-\nu}v^3_{-\nu} + v^1_{-\mu}v^2_{+\nu}v^3_{+\nu}\right)\\ &\qquad - c_3\left(v^1_{+\mu}v^2_{+\nu}v^3_{+\nu} + v^1_{-\mu}v^2_{-\nu}v^3_{-\nu}\right),\end{aligned} \quad (4.20)$$

which expressions determine $r^{11}_{+\mu\nu}$ and $r^{22}_{-\mu\nu}$. The rest of the coefficients are determined by a cyclic permutations in 1, 2 and 3. Although cyclicity is not profound in the above expressions, we can restore it by adding the corresponding equivalent expressions evaluated by the other components.

Finally, as it was stated in (4.15), $r_{\pm\mu\nu}$ satisfy the modified classical Yang–Baxter equation, which in our case reduces to six equations given compactly by

$$r^{AA}_{+\nu_1\nu_2} r^{CC}_{+\nu_2\nu_3} = r^{BB}_{-\nu_1\nu_2} r^{CC}_{+\nu_1\nu_3} + r^{AA}_{+\nu_1\nu_3} r^{BB}_{+\nu_2\nu_3}, \quad A \neq B \neq C. \quad (4.21)$$

The explicit form of the equations can be extracted from the coefficients of the combination $\varepsilon_{ABC} t_A \otimes t_B \otimes t_C$. We have checked that this condition is indeed satisfied through a heavy use of (4.5).

5. Conclusion and outlook

In this paper we proved that the σ-model action (1.1) for the group $SU(2)$ and for a diagonalizable coupling matrix λ_{AB} is classically integrable. We achieved this by explicitly constructing the spectral depending Lax pair (4.4) and thus giving rise to an infinite number of conserved charges. We computed the Poisson bracket of the spatial part L_1 of the Lax pair and demonstrated that it assumes the *Maillet-type* form [23,24] from which we read off the r and s matrices satisfying the modified Yang–Baxter equation, arising from the Jacobi identity for these Poisson brackets. Our result establish an integrable interpolation between the WZW model (CFT) and the non-Abelian T-dual for the anisotropic PCM for $SU(2)$.

In the context of λ-deformations, integrability has been proven so far for three cases: The isotropic case, i.e. single coupling and any group G, the symmetric coset case G/H again for a single coupling, and finally for the anisotropic $SU(2)$ case with a diagonalizable coupling matrix in the present paper. The latter case is special as it possesses only Abelian subgroups which seems to be at the root of the integrability proof we have achieved. One may wonder if there exist other cases, based either on groups or on (non) symmetric cosets for which specific choices of the matrix λ may render the corresponding σ-model as classically integrable. A starting point in this direction could be to examine if with the right amount of torsion non-symmetric coset spaces may prove integrable. In fact the $U(3)/U(1)^3$ non-symmetric coset was recently shown to belong in this category [25], although the two-form takes imaginary values.

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[8] In brief:

$$M^{(1)} = M \otimes \mathbb{I}, \qquad M^{(2)} = \mathbb{I} \otimes M, \qquad M = M_A t_A,$$

$$m^{(12)} = m_{AB} t_A \otimes t_B \otimes \mathbb{I}, \qquad m^{(13)} = m_{AB} t_A \otimes \mathbb{I} \otimes t_B,$$

$$m^{(23)} = m_{AB} \mathbb{I} \otimes t_A \otimes t_B,$$

for an arbitrary matrix $m = m_{AB} t_A \otimes t_B$.

Appendix A. Anisotropic PCM and its non-abelian T-dual

In this appendix we prove that the equations of motion and the Bianchi identities of the anisotropic PCM and its non-abelian T-dual are mapped to each other.

The anisotropic PCM action (2.2) can be reformulated as

$$S=\frac{1}{2\pi}\int \mathrm{Tr}\left(j\wedge\star Gj - j\wedge Bj\right), \quad j=g^{-1}\mathrm{d}g, \quad E=G+B. \tag{A.1}$$

Varying with respect to g we find the eom

$$G\,\mathrm{d}\star j = B\,\mathrm{d}j-(G\star j-Bj)\wedge j - j\wedge(G\star j-Bj), \tag{A.2}$$

plus the flatness condition for j

$$\mathrm{d}j+j\wedge j=0. \tag{A.3}$$

We would like to show that these follow from (2.14) by letting $k\gg 1$

$$\lambda=\mathbb{I}-\frac{E}{k}+\mathcal{O}\left(\frac{1}{k^2}\right), \tag{A.4}$$

and keeping the leading term in the $\frac{1}{k}$ expansion. Indeed one easily obtains that

$$\begin{aligned}\partial_+A_- &= (E+E^T)^{-1}\left(E^T[A_+,A_-]+[A_+,EA_-]-[E^TA_+,A_-]\right),\\ \partial_-A_+ &= (E+E^T)^{-1}\Big(-E[A_+,A_-]+[A_+,EA_-]\\ &\qquad -[E^TA_+,A_-]\Big).\end{aligned} \tag{A.5}$$

These can be rewritten as

$$\partial_+A_- - \partial_-A_+ = [A_+,A_-] \quad \text{or} \quad \mathrm{d}A=A\wedge A \tag{A.6}$$

and

$$E\partial_+A_- + E^T\partial_-A_+ = [A_+,EA_-]-[E^TA_+,A_-]. \tag{A.7}$$

It is elementary to prove that these can be mapped to (A.2), (A.3) for $A=-j$.

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