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# Eigenvalue Estimates for Non-Selfadjoint Dirac Operators on the Real Line

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**Abstract.** We show that the non-embedded eigenvalues of the Dirac operator on the real line with complex mass and non-Hermitian potential $V$ lie in the disjoint union of two disks, provided that the $L^1$-norm of $V$ is bounded from above by the speed of light times the reduced Planck constant. The result is sharp; moreover, the analogous sharp result for the Schrödinger operator, originally proved by Abramov, Aslanyan and Davies, emerges in the nonrelativistic limit. For massless Dirac operators, the condition on $V$ implies the absence of non-real eigenvalues. Our results are further generalized to potentials with slower decay at infinity. As an application, we determine bounds on resonances and embedded eigenvalues of Dirac operators with Hermitian dilation-analytic potentials.

## 1. Introduction

There has been an increasing interest in the spectral theory of non-selfadjoint differential operators during the past few years. In particular, eigenvalue estimates for Schrödinger operators with complex potentials have recently been investigated by various authors, [1,6,9,10,17,19]. Corresponding results for non-selfadjoint Dirac operators are much more sparse, [22,23], although operators of this type arise for example as Lax operators in the focusing nonlinear Schrödinger equation [3].

In this paper we derive the first eigenvalue enclosures for Dirac operators with non-Hermitian potentials. We consider one-dimensional Dirac operators $H = H_0 + V$ in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where the free Dirac operator is of the form

$$H_0 = -\mathrm{i}c\hbar \frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 + mc^2\sigma_3, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

with $c$ denoting the speed of light, $\hbar$ the reduced Planck constant, $m$ the particle mass and where $V$ is a $2 \times 2$ matrix-valued function with entries in $L^1(\mathbb{R})$. Since we do not assume $V(x)$ to be Hermitian, the operator $H$ is not selfadjoint, in general. In fact, in our main result, Theorem 2.1, we do not even require the free Dirac operator $H_0$ to be selfadjoint since we allow the mass $m$ to be complex. In this case, the (possibly non-real) spectrum of $H_0$ is given by $\sigma(H_0) = \{\pm(p^2 + m^2)^{1/2} : p \in \mathbb{R}\}$. We prove that if the potential $V$ satisfies

$$\|V\|_1 := \int_{\mathbb{R}} \|V(x)\| \, \mathrm{d}x < \hbar c, \tag{2}$$

where $\|V(x)\|$ is the operator norm of $V(x)$ in $\mathbb{C}^2$ with Euclidean norm, then the non-embedded eigenvalues $z \in \mathbb{C} \setminus \sigma(H_0)$ of $H$ lie in the union of two disjoint disks,

$$z \in K_{|m|r_0}(mx_0) \,\dot{\cup}\, K_{|m|r_0}(-mx_0); \tag{3}$$

the radii $|m|r_0$, as well as the points $mx_0$ determining the centres, diverge to $\infty$ as $\|V\|_1 \to \hbar c$. In particular, our theorem implies that the massless Dirac operator [i.e. $m = 0$ in (1)] with non-Hermitian potential $V$ has no complex eigenvalues at all.

The second main result of this paper is an enclosure for resonances of Dirac operators with Hermitian potentials under some analyticity assumptions on $V$. While the literature on the theory of resonances of Schrödinger operators is vast, see e.g. [21,28] and the references therein, much less is known for the Dirac operator; we only mention [20] where the complex scaling method was employed. We use the interplay of this method with Theorem 2.1 for the scaled Dirac operators $H_\theta$ to describe a region in the complex plane where the uncovered resonances may lie in terms of $L_1$-norms of the scaled potentials $V(\mathrm{e}^{\mathrm{i}\theta}\cdot)$. Moreover, for the massless Dirac operator, we show that there are no resonances near the real axis.

Further results concern the sharpness of our eigenvalue enclosures and generalizations to more slowly decaying potentials. Finally, in the nonrelativistic limit ($c \to \infty$), our main result reproduces [1, Theorem 4] for the one-dimensional Schrödinger operator

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V \tag{4}$$

in $L^2(\mathbb{R})$ with complex-valued potential $V \in L^1(\mathbb{R})$ whose eigenvalues $\lambda \in \mathbb{C} \setminus [0, \infty)$ lie in a disk around the origin:

$$\frac{\hbar^2}{2m}|\lambda| \le \frac{1}{4}\left(\int_{\mathbb{R}} |V(x)| \, \mathrm{d}x\right)^2. \tag{5}$$

Inequality (5) is sharp; in the case of symmetric $V \le 0$ it may be interpreted as the sharp Lieb-Thirring inequality (phase volume type inequality) for the square root of the modulus of the negative eigenvalue for a class of potentials with one bound state. We find it rather surprising that the $L^1$-norm of the potential appears, in a rather complicated way, even in the sharp estimate (3) of the eigenvalues of one-dimensional Dirac operators which, by no means, is a phase volume type estimate. Note that the fact that (5) is obtained from our Theorem 2.1 as $c \to \infty$ also confirms the sharpness of our results.

Our proofs are based on the so-called Birman–Schwinger principle. Although the latter is not bound to one dimension, the generalization to higher dimensions poses a major challenge; the reason for this is the intrinsically different behaviour of the resolvent kernel of $H_0$ which already in the case of Schrödinger operators requires sophisticated analytical estimates [9].

The outline of the paper is as follows. In Theorem 2.1 of Sect. 2, we prove the enclosure (3) and show that, for $m \neq 0$, the eigenvalue bound (5) for the Schrödinger operator emerges in the nonrelativistic limit ($c \to \infty$).

In Sect. 3, we demonstrate the sharpness of Theorem 2.1 by considering a family of delta-potentials. Moreover, we show that assumption (2) may be weakened if the potential has additional structure, e.g. if $m \geq 0$ and $V$ is purely imaginary.

In Sect. 4, we extend Theorem 2.1 to potentials with slower decay at infinity; in this case (2) has to be replaced by more complicated conditions. From this we derive eigenvalue estimates in terms of higher $L^p$-norms of $V$, see Corollary 4.6. We also prove that if $m$ is real, $p \in [2, \infty]$ and an additional smallness assumption holds, then $H$ is similar to a block-diagonal matrix operator; see Theorem 4.9.

In Sect. 5, we establish enclosures for resonances and embedded eigenvalues of $H$ with real $m$ and Hermitian $V(x)$. For this purpose, we use the well-known method of complex scaling where resonances are characterized as eigenvalues of non-selfadjoint operators and apply Theorem 2.1 to the scaled Dirac operators $H_\theta$. To this end, a careful analysis of the dependence of the corresponding balls $K_{mr_\theta}(\pm m x_\theta)$ on the scaling angle $\theta$ is required.

To avoid overly technical discussions, we prove all results in Sects. 2, 3, 4 and 5 for the case of bounded $V$, i.e. $V_{ij} \in L^\infty(\mathbb{R})$, $i, j = 1, 2$; it will be evident, however, that the boundedness does not play an essential role, and we will show in Sect. 6 how to dispense with it.

The following notation will be used throughout this paper. For $z_0 \in \mathbb{C}$ and $r > 0$, let $K_r(z_0)$ be the closed disk centred at $z_0$ with radius $r$; for $r = 0$, we use the convention that $K_r(z_0) = \emptyset$. For a closed densely defined linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, we denote by $\mathcal{D}(T)$, $\ker(T)$, $\rho(T)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$ its domain, kernel, resolvent set, spectrum, and set of eigenvalues, respectively. Let $\mathrm{L}(\mathcal{H})$ denote the algebra of bounded linear operators with domain equal to $\mathcal{H}$ and by $\|\cdot\|$ the operator norm on $\mathrm{L}(\mathcal{H})$; the norm on the ideal of Hilbert–Schmidt operators is denoted by $\|\cdot\|_{\mathrm{HS}}$. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. We shall use the abbreviation $T - z$ for the operator $T - z\, I_{\mathcal{H}}$, $z \in \mathbb{C}$. Throughout Sects. 2, 3, 4 and 5 we work in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$. By tr we denote the trace in this Hilbert space, while Tr is the trace in $\mathbb{C}^2$. By abuse of notation, we shall denote integral operators on $\mathcal{H}$ and their kernels by the same symbol. For example, we write $R_0(z) = (H_0 - z)^{-1}$ for the resolvent of the free Dirac operator $H_0$ and $R_0(x, y; z)$ for its resolvent kernel. For a measurable matrix-valued function $V = (V_{ij})_{i,j=1}^2$ we shall always identify the function $V$ with the closed maximal multiplication operator in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$.

The most general potentials in this paper, considered in Sect. 4, are of the form $V = W + X$ with $W_{ij} \in L^1(\mathbb{R})$, $i, j = 1, 2$, and $X$ bounded. These potentials leave the essential spectrum invariant,

$$\sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{e}}(H_0) = \{\pm(p^2 + m^2)^{1/2} : p \in \mathbb{R}\}; \tag{6}$$

see Proposition 6.6. Note that there are at least five different notions of essential spectrum for a non-selfadjoint closed operator $T$; here we use the following one:

$$\sigma_{\mathrm{e}}(T) := \{z \in \mathbb{C} : T - z \text{ is not a Fredholm operator}\}.$$

The discrete spectrum of $T$ is defined as

$$\sigma_{\mathrm{d}}(T) := \{z \in \mathbb{C} : z \text{ is an isolated eigenvalue of } T \text{ of finite multiplicity}\}.$$

If $T$ is not selfadjoint, then, in general, $\sigma(T)$ is not the disjoint union of $\sigma_{\mathrm{e}}(T)$ and $\sigma_{\mathrm{d}}(T)$. However, for the Dirac operators $H = H_0 + V$ considered here, $\mathbb{C} \setminus \sigma_{\mathrm{e}}(H_0) = \rho(H_0)$ has either one or two (for $m = 0$) connected components, each of which contains points of $\rho(H)$. Hence [12, Theorem XVII.2.1] implies that

$$\sigma(H) \setminus \sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{d}}(H). \tag{7}$$

For simplicity, we will use units where $\hbar = c = 1$ from now on. The correct values in other units may simply be restored by dimensional analysis.

## 2. Integrable Potentials

In this section we derive sharp bounds on the eigenvalues of the perturbed Dirac operator $H$ in (1), with non-Hermitian potential $V = (V_{ij})_{i,j=1}^2$, $V_{ij} \in L^1(\mathbb{R})$ and complex mass $m$. For eigenvalue bounds in terms of higher $L^p$-norms, see Corollary 4.6 as well as the forthcoming paper [5].

**Theorem 2.1.** *Let $V = (V_{ij})_{i,j=1}^2$ with $V_{ij} \in L^1(\mathbb{R})$ for $i, j = 1, 2$ be such that*

$$\|V\|_1 < 1. \tag{8}$$

*Then every non-embedded eigenvalue $z \in \mathbb{C} \setminus \sigma(H_0)$ of $H$ lies in the union of two disks,*

$$z \in K_{|m|r_0}(mx_0) \cup K_{|m|r_0}(-mx_0), \tag{9}$$

*where*

$$x_0 := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} + \frac{1}{2}}, \quad r_0 := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} - \frac{1}{2}}; \tag{10}$$

*in particular, the spectrum of the massless Dirac operator ($m = 0$) with non-Hermitian potential $V$ is $\mathbb{R}$.*

The two disks of Theorem 2.1 for $m = 1$ are shown in Fig. 1.

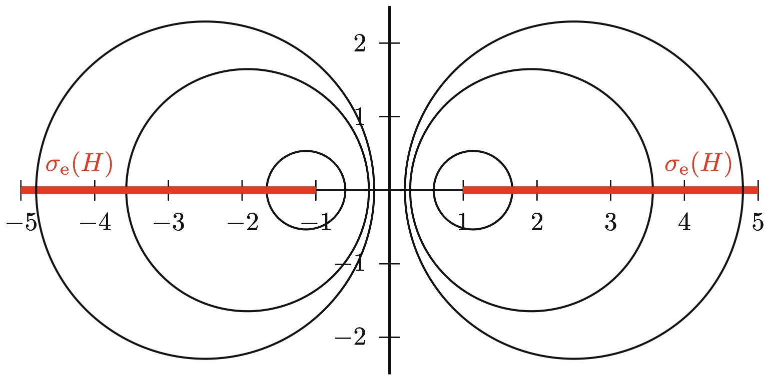


FIGURE 1. The two disks of Theorem 2.1 for three different values of $\|V\|_1 \in (0, 1)$ and $m = 1$

*Proof.* In this section we prove Theorem 2.1 under the assumption that $V$ is bounded in which case $H = H_0 + V$ is a closed operator. The only additional obstruction in the general case is the construction of a closed extension $H$ of $H_0 + V$, a technical point which we postpone to Sect. 6.

The proof of Theorem 2.1 is based on the Birman–Schwinger principle: Let $U$ be the partial isometry in the polar decomposition of $V = U|V|$. We shall factorize $V$ according to

$$V = BA, \quad B := U|V|^{1/2}, \quad A := |V|^{1/2}. \tag{11}$$

We denote by $R_0(\cdot)$ the resolvent of $H_0$, i.e.

$$R_0(z) := (H_0 - z)^{-1}, \quad z \in \rho(H_0).$$

Let $z \in \rho(H_0)$. It is easy to verify that $z$ is an eigenvalue of $H$ if and only if $-1$ is an eigenvalue of $VR_0(r)$. Since the nonzero eigenvalues of $BAR_0(z)$ and $AR_0(z)B$ are the same, this is thus equivalent to $-1$ being an eigenvalue of the operator

$$Q(z) := AR_0(z)B : \mathcal{H} \to \mathcal{H}, \quad z \in \rho(H_0). \tag{12}$$

Hence, if $z$ is an eigenvalue of $H$, then $\|Q(z)\| \geq 1$. On the other hand, since the spectrum of $H$ in the complement of $\sigma_e(H_0)$ is discrete by (6) and (7), $z \in \rho(H)$ whenever $\|Q(z)\| < 1$.

It is well known that the resolvent kernel of the free Dirac operator is given by

$$R_0(x, y; z) = M(x, y; z)\, \mathrm{e}^{\mathrm{i}k(z)|x-y|},$$
$$M(x, y; z) := \frac{\mathrm{i}}{2} \begin{pmatrix} \zeta(z) & \operatorname{sgn}(x-y) \\ \operatorname{sgn}(x-y) & \zeta(z)^{-1} \end{pmatrix},$$

where

$$\zeta(z) := \frac{z+m}{k(z)}, \quad k(z) := \sqrt{z^2 - m^2}, \quad z \in \rho(H_0), \tag{13}$$

and the branch of the square root on $\mathbb{C}\setminus[0,\infty)$ is chosen such that $\operatorname{Im} k(z) > 0$. We set

$$\begin{aligned}\Phi(z) &:= \zeta(z)^2 = \frac{z+m}{z-m} \in \mathbb{C}\setminus[0,\infty), \quad z \in \rho(H_0),\\ \eta(s) &:= \sqrt{\frac{1}{2} + \frac{1}{4}\,(s+s^{-1})}, \quad s > 0.\end{aligned} \tag{14}$$

Observing that

$$\|M(x,y;z)\| = \|M(x,y;z)\|_{\mathrm{HS}} = \eta(|\Phi(z)|),$$

we obtain that for $z \in \rho(H_0)$, $f,g \in \mathcal{H}$,

$$\begin{aligned}&|(AR_0(z)Bf,g)|\\ &\quad\le \eta(|\Phi(z)|)\int_{\mathbb{R}}\int_{\mathbb{R}} \|A(x)\|\,\|B(y)\|\,\|f(y)\|_{\mathbb{C}^2}\|g(x)\|_{\mathbb{C}^2}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\le \eta(|\Phi(z)|)\left(\int_{\mathbb{R}}\|A(x)\|^2\,\mathrm{d}x\right)^{1/2}\|g\|_{\mathcal{H}}\left(\int_{\mathbb{R}}\|B(y)\|^2\,\mathrm{d}y\right)^{1/2}\|f\|_{\mathcal{H}}\\ &\quad= \eta(|\Phi(z)|)\left(\int_{\mathbb{R}}\|V(x)\|\,\mathrm{d}x\right)\|g\|_{\mathcal{H}}\,\|f\|_{\mathcal{H}}.\end{aligned} \tag{15}$$

Here, we used $\exp(-\operatorname{Im} k(z)\,|x-y|) \le 1$ in the first line, the Cauchy–Schwarz inequality in the second line, and the equality

$$\|B(x)\| = \|A(x)\| = \|\,|V(x)|^{1/2}\| = \|V(x)\|^{1/2}, \quad x \in \mathbb{R},$$

in the last line. It follows that

$$\|Q(z)\| \le \eta(|\Phi(z)|)\,\|V\|_1. \tag{16}$$

Hence, $\|Q(z)\| < 1$ whenever

$$w := \Phi(z) \in B_{\rho^2,\rho^{-2}} := \{w \in \mathbb{C} : \rho^{-2} < |w| < \rho^2\}, \quad \rho := \frac{1+\sqrt{1-\|V\|_1^2}}{\|V\|_1}. \tag{17}$$

Observing that $\Phi$ is a Möbius transformation for $m \neq 0$ with inverse

$$z = \Phi^{-1}(w) = m\,\frac{w+1}{w-1},$$

we see that the complement of the annulus $B_{\rho^2,\rho^{-2}}$ in the $w$-plane is mapped onto the union of the disks $K_{|m|r_0}(mx_0)$ and $K_{|m|r_0}(-mx_0)$ in the $z$-plane. Indeed, $\Phi^{-1}$ maps (generalized) circles to (generalized) circles, and, by virtue of the equality

$$\overline{\mathrm{e}^{-\mathrm{i}\arg(m)}\Phi^{-1}(w)} = \mathrm{e}^{-\mathrm{i}\arg(m)}\Phi^{-1}(\overline{w}), \quad w \in \mathbb{C}\cup\{\infty\},$$

the image of a circle with centre at the origin is symmetric with respect to $\mathrm{e}^{-\mathrm{i}\arg(m)}\mathbb{R}$. The outer boundary of $B_{\rho^2,\rho^{-2}}$ is mapped to the circle with centre $mx_0$ and radius $|m|r_0$ given by

$$x_0 = \frac{1}{2}\left(\frac{\rho^2+1}{\rho^2-1} + \frac{-\rho^2+1}{-\rho^2-1}\right) = \frac{\rho^4+1}{\rho^4-1},$$
$$r_0 = \frac{1}{2}\left(\frac{\rho^2+1}{\rho^2-1} - \frac{-\rho^2+1}{-\rho^2-1}\right) = \sqrt{x_0^2-1}.$$

On the other hand, since

$$\Phi^{-1}(w^{-1}) = -\Phi^{-1}(w), \quad w \in \mathbb{C} \cup \{\infty\},$$

the inner boundary of $B_{\rho^2,\rho^{-2}}$ is mapped to the circle with centre $-mx_0$ and radius $|m|r_0$. Since $\Phi^{-1}$ is biholomorphic and $\mathbb{C}\backslash\big(B_{\rho^2,\rho^{-2}}\big)$ is doubly connected, its image must also be so, so it fills the regions inside the two circles. Observing that

$$\frac{\rho^4+1}{\rho^4-1} = \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} + \frac{1}{2}},$$

the spectral inclusion (9) is proved for the case $m \neq 0$. If $m = 0$, then $\Phi(z) = 1$ and $\eta(|\Phi(z)|) = 1$ for all $z \in \mathbb{C}$. Hence, (16) implies that $\|Q(z)\| < 1$ for $z \in \rho(H_0) = \mathbb{C} \setminus \mathbb{R}$. This proves the limiting case $m = 0$ in (9). □

*Remark* 2.2. The eigenvalue bound (5) of [1] for the Schrödinger operator with complex potential $V$ emerges from the corresponding bounds for the Dirac operator (9) in the nonrelativistic limit since

$$\lim_{c\to\infty} (H(c) - mc^2 - z)^{-1} = \begin{pmatrix} \left(-\frac{1}{2m}\,\Delta + V - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

see e.g. [24, Theorem 6.4]. Here, we have restored $c$ (the speed of light) by replacing $m$ by $mc^2$ and $\|V\|_1$ by $c^{-1}\|V\|_1$. It follows from Theorem 2.1 that the non-embedded eigenvalues of $(H(c)-mc^2)$ lie in the union of two disks with radius $|m|c^2r_0(c)$ and centres $mc^2(x_0(c) \pm 1)$, where $x_0(c)$, $r_0(c)$ now depend on $c$ via $c^{-1}\|V\|_1$. An easy calculation shows that, in the limit $c \to \infty$, one of the disks disappears at minus infinity, while the other converges to the closed disk with radius $|m|/2\,\|V\|_1^2$ and centre at the origin, compare (5) (recall that $\hbar = 1$ here).

*Remark* 2.3. For the massless Dirac operator ($m = 0$), it is not difficult to show that $|V|^{1/2}$ is $H_0$-smooth in the sense of Kato [13]. This means that for all $u \in L^2(\mathbb{R}, \mathbb{C}^2)$,

$$\sup_{\varepsilon>0} \int_{-\infty}^{\infty} \left(\||V|^{1/2}R_0(\lambda+\mathrm{i}\varepsilon)u\|^2 + \||V|^{1/2}R_0(\lambda-\mathrm{i}\varepsilon)u\|^2\right)\,\mathrm{d}\lambda \leq C\|u\|^2.$$

It then follows from Theorem 2.1 and [13, Theorem 1.5] that if $\|V\|_1 < 1$, then $|V|^{1/2}$ is also $H$-smooth, and $H$ is similar to $H_0$ by means of the Kato wave operators $W_\pm = \mathrm{s}\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}$, see also [22]. The absence of non-real eigenvalues is an immediate consequence of this similarity. Moreover, if $V$ is an electric potential (i.e. a scalar multiple of the identity matrix) $V = q\,I$ with a complex-valued function $q \in L^1(\mathbb{R})$, then the similarity of $H$ and $H_0$ (with $m = 0$) holds without the assumption $\|V\|_1 < 1$. Indeed, if $U$ is the operator of multiplication with

$$U(x) = \exp\left( \mathrm{i}\sigma_1 \int_{-\infty}^{x} q(y)\,\mathrm{d}y \right), \quad x \in \mathbb{R},$$

then $U$ is bounded and boundedly invertible in $\mathcal{H}$, and $U^{-1}H_0U = H$.

As a supplement to Theorem 2.1, the following proposition provides an estimate for the norm of the resolvent $R(z)$ of $H$.

**Proposition 2.4.** *Let $V = (V_{ij})_{i,j=1}^2$ with $V_{ij} \in L^1(\mathbb{R})$ for $i, j = 1, 2$ be such that $\|V\|_1 < 1$. Then, for $z \in \rho(H_0) = \mathbb{C} \setminus \{\pm(p^2 + m^2)^{1/2} : p \in \mathbb{R}\}$ outside the union of the two disks $K_{|m|r_0}(mx_0)$ and $K_{|m|r_0}(-mx_0)$ in Theorem 2.1,*

$$\|R(z)\| \leq \|R_0(z)\| + \frac{\eta(|\Phi(z)|)^2}{\operatorname{Im} k(z)} \frac{\|V\|_1}{1 - \eta(|\Phi(z)|)\|V\|_1} \tag{18}$$

*Remark* 2.5. Since $R_0(z)$ is a Fourier multiplier, its norm in $L^2(\mathbb{C}^2)$ is given by

$$\|R_0(z)\| = \sup_{p \in \mathbb{R}} \left\| \frac{p\,\sigma_1 + m\,\sigma_3 + z\,I}{p^2 + m^2 - z^2} \right\| \tag{19}$$

where the above norm is the operator norm in $\mathbb{C}^2$, equipped with Euclidean norm; in particular, if $m \geq 0$, then $H_0$ is selfadjoint and the supremum is equal to $(\operatorname{dist}(z, \sigma(H_0)))^{-1}$. If $m$ is non-real, then $H_0$ is not even a normal operator and the supremum is a more complicated expression.

*Proof of Proposition 2.4.* By iterating the second resolvent identity,

$$R(z) = R_0(z) - R_0(z)VR(z),$$

we infer that

$$R(z) = R_0(z) - R_0(z)B(I_{\mathcal{H}} + Q(z))^{-1}AR_0(z). \tag{20}$$

A straightforward computation shows that

$$\max\{\|AR_0(z)\|_{\mathrm{HS}}, \|R_0(z)B\|_{\mathrm{HS}}\} \leq \frac{\eta(|\Phi(z)|)}{\sqrt{\operatorname{Im} k(z)}} \|V\|_1^{1/2}. \tag{21}$$

From (20) and the Neumann series, it follows that

$$\|R(z)\| \leq \|R_0(z)\| + \|R(z) - R_0(z)\| \leq \|R_0(z)\| + \frac{\|AR_0(z)\|\,\|R_0(z)B\|}{1 - \|Q(z)\|}.$$

If we combine this with (21) and (16), the claim is proved. $\square$

## 3. Sharpness of Theorem 2.1 and Purely Imaginary Potentials

In this section we provide an example which suggests that the eigenvalue enclosures of Theorem 2.1 are sharp and that the assumption $\|V\|_1 < 1$ cannot be omitted. Moreover, we show how additional structure of the potential may be used to improve the bounds of Theorem 2.1.

*Example* 3.1. We consider the family of delta-potentials

$$V_\tau = \mathrm{i}\,\kappa\,\delta_0\,W_\tau, \quad W_\tau := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\tau} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\tau} \end{pmatrix}, \quad \kappa > 0, \quad -\pi \le \tau < \pi, \tag{22}$$

for which the operator $Q(z)$ in (12) reduces to the matrix

$$Q(z) = -\frac{\kappa}{2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\tau}\,\zeta(z) & \mathrm{e}^{-\mathrm{i}\,\tau} \\ \mathrm{e}^{\mathrm{i}\,\tau} & \mathrm{e}^{-\mathrm{i}\,\tau}\,\zeta(z)^{-1} \end{pmatrix} \tag{23}$$

in $\mathbb{C}^2$ if we define $\operatorname{sgn}(0) = 1$. The perturbed operator $H_\tau$ may be rigorously defined as a rank two perturbation of $H_0$. Alternatively, it may be described in terms of boundary conditions, viz.

$$\begin{aligned} \mathcal{D}(H_\tau) = \{ & f \in L^2(\mathbb{R},\mathbb{C}^2) \cap H^1(\mathbb{R}\setminus\{0\},\mathbb{C}^2) \\ & : \sigma_1(f(0+) - f(0-)) - \kappa\,W_\tau f(0+) = 0\}, \\ (H_\tau f)(x) = & -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\sigma_1\,f(x) + m\,\sigma_3\,f(x), \quad x \in \mathbb{R}\setminus\{0\}, \quad f \in \mathcal{D}(H_\tau). \end{aligned}$$

It follows that

$$\ker(H_\tau - z) \subset \left\{ \begin{pmatrix} \zeta(z) \\ \operatorname{sgn}(\cdot) \end{pmatrix} \mathrm{e}^{\mathrm{i}\,k(z)\,|\cdot|}, \begin{pmatrix} \operatorname{sgn}(\cdot) \\ \zeta(z)^{-1} \end{pmatrix} \mathrm{e}^{\mathrm{i}\,k(z)\,|\cdot|} \right\},$$

and the boundary conditions imply that $\ker(H_\tau - z)$ is nontrivial if and only if

$$\det(I + Q(z)) = \det\begin{pmatrix} 1 - \kappa/2\,\mathrm{e}^{\mathrm{i}\,\tau}\,\zeta(z) & -\kappa/2\,\mathrm{e}^{-\mathrm{i}\,\tau} \\ -\kappa/2\,\mathrm{e}^{\mathrm{i}\,\tau} & 1 - \kappa/2\,\mathrm{e}^{-\mathrm{i}\,\tau}\,\zeta(z)^{-1} \end{pmatrix} = 0.$$

Solving this equation for $\zeta(z)$, we find the solutions

$$\zeta(z) = \zeta_\pm := \mathrm{e}^{-\mathrm{i}\,\tau}\,\frac{1 \pm \sqrt{1-\kappa^2}}{\kappa}. \tag{24}$$

Recalling (13), (14), it is seen that we must have $\operatorname{Im}\zeta(z) < 0$ for $z$ to be an eigenvalue of $H_\tau$.

If $\kappa < 1$, then $\operatorname{Im}\zeta_\pm < 0$ if and only if $0 < \tau < \pi$; in this case, as $\tau$ varies from 0 to $\pi$, the points $w_\pm := \zeta_\pm^2$ trace out the boundary of the annulus $B_{\rho^2,\rho^{-2}}$ with

$$\rho := \frac{1 + \sqrt{1-\kappa^2}}{\kappa},$$

which is precisely $\rho$ in (17) with $\|V\|_1$ replaced by $\kappa$ ($< 1$). This implies that the two eigenvalues of $H_\tau$, $0 < \tau < \pi$, lie on the boundaries of the disks $K_{|m|r_0}(\pm m x_0)$ of Theorem 2.1. In the case $-\pi \le \tau \le 0$, there are no eigenvalues.

If $\kappa \geq 1$, then the square root in (24) becomes imaginary, and it is easily verified that $\zeta_\pm$ lie on the unit circle, with

$$\operatorname{Im} \zeta_\pm = \frac{1}{\kappa}\left(-\sin(\tau) \pm \cos(\tau)\sqrt{\kappa^2 - 1}\right).$$

Hence, for $m \neq 0$, there are either zero, one, or two eigenvalues; as $\tau$ varies, they cover the imaginary axis.

A straightforward calculation shows that

$$\zeta_+ = 1 \iff \tau = \arccos(1/\kappa),$$
$$\zeta_- = -1 \iff \tau = \pi - \arccos(1/\kappa).$$

Hence, for $m = 0$,

$$\sigma(H_\tau) \cap (\mathbb{C} \setminus \mathbb{R}) = \begin{cases} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, & \tau = \arccos(1/\kappa), \\ \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, & \tau = \pi - \arccos(1/\kappa), \\ \quad \emptyset, & \text{otherwise}. \end{cases}$$

Hence, for $\kappa \geq 1$, the eigenvalues of $H_\tau$ need not lie in a bounded set, and hence an enclosure as in Theorem 2.1 cannot hold.

Incidentally, this example (with $m = 0$) illustrates two typical non-selfadjoint phenomena: First, since $H_\tau$ is a rank two resolvent perturbation of $H_0$, the essential spectra are clearly the same, $\sigma_{\mathrm{e}}(H_\tau) = \sigma_{\mathrm{e}}(H_0) = \mathbb{R}$. However, for $\tau = \arccos(1/\kappa)$ and $\tau = \pi - \arccos(1/\kappa)$, the spectrum in $\mathbb{C} \setminus \mathbb{R}$ is not discrete, but consists of dense point spectrum in the upper or lower half plane; this is not a contradiction to [12, Theorem 3.1] since $\mathbb{C} \setminus \mathbb{R}$ is not connected. Secondly, although it can be shown that the mapping $\tau \mapsto H_\tau$ is continuous in the norm resolvent topology, for $m = 0$ the spectrum $\sigma(H_\tau)$ is lower-semidiscontinuous as a function of $\tau$ at the points $\tau = \arccos(1/\kappa)$ and $\tau = \pi - \arccos(1/\kappa)$; compare e.g. [14, IV.3.2].

If the potential has additional structure, the assumption $\|V\|_1 < 1$ may be weakened in some cases. As an example, we consider perturbations of the self-adjoint free Dirac operator ($m \geq 0$) by purely imaginary potentials $V = \mathrm{i}\widetilde{V}$ with $\widetilde{V} \geq 0$. Such potentials have been studied in [17] in the framework of Schrödinger operators.

**Theorem 3.2.** *Assume that $m \geq 0$ and let $V = \mathrm{i}\widetilde{V}$, with $\widetilde{V} = (\widetilde{V}_{ij})_{i,j=1}^2$ such that $\widetilde{V} \geq 0$ and $\widetilde{V}_{ij} \in L^1(\mathbb{R})$ for $i, j = 1, 2$. Then $\sigma_{\mathrm{d}}(H)$ lies in the open upper half plane; if $z \in \rho(H_0) = \mathbb{C} \setminus (-\infty, -m] \cup [m, \infty)$ and*

$$\left(\operatorname{Re} \frac{z + m}{\sqrt{z^2 - m^2}}\right) \|\widetilde{V}_{11}\|_1 + \left(\operatorname{Re} \frac{\sqrt{z^2 - m^2}}{z + m}\right) \|\widetilde{V}_{22}\|_1 < 2, \tag{25}$$

*then $z \notin \sigma(H)$. In particular, if $m = 0$ and*

$$\|\widetilde{V}_{11}\|_1 + \|\widetilde{V}_{22}\|_1 < 2, \tag{26}$$

*then the spectrum of $H$ is $\mathbb{R}$.*

*Remark* 3.3. The set of points satisfying (25) does not have such a simple form as the disks in Theorem 2.1. However, (25) implies e.g. that for $m > 0$

$$\sigma(H) \cap \mathrm{i}\mathbb{R} \subset \left\{ \mathrm{i}\,\mu : \mu > 0,\ \frac{\sqrt{\mu^2+m^2}}{\mu} \geq \frac{\|\widetilde{V}_{11}\|_1 + \|\widetilde{V}_{22}\|_1}{2} \right\}.$$

*Proof.* We follow the lines of the proof of [17, Theorem 9]. Like in the proof of Theorem 2.1 we assume that $V$ is bounded; for the proof of the general case, see Sect. 6.

Let $z \in \rho(H_0)$ and $Q(z)$ be given by (12), i.e.

$$Q(z) = \mathrm{i}\,\widetilde{V}^{1/2} R_0(z) \widetilde{V}^{1/2}.$$

Using the first resolvent identity, we find

$$\operatorname{Re} Q(z) = -(\operatorname{Im} z)(R_0(z)\widetilde{V}^{1/2})^*(R_0(z)\widetilde{V}^{1/2}). \tag{27}$$

If $\operatorname{Im} z \leq 0$, this implies that $\operatorname{Re} Q(z) \geq 0$. Hence the numerical range

$$W(I+Q(z)) := \{((I+Q(z))f, f) : f \in \mathcal{H},\ \|f\| = 1\},$$

satisfies

$$W(I+Q(z)) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 1\}.$$

Since the spectrum of a bounded operator is contained in the closure of its numerical range, see [14, Corollary V.3.3], it follows that $0 \in \rho(I+Q(z))$, i.e. $z \in \rho(H)$ for $\operatorname{Im} z \leq 0$.

To prove the second claim, assume to the contrary that $z \in \rho(H_0)$ with $\operatorname{Im} z > 0$ satisfies condition (25), and $z \in \sigma(H)$. Then (27) implies that $\operatorname{Re} Q(z) \leq 0$, i.e. the spectrum of $Q(z)$ lies in the left half plane, and $-1$ is an eigenvalue of $Q(z)$. Hence the eigenvalues $\lambda_j(Q(z))$ of $Q(z)$ satisfy

$$\sum_{j=1}^{\infty} \operatorname{Re} \lambda_j(Q(z)) \leq -1.$$

It follows that

$$1 \leq -\sum_{j=1}^{\infty} \operatorname{Re} \lambda_j(Q(z)) \leq -\operatorname{tr}(\operatorname{Re} Q(z)) = -\int_{\mathbb{R}} \operatorname{Tr}(\operatorname{Re} Q)(x,x;z)\,\mathrm{d}x, \tag{28}$$

where $(\operatorname{Re} Q)(\cdot,\cdot;z)$ is the kernel of the operator $\operatorname{Re} Q(z)$; for the proof of the second inequality, we refer to [17, Corollary 1] or [2, Theorem 1]; see also [10, Lemma 1] for a different idea of the proof. Since

$$\operatorname{Re} Q(z) = -\widetilde{V}^{1/2} \operatorname{Im} R_0(z) \widetilde{V}^{1/2},$$

we have

$$(\operatorname{Re} Q)(x,x;z) = -\frac{1}{2} \widetilde{V}(x)^{1/2} \begin{pmatrix} \operatorname{Re} \zeta(z) & 0 \\ 0 & \operatorname{Re} \zeta(z)^{-1} \end{pmatrix} \widetilde{V}(x)^{1/2}.$$

Together with assumption (25), this implies

$$-\operatorname{tr}(\operatorname{Re} Q(z)) = \frac{1}{2}\left(\operatorname{Re}\zeta(z)\int_{\mathbb{R}} \widetilde{V}_{11}(x)\,\mathrm{d}x + \operatorname{Re}\zeta(z)^{-1}\int_{\mathbb{R}} \widetilde{V}_{22}(x)\,\mathrm{d}x\right) < 1,$$

a contradiction to (28). The last claim is immediate since (25) reduces to (26) in the case $m = 0$. □

## 4. Slowly Decaying Potentials

In this section we consider potentials decaying more slowly at infinity than just $V_{ij} \in L^1(\mathbb{R})$ as in Theorem 2.1. We assume that $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$, i.e. there exists a decomposition $V = W + X$ such that $W_{ij} \in L^1(\mathbb{R})$ and $X_{ij} \in L_0^\infty(\mathbb{R})$; here, $L_0^\infty(\mathbb{R})$ is the space of bounded functions that vanish at infinity. Schrödinger operators with this type of potentials have been studied in [6].

It is well known, and easy to see, that if $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$ and $\varepsilon > 0$, then there exists a (generally non-unique) decomposition $V = W + X$ with $W_{ij} \in L^1(\mathbb{R})$ and $\|X\| \le \varepsilon$, see [6]. We set

$$C_\varepsilon := \inf\left\{\int_{\mathbb{R}} \|W(x)\|\,\mathrm{d}x : V = W + X,\ W_{ij} \in L^1(\mathbb{R}),\ \|X\| \le \varepsilon\right\} \in [0,\infty). \tag{29}$$

**Theorem 4.1.** *Let $V = (V_{ij})_{i,j=1}^2$ with $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$ for $i,j = 1,2$. Let $z \in \rho(H_0) = \mathbb{C} \setminus \{\pm(p^2+m^2)^{1/2} : p \in \mathbb{R}\}$ and let $\eta$, $\Phi$ be defined as in (14), i.e.*

$$\eta(|\Phi(z)|) = \sqrt{\frac{1}{2} + \frac{1}{4}\left(\left|\frac{z+m}{z-m}\right| + \left|\frac{z-m}{z+m}\right|\right)}, \tag{30}$$

*and $C_\varepsilon$ as in (29). If for some $\varepsilon > 0$*

$$C_\varepsilon < \eta(|\Phi(z)|)^{-1} \tag{31}$$

*and*

$$\|R_0(z)\| + \frac{\eta(|\Phi(z)|)^2}{\operatorname{Im}\sqrt{z^2-m^2}}\,\frac{C_\varepsilon}{1-\eta(|\Phi(z)|)\,C_\varepsilon} < \frac{1}{\varepsilon}, \tag{32}$$

*then $z \notin \sigma(H)$.*

*Remark* 4.2. Recall that $\|R_0(z)\|$ is explicitly given by (19) and that $\|R_0(z)\| = (\operatorname{dist}(z,\sigma(H_0)))^{-1}$ if $m \ge 0$. Moreover, if $V_{ij} \in L^1(\mathbb{R})$, then, in the limit $\varepsilon \to 0$, the condition (31) becomes (8) since $\lim_{\varepsilon\to 0} C_\varepsilon = \|V\|_1$ [compare (16)], and (32) is automatically satisfied. Hence, Theorem 2.1 is a special case of Theorem 4.1.

*Proof.* Again, in order to avoid technical complications we shall assume that $V$ is bounded. This restriction does not play a role for the eigenvalue bounds and may be omitted if the construction of Sect. 6 is used.

It can be shown that the infimum in (29) is in fact a minimum; see [6]. Let $W$ be the corresponding minimizing element, and set $X := V - W$. Let

$$A_W := |W|^{1/2}, \quad B_W := U_W|W|^{1/2},$$
$$A_X := |X|^{1/2}, \quad B_X := U_X|X|^{1/2},$$

where $U_W$ and $U_X$ are the partial isometries in the polar decompositions of $W$ and $X$, respectively. Set $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ and define the operators

$$A := \begin{pmatrix} A_W \\ A_X \end{pmatrix} : \mathcal{H} \to \mathcal{K}, \quad B := \begin{pmatrix} B_W B_X \end{pmatrix} : \mathcal{K} \to \mathcal{H}. \tag{33}$$

Then $V = BA$ and $z \in \rho(H_0)$ is an eigenvalue of $H$ if and only if $-1$ is an eigenvalue of $Q(z)$,

$$Q(z) := AR_0(z)B = \begin{pmatrix} A_W R_0(z) B_W & A_W R_0(z) B_X \\ A_X R_0(z) B_W & A_X R_0(z) B_X \end{pmatrix}, \quad z \in \rho(H_0).$$

Since $\|A_X\| = \|B_X\| = \varepsilon^{1/2} < \|R_0(z)\|^{1/2}$ by (32), it follows that the operator $I_{\mathcal{H}} + A_X R_0(z) B_X$ has a bounded inverse. By the well-known Schur–Frobenius factorization (see e.g. [25, Proposition 1.6.2]), $I_{\mathcal{K}} + Q(z)$ has a bounded inverse if and only if so does its Schur complement $S(z)$,

$$S(z) := I_{\mathcal{H}} + A_W R_0(z) B_W - A_W R_0(z) B_X \left(I_{\mathcal{H}} + A_X R_0(z) B_X\right)^{-1} A_X R_0(z) B_W.$$

By a Neumann series argument, the latter holds whenever

$$\omega(z) := \frac{\|A_W R_0(z) B_X\| \, \|A_X R_0(z) B_W\|}{(1 - \|A_W R_0(z) B_W\|)(1 - \|A_X R_0(z) B_X\|)} < 1, \tag{34}$$

provided that $I_{\mathcal{H}} + A_W R_0(z) B_W$ has a bounded inverse as well. By the estimates used in the proof of Theorem 2.1, we have

$$\|A_W R_0(z) B_W\| \leq \eta(|\Phi(z)|)\, C_\varepsilon < 1$$

by (31). Together with (21) this yields

$$\omega(z) \leq \frac{\varepsilon\, C_\varepsilon \eta(|\Phi(z)|)^2}{(\operatorname{Im} \sqrt{z^2 - m^2})(1 - \eta(|\Phi(z)|)\, C_\varepsilon)\, (1 - \varepsilon \|R_0(z)\|)}.$$

It is not difficult to check that the right hand side above is $< 1$ if (and only if) (32) holds. $\square$

Theorem 4.1 is the analogue of [6, Theorem 1.5] for Dirac operators. The next theorem is the counterpart to [6, Theorem 2.9]. Keeping the same notation as in [6], we define the positive, decreasing convex function

$$F_V(s) := \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \|V(x)\|\, \mathrm{e}^{-s|x-y|}\, \mathrm{d}x, \quad s > 0.$$

**Theorem 4.3.** *Let* $V=(V_{ij})_{i,j=1}^2$ *with* $V_{ij}\in L^1(\mathbb{R})+L_0^\infty(\mathbb{R})$ *for* $i,j=1,2$. *Let* $z\in\rho(H_0)=\mathbb{C}\setminus\{\pm(p^2+m^2)^{1/2}:p\in\mathbb{R}\}$ *and let* $\eta$, $\Phi$ *be defined as in* (30). *If*

$$\eta(|\Phi(z)|)\,F_V\left(\operatorname{Im}\sqrt{z^2-m^2}\right)<1, \tag{35}$$

*then* $z\notin\sigma(H)$. *If* $m>0$ *and the equation* $F_V(\mu)=\mu/m$ *has a solution* $\mu_0\in(-m,m)$, *it is unique and*

$$\sigma(H)\cap\left(-\sqrt{m^2-\mu_0^2},\sqrt{m^2-\mu_0^2}\right)=\emptyset.$$

*Remark* 4.4. If $V_{ij}\in L^1(\mathbb{R})$, then by [6, Lemma 2.1]

$$F_V(s)\le\|V\|_1,\quad s>0.$$

Hence, Theorem 2.1 is a special case of Theorem 4.3.

*Proof.* As in the proof of Theorem 2.1, we assume that $V$ is bounded and use the factorization $V=BA$ with $A=|V|^{1/2}$, $B=U|V|^{1/2}$ [see (11)]. As before, we set $Q(z)=AR_0(z)B$ [see (12)].

Using a straightforward generalization of the Schur inequality to matrix-valued kernels, we obtain

$$\|Q(z)\|\le\left(\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\|Q(x,y;z)\|\frac{\mathrm{d}y}{\rho(x,y)}\right)^{1/2}\times\left(\sup_{y\in\mathbb{R}}\int_{\mathbb{R}}\|Q(x,y;z)\|\rho(x,y)\,\mathrm{d}x\right)^{1/2},$$

where $Q(x,y;z)$ is the kernel of $Q(z)$ and $\rho(x,y)$ is a positive weight. Choosing $\rho(x,y):=\|V(x)\|^{1/2}\|V(y)\|^{-1/2}$ and using $|R_0(x,y;z)|\le\eta(|\Phi(z)|)\mathrm{e}^{\operatorname{Im}k(z)}$, we arrive at

$$\|Q(z)\|\le\eta(|\Phi(z)|)\,F_V(\operatorname{Im}\sqrt{z^2-m^2}).$$

This proves the first part of the theorem.

Assume now that $m>0$ and let $z\in(-m,m)$. Observing that by (30),

$$\eta(|\Phi(z)|)=\frac{1}{\sqrt{2}}\sqrt{1+\frac{m^2+z^2}{m^2-z^2}}=\frac{m}{\sqrt{m^2-z^2}},$$

we infer that

$$\eta(|\Phi(z)|)\,F_V\left(\operatorname{Im}\sqrt{z^2-m^2}\right)=1\quad\Longleftrightarrow\quad F_V\left(\sqrt{m^2-z^2}\right)=\frac{\sqrt{m^2-z^2}}{m}.$$

Since the function $\mu\mapsto F_V(\mu)$ is decreasing [6, Lemma 2.1] and $\mu\mapsto\mu/m$ is increasing, the solution $\mu_0\in(-m,m)$ of the latter equation (which exists by assumption) is unique, and $F_V(\mu)<\mu/m$ for $\mu>\mu_0$. Therefore,

$$\eta(|\Phi(z)|)\,F_V\left(\operatorname{Im}\sqrt{z^2-m^2}\right)<1,\quad |z|<\sqrt{m^2-\mu_0^2},$$

and hence $z\notin\sigma(H)$ by the first part of the theorem. $\square$

*Remark* 4.5. Using different factorizations of $V$, one infers from the proof of Theorem 4.3 that for any factorization $V = B'A'$,

$$\eta(|\Phi(z)|)\, F_{A'^2}(\operatorname{Im}\sqrt{z^2-m^2})^{1/2}\cdot F_{B'^2}(\operatorname{Im}\sqrt{z^2-m^2})^{1/2} < 1 \implies z\in\rho(H),$$

However, Hölder's inequality applied to the positive measures $\mathrm{e}^{-s|x-y|}\,\mathrm{d}x$, $y\in\mathbb{R}$, yields

$$F_V(s)\le F_{A'^2}(s)^{1/2}F_{B'^2}(s)^{1/2}.$$

Theorem 4.1 enables us to obtain eigenvalue bounds in terms of higher $L^p$-norms of the potential $V$.

**Corollary 4.6.** *Suppose $V_{ij}\in L^p(\mathbb{R})$ for $i,j=1,2$ and some $p\in(1,\infty)$, and set*

$$\|V\|_p := \left(\int_{\mathbb{R}} \|V(x)\|^p\,\mathrm{d}x\right)^{1/p}.$$

*Let $z\in\rho(H_0)=\mathbb{C}\setminus\{\pm(p^2+m^2)^{1/2} : p\in\mathbb{R}\}$ and let $\eta$, $\Phi$ be defined as in (30). If*

$$\eta(|\Phi(z)|)\left(\frac{2(p-1)}{p}\right)^{(p-1)/p}\left(\operatorname{Im}\sqrt{z^2-m^2}\right)^{-(p-1)/p}\|V\|_p<1, \tag{36}$$

*then $z\notin\sigma(H)$.*

*Proof.* This is a consequence of Theorem 4.3 and the inequality

$$F_V(s)\le\left(\frac{2(p-1)}{p}\right)^{(p-1)/p} s^{-(p-1)/p}\,\|V\|_p\,,$$

see [6, Corollary 2.17]. □

Although the conditions in the above theorems seem to be very complicated, they may still provide explicit eigenvalue bounds as the following example shows.

*Example* 4.7. Let $\mu\in\mathbb{C}$, $\operatorname{Re}\mu\neq 0$, and consider the massless Dirac operator $H_\mu = H_0 + V_\mu$ with potential

$$V_\mu(x)=\frac{2\mu}{\sinh(2\mu x+\mathrm{i})}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad x\in\mathbb{R},$$

see [23]. Since

$$\|V_\mu\|_p^p=(2|\mu|)^{p-1}\int_{\mathbb{R}}\frac{1}{|\sinh(\mathrm{e}^{\mathrm{i}\arg(\mu)}x+\mathrm{i})|^p}\,\mathrm{d}x$$

and $\eta(|\Phi(z)|) = 1$ for $m = 0$ by (30), Corollary 4.6 implies that for every $p > 1$, all eigenvalues of $H_\mu$ are contained in the strip

$$\sigma_{\mathrm{d}}(H_\mu) \subset \left\{ z \in \mathbb{C} : 0 < |\mathrm{Im}\, z| \le |\mu| \, \frac{4(p-1)}{p} \right. \\ \left. \times \left( \int_{\mathbb{R}} \frac{1}{|\sinh(\mathrm{e}^{\mathrm{i} \arg(\mu)} x + \mathrm{i})|^p} \, \mathrm{d}x \right)^{1/(p-1)} \right\}.$$

For $p = 1$, one can check that

$$\|V_\mu\|_1 = \int_{\mathbb{R}} \frac{1}{|\sinh(\mathrm{e}^{\mathrm{i} \arg(\mu)} x + \mathrm{i})|} \, \mathrm{d}x \ge \int_{\mathbb{R}} \frac{1}{|\sinh(x + \mathrm{i})|} \, \mathrm{d}x \ (\approx 3.4184)$$

is greater than one (and independent of $|\mu|$) so that Theorem 2.1 cannot exclude the occurrence of non-real eigenvalues. In fact, it was shown in [23] that $H_\mu$ does have the non-real eigenvalue $\mathrm{i}\mu$.

*Remark* 4.8. Similar estimates as in (36) have been derived in [5] by a more abstract approach. For example, for $m > 0$ and $p = 2$, the results of [5] imply that

$$\sigma(H) \subset \left\{ z \in \mathbb{C} : |\mathrm{Im}\, z| \le 2 \, \|V\|_2^2 \, (1 + |z|)^{1/2} \right\}. \tag{37}$$

In comparison, (36) above implies that

$$\sigma(H) \subset \left\{ z \in \mathbb{C} : \mathrm{Im} \sqrt{z^2 - m^2} \le \eta(|\Phi(z)|)^2 \|V\|_2^2 \right\}. \tag{38}$$

Asymptotically, (37) and (38) yield that for $z \in \sigma(H)$

$$|\mathrm{Im}\, z| \le 2 \, \|V\|_2^2 \, |z|^{1/2} \quad \text{and} \quad |\mathrm{Im}\, z| \le \|V\|_2^2, \quad |z| \to \infty,$$

respectively. The second estimate is clearly superior, which is not surprising since the results of [5] are of much more general nature. They are applicable to Dirac operators in arbitrary dimension as well as to abstract Hilbert space operators.

The result of Corollary 4.6 may also be used to prove that $H$ is similar to a block diagonal matrix operator if the $L^p$-norm is sufficiently small and $p \in [2, \infty]$. For more results on block diagonalization of Dirac operators as well as abstract Hilbert space operators, the reader is referred to [4].

**Theorem 4.9.** *Let $m > 0$, $V_{ij} \in L^p(\mathbb{R})$ for $i, j = 1, 2$ and some $p \in [2, \infty)$. If*

$$\|V\|_p < \left( \frac{mp}{2(p-1)} \right)^{(p-1)/p}, \tag{39}$$

*then $H$ is similar to a block-diagonal operator,*

$$SHS^{-1} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad \sigma(H_\pm) = \sigma(H) \cap \{z \in \mathbb{C} : \pm \mathrm{Re}\, z > 0\}.$$

*Proof.* If $z = \mathrm{i}\,t$, $t \in \mathbb{R}$, then (39) ensures that (36) holds and thus

$$\|Q(\mathrm{i}\,t))\| < \left(\frac{2(p-1)}{p}\right)^{(p-1)/p} \left(\sqrt{t^2+m^2}\right)^{-(p-1)/p} \left(\frac{mp}{2(p-1)}\right)^{(p-1)/p} \leq 1; \tag{40}$$

hence, $\mathrm{i}\mathbb{R} \subset \rho(H)$. Let again $A := |V|^{1/2}$, $B := U|V|^{1/2}$, and set $Y := A^p$. Since $A_{ij} \in L^{2p}(\mathbb{R})$, it follows that $Y_{ij} \in L^2(\mathbb{R})$; hence $Y$ is $H_0$-bounded (see for instance [27, Satz 17.7]). By Heinz' inequality, $Y^\alpha$ is $|H_0|^\alpha$-bounded for any $\alpha \in (0,1)$. In particular, for $\alpha = 1/p$, $A$ is $|H_0|^{1/p}$-bounded. Thus, since $|H_0|^{1/p} \geq (m)^{1/p}$, there exists a constant $\delta_m < \infty$ such that for all $z \in \rho(H_0)$

$$\|AR_0(z)\| \leq \delta_m \,\||H_0|^{1/p} R_0(z)\|. \tag{41}$$

Analogously, one can show that

$$\|R_0(z)B\| \leq \delta_m \,\||H_0|^{1/p} R_0(z)\|. \tag{42}$$

For $\chi \in \mathbb{C}$, $|\chi| < 1$, let $H(\chi) := H_0 + \chi V$. By inspection of the resolvent of $H(\chi)$,

$$(H(\chi)-z)^{-1} = R_0(z) - \chi R_0(z) B \left(I_{\mathcal{K}} + \chi Q(z)\right)^{-1} AR_0(z),$$

it is easily seen that $H(\chi)$, $|\chi| < 1$, is a holomorphic family. For $f \in \mathcal{H}$, we define

$$P(\chi)f := \frac{1}{2}\,f + \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} (H(\chi) - \mathrm{i}t)^{-1} f \,\mathrm{d}t, \quad |\chi| < 1. \tag{43}$$

We shall show that the limit exists and that $P(\chi)$ is a bounded-holomorphic family of projections. By [14, II.4.2], it then follows that there exists a bounded-holomorphic family of isomorphisms $U(\chi)$ such that

$$U(\chi)P(\chi)U(\chi)^{-1} = P(0), \quad \chi \in \mathbb{C}, \quad |\chi| < 1.$$

On the other hand, by the standard Foldy–Wouthuysen transformation (i.e. diagonalizing $H_0$ in momentum space, see e.g. [24]), there exists a unitary operator $\widetilde{U}$ such that

$$\widetilde{U} P(0) \widetilde{U}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The claim thus follows with $S := \widetilde{U}U(1)$.

Since $H_0$ is selfadjoint, the right hand side of (43) exists for $\chi = 0$ and coincides with the spectral projection onto the positive spectral subspace of $H_0$, by the spectral theorem. It is thus sufficient to show the convergence of the integral

$$\lim_{R\to\infty} \int_{-R}^{R} \left(\left((H(\chi)-\mathrm{i}t)^{-1} - R_0(\mathrm{i}t)\right) f, g\right) \,\mathrm{d}t$$

uniformly in $g \in \mathcal{H}$, $\|g\| = 1$, and locally uniformly in $\chi \in \mathbb{C}$, $|\chi| < 1$. Indeed, since by (40),

$$q_0 := \sup_{t\in\mathbb{R}} \|Q(\mathrm{i}t)\| < 1,$$

the estimates (41), (42) imply, for $|\chi| < 1$,

$$\begin{aligned}
&\int_{-R}^{R} \left|\left(\left(H(\chi) - \mathrm{i}t\right)^{-1} - R_0(\mathrm{i}t)\right) f, g\right)\right| \mathrm{d}t \\
&\quad\le (1-q_0)^{-1} \int_{-R}^{R} \|AR_0(\mathrm{i}t)f\| \|R_0(\mathrm{i}t)Bg\| \,\mathrm{d}t \\
&\quad\le (1-q_0)^{-1} \int_{-R}^{R} \||H_0|^{1/p} R_0(\mathrm{i}t)f\| \||H_0|^{1/p} R_0(\mathrm{i}t)g\| \,\mathrm{d}t \\
&\quad\le (1-q_0)^{-1} \left(\int_{-R}^{R} \||H_0|^{1/p} R_0(\mathrm{i}t)f\|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_{-R}^{R} \||H_0|^{1/p} R_0(\mathrm{i}t)g\|^2 \,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Denoting by $E(\cdot)$ the spectral function of $H_0$, we can estimate

$$\begin{aligned}
\int_{-R}^{R} \||H_0|^{1/p} R_0(\mathrm{i}t)f\|^2 \,\mathrm{d}t &\le \int_{\sigma(H_0)} \int_{-\infty}^{\infty} \frac{|s|^{2/p}}{s^2+t^2} \,\mathrm{d}t \,\mathrm{d}\|E(s)f\|^2 \\
&= \pi \int_{\sigma(H_0)} |s|^{(2/p)-1} \,\mathrm{d}\|E(s)f\|^2 \le \pi(m)^{(2/p)-1} \|f\|^2.
\end{aligned}$$

The fact that $P(\chi)$ is the spectral projection corresponding to the right half plane may be deduced from [12, Theorem 3.1] in combination with the residue theorem, see also [16, Theorem 1.1], [4, Theorem 2.4]. In order to apply the latter, it remains to be shown that

$$\lim_{t\to\infty} \|(H - \mathrm{i}t)^{-1}\| = 0. \tag{44}$$

By the spectral theorem for $H_0$,

$$\begin{aligned}
\|(H-\mathrm{i}t)^{-1}\| &\le \|(H_0-\mathrm{i}t)^{-1}\| + \|(H-\mathrm{i}t)^{-1} - (H_0-\mathrm{i}t)^{-1}\| \\
&\le \frac{1}{|t|} + (1-q_0)^{-1} \||H_0|^{1/p} R_0(\mathrm{i}t)\|^2 \le \frac{1}{|t|} + \frac{C}{|t|^{1-1/p}}
\end{aligned}$$

for some $C > 0$. This proves (44). □

## 5. Embedded Eigenvalues and Resonances

In this section we show how the previous results may be applied to locate the embedded eigenvalues and resonances of selfadjoint Dirac operators using the method of complex scaling. To this end, we assume that $m > 0$, $V$ is Hermitian valued and dilation analytic.

For simplicity, we assume that $V$ is bounded and restrict ourselves to the case $\|V\|_1 < 1$ (see Theorem 2.1).

Let $U(\theta)$ be the unitary dilation in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, given by

$$(U(\theta)f)(x) := \mathrm{e}^{\theta/2} f(\mathrm{e}^{\theta} x), \quad x, \theta \in \mathbb{R}.$$

For $\alpha \in (0, \pi/2)$ let $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$ with $-\pi < \arg(z) < \pi$.

**Hypothesis 5.1.** Assume that there exists $\alpha \in (0, \pi/2)$ such that:

(i) $V : \Sigma_\alpha \cup (-\Sigma_\alpha) \to \mathbb{C}^{2\times 2}$ is a bounded analytic function;
(ii) The restriction of $V$ to the real axis is Hermitian valued;
(iii) For each $\beta \in (0, \alpha)$ the functions $V(\mathrm{e}^{\mathrm{i}\varphi}\cdot)$, $|\varphi| \le \beta$, are in $L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$ with uniformly bounded $L^1$-norms.

We define the complex-dilated operators

$$\begin{aligned} H_0(\theta) &:= U(\theta) H_0 U(\theta)^{-1} = -\mathrm{i}\mathrm{e}^{-\theta} \frac{\mathrm{d}}{\mathrm{d}x} \sigma_1 + m\sigma_3, \\ V(\theta) &:= U(\theta) V U(\theta)^{-1} = V(\mathrm{e}^{\theta}\cdot), \\ H(\theta) &:= U(\theta)(H_0 + V) U(\theta)^{-1} = H_0(\theta) + V(\theta). \end{aligned}$$

It is straightforward to check that $H_0(\theta)$ has an extension to an entire family of type (A) in the sense of Kato [14, VII.2]; see e.g. [26, Lemma 1].

**Proposition 5.2.** *Assume that $m > 0$ and that $V$ is bounded and satisfies Hypothesis 5.1 for some $\alpha \in (0, \pi/2)$. Then the following hold:*

(i) *$V(\theta)$ has an extension to an analytic bounded operator-valued function in the strip $S_\alpha := \{\theta \in \mathbb{C} : |\mathrm{Im}\,\theta| < \alpha\}$;*
(ii) *for $\mu \in \mathbb{R}$, $|\mu|$ sufficiently large, $\mathrm{i}\,\mu \in \rho(H(\theta))$ for all $\theta \in S_\alpha$, and for $\mathrm{i}\mu \in \rho(H(\theta))$ fixed, $(H(\theta) - \mathrm{i}\,\mu)^{-1}$ is an analytic bounded operator-valued function in $S_\alpha$;*
(iii) *$U(\varphi)H(\theta)U(\varphi)^{-1} = H(\theta + \varphi)$ for all $\varphi \in \mathbb{R}$, $\theta \in S_\alpha$;*
(iv) *$\sigma(H(\theta))$ depends only on $\mathrm{Im}\,\theta$;*
(v) *$\sigma_{\mathrm{e}}(H_0(\theta)) = \{\pm\sqrt{\mathrm{e}^{-2\theta}p^2 + m^2} : p \in \mathbb{R}\}$;*
(vi) *$\sigma_{\mathrm{d}}(H(\theta)) \cap \mathbb{R} = \sigma_{\mathrm{p}}(H) \setminus \{-m, m\}$;*
(vii) *for $\mathrm{Im}\,\theta \in (0, \alpha)$, all non-real eigenvalues of $H(\theta)$ lie in the region*

$$D_\theta := \{\pm\sqrt{\mathrm{e}^{-2\omega}p^2 + m^2} : p \in \mathbb{R},\ \mathrm{Im}\,\omega \in [0, \mathrm{Im}\,\theta]\},$$

*see Fig. 2. If $0 < \mathrm{Im}\,\theta_1 < \mathrm{Im}\,\theta_2 < \alpha$, then $\sigma_{\mathrm{d}}(H(\theta_1)) \subset \sigma_{\mathrm{d}}(H(\theta_2))$.*
(viii) *for $\beta \in (0, \alpha)$, the function $\varphi \mapsto \|V(\mathrm{e}^{\mathrm{i}\varphi}\cdot)\|_1$ is logarithmically convex in the interval $[-\beta, \beta]$.*

*Proof.* (i) Since $S_\alpha$ is mapped onto $\Sigma_\alpha$ under the mapping $\theta \mapsto \mathrm{e}^{\theta}$, it follows that $V(\theta) \in \mathrm{L}(\mathcal{H})$. It is easy to see that $V(\theta)$, $\theta \in S_\alpha$, is weakly analytic, and hence analytic in norm; see e.g. [14, Theorem III.1.3.7].

(ii) Since $V(\theta)$ is uniformly bounded in the operator norm, $\|V(\theta)\| \le M < \infty$, the spectrum of $H(\theta)$ is contained in the $M$-neighbourhood of

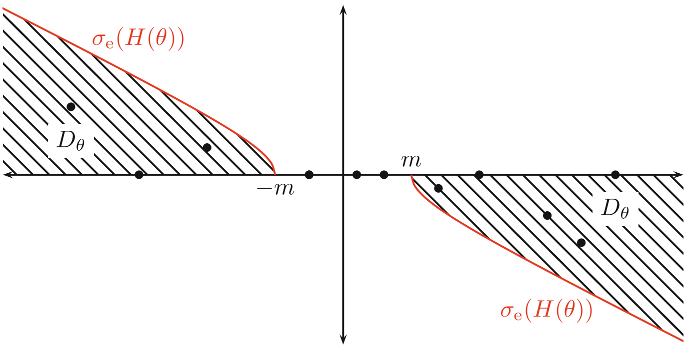


FIGURE 2. Eigenvalues of $H$ and the set $D_\theta$ enclosing resonances of $H$

$\sigma(H_0(\theta))$ by the stability of bounded invertibility. Hence, $\mathrm{i}\,\mu \in \rho(H(\theta))$ for $|\mu|$ sufficiently large. The analyticity of $(H(\theta) - \mathrm{i}\,\mu)^{-1}$ follows from the formula

$$(H(\theta) - \mathrm{i}\,\mu)^{-1} = (H_0(\theta) - \mathrm{i}\,\mu)^{-1}(I + V(\theta)(H_0(\theta) - \mathrm{i}\,\mu)^{-1})^{-1}$$

and from the observation that $H_0(\theta)$ is a normal operator, whence for $|\mu|$ sufficiently large,

$$\|(H_0(\theta) - \mathrm{i}\,\mu)^{-1}\| = \mathrm{dist}(\mathrm{i}\,\mu, \sigma(H_0(\theta)) < 1/M.$$

(iii) is clearly valid for real $\theta$, and since both sides of the equation are analytic, the claim follows from the identity theorem. (iv) is a direct consequence of (iii).

For the proof of (v)–(vii), we refer to [20, Theorem 1]; compare also [18, XIII.36]. Unlike in [20], we do not assume that $V$ is $H_0$-compact; however, by Proposition 6.6 below the essential spectra of $H$ and $H_0$ are the same. Since

$$(H(\theta) - z)^{-1} - (H_0(\theta) - z)^{-1} = U(\theta)((H - z)^{-1} - (H_0 - z)^{-1})U(\theta)^{-1},$$

the same applies to the essential spectra of $H(\theta)$ and $H_0(\theta)$ and thus the proof of [20, Theorem 1] carries through in the case considered here.

(viii) Let $g \in L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} V_{ij}(\mathrm{e}^\theta x)g(x)\ \mathrm{d}x$$

depends analytically on $\theta \in S_\alpha$ since on any compact subset $K \subset S_\alpha$ the absolute value of the integral is bounded by

$$\rho \cdot \sup_{|\varphi|\le\beta} \|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1 \cdot \|g\|_\infty \quad \text{where} \quad \rho := \min_{\theta\in K} \mathrm{e}^{-\mathrm{Re}\,\theta}, \quad \beta := \max_{\theta\in K} |\mathrm{Im}\,\theta|.$$

Hence, the map $(\theta \mapsto V(\mathrm{e}^\theta\cdot)) : S_\alpha \to L^1(\mathbb{R}, \mathbb{C}^{2\times 2})$ is weakly (and hence strongly) analytic. For $\beta \in (0, \alpha)$ consider the map

$$F : S_\beta \to L^1(\mathbb{R}, \mathbb{C}^{2\times 2}), \quad F(\theta) := \mathrm{e}^\theta V(\mathrm{e}^\theta\cdot)$$

which is analytic, continuous up to the boundary of $S_\beta$, and uniformly bounded in $\overline{S_\beta}$. The claim follows by applying Hadamard's three lines theorem for analytic functions with values in a Banach space, see e.g. [7, III.14], to $F$ and noting that $\|F(\mathrm{i}\,\varphi)\|_1 = \|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1$. $\square$

It may be shown, see [20, Theorem 2], that the resolvent $(H-z)^{-1}$ has a (many-sheeted) analytic continuation to the set $\rho(H_\theta)$. The poles of the analytically continued resolvent are called the *resonances* of $H$, and they are located precisely at the eigenvalues of $H_\theta$. We denote the set of resonances of $H$ by $\mathcal{R}(H)$.

**Theorem 5.3.** *Assume that $m>0$ and that $V$ is bounded and satisfies Hypothesis 5.1 with $\alpha \in (0,\pi/2)$.*

(i) *If $\operatorname{Im}\theta \in [0,\alpha)$ and*

$$v_\theta := \inf_{\operatorname{Im}\theta \le \varphi < \alpha} \|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1 < 1,$$

*then the resonances of $H$ satisfy the inclusion*

$$\mathcal{R}(H) \cap D_\theta \subset K_{mr_\theta}(mx_\theta) \cup K_{mr_\theta}(-mx_\theta) \tag{45}$$

*where*

$$x_\theta := \sqrt{\frac{v_\theta^4 - 2v_\theta^2 + 2}{4(1-v_\theta^2)} + \frac{1}{2}}, \quad r_\theta := \sqrt{\frac{v_\theta^4 - 2v_\theta^2 + 2}{4(1-v_\theta^2)} - \frac{1}{2}}. \tag{46}$$

(ii) *Assume that $\|V\|_1 < 1$. Then all eigenvalues of $H$ (including the embedded ones) are contained in the intervals*

$$\big(-m(x_0+r_0), -m(x_0-r_0)\big) \cup \big(m(x_0-r_0), m(x_0+r_0)\big), \tag{47}$$

*where $x_0$, $r_0$ are given in (10) (i.e. (46) with $v_\theta = v_0 = \|V\|_1$).*

(iii) *If $m=0$ and $\|V\|_1 < 1$, then there are no resonances close to the real axis; more precisely, if we set*

$$\varphi_0 := \sup\{\operatorname{Im}\theta \in [0,\alpha) : v_\theta < 1\} > 0,$$

*then*

$$\mathcal{R}(H) \cap \Big\{ \pm\sqrt{\mathrm{e}^{-2\omega}p^2 + m^2} : p \in \mathbb{R},\, \operatorname{Im}\omega \in [0,\varphi_0] \Big\} = \emptyset.$$

*Proof.* (i) Let $(\theta_n)_{n\in\mathbb{N}} \subset S_\alpha$ be such that $\varphi_n := \operatorname{Im}\theta_n \ge \operatorname{Im}\theta$, $n \in \mathbb{N}$, and

$$\|V(\mathrm{e}^{\mathrm{i}\operatorname{Im}\theta_n}\cdot)\|_1 \longrightarrow v_\theta, \quad n \to \infty.$$

Then there exists $N \in \mathbb{N}$ such that $\|V(\mathrm{e}^{\mathrm{i}\operatorname{Im}\theta_n}\cdot)\|_1 < 1$ for all $n \ge N$. Since

$$\mathrm{e}^{\mathrm{i}\,\varphi_n} H(\mathrm{i}\,\varphi_n) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 + m\mathrm{e}^{\mathrm{i}\,\varphi_n}\sigma_3 + \mathrm{e}^{\mathrm{i}\,\varphi_n} V(\mathrm{e}^{\mathrm{i}\,\varphi_n}\cdot)$$

and $|\mathrm{e}^{\mathrm{i}\,\varphi_n}| = 1$, Theorem 2.1 and Proposition 5.2 (iii) imply that for all $n \ge N$, the non-embedded eigenvalues of $\mathrm{e}^{\mathrm{i}\,\varphi_n} H(\theta_n)$ lie in the disks

$$K_{mr_{\theta_n}}(m\mathrm{e}^{\mathrm{i}\,\varphi_n} x_{\theta_n}) \cup K_{mr_{\theta_n}}(-m\mathrm{e}^{\mathrm{i}\,\varphi_n} x_{\theta_n}). \tag{48}$$

By Proposition 5.2 (vii) and (48), it follows that for all $n \geq N$,

$$\mathcal{R}(H) \cap D_\theta \subset K_{mr_{\theta_n}}(mx_{\theta_n}) \cup K_{mr_{\theta_n}}(-mx_{\theta_n}).$$

Letting $n \to \infty$ proves (45).

(ii) By the proof of Proposition 5.2 (viii), $\|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1$ is continuous, so that

$$\lim_{\varphi \searrow 0} \|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1 = \|V\|_1.$$

Let $(\theta_n)_{n\in\mathbb{N}} \subset S_\alpha$ be such that $\varphi_n := \operatorname{Im}\theta_n \to 0$ and $\|V(\mathrm{e}^{\mathrm{i}\,\varphi_n}\cdot)\|_1 \to \|V\|_1$, $n \to \infty$. Moreover, let $N \in \mathbb{N}$ be such that $\|V(\mathrm{e}^{\mathrm{i}\,\varphi_n}\cdot)\|_1 < 1$, $n \geq N$. If $\lambda \in \mathbb{R}\backslash\{\pm m\}$ is an eigenvalue of $H$, then by Proposition 5.2 (vi), $\lambda \in \sigma(H(\theta_n))$ for all $n \geq N$. The inclusion (47) now follows from (48) if we take $n \to \infty$.

(iii) is immediate from i) since then $mr_\theta = 0$ (recall that we use the convention $K_0(z_0) = \emptyset$).

□

*Remark* 5.4. The resonance enclosure (45) in Theorem 5.3 may be used for every $\theta$, with $v_\theta < 1$. However, increasing $\operatorname{Im}\theta$ in order to enlarge the set $D_\theta$ revealing the resonances increases the size of the resonance-enclosing disks $K_{mr_\theta(\pm mx_\theta)}$. For every $\theta$, the disks $K_{mr_\theta(\pm mx_\theta)}$ intersect the boundary $\sigma_{\mathrm{e}}(H(\theta))$ of $D_\theta$ in only one point each. The set of intersection points consists of two curves parameterized by $\operatorname{Im}\theta$. All resonances in $D_\alpha$ in the lower half plane lie between these two curves (see Fig. 3) and analogously in the upper half plane.

*Example* 5.5. Consider the resonances and embedded eigenvalues for the potential

$$V(x) = a\,\mathrm{e}^{-b\,x^2}\,I_{\mathbb{C}^2}$$

with $a \in \mathbb{R}$, $b > 0$. Clearly, $V$ has an analytic continuation to an entire function, bounded on $\overline{\Sigma_{\pi/4}}$. Moreover, for $|\varphi| < \pi/4$, the function $V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)$ is in $L^1(\mathbb{R})$ with norm

$$\|V(\mathrm{e}^{\mathrm{i}\,\varphi}\cdot)\|_1 = \frac{|a|\sqrt{\pi}}{\sqrt{b\,\cos(2\varphi)}},$$

hence it is uniformly bounded for $|\varphi| \leq \beta < \pi/4$. Since $V(x) \geq 0$, $x \in \mathbb{R}$, by Theorem 5.3 (ii), $v_\theta = \|V(\mathrm{e}^{\mathrm{i}\,\operatorname{Im}\theta}\cdot)\|_1$. Hence, if $|a|\sqrt{\pi}/\sqrt{b} < 1$, then $v_\theta < 1$ for all $\theta \in [0, \pi/4)$ with

$$\operatorname{Im}\theta < \frac{1}{2}\arccos\left(\frac{|a|^2\pi}{b}\right).$$

Therefore, for these $\theta$, Theorem 5.3 (i) and (iii) apply; for example, the resonances in $D_{\pi/6}$ lie in the union of the two disks $K_{mr_{\pi/6}}(\pm mx_{\pi/6})$ with

$$x_{\pi/6} = \frac{b - a^2\pi}{\sqrt{b(b-2a^2\pi)}}, \quad r_{\pi/6} = \frac{a^2\pi}{\sqrt{b(b-2a^2\pi)}},$$

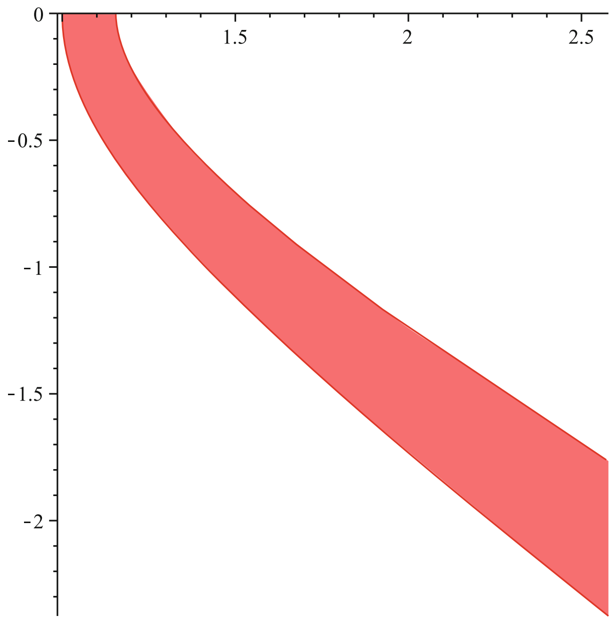


Figure 3. The resonances of Example 5.5 in the *lower half plane* are situated in the area between the *two red curves* (colour figure online)

the eigenvalues of $H$ (including the embedded ones) lie in the two intervals

$$\left(-m\left(1-\frac{a^2\pi}{b}\right)^{-1/2}, -m\left(1-\frac{a^2\pi}{b}\right)^{1/2}\right)$$
$$\cup\left(m\left(1-\frac{a^2\pi}{b}\right)^{1/2}, m\left(1-\frac{a^2\pi}{b}\right)^{-1/2}\right).$$

Figure 3 shows the region of resonance enclosure in the lower half plane; the picture in the upper half plane is just the mirror image.

## 6. Construction of H for Potentials in $\mathbf{L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})}$

In Sects. 2, 3, 4 and 5 we assumed in all proofs that $V$ is bounded so that we could conveniently define the sum of $H_0$ and $V$. In this final section we show how to construct a closed extension $H$ of $H_0 + V$ for $V \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$.

One might first try to approximate $V \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$ by bounded potentials $V_n$, and then show that the operators $H_n = H_0 + V_n$ converge in the norm-resolvent topology to some operator $H$. If $V$ were Hermitian valued (and thus $H_n$, $H$ selfadjoint), we could conclude that the eigenvalue estimates also hold for the limit operator $H$. However, for non-Hermitian potentials, this need not be true since the spectrum is not lower-semicontinuous on the metric space of closed operators; see [14, IV.3.2].

Therefore, we need a more direct access to the perturbed operator $H$. If we define it via its resolvent by Eq. (20), then it will turn out to be a closed extension of $H_0 + V$. The precise statement is given in the subsequent abstract theorem, which includes the general version of the Birman–Schwinger principle. We note that this construction is more general than a quadratic form approach or even an operator perturbation approach; see [11, Remark 2.4 iii)].

**Theorem 6.1.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $H_0 : \mathcal{H} \to \mathcal{H}$, $A : \mathcal{H} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{H}$ be closed densely defined operators. Suppose that $\rho(H_0) \neq \emptyset$ and that the following hold:*

(a) $AR_0(z) \in \mathrm{L}(\mathcal{H}, \mathcal{K})$ *and* $\overline{R_0(z)B} \in \mathrm{L}(\mathcal{K}, \mathcal{H})$.
(b) *For some (and hence for all ) $z \in \rho(H_0)$, the operator $AR_0(z)B$ has bounded closure*

$$Q(z) := \overline{AR_0(z)B} \in \mathrm{L}(\mathcal{K}).$$

(c) $-1 \in \rho(Q(z_0))$ *for some* $z_0 \in \rho(H_0)$.

*Then there exists a closed densely defined extension $H$ of $H_0 + BA$ whose resolvent $R(z) = (H - z)^{-1}$, $z \in \rho(H)$, is given by*

$$R(z) = R_0(z) - \overline{R_0(z)B}\,(I_{\mathcal{K}} + Q(z))^{-1} AR_0(z) \in \mathrm{L}(\mathcal{H}), \quad z \in \rho(H_0) \cap \rho(H), \tag{49}$$

*with*

$$\rho(H) \cap \rho(H_0) = \{z \in \rho(H_0) : -1 \in \rho(Q(z))\}.$$

*Moreover, for $z \in \rho(H_0)$, the subspaces* $\ker(H - z)$ *and* $\ker(I + Q(z))$ *are isomorphic.*

*Proof.* The proof may be found e.g. in [11]; compare also [13,15]. □

*Remark* 6.2. If $H_0 + V$ has non-empty resolvent set, and is, hence, closed, then $H = H_0 + V$. In particular, this is the case whenever $V$ is bounded, or, more generally, $H_0$-bounded with relative bound less than one. For example, this holds if $V_{i,j} \in L^p(\mathbb{R})$ for some $p \in [2, \infty]$; see e.g. [27, Satz 17.7]. Note that the whole $L^p$-scale, $p \in [1, \infty]$, is contained in the class $L^1(\mathbb{R}) + L^\infty_0(\mathbb{R})$ considered in Sect. 4.

Since the proofs of Sects. 2, 3, 4 and 5 only involve the resolvent $R_0(z)$, they admit straightforward generalizations to the case where $V$ is unbounded and $H$ is the operator given by Theorem 6.1; one just has to replace $R_0(z)B$ and $AR_0(z)B$ by their bounded closures everywhere. Indeed, (16) and (21) guarantee that the conditions (a)–(c) of Theorem 6.1 are satisfied. What remains to be shown is that

1. the different factorizations of $V$ used in Sect. 4 lead to the same extension $H$;
2. we still have $\sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{e}}(H_0)$.

To address (1) we introduce the following definition.

**Definition 6.3.** Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}'$ be Hilbert spaces, and let $H_0 : \mathcal{H} \to \mathcal{H}$, $A : \mathcal{H} \to \mathcal{K}$, $B : \mathcal{K} \to \mathcal{H}$, $A' : \mathcal{H} \to \mathcal{K}'$, $B' : \mathcal{K}' \to \mathcal{H}$ be such that $BA = B'A'$. Suppose that the triples $(H_0, A, B)$ and $(H_0, A', B')$ satisfy the assumptions of Theorem 6.1. The two factorizations $V := BA = B'A'$ are called *compatible* if the following hold:

(i) The operators $A'R_0(z)B$ and $AR_0(z)B'$ have bounded closure for one (and hence for all) $z \in \rho(H_0)$,

$$F(z) := \overline{A'R_0(z)B} \in \mathrm{L}(\mathcal{K}, \mathcal{K}'), \quad G(z) := \overline{AR_0(z)B'} \in \mathrm{L}(\mathcal{K}', \mathcal{K}).$$

(ii) There exist dense linear manifolds $\mathcal{C} \subset \mathcal{H}$, $\mathcal{D} \subset \mathcal{K}$ and $\mathcal{D}' \subset \mathcal{K}'$ such that for all $z \in \rho(H_0)$,

$$\begin{aligned}
\mathcal{C} &\subset \{f \in \mathcal{H} : R_0(z)f \in \mathcal{D}(V),\ R_0(z)VR_0(z)f \in \mathcal{D}(V)\},\\
\mathcal{D} &\subset \{f \in \mathcal{D}(B) : R_0(z)Bf \in \mathcal{D}(V)\},\\
\mathcal{D}' &\subset \{f \in \mathcal{D}(B') : R_0(z)B'f \in \mathcal{D}(V)\}.
\end{aligned}$$

**Proposition 6.4.** *If $V = BA = B'A'$ are two compatible factorizations, then the corresponding extensions $H$ and $H'$ of $H_0 + V$ in Theorem 6.1 coincide.*

*Proof.* By the first resolvent identity for $H_0$, for $z_1, z_2 \in \rho(H_0)$,

$$A'R_0(z_1)B - A'R_0(z_2)B = (z_2 - z_1)\, A'R_0(z_2)R_0(z_1)B.$$

Since the right hand side has bounded (everywhere defined) closure by assumption (i), it follows that $A'R_0(z_1)B$ has bounded closure if and only if $A'R_0(z_2)B$ does. Denote

$$Q(z) := \overline{AR_0(z)B}, \quad Q'(z) := \overline{A'R_0(z)B'}, \quad z \in \rho(H_0).$$

For $f \in \mathcal{D}$, $g \in \mathcal{D}'$, $z \in \rho(H_0)$, we then have the identities

$$\begin{aligned}
F(z)Q(z)f &= A'R_0(z)B\,AR_0(z)Bf = A'R_0(z)B'\,A'R_0(z)Bf = Q'(z)F(z)f,\\
G(z)Q'(z)g &= AR_0(z)B'\,A'R_0(z)B'g = AR_0(z)B\,AR_0(z)B'g = Q(z)G(z)g,
\end{aligned}$$

which extend to all $f \in \mathcal{K}$, $g \in \mathcal{K}'$ by continuity, due to (ii). In particular, for all $z \in \rho(H)$,

$$\begin{aligned}
F(z)(I_{\mathcal{K}} \pm Q(z)) &= (I_{\mathcal{K}'} \pm Q'(z))F(z),\\
G(z)(I_{\mathcal{K}'} \pm Q'(z)) &= (I_{\mathcal{K}} \pm Q(z))G(z).
\end{aligned}$$

Using the identities above, one can check that if $-1 \in \rho(Q(z))$, then $-1 \in \rho(Q'(z))$ and vice versa, and

$$(I_{\mathcal{K}'} + Q'(z))^{-1} = (I_{\mathcal{K}'} - Q'(z)) + F(z)(I_{\mathcal{K}} + Q(z))^{-1}G(z), \tag{50}$$

$$(I_{\mathcal{K}} + Q(z))^{-1} = (I_{\mathcal{K}} - Q(z)) + G(z)(I_{\mathcal{K}'} + Q'(z))^{-1}F(z). \tag{51}$$

This proves that

$$\rho(H) \cap \rho(H_0) = \rho(H') \cap \rho(H_0) \neq \emptyset.$$

Using formula (50) and the equality $BA = B'A'$, we infer that on the linear manifold $\mathcal{C} \subset \mathcal{H}$, for all $z \in \rho(H_0) \cap \rho(H)$,

$$\begin{aligned}
&R_0(z)B\left(I_{\mathcal{K}} + Q(z)\right)^{-1} AR_0(z)\\
&\quad = R_0(z)VR_0(z) - R_0(z)VR_0(z)VR_0(z)\\
&\qquad + R_0(z)VR_0(z)B'\left(I_{\mathcal{K}'} + Q'(z)\right)^{-1} A'R_0(z)VR_0(z)\\
&\quad = R_0(z)B'(I_{\mathcal{K}'} - Q'(z) + Q'(z)\left(I_{\mathcal{K}'} + Q'(z)\right)^{-1} Q'(z))A'R_0(z)\\
&\quad = R_0(z)B'\left(I_{\mathcal{K}'} + Q'(z)\right)^{-1} A'R_0(z).
\end{aligned}$$

Since $\mathcal{C}$ is dense in $\mathcal{H}$, this identity extends to all of $\mathcal{H}$ by continuity if we replace $R_0(z)B$ and $R_0(z)B'$ by their (bounded) closures, and hence formula (49) for the resolvents of $H$ and $H'$ shows that

$$(H - z)^{-1} = (H' - z)^{-1}, \quad z \in \rho(H) \cap \rho(H_0) = \rho(H') \cap \rho(H_0). \qquad \square$$

**Proposition 6.5.** *Let $H_0$ be the free Dirac operator (1) on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, and let $V = (V_{ij})_{i,j=1}^2$ with $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$ for $i, j = 1, 2$. For any decomposition*

$$V = W + X, \quad W_{ij} \in L^1(\mathbb{R}), \quad X_{ij} \in L_0^\infty(\mathbb{R}), \tag{52}$$

*define $A$, $B$ as in (33) on their natural domain. Then all decompositions of the form (52) give rise to compatible factorizations $V = BA$. Moreover, these factorizations are also compatible with the one in (11).*

*Proof.* We only prove the first claim. The proof of the second one is analogous. Let $W, W' \in (L^1(\mathbb{R}))^4$ and $X, X' \in (L_0^\infty(\mathbb{R}))^4$ be such that

$$V = W + X = W' + X'.$$

It is easy to see that $A^\sharp R_0(z)$, $\overline{R_0(z)B^\sharp}$ and $\overline{A^\sharp R_0(z)B^\sharp}$ are all bounded; here, $A^\sharp$ stands for $A$ or $A'$, and $B^\sharp$ stands for $B$ or $B'$. This shows that the condition (i) of Definition 6.3 is satisfied.

In order to check condition (ii) of Definition 6.3, let $\Xi(\mathbb{R}) \subset L^2(\mathbb{R})$ denote the linear submanifold of step functions $f : \mathbb{R} \to \mathbb{C}$. We set

$$\mathcal{C} := \Xi(\mathbb{R}) \otimes \mathbb{C}^2, \quad \mathcal{D} := \Xi(\mathbb{R}) \otimes \mathbb{C}^4, \quad \mathcal{D}' := \Xi(\mathbb{R}) \otimes \mathbb{C}^4.$$

Clearly, $\mathcal{C} \subset \mathcal{H}$, $\mathcal{D} \subset \mathcal{K}$, $\mathcal{D}' \subset \mathcal{K}$ are dense. Here, we only show that

$$\mathcal{D} \subset \{f \in \mathcal{D}(B) : R_0(z)Bf \in \mathcal{D}(V)\}, \quad z \in \rho(H_0); \tag{53}$$

the proofs of the other two inclusions in Definition 6.3 (ii) are similar. Note that, since $X$ is bounded, we have

$$\mathcal{D}(B) = \mathcal{D}(B_W) \oplus \mathcal{H}, \quad \mathcal{D}(V) = \mathcal{D}(W).$$

Let $f := \chi_{[a,b]} \otimes (\alpha, \beta)^t$ for some $a < b$ and $\alpha, \beta \in \mathbb{C}^2$. Then $f = f_1 + f_2$ with $f_1 = \chi_{[a,b]} \otimes (\alpha, 0)^t$, $f_2 = \chi_{[a,b]} \otimes (0, \beta)^t$ and for any $\varepsilon > 0$

$$\int_{\mathbb{R}} \|B(x)f_1(x)\|_{\mathbb{C}^2}^2 \,\mathrm{d}x \le |\alpha|^2 \int_a^b \|V(x)\| \,\mathrm{d}x \le |\alpha|^2 \left(C_\varepsilon + (b - a)\,\varepsilon\right),$$

whence $f \in \mathcal{D}(B)$. Now let $z \in \rho(H_0)$ and set $g := R_0(z)Bf$. Then

$$\|g(x)\|_{\mathbb{C}^2} \leq \eta|\alpha| \int_a^b \mathrm{e}^{-\mathrm{Im} k(z)|x-y|} \|W(y)\|^{1/2} \, \mathrm{d}y + \eta|\beta| \|X\| \int_a^b \mathrm{e}^{-\mathrm{Im} k(z)|x-y|} \, \mathrm{d}y$$

where we abbreviated $\eta(|\Phi(z)|)$ by $\eta$. For $h \in \mathcal{D}(W^*)$, we have

$$|(W^*h, g)| \leq \int_{\mathbb{R}} \|W(x)\| \, \|h(x)\|_{\mathbb{C}^2} \, \|g(x)\|_{\mathbb{C}^2} \, \mathrm{d}x \leq \eta \, |\alpha| \, I_1(h) + \eta|\beta| \, \|X\| \, I_2(h)$$

where

$$\begin{aligned} I_1(h) &= \int_{\mathbb{R}} \int_a^b \|W(x)\| \, \|h(x)\|_{\mathbb{C}^2} \, \mathrm{e}^{-\mathrm{Im}\, k(z)\, |x-y|} \, \|W(y)\|^{1/2} \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \eta \, \|h\| \int_a^b \left( \int_{\mathbb{R}} \|W(x)\|^2 \, \mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}x \right)^{1/2} \|W(y)\|^{1/2} \, \mathrm{d}y \\ &\leq \eta \, \|h\| \left( \sup_{a \leq y \leq b} \int_{\mathbb{R}} \|W(x)\|^2 \, \mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}x \right)^{1/2} \int_a^b \|W(y)\|^{1/2} \, \mathrm{d}y \\ &\leq \eta \|h\| \left( \sup_{a \leq y \leq b} \int_{\mathbb{R}} \|W(x)\|^2 \mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}x \right)^{1/2} (b-a) \int_a^b \|W(y)\| \, \mathrm{d}y, \end{aligned}$$

and, similarly,

$$\begin{aligned} I_2(h) &= \int_{\mathbb{R}} \int_a^b \|W(x)\| \, \|h(x)\|_{\mathbb{C}^2} \, \mathrm{e}^{-\mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \eta \, \|h\| \, (b-a) \left( \sup_{a \leq y \leq b} \int_{\mathbb{R}} \|W(x)\|^2 \, \mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}x \right)^{1/2}. \end{aligned}$$

The supremum in the above two estimates is finite; indeed, repeated application of Young's inequality yields

$$\sup_{a \leq y \leq b} \int_{\mathbb{R}} \|W(x)\|^2 \, \mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |x-y|} \, \mathrm{d}x \leq \|W\|_1^4 \, \|\mathrm{e}^{-2\, \mathrm{Im}\, k(z)\, |\cdot|}\|_{6/7}.$$

This shows that $g \in \mathcal{D}(W^{**}) = \mathcal{D}(W)$. The claim now follows from Proposition 6.4. $\square$

It remains to prove the invariance of the essential spectrum under perturbations $V \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$.

**Proposition 6.6.** *Let $H_0$ be the free Dirac operator (1) on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, and let $V = (V_{ij})_{i,j=1}^2$ with $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$ for $i, j = 1, 2$. Then*

$$\sigma_e(H) = \sigma_e(H_0) = \{\pm(p^2 + m^2)^{1/2} : p \in \mathbb{R}\}.$$

*Proof.* Suppose first that $V_{ij} \in L^1(\mathbb{R})$, and let $V = BA$ with $A$ and $B$ given by (11). By (21), $AR_0(z)$ and $\overline{R_0(z)B}$ are Hilbert–Schmidt operators, which implies that the resolvent difference $R(z)-R_0(z)$ is compact (even trace class), by (49). The equality of the essential spectra of $H_0$ and $H$ thus follows from [8, Theorem IX.2.4].

If $V_{ij} \in L^1(\mathbb{R}) + L_0^\infty(\mathbb{R})$, we choose sequences $(W_n)_{n\in\mathbb{N}} \subset (L^1(\mathbb{R}))^4$ and $(X_n)_{n\in\mathbb{N}} \subset (L_0^\infty(\mathbb{R}))^4$ such that $V = W_n + X_n$ for all $n \in \mathbb{N}$ and $\|X_n\| \to 0$, $n \to \infty$. Furthermore, let

$$A_n := \begin{pmatrix} A_{W_n} \\ A_{X_n} \end{pmatrix}, \quad B_n := \begin{pmatrix} B_{W_n} B_{X_n} \end{pmatrix}, \quad Q_n(z) := \overline{A_n R_0(z) B_n},$$

where e.g. $A_{W_n} := |W_n|^{1/2}$, $B_{W_n} := U_{W_n}|W_n|^{1/2}$, and $U_{W_n}$ is the partial isometry in the polar decomposition of $W_n$. By Proposition 6.5 it follows that

$$R(z) = R_0(z) - \overline{R_0(z)B_n}\,(I_{\mathcal{K}} + Q_n(z))^{-1}\, A_n R_0(z)$$

is independent of $n$. Using the relation (50) or (51), we obtain

$$R(z) - R_0(z) = S_n + T_n$$

where each summand of $S_n$ contains at least one factor of $A_{W_n}R_0(z)$, $\overline{R_0(z)B_{W_n}}$ or $\overline{A_{W_n}R_0(z)B_{W_n}}$, and each summand of $T_n$ contains only factors of $A_{X_n}$, $B_{X_n}$ or $R_0(z)$. This means that $S_n$ is compact (even Hilbert–Schmidt), while $\|T_n\| \to 0$ as $n \to \infty$. Therefore, $R(z) - R_0(z)$ is the norm limit of compact operators and hence compact itself. □

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