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Second-order elliptic PDEs with discontinuous boundary data

PAUL HOUSTON*
School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK
*Corresponding author: paul.houston@nottingham.ac.uk

AND

THOMAS P. WIHLER
Mathematisches Institut, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland
wihler@math.unibe.ch

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We consider the weak formulation of a linear elliptic model problem with discontinuous Dirichlet boundary conditions. Since such problems are typically not well defined in the standard H^1−H^1 setting we introduce a suitable saddle point formulation in terms of weighted Sobolev spaces. Furthermore, we discuss the numerical solution of such problems. Specifically, we employ an hp-discontinuous Galerkin method and derive (enhanced) L^2-norm upper and local lower *a posteriori* error bounds. Numerical experiments demonstrate the effectiveness of the proposed error indicator in both the h- and the hp-version setting. Indeed, in the latter case, exponential convergence of the error is attained as the mesh is adaptively refined.

Keywords: second-order elliptic PDE; discontinuous Dirichlet boundary conditions; inf-sup condition; hp-discontinuous Galerkin FEM; L^2-norm *a posteriori* error analysis; exponential convergence.

1. Introduction

On a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with straight edges and connected boundary $\Gamma = \partial\Omega$, we consider the linear diffusion–reaction problem

$$-\Delta u + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \Gamma, \tag{1.2}$$

where $c \in L^\infty(\Omega)$ is a non-negative function, $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ is a possibly discontinuous function on Γ whose precise regularity will be specified later. Throughout the paper we shall use the following notation. For a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$) we denote by $L^2(D)$ the space of all square-integrable functions on D, with norm $\|\cdot\|_{0,D}$. Furthermore, for an integer $k \in \mathbb{N}_0$, we let $H^k(D)$ be the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and seminorm $|\cdot|_{k,D}$. The space $\mathring{H}^1(\Omega)$ is defined as the subspace of $H^1(\Omega)$ consisting of functions with zero trace on Γ.

Several variational formulations for elliptic problems with discontinuous Dirichlet boundary conditions exist. We mention the *very weak formulation* which is to find a solution $u \in L^2(\Omega)$ such that

$$-\int_\Omega u\,\Delta v\,\mathrm{d}\boldsymbol{x} + \int_\Omega cuv\,\mathrm{d}\boldsymbol{x} = \int_\Omega fv\,\mathrm{d}\boldsymbol{x} - \int_\Gamma g\nabla v\cdot\mathbf{n}\,\mathrm{d}s$$

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for any $v \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$, where $\mathbf{n}$ denotes the unit outward normal vector to the boundary Γ. It is based on twofold integration by parts of (1.1) and incorporates the Dirichlet boundary data in a natural way. On the other hand, however, the numerical solution by means of a conforming finite element discretization would require continuously differentiable test functions. In order to avoid this problem the following saddle point formulation can be used (see Nečas, 1962): provided that $g \in H^{1/2-\varepsilon}(\Gamma)$, for some $\varepsilon \in [0, 1/2)$, find $u \in H^{1-\varepsilon}(\Omega)$ with $u|_\Gamma = g$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} + \int_\Omega cuv \, \mathrm{d}\boldsymbol{x} = \int_\Omega fv \, \mathrm{d}\boldsymbol{x} \tag{1.3}$$

for all $v \in H^{1+\varepsilon}(\Omega) \cap \mathring{H}^1(\Omega)$. We note that the bilinear form on the left-hand side is formally symmetric and corresponds to the standard form for the Poisson equation. For results dealing with related finite element approximations we refer to Babuška (1971).

In the present paper a new variational formulation for (1.1)–(1.2) is presented and analysed. Here the emphasis shall be on Dirichlet boundary conditions that may exhibit (isolated) *discontinuities* and are essentially continuous otherwise. The formulation in this article is closely related to the saddle point formulation (1.3), however, it features Sobolev spaces that describe the local singularities in the analytical solution resulting from the discontinuities in the boundary data in a more specific way. More precisely, *weighted* Sobolev spaces that have been used in the context of regularity statements for second-order elliptic boundary value problems, see, e.g., Babuška & Guo (1988, 1989) and Guo & Schwab (2006), will be used. The idea of applying weights for problems with discontinuous boundary data has been employed previously in Bernardi & Karageorghis (1999). For the formulation in the current paper we will establish well-posedness of the weak formulation in terms of an appropriate *inf-sup condition*.

In order to discretize the underlying partial differential equation (PDE) problem we employ a framework that allows possible singularities in the solution to be resolved efficiently (see, e.g., Babuška *et al.*, 1979; Guo & Babuška, 1986a,b; Schwab, 1998; Nicaise, 2000 for results on the approximation of singularities in weighted Sobolev spaces). Specifically, in this paper, we shall exploit the hp-version of the symmetric interior penalty discontinuous Galerkin (DG) finite element method, cf. Arnold *et al.* (2001), and the references cited therein. DG methods are ideally suited for realizing hp-adaptivity for second-order boundary value problems, an advantage that has been noted early on in the recent development of these methods; see, for example, Baumann & Oden (1999), Rivière *et al.* (1999), Cockburn *et al.* (2000), Perugia & Schötzau (2002), Wihler *et al.* (2003), Houston *et al.* (2002, 2007, 2008), Stamm & Wihler (2010) and the references therein. Indeed, working with discontinuous finite element spaces easily facilitates the use of variable polynomial degrees and local mesh refinement techniques on possibly irregularly refined meshes—the two key ingredients for hp-adaptive algorithms. A further advantage of interior penalty DG formulations is that they incorporate Dirichlet boundary conditions in a natural way irrespective of their smoothness (in fact, L^1-regularity is sufficient for well-posedness). With this in mind, we shall derive computable upper and local lower *a posteriori* bounds for the error measured in terms of an enhanced L^2-norm on Ω. On the basis of the resulting computable error indicators, adaptive h- and hp-mesh adaptation strategies will be investigated for a model second-order elliptic PDE with discontinuous boundary conditions. In particular, we shall show numerically that exploiting hp-refinement leads to exponential convergence of the (enhanced) L^2-norm of the error as the finite element space is enriched.

The article is organized as follows. In Section 2 the new variational formulation of (1.1)–(1.2) will be presented. In addition, its well-posedness will be proved. Then, in Section 3, we will briefly review hp-version DG discretizations for the Laplace operator and derive L^2-norm *a posteriori* error estimates.

Additionally, the performance of the corresponding local error indicators is shown with a number of numerical experiments within an h- and hp-version adaptive framework. Finally, a few concluding remarks are made in Section 4.

2. Variational formulation

2.1 *Weighted Sobolev spaces*

Let $\mathcal{A} = \{A_i\}_{i=1}^M \subset \Gamma$, $A_i \neq A_j$ for $i \neq j$, be a finite set of points on the boundary of the polygonal domain Ω which are numbered in a counter clockwise direction along Γ; the points in $\mathcal{A}$ will signify the locations of the discontinuities in the Dirichlet boundary condition g in (1.2). Furthermore, we denote by $\Gamma_i \subset \Gamma$, $i = 1, 2, \ldots, M$, the (open) subset of Γ, which connects the two points A_i and A_{i+1}; here we set $A_{M+1} = A_1$. Moreover, let $\omega_i \in (0, 2\pi]$ signify the interior angle of the polygon Ω at A_i. To each $A_i \in \mathcal{A}$, $i = 1, 2, \ldots, M$, we associate a weight $\alpha_i \in \mathbb{R}$. These numbers are stored in a weight vector

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M) \in \mathbb{R}^M. \tag{2.1}$$

Moreover, for any number $k \in \mathbb{R}$, we use the notation $k\boldsymbol{\alpha} = (k\alpha_1, k\alpha_2, \ldots, k\alpha_M)$ and $\boldsymbol{\alpha} + k = (\alpha_1 + k, \alpha_2 + k, \ldots, \alpha_M + k)$. Furthermore, for a fixed number

$$\eta > 0, \tag{2.2}$$

we introduce the following weight function on Ω:

$$\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \prod_{i=1}^{M} r_i(\boldsymbol{x})^{\alpha_i}, \quad r_i(\boldsymbol{x}) = \min\left\{\eta^{-1}|\boldsymbol{x} - A_i|, 1\right\}.$$

Here we assume that η is small enough, so that the open sectors

$$S_i = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x} - A_i| < \eta\}, \quad i = 1, 2, \ldots, M, \tag{2.3}$$

do not intersect, i.e., $\overline{S}_i \cap \overline{S}_j = \emptyset$ if $i \neq j$. There holds, for $\boldsymbol{x} \in \Omega$, that

$$r_i(\boldsymbol{x}) = \begin{cases} \eta^{-1}|\boldsymbol{x} - A_i| & \text{if } \boldsymbol{x} \in S_i, \\ 1 & \text{if } \boldsymbol{x} \in \Omega \setminus S_i, \end{cases}$$

and $r_i \in C^0(\Omega)$, $i = 1, 2, \ldots, M$. Furthermore, setting

$$\mathscr{S} = \bigcup_{i=1}^{M} S_i, \quad \Omega_0 = \Omega \setminus \overline{\mathscr{S}},$$

we have

$$\Phi_{\boldsymbol{\alpha}} = \begin{cases} r_i^{\alpha_i} & \text{if } \boldsymbol{x} \in S_i \text{ for some } i = 1, 2, \ldots M, \\ 1 & \text{if } \boldsymbol{x} \in \Omega_0. \end{cases} \tag{2.4}$$

Note that $\Phi_{\boldsymbol{\alpha}}$ is continuous on Ω. Furthermore, for $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^M$, we have

$$\Phi_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} = \Phi_{\boldsymbol{\alpha}_1}\Phi_{\boldsymbol{\alpha}_2}, \quad \Phi_{\boldsymbol{\alpha}}^{-1} = \Phi_{-\boldsymbol{\alpha}}.$$

Then, for any integers $m \geqslant l \geqslant 0$, we define the weighted Sobolev spaces $H_{\boldsymbol{\alpha}}^{m,l}(\Omega)$ as the completion of the space $C^\infty(\overline{\Omega})$ with respect to the weighted Sobolev norms

$$\|u\|^2_{H_{\boldsymbol{\alpha}}^{m,l}(\Omega)} = \|u\|^2_{l-1,\Omega} + \sum_{k=l}^{m} |u|^2_{H_{\boldsymbol{\alpha}}^{k,l}(\Omega)}, \quad l \geqslant 1,$$

$$\|u\|^2_{H_{\boldsymbol{\alpha}}^{m,0}(\Omega)} = \sum_{k=0}^{m} |u|^2_{H_{\boldsymbol{\alpha}}^{k,0}(\Omega)}.$$

Here

$$|u|^2_{H_{\boldsymbol{\alpha}}^{k,l}(\Omega)} = \sum_{|\boldsymbol{\lambda}|=k} \left\| \Phi_{\boldsymbol{\alpha}+k-l} \left| \mathsf{D}^{\boldsymbol{\lambda}} u \right| \right\|^2_{0,\Omega}$$

is the $H_{\boldsymbol{\alpha}}^{k,l}$-seminorm in Ω, where

$$\mathsf{D}^{\boldsymbol{\lambda}} u = \frac{\partial^{|\boldsymbol{\lambda}|} u}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}},$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{N}_0^2$ and $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2$.

In addition, for $m \geqslant l \geqslant 1$, let us define the space $H_{\boldsymbol{\alpha}}^{m-\frac{1}{2},l-\frac{1}{2}}(\Gamma)$ as the trace space of $H_{\boldsymbol{\alpha}}^{m,l}(\Omega)$, equipped with the norm

$$\|u\|_{H_{\boldsymbol{\alpha}}^{m-\frac{1}{2},l-\frac{1}{2}}(\Gamma)} = \inf_{\substack{v \in H_{\boldsymbol{\alpha}}^{m,l}(\Omega) \\ v|_{\partial\Omega}=u}} \|v\|_{H_{\boldsymbol{\alpha}}^{m,l}(\Omega)}.$$

Moreover, we denote by $\mathring{H}_{\boldsymbol{\alpha}}^{m,l}(\Omega)$ the subspace of $H_{\boldsymbol{\alpha}}^{m,l}(\Omega)$ consisting of functions with zero trace on Γ.

2.2 *Inequalities in $H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$*

In order to describe the well-posedness of (1.1)–(1.2) the weighted Sobolev space $H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ will play an important role. In the sequel we shall collect a few inequalities that will be used for the analysis in this paper.

LEMMA 2.1 Let $I = (a, b) \subset \mathbb{R}$, $a < b$, be an open interval. Then there holds the Poincaré–Friedrichs inequality

$$\int_a^b \phi(x)^2 \, \mathrm{d}x \leqslant \frac{(b-a)^2}{\pi^2} \int_a^b (\phi'(x))^2 \, \mathrm{d}x$$

for all $\phi \in H^1(a, b)$ with $\phi(a) = \phi(b) = 0$.

Proof. The bound follows from Hardy *et al.* (1952, Theorem 257) and a scaling argument. □

Applying the previous lemma we shall prove the following result.

LEMMA 2.2 Consider a sector $S = \{(r,\theta)\colon 0 < r < R, \theta_0 < \theta < \theta_1\} \subset \mathbb{R}^2$, where (r,θ) denote polar coordinates in $\mathbb{R}^2$, and $R > 0$, $0 \leqslant \theta_0 < \theta_1 \leqslant 2\pi$ are constants. Furthermore, let $u \in L^2(S)$ with $\|r^\alpha \nabla u\|_{0,S} < \infty$ for some $\alpha \in [0,1)$, and $u|_{\partial S_<} = 0$, where $\partial S_< = \{(r,\theta)\colon 0 < r < R, \theta \in \{\theta_0,\theta_1\}\}$. Then there holds

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_S r^{2\alpha} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}.$$

Proof. Using integration in polar coordinates we get

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} = \int_0^R r^{2\alpha-1} \int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \,\mathrm{d}r. \tag{2.5}$$

Then since for any $r \in (0,R)$ there holds $u(r,\theta_0) = u(r,\theta_1) = 0$ we can apply Lemma 2.1. This implies

$$\int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_{\theta_0}^{\theta_1} |\partial_\theta u|^2 \,\mathrm{d}\theta, \quad 0 < r < R.$$

Furthermore, noting that $|\partial_\theta u| \leqslant r|\nabla_{\boldsymbol{x}} u|$, we obtain

$$\int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} r^2 \int_{\theta_0}^{\theta_1} |\nabla_{\boldsymbol{x}} u|^2 \,\mathrm{d}\theta, \quad 0 < r < R.$$

Inserting this estimate into (2.5) leads to

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_0^R r^{2\alpha+1} \int_{\theta_0}^{\theta_1} |\nabla_{\boldsymbol{x}} u|^2 \,\mathrm{d}\theta \,\mathrm{d}r.$$

Changing back to Cartesian coordinates $\boldsymbol{x}$ completes the proof. □

LEMMA 2.3 Given a weight vector $\boldsymbol{\alpha} \in [0,1)^M$. Then there holds

$$\|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\Omega} \leqslant C \|u\|_{1,\Omega}$$

for any $u \in H^1(\Omega)$, where the constant $C > 0$ only depends on $\boldsymbol{\alpha}$ and Ω.

Proof. Let S_i, $i = 1,2,\ldots,M$, be the (sufficiently small) sectors from (2.3). Then we recall the property (2.4) to write

$$\|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\Omega}^2 = \|u\|_{0,\Omega_0}^2 + \|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\mathscr{S}}^2 = \|u\|_{0,\Omega_0}^2 + \sum_{i=1}^M \left\| r_i^{-\alpha_i} u \right\|_{0,S_i}^2. \tag{2.6}$$

If, for some $1 \leqslant i \leqslant M$, we have that $\alpha_i > 0$, then

$$\left\| r_i^{-\alpha_i} u \right\|_{0,S_i}^2 \leqslant C \left(\|u\|_{0,S_i}^2 + \left\| r_i^{1-\alpha_i} \nabla u \right\|_{0,S_i}^2 \right) \leqslant C \|u\|_{1,S_i}^2;$$

this follows from expressing the norms in terms of polar coordinates and from applying Hardy *et al.* (1952, Theorem 330). Inserting this into (2.6) gives the desired inequality. □

LEMMA 2.4 Consider a function $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, where $\alpha_i \in [0, 1)$, $i = 1, 2, \ldots, M$. Then there holds

$$\||\nabla(\Phi_{\boldsymbol{\alpha}})|u\|_{0,\Omega} \leqslant \frac{1}{\pi} \max_{1\leqslant i\leqslant M} \alpha_i \omega_i \, |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} .$$

Proof. Let S_i, $i = 1, 2, \ldots, M$, be the (sufficiently small) sectors from (2.3). Then, due to (2.4), we have

$$|\nabla(\Phi_{\boldsymbol{\alpha}})| = \begin{cases} \left|\nabla(r_i^{\alpha_i})\right| = \alpha_i \eta^{-1} r_i^{\alpha_i - 1} & \text{if } \boldsymbol{x} \in S_i \text{ for some } i = 1, 2, \ldots, M, \\ 0 & \text{if } \boldsymbol{x} \in \Omega_0. \end{cases} \tag{2.7}$$

Hence,

$$\int_\Omega |\nabla(\Phi_{\boldsymbol{\alpha}})|^2 u^2 \, \mathrm{d}\boldsymbol{x} = \eta^{-2} \sum_{i=1}^{M} \alpha_i^2 \int_{S_i} r_i^{2\alpha_i - 2} u^2 \, \mathrm{d}\boldsymbol{x}. \tag{2.8}$$

Then, applying Lemma 2.2, we have

$$\int_{S_i} r_i^{2\alpha_i - 2} u^2 \, \mathrm{d}\boldsymbol{x} \leqslant \eta^2 \frac{\omega_i^2}{\pi^2} \int_{S_i} r_i^{2\alpha_i} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}.$$

Thus,

$$\int_\Omega |\nabla(\Phi_{\boldsymbol{\alpha}})|^2 u^2 \, \mathrm{d}\boldsymbol{x} \leqslant \sum_{i=1}^{M} \frac{\alpha_i^2 \omega_i^2}{\pi^2} \int_{S_i} r_i^{2\alpha_i} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \leqslant \frac{\max_{1\leqslant i\leqslant M} \left(\alpha_i^2 \omega_i^2\right)}{\pi^2} \int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}$$

as required. □

Furthermore, there holds the following Poincaré–Friedrichs inequality.

LEMMA 2.5 Consider a weight vector $\boldsymbol{\alpha} \in [0, 1)^M$ and $\gamma \subseteq \Gamma$ with $\int_\gamma \mathrm{d}s > 0$. Then there exists a constant $C > 0$ depending only on γ, Ω and $\boldsymbol{\alpha}$ such that

$$\|u\|_{0,\Omega} \leqslant C |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}$$

for all functions $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ with $u|_\gamma = 0$ (in the trace sense). In particular, we have that $|\cdot|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}$ is a norm on $\mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$.

Proof. We first note that the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous for Lipschitz polygons in $\mathbb{R}^2$ (cf., e.g., Adams & Fournier, 2003, Theorem 4.12). Hence, there exists a constant $C > 0$ depending on Ω such that

$$\|u\|_{0,\Omega} \leqslant C \|u\|_{W^{1,1}(\Omega)}.$$

Moreover, applying the Poincaré–Friedrichs inequality in $W^{1,1}(\Omega)$, it follows that

$$\|u\|_{0,\Omega} \leqslant C \|u\|_{W^{1,1}(\Omega)} \leqslant C' \|\nabla u\|_{L^1(\Omega)}$$

for a constant $C' > 0$ depending on γ and Ω. Therefore, using Hölder's inequality, we obtain

$$\|u\|_{0,\Omega} \leqslant C' \int_\Omega |\nabla u| \, \mathrm{d}\mathbf{x} \leqslant C' \left(\int_\Omega \Phi_{\boldsymbol{\alpha}}^{-2} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \left(\int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}}.$$

Then employing (2.4) yields

$$\int_\Omega \Phi_{\boldsymbol{\alpha}}^{-2}\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{M}\int_{S_i} r_i^{-2\alpha_i}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_0} 1\,\mathrm{d}\boldsymbol{x},$$

and using integration in polar coordinates it follows that the above integrals are all bounded for $\alpha_i < 1$, $i = 1, 2, \ldots, M$. This completes the proof. □

To close this section we shall prove the following Green type formulae.

LEMMA 2.6 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector and consider two functions $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ and $\phi \in H^2(\Omega)$. In addition, suppose that the trace of $u|_\Gamma \in L^2(\Gamma)$. Then

$$\int_\Omega \Delta\phi u\,\mathrm{d}\boldsymbol{x} = \int_\Gamma (\nabla\phi\cdot \mathrm{n})u\,\mathrm{d}s - \int_\Omega \nabla\phi\cdot\nabla u\,\mathrm{d}\boldsymbol{x} \tag{2.9}$$

holds true, where $\mathbf{n}$ denotes the outward unit vector to Γ.

Proof. Due to the density of $C^\infty(\overline{\Omega})$ in $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ we can choose a sequence $\{u_n\}_{n\geqslant 0} \subset C^\infty(\overline{\Omega})$ such that $\lim_{n\to\infty}\|u-u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} = 0$. Then, using Green's formula for smooth functions, we have

$$\int_\Omega \Delta\phi u_n\,\mathrm{d}\boldsymbol{x} = \int_\Gamma (\nabla\phi\cdot\boldsymbol{n})u_n\,\mathrm{d}s - \int_\Omega \nabla\phi\cdot\nabla u_n\,\mathrm{d}\boldsymbol{x}$$

for any function $\phi \in C^\infty(\overline{\Omega})$. Furthermore, there holds

$$\left|\int_\Omega \Delta\phi\,(u_n - u)\,\mathrm{d}\boldsymbol{x}\right| \leqslant \|\phi\|_{2,\Omega}\|u-u_n\|_{0,\Omega} \overset{n\to\infty}{\longrightarrow} 0,$$

and, using Lemma 2.3,

$$\begin{aligned}\left|\int_\Omega \nabla\phi\cdot\nabla(u_n-u)\,\mathrm{d}\boldsymbol{x}\right| &\leqslant \|\Phi_{-\boldsymbol{\alpha}}\nabla\phi\|_{0,\Omega}\|\Phi_{\boldsymbol{\alpha}}\nabla(u-u_n)\|_{0,\Omega}\\ &\leqslant C\|\phi\|_{2,\Omega}\|u-u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Furthermore, applying the trace theorem in $W^{1,1}(\Omega)$ yields

$$\begin{aligned}\left|\int_\Gamma (\nabla\phi\cdot\boldsymbol{n})(u_n-u)\,\mathrm{d}s\right| &\leqslant \sup_{\overline{\Omega}}|\nabla\phi|\|u-u_n\|_{L^1(\Gamma)}\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|\big(\|u-u_n\|_{L^1(\Omega)} + \|\nabla(u-u_n)\|_{L^1(\Omega)}\big)\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|(\|u-u_n\|_{0,\Omega} + \|\Phi_{-\boldsymbol{\alpha}}\|_{0,\Omega}\|\Phi_{\boldsymbol{\alpha}}\nabla(u-u_n)\|_{0,\Omega})\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|\|u-u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

This implies the identity (2.9) for $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ and $\phi \in C^\infty(\overline{\Omega})$.

For $\phi \in H^2(\Omega)$ the density of $C^\infty(\overline{\Omega})$ in $H^2(\Omega)$ guarantees the existence of a sequence $\{\phi_n\}_{n\geqslant 0} \subset C^\infty(\overline{\Omega})$ with $\lim_{n\to\infty} \|\phi_n - \phi\|_{2,\Omega} = 0$. Then

$$\int_\Omega \Delta\phi_n u \,\mathrm{d}\boldsymbol{x} = \int_\Gamma (\nabla\phi_n \cdot \boldsymbol{n}) u \,\mathrm{d}s - \int_\Omega \nabla\phi_n \cdot \nabla u \,\mathrm{d}\boldsymbol{x}$$

for all $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$. Similarly, as before, we have

$$\left|\int_\Omega \Delta(\phi_n - \phi) u \,\mathrm{d}\boldsymbol{x}\right| \leqslant \|\phi_n - \phi\|_{2,\Omega} \, \|u\|_{0,\Omega} \overset{n\to\infty}{\longrightarrow} 0,$$

and, with Lemma 2.3,

$$\begin{aligned}\left|\int_\Omega \nabla(\phi_n - \phi) \cdot \nabla u \,\mathrm{d}\boldsymbol{x}\right| &\leqslant \|\Phi_{-\boldsymbol{\alpha}} \nabla(\phi_n - \phi)\|_{0,\Omega} \|\Phi_{\boldsymbol{\alpha}} \nabla u\|_{0,\Omega} \\ &\leqslant \|\phi_n - \phi\|_{2,\Omega} \|u\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Moreover, using the trace theorem again, we obtain

$$\begin{aligned}\left|\int_\Gamma (\nabla(\phi_n - \phi) \cdot \boldsymbol{n}) u \,\mathrm{d}s\right| &\leqslant \|\nabla(\phi_n - \phi)\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)} \\ &\leqslant C\|\phi_n - \phi\|_{2,\Omega} \|u\|_{L^2(\Gamma)} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

This completes the proof. □

LEMMA 2.7 Let $\boldsymbol{\alpha} \in [0,1)^M$, and $\Omega_0 \subseteq \Omega$ a connected subset with Lipschitz boundary. Furthermore, consider $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega_0)$ with $\Delta u \in L^2(\Omega_0)$, and $v \in W^{1,\infty}(\Omega_0)$ with $v|_\Gamma = 0$. Then there holds

$$\int_{\Omega_0} v \Delta u \,\mathrm{d}\boldsymbol{x} + \int_{\Omega_0} \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = 0.$$

Here the space $H^{1,1}_{\boldsymbol{\alpha}}(\Omega_0)$ is defined as the restriction of $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ to Ω_0.

Proof. This follows again by density and from the fact that $W^{1,\infty}(\Omega_0) \hookrightarrow H^{1,1}_{-\boldsymbol{\alpha}}(\Omega_0)$ continuously for $\boldsymbol{\alpha} \in [0,1)^M$; in particular, all integrals are well defined. □

2.3 *Weak formulation*

The aim of this section is to introduce a weak formulation for the boundary value problem (1.1)–(1.2) and to discuss its well-posedness.

Let $g \in H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma)$ in (1.2), where $\boldsymbol{\alpha}$ is the weight vector from (2.1) with $\alpha_i \in [0,1)$, $i = 1, 2, \ldots, M$. Then we call $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ with $u|_\Gamma = g$ a weak solution of (1.1)–(1.2) if

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega c u v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \quad \forall\, v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega). \tag{2.10}$$

Writing the solution in the form $u = u_0 + G$, where $u_0 \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ and $G \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ is a lifting of the boundary data g, i.e., $G|_\Gamma = g$, there holds

$$\int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega c u_0 v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} - \int_\Omega \nabla G \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega c G v \,\mathrm{d}\boldsymbol{x} \quad \forall\, v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega).$$

We note that this is a saddle point formulation on $\mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega) \times \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$. Its well-posedness will be discussed in the following.

We first show that the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} + \int_\Omega cuv \, \mathrm{d}\boldsymbol{x}$$

and the linear functional

$$\ell(v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \nabla G \cdot \nabla v \, \mathrm{d}\boldsymbol{x} - \int_\Omega cGv \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} - a(G, v)$$

are continuous. Here we suppose that the lifting G is chosen such that

$$\|G\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \leqslant C \|g\|_{H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma)} \tag{2.11}$$

for some fixed constant $C > 1$ independent of g.

PROPOSITION 2.8 Let $\boldsymbol{\alpha} \in [0, 1)^M$ be a weight vector. There is a constant $C > 0$ (depending on Ω and $\boldsymbol{\alpha}$) such that

$$|a(u, v)| \leqslant C |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}$$

for all $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, $v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$. Furthermore, for $f \in L^2(\Omega)$ and $g \in H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma)$, we have

$$|\ell(v)| \leqslant C \left(\|f\|_{0,\Omega} + \|g\|_{H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma)} \right) |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}$$

for any $v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$.

Proof. There holds

$$\begin{aligned} |a(u, v)| &\leqslant \|\Phi_{\boldsymbol{\alpha}} \nabla u\|_{0,\Omega} \|\Phi_{-\boldsymbol{\alpha}} \nabla v\|_{0,\Omega} + \|c\|_{L^\infty(\Omega)} \|u\|_{0,\Omega} \|v\|_{0,\Omega} \\ &\leqslant C \left(|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} + \|u\|_{0,\Omega} \|v\|_{0,\Omega} \right). \end{aligned}$$

Furthermore, using the Poincaré–Friedrichs inequality and Lemma 2.5, we get

$$\|u\|_{0,\Omega} \|v\|_{0,\Omega} \leqslant C |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{1,\Omega} \leqslant C |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}.$$

Hence,

$$|a(u, v)| \leqslant C |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}.$$

Moreover, employing the previous estimate and proceeding as before to estimate the L^2-norm, we obtain

$$|\ell(v)| \leqslant \|f\|_{0,\Omega} \|v\|_{0,\Omega} + |a(G, v)| \leqslant \|f\|_{0,\Omega} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} + C |G|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}.$$

Then applying (2.11) yields the stability bound for ℓ. □

Furthermore, the following inf-sup stability holds.

PROPOSITION 2.9 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector. Suppose that the weights α_i, $i = 1, 2, \dots M$, are sufficiently small so that

$$\mu := \frac{1}{\pi} \max_{1 \leqslant i \leqslant M} \alpha_i \omega_i < \frac{1}{2}.$$

Then there holds

$$\inf_{0 \not\equiv u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \sup_{0 \not\equiv v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \frac{a(u,v)}{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \geqslant \delta, \tag{2.12}$$

where

$$\delta = \frac{1 - 2\mu}{\sqrt{2(4\mu^2 + 1)}}.$$

Furthermore, we have that

$$\sup_{u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)} a(u,v) > 0 \quad \forall v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega), \quad v \not\equiv 0. \tag{2.13}$$

Proof. For $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ we define $\widetilde{v} = \Phi^2_{\boldsymbol{\alpha}} u$. Then there holds

$$\begin{aligned} |\widetilde{v}|^2_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} &= \int_\Omega \Phi^2_{-\boldsymbol{\alpha}} |\nabla \widetilde{v}|^2 \, \mathrm{d}\boldsymbol{x} \leqslant 2 \int_\Omega \Phi^{-2}_{\boldsymbol{\alpha}} \left(\left| \nabla \left(\Phi^2_{\boldsymbol{\alpha}} \right) \right|^2 u^2 + \Phi^4_{\boldsymbol{\alpha}} |\nabla u|^2 \right) \mathrm{d}\boldsymbol{x} \\ &\leqslant 2 \left(4 \int_\Omega |\nabla \Phi_{\boldsymbol{\alpha}}|^2 u^2 \, \mathrm{d}\boldsymbol{x} + |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \right). \end{aligned}$$

Hence, applying Lemma 2.4, results in

$$|\widetilde{v}|^2_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \leqslant 2\left(4\mu^2 + 1\right) |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}. \tag{2.14}$$

In particular, it follows that $\widetilde{v} \in H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$.

Moreover, we observe that

$$a(u, \widetilde{v}) = \int_\Omega \nabla u \cdot \nabla \widetilde{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega c u \widetilde{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla u \cdot \nabla (\Phi^2_{\boldsymbol{\alpha}} u) \, \mathrm{d}\boldsymbol{x} + \int_\Omega c \Phi^2_{\boldsymbol{\alpha}} u^2 \, \mathrm{d}\boldsymbol{x}.$$

Thus, since $c \geqslant 0$, we get

$$\begin{aligned} a(u, \widetilde{v}) &\geqslant \int_\Omega \left(\nabla u \cdot \nabla \left(\Phi^2_{\boldsymbol{\alpha}} \right) u + \Phi^2_{\boldsymbol{\alpha}} |\nabla u|^2 \right) \mathrm{d}\boldsymbol{x} \\ &= 2 \int_\Omega \Phi_{\boldsymbol{\alpha}} \nabla u \cdot \nabla (\Phi_{\boldsymbol{\alpha}}) u \, \mathrm{d}\boldsymbol{x} + \int_\Omega \Phi^2_{\boldsymbol{\alpha}} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \\ &\geqslant -\frac{1}{\mu} \int_\Omega |\nabla (\Phi_{\boldsymbol{\alpha}})|^2 u^2 \, \mathrm{d}\boldsymbol{x} + (1 - \mu) \int_\Omega \Phi^2_{\boldsymbol{\alpha}} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Recalling Lemma 2.4 leads to

$$a(u,\widetilde{v}) \geqslant -\mu|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} + (1-\mu)|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \geqslant (1-2\mu)|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}. \tag{2.15}$$

Now, combining (2.14) and (2.15), it follows that

$$\sup_{v\in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \frac{a(u,v)}{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \geqslant \frac{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}}{|\widetilde{v}|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \frac{a(u,\widetilde{v})}{|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}} \geqslant \delta$$

for any $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, $u \not\equiv 0$. Taking the infimum over all $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ results in (2.12).

In addition, let $v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$, $v \not\equiv 0$. Then

$$\sup_{u\in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)} a(u,v) \geqslant a(v,v) \geqslant \int_\Omega |\nabla v|^2 \,\mathrm{d}\boldsymbol{x}.$$

Due to $v|_\Gamma = 0$ and $v \not\equiv 0$, there holds $\|\nabla v\|_{0,\Omega} > 0$, and hence (2.13) holds. □

The above results, Propositions 2.8 and 2.9, imply the well-posedness of the variational formulation (2.10); cf., e.g., Schwab (1998, Theorem 1.15).

THEOREM 2.10 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector, with α_i, $i = 1,2,\ldots,M$, sufficiently small such that

$$\max_{1\leqslant i\leqslant M} \alpha_i\omega_i < \frac{\pi}{2}$$

is satisfied. Furthermore, suppose that $g \in H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma)$ and $f \in L^2(\Omega)$ in (1.1)–(1.2). Then there exists exactly one solution of the weak formulation (2.10) in $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$.

3. Numerical approximation

We shall now discuss the numerical approximation of the problem (1.1)–(1.2). To this end, we will consider *hp*-version interior penalty DG finite element methods. Particularly, we will derive an (enhanced) L^2-norm *a posteriori* error estimate that can be applied for adaptive purposes.

3.1 *Meshes, spaces and element edge operators*

We consider shape-regular meshes $\mathscr{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint triangles and/or parallelograms $\{K\}_{K\in\mathscr{T}_h}$, i.e., $\overline{\Omega} = \bigcup_{K\in\mathscr{T}_h} \overline{K}$. Each element $K \in \mathscr{T}_h$ can then be affinely mapped onto the reference triangle $\widehat{T} = \{(\widehat{x},\widehat{y})\colon -1 < \widehat{x} < 1, -1 < \widehat{y} < -\widehat{x}\}$ or the reference square $\widehat{S} = (-1,1)^2$, respectively. We allow the meshes to be 1-irregular, i.e., elements may contain hanging nodes. By h_K, we denote the diameter of an element $K \in \mathscr{T}_h$. We assume that these quantities are of bounded variation, i.e., there is a constant $\rho_1 \geqslant 1$ such that

$$\rho_1^{-1} \leqslant h_{K_\sharp}/h_{K_\flat} \leqslant \rho_1, \tag{3.1}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge. We store the elemental diameters in a vector $\boldsymbol{h}$ given by $\boldsymbol{h} = \{h_K\colon K \in \mathscr{T}_h\}$. Similarly, to each element $K \in \mathscr{T}_h$ we assign a polynomial degree $p_K \geqslant 1$ and

define the degree vector $\boldsymbol{p} = \{p_K \colon K \in \mathscr{T}_h\}$. We suppose that $\boldsymbol{p}$ is also of bounded variation, i.e., there is a constant $\rho_2 \geqslant 1$ such that

$$\rho_2^{-1} \leqslant p_{K_\sharp}/p_{K_\flat} \leqslant \rho_2, \tag{3.2}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge.

Moreover, we shall define some suitable element edge operators that are required for the DG method. To this end, we denote by $\mathscr{E}_{\mathscr{I}}$ the set of all interior edges of the partition $\mathscr{T}_h$ of Ω and by $\mathscr{E}_{\mathscr{B}}$ the set of all boundary edges of $\mathscr{T}_h$. In addition, let $\mathscr{E} = \mathscr{E}_{\mathscr{I}} \cup \mathscr{E}_{\mathscr{B}}$. The boundary ∂K of an element K and the sets $\partial K \setminus \Gamma$ and $\partial K \cap \Gamma$ will be identified in a natural way with the corresponding subsets of $\mathscr{E}$.

Let $K_\sharp$ and $K_\flat$ be two adjacent elements of $\mathscr{T}_h$, $\partial K_\sharp \cap \partial K_\flat = e$ for some $e \in \mathscr{E}_{\mathscr{I}}$ and $\boldsymbol{x}$ an arbitrary point on e. Furthermore, let v and $\boldsymbol{q}$ be scalar- and vector-valued functions, respectively, that are sufficiently smooth inside each element $K_{\sharp/\flat}$. By $(v_{\sharp/\flat}, \boldsymbol{q}_{\sharp/\flat})$ we denote the traces of $(v, \boldsymbol{q})$ on e taken from within the interior of $K_{\sharp/\flat}$, respectively. Then the averages of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$\langle\!\langle v \rangle\!\rangle = \frac{1}{2}(v_\sharp + v_\flat), \quad \langle\!\langle \boldsymbol{q} \rangle\!\rangle = \frac{1}{2}(\boldsymbol{q}_\sharp + \boldsymbol{q}_\flat),$$

respectively. Similarly, the jumps of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$[\![v]\!] = v_\sharp \boldsymbol{n}_{K_\sharp} + v_\flat \boldsymbol{n}_{K_\flat}, \quad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_\sharp \cdot \boldsymbol{n}_{K_\sharp} + \boldsymbol{q}_\flat \cdot \boldsymbol{n}_{K_\flat},$$

respectively, where we denote by $\boldsymbol{n}_{K_{\sharp/\flat}}$ the unit outward normal vector on $\partial K_{\sharp/\flat}$, respectively. On a boundary edge $e \in \mathscr{E}_{\mathscr{B}}$, we set $\langle\!\langle v \rangle\!\rangle = v$, $\langle\!\langle \boldsymbol{q} \rangle\!\rangle = \boldsymbol{q}$, $[\![v]\!] = v\boldsymbol{n}$ and $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ denoting the unit outward normal vector on the boundary Γ.

Given a finite element mesh $\mathscr{T}_h$ and an associated polynomial degree vector $\boldsymbol{p} = (p_K)_{K \in \mathscr{T}_h}$, with $p_K \geqslant 1$ for all $K \in \mathscr{T}_h$, consider the hp-discretization space

$$V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p}) = \left\{ v \in L^2(\Omega) \colon v|_K \in \mathbb{S}_{p_K}(K),\, K \in \mathscr{T}_h \right\} \tag{3.3}$$

for the DG method. Here, for $K \in \mathscr{T}_h$, $\mathbb{S}_{p_K}(K)$ is either the space $\mathbb{P}_{p_K}(K)$ of all polynomials of total degree at most p_K on K or the space $\mathbb{Q}_{p_K}(K)$ of all polynomials of degree at most p_K in each coordinate direction on K.

Finally, let us introduce the *enhanced* L^2-norm

$$|\!|\!|v|\!|\!|_{0,h,\Omega}^2 := \|v\|_{0,\Omega}^2 + \int_{\mathscr{E}} \mathsf{hp}\, |[\![v]\!]|^2 \, \mathrm{d}s, \tag{3.4}$$

for any $v \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega) + V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$, $\boldsymbol{\alpha} \in [0,1)^M$. Here the two functions $\mathsf{h} \in L^\infty(\mathscr{E})$ and $\mathsf{p} \in L^\infty(\mathscr{E})$ are given by

$$\mathsf{h}(\boldsymbol{x}) = \begin{cases} \min(h_{K_\sharp}, h_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathscr{E}_{\mathscr{I}}, \\ h_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega \in \mathscr{E}_{\mathscr{B}}, \end{cases}$$

$$\mathsf{p}(\boldsymbol{x}) = \begin{cases} \max(p_{K_\sharp}, p_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathscr{E}_{\mathscr{I}}, \\ p_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega \in \mathscr{E}_{\mathscr{B}}, \end{cases}$$

respectively.

3.2 *hp*-DG *discretization*

We will now consider the following *hp*-DG formulation for the numerical approximation of (1.1)–(1.2): find $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ such that

$$a_{\mathrm{DG}}(u_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \quad \forall\, v \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p}). \tag{3.5}$$

Here

$$\begin{aligned} a_{\mathrm{DG}}(w, v) = &\int_{\Omega} (\nabla_h w \cdot \nabla_h v + c w v)\, \mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}} \langle\!\langle \nabla_h w \rangle\!\rangle \cdot [\![v]\!]\, \mathrm{d}s \\ &- \int_{\mathscr{E}} [\![w]\!] \cdot \langle\!\langle \nabla_h v \rangle\!\rangle\, \mathrm{d}s + \gamma \int_{\mathscr{E}} \sigma [\![w]\!] \cdot [\![v]\!]\, \mathrm{d}s \end{aligned} \tag{3.6}$$

is an *hp*-version symmetric interior penalty DG form, and

$$\ell_{\mathrm{DG}}(v) = \int_{\Omega} f v\, \mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla_h v \cdot \boldsymbol{n}) g\, \mathrm{d}s + \gamma \int_{\mathscr{E}_{\mathscr{B}}} \sigma g v\, \mathrm{d}s. \tag{3.7}$$

In these forms ∇_h denotes the elementwise gradient operator, $\gamma > 0$ is a stability constant, and the function σ is defined by

$$\sigma = \frac{\mathsf{p}^2}{\mathsf{h}}. \tag{3.8}$$

REMARK 3.1 Provided that $\gamma > 0$ is chosen sufficiently large (independently of the local element sizes and polynomial degrees), it is well known that the DG form a_{DG} is coercive. More precisely, there is a constant $C > 0$ independent of $\mathscr{T}_h$ and $\boldsymbol{p}$ such that

$$a_{\mathrm{DG}}(v, v) \geqslant C \left(\|\nabla_h v\|_{0,\Omega}^2 + \gamma \int_{\mathscr{E}} \sigma |[\![v]\!]|^2\, \mathrm{d}s \right)$$

for any $v \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$. In particular, the DG method (3.5) admits a unique solution $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$; see, e.g., Stamm & Wihler (2010) and the references therein.

3.3 *A posteriori error estimation in the L^2-norm*

We shall now derive upper and local lower residual-based *hp-a posteriori* error estimates in the enhanced L^2-norm from (3.4) for the DG formulation (3.5).

3.3.1 *Upper bound.* Let us consider the dual problem

$$-\Delta\phi + c\phi = e_{\mathrm{DG}} \quad \text{in } \Omega, \tag{3.9}$$

$$\phi = 0 \quad \text{on } \Gamma. \tag{3.10}$$

Here

$$e_{\mathrm{DG}} = u - u_{\mathrm{DG}} \tag{3.11}$$

denotes the error, where $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, $\boldsymbol{\alpha} \in [0,1)^M$, is the weak solution of (1.1)–(1.2) defined in (2.10), and $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ is the DG solution defined in (3.5). Throughout this section we suppose that this problem has a solution $\phi \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ with continuous dependence on the data, i.e., there exists a constant $C > 0$ such that

$$\|\phi\|_{H^2(\Omega)} \leqslant C\|e_{\mathrm{DG}}\|_{0,\Omega}. \tag{3.12}$$

This is the case, for example, if Ω is a convex polygon since then $\Delta\colon H^2(\Omega) \cap \mathring{H}^1(\Omega) \to L^2(\Omega)$ is an isomorphism; cf. Grisvard (1985, Theorem 3.2.1.2), Babuška & Guo (1988), Dauge (1988). Furthermore, we assume that the Dirichlet boundary data satisfies

$$g = u|_\Gamma \in H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma).$$

Then if $\boldsymbol{\alpha} \in [0, 1/2]^M$, the embedding $H^{1/2,1/2}_{\boldsymbol{\alpha}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous (this follows from Kufner 1985, Theorem 9.15), and hence, $g \in L^2(\Gamma)$.

We start the development of the (enhanced) L^2-norm *a posteriori* error estimate by writing

$$\|e_{\mathrm{DG}}\|^2_{0,\Omega} = \int_\Omega (-\Delta\phi + c\phi)e_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega (-\Delta\phi + c\phi)u\,\mathrm{d}\boldsymbol{x} - \int_\Omega (-\Delta\phi + c\phi)u_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x}.$$

Applying Lemma 2.6 in the first integral and integrating by parts elementwise in the second integral, noting that $[\![\nabla\phi]\!] = 0$ on $\mathscr{E}_{\mathscr{I}}$ (since $\phi \in H^2(\Omega)$), results in

$$\begin{aligned}
\|e_{\mathrm{DG}}\|^2_{0,\Omega} &= \int_\Omega (\nabla u \cdot \nabla\phi + cu\phi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}} \cdot \nabla\phi + cu_{\mathrm{DG}}\phi)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\mathscr{E}_{\mathscr{I}}} \nabla\phi \cdot [\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi \cdot \boldsymbol{n})(u - u_{\mathrm{DG}})\,\mathrm{d}s \\
&= \int_\Omega f\phi\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}} \cdot \nabla\phi + cu_{\mathrm{DG}}\phi)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle \nabla\phi \rangle\!\rangle \cdot [\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi \cdot \boldsymbol{n})(g - u_{\mathrm{DG}})\,\mathrm{d}s.
\end{aligned}$$

Moreover, for an arbitrary function $\phi_h \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$, exploiting (3.5) with $v = \phi_h$, gives

$$\begin{aligned}
\|e_{\mathrm{DG}}\|^2_{0,\Omega} &= \int_\Omega f(\phi - \phi_h)\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}} \cdot \nabla_h(\phi - \phi_h) + cu_{\mathrm{DG}}(\phi - \phi_h))\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle \nabla\phi \rangle\!\rangle \cdot [\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi \cdot \boldsymbol{n})(g - u_{\mathrm{DG}})\,\mathrm{d}s \\
&\quad + \int_{\mathscr{E}_{\mathscr{B}}} (\nabla_h\phi_h \cdot \boldsymbol{n})g\,\mathrm{d}s - \gamma \int_{\mathscr{E}_{\mathscr{B}}} \sigma g\phi_h\,\mathrm{d}s - \int_{\mathscr{E}} \langle\!\langle \nabla_h u_{\mathrm{DG}} \rangle\!\rangle \cdot [\![\phi_h]\!]\,\mathrm{d}s \\
&\quad - \int_{\mathscr{E}} \langle\!\langle \nabla_h\phi_h \rangle\!\rangle \cdot [\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s + \gamma \int_{\mathscr{E}} \sigma [\![u_{\mathrm{DG}}]\!] \cdot [\![\phi_h]\!]\,\mathrm{d}s.
\end{aligned}$$

Using Green's formula in the second integral leads to

$$\begin{aligned}\int_\Omega \nabla_h u_{\mathrm{DG}} \cdot \nabla_h(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \Delta_h u_{\mathrm{DG}}(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} + \sum_{K\in\mathscr{T}_h}\int_{\partial K}(\nabla_h u_{\mathrm{DG}}\cdot \boldsymbol{n}_K)(\phi-\phi_h)\,\mathrm{d}s\\ &= -\int_\Omega \Delta_h u_{\mathrm{DG}}(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} + \int_{\mathscr{E}} \langle\!\langle \nabla_h u_{\mathrm{DG}}\rangle\!\rangle \cdot [\![\phi-\phi_h]\!]\,\mathrm{d}s\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\mathrm{d}s,\end{aligned}$$

where Δ_h is the elementwise Laplace operator. Hence, using that $[\![\phi]\!]=\mathbf{0}$ on $\mathscr{E}$, yields

$$\begin{aligned}\|e_{\mathrm{DG}}\|_{0,\Omega}^2 &= \int_\Omega (f+\Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}})(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\,\mathrm{d}s\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla\phi\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}}(\nabla\phi\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s + \int_{\mathscr{E}_{\mathscr{B}}}(\nabla_h\phi_h\cdot\boldsymbol{n})g\,\mathrm{d}s\\ &\quad - \gamma\int_{\mathscr{E}_{\mathscr{B}}}\sigma g\phi_h\,\mathrm{d}s - \int_{\mathscr{E}}\langle\!\langle\nabla_h\phi_h\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s + \gamma\int_{\mathscr{E}}\sigma[\![u_{\mathrm{DG}}]\!]\cdot[\![\phi_h]\!]\,\mathrm{d}s\\ &= \int_\Omega (f+\Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}})(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\mathrm{d}s\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla_h(\phi-\phi_h)\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}}(\nabla_h(\phi-\phi_h)\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s\\ &\quad - \gamma\int_{\mathscr{E}_{\mathscr{B}}}\sigma(g-u_{\mathrm{DG}})(\phi_h-\phi)\,\mathrm{d}s + \gamma\int_{\mathscr{E}_{\mathscr{I}}}\sigma[\![u_{\mathrm{DG}}]\!]\cdot[\![\phi_h-\phi]\!]\,\mathrm{d}s.\end{aligned}$$

Now, applying the Cauchy–Schwarz inequality and noting that $p_K \geqslant 1$, $K\in\mathscr{T}_h$, gives

$$\begin{aligned}\|e_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant \Bigg(&\sum_{K\in\mathscr{T}_h} h_K^4 p_K^{-4}\|f+\Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}}\|_{0,K}^2 + \sum_{K\in\mathscr{T}_h} h_K^3 p_K^{-3}\|[\![\nabla_h u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\Gamma}^2\\ &+(\gamma^2+1)\sum_{K\in\mathscr{T}_h} h_K p_K\|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\Gamma}^2\\ &+(\gamma^2+1)\sum_{K\in\mathscr{T}_h} h_K p_K\|g-u_{\mathrm{DG}}\|_{0,\partial K\cap\Gamma}^2\Bigg)^{\frac{1}{2}}\\ \times\Bigg(&\sum_{K\in\mathscr{T}_h} h_K^{-4}p_K^4\,\|\phi-\phi_h\|_{0,K}^2 + \sum_{K\in\mathscr{T}_h} h_K^{-3}p_K^3\,\|\phi-\phi_h\|_{0,\partial K}^2\\ &+\sum_{K\in\mathscr{T}_h} h_K^{-1}p_K\,\|\nabla_h(\phi-\phi_h)\|_{0,\partial K}^2\Bigg)^{\frac{1}{2}}.\end{aligned}$$

Then choosing $\phi_h \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ to be an elementwise optimal hp-interpolant (see, e.g., Babuška & Suri, 1987a,b), i.e., for any $K \in \mathscr{T}_h$,

$$h_K^{-4} p_K^4 \|\phi - \phi_h\|_{0,K}^2 + h_K^{-3} p_K^3 \|\phi - \phi_h\|_{0,\partial K}^2 + h_K^{-1} p_K \|\nabla_h(\phi - \phi_h)\|_{0,\partial K}^2 \leqslant C \, \|\phi\|_{H^2(K)}^2 ,$$

and recalling the regularity estimate (3.12) gives

$$\|e_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant C \|e_{\mathrm{DG}}\|_{0,\Omega} \left(\sum_{K \in \mathscr{T}_h} \widetilde{\eta}_K^2 \right)^{\frac{1}{2}} ,$$

with

$$\begin{aligned} \widetilde{\eta}_K^2 = {} & h_K^4 p_K^{-4} \| f + \Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}} \|_{0,K}^2 + h_K^3 p_K^{-3} \| [\![\nabla_h u_{\mathrm{DG}}]\!] \|_{0,\partial K \setminus \Gamma}^2 \\ & + h_K p_K \| [\![u_{\mathrm{DG}}]\!] \|_{0,\partial K \setminus \Gamma}^2 + h_K p_K \| g - u_{\mathrm{DG}} \|_{0,\partial K \cap \Gamma}^2 . \end{aligned}$$

Hence, dividing both sides of the above inequality by $\|e_{\mathrm{DG}}\|_{0,\Omega}$ leads to

$$\|e_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant C \sum_{K \in \mathscr{T}_h} \widetilde{\eta}_K^2 .$$

Furthermore, since $u \in H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ it holds that $[\![u]\!] = 0$ on all interior edges of $\mathscr{T}_h$, see Wihler (2002, Lemma 1.3.4). Consequently,

$$\| [\![e_{\mathrm{DG}}]\!] \|_{0,e} = \| [\![u_{\mathrm{DG}} - u]\!] \|_{0,e} = \| [\![u]\!] - [\![u_{\mathrm{DG}}]\!] \|_{0,e} = \| [\![u_{\mathrm{DG}}]\!] \|_{0,e} \tag{3.13}$$

for any $e \in \mathscr{E}_{\mathscr{I}}$, and

$$\| [\![e_{\mathrm{DG}}]\!] \|_{0,e} = \| [\![u_{\mathrm{DG}} - g]\!] \|_{0,e} \tag{3.14}$$

if $e \in \mathscr{E}_{\mathscr{B}}$.

Hence, denoting by $\Pi_h f$ the elementwise L^2-projection into $V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$, we obtain the following result.

THEOREM 3.2 Suppose that the dual problem (3.9)–(3.10) fulfils (3.12), and that the Dirichlet boundary data $g \in H_{\boldsymbol{\alpha}}^{1/2,1/2}(\Gamma)$, for some weight vector $\boldsymbol{\alpha} \in [0, 1/2]^M$. Furthermore, let $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ denote the hp-DG solution from (3.5), and $u \in H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ the analytical weak solution of (1.1)–(1.2). Then the following *a posteriori* error estimate holds

$$|\!|\!| u - u_{\mathrm{DG}} |\!|\!|_{0,h,\Omega}^2 \leqslant C \left(\sum_{K \in \mathscr{T}_h} \eta_K^2 + \sum_{K \in \mathscr{T}_h} h_K^4 p_K^{-4} \| f - \Pi_h f \|_{0,K}^2 \right) ,$$

where the local error indicators η_K, $K \in \mathscr{T}_h$, are defined by

$$\begin{aligned} \eta_K^2 = {} & h_K^4 p_K^{-4} \| \Pi_h f + \Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}} \|_{0,K}^2 + h_K^3 p_K^{-3} \| [\![\nabla_h u_{\mathrm{DG}}]\!] \|_{0,\partial K \setminus \Gamma}^2 \\ & + h_K p_K \| [\![u_{\mathrm{DG}}]\!] \|_{0,\partial K \setminus \Gamma}^2 + h_K p_K \| g - u_{\mathrm{DG}} \|_{0,\partial K \cap \Gamma}^2 , \end{aligned} \tag{3.15}$$

and $C > 0$ is a constant independent of the local element sizes $\boldsymbol{h}$ and polynomial degrees $\boldsymbol{p}$.

REMARK 3.3 We point out that Theorem 3.2 provides an upper bound on the error $u - u_{\mathrm{DG}}$ measured in terms of the enhanced L^2-norm $|\!|\!| \cdot |\!|\!|_{0,h,\Omega}$ from (3.4). This norm has been exploited since it allows for the derivation of local lower bounds; this topic will be addressed in the Section 3.3.2.

3.3.2 *Local lower bound.* In order to derive local lower bounds we make the simplifying assumptions that the mesh $\mathscr{T}_h$ consists of quadrilaterals only and that the coefficient c from (1.1) is elementwise constant, i.e.,

$$c|_K = c_K \in \mathbb{Q}_0(K) \quad \forall\, K \in \mathscr{T}_h.$$

THEOREM 3.4 Let $K, K' \in \mathscr{T}_h$ be any two neighbouring elements, $e = \partial K \cap \partial K' \in \mathscr{E}_{\mathscr{I}}$, and $\omega_e = (\overline{K} \cup \overline{K'})^\circ$. Furthermore, $\epsilon \in (0, 1/2]$. Then the following local hp-version *a posteriori* lower bounds on the error e_{DG} from (3.11) hold:

(a) $h_K^2\, p_K^{-2} \| \Delta u_{\mathrm{DG}} - c_K u_{\mathrm{DG}} + \Pi_h f \|_{0,K} \leqslant C_\epsilon \left(p_K^{7/2+\epsilon} \|e_{\mathrm{DG}}\|_{0,K} + h_K^2\, p_K^{1+2\epsilon} \| f - \Pi_h f \|_{0,K} \right)$;

(b) $h_K^{3/2}\, p_K^{-3/2} \| [\![\nabla_h u_{\mathrm{DG}}]\!] \|_{0,e}$
$\leqslant C_\epsilon \left(p_K^{11/2+2\epsilon} \|e_{\mathrm{DG}}\|_{0,\omega_e} + h_K^{1/2}\, p_K^{1/2} \| [\![e_{\mathrm{DG}}]\!] \|_{0,e} + h_K^2\, p_K^{3+3\epsilon} \| f - \Pi_h f \|_{0,\omega_e} \right)$;

(c) for any $e \in \mathscr{E}_{\mathscr{B}}$ we have that

$$p_K^{1/2} h_K^{1/2} \|u_{\mathrm{DG}} - g\|_{0,e} = p_K^{1/2} h_K^{1/2} \|e_{\mathrm{DG}}\|_{0,e},$$

and for $e \in \mathscr{E}_{\mathscr{I}}$ that

$$p_K^{1/2} h_K^{1/2} \| [\![u_{\mathrm{DG}}]\!] \|_{0,e} = p_K^{1/2} h_K^{1/2} \| [\![e_{\mathrm{DG}}]\!] \|_{0,e}.$$

Here the constant $C_\epsilon > 0$ is independent of $\boldsymbol{h}$ and $\boldsymbol{p}$.

Before proving these estimates we introduce the following auxiliary results.

LEMMA 3.5 Let $\widehat{K} = (0, 1)^2$ be the unit square, and $\epsilon \in (0, 5/2]$. We define the cut-off function

$$\widehat{B}_{\widehat{K}}^{\beta}(\widehat{x}, \widehat{y}) = \widehat{x}^{\beta}(1 - \widehat{x})^{\beta} \widehat{y}^{\beta}(1 - \widehat{y})^{\beta},$$

where $\beta = \frac{3}{2} + \epsilon$. Then $\widehat{B}_{\widehat{K}}^{\beta} = 0$ and $\nabla \widehat{B}_{\widehat{K}}^{\beta} = \mathbf{0}$ on $\partial \widehat{K}$, and $\Delta \widehat{B}_{\widehat{K}}^{\beta} \in L^2(\widehat{K})$. Furthermore, for any $\widehat{v} \in \mathbb{Q}_p(\widehat{K})$, $p \geqslant 1$, we have

$$\left\| \Delta \left(\widehat{B}_{\widehat{K}}^{\beta} \widehat{v} \right) \right\|_{0,\widehat{K}} \leqslant C_\epsilon\, p^{4-\beta} \left\| \widehat{B}_{\widehat{K}}^{\beta/2} \widehat{v} \right\|_{0,\widehat{K}},$$

and

$$\|\widehat{v}\|_{0,\widehat{K}} \leqslant C_\epsilon\, p^{2\beta} \left\| \widehat{B}_{\widehat{K}}^{\beta/2} \widehat{v} \right\|_{0,\widehat{K}},$$

with a constant $C_\epsilon > 0$ independent of p and of $\widehat{v}$.

Proof. The vanishing boundary value properties follow immediately from the definition of $\widehat{B}_{\widehat{K}}^{\beta}$. Furthermore, the estimates result from tensorizing corresponding one-dimensional (1D) results; see, e.g., Bernardi *et al.* (2001, Lemmas 4 and 5). □

LEMMA 3.6 We consider the unit square $\widehat{K} = (0, 1)^2$, and $0 < \widehat{a} < 1$. Furthermore, let $\widehat{v} \in \mathbb{P}_p(0, 1)$, $p \geqslant 1$, and

$$\widehat{b}^{\beta}(\widehat{y}) = \widehat{y}^{\beta}(1 - \widehat{y})^{\beta}$$

a bubble function, with $\beta = \frac{3}{2} + \epsilon$, $\epsilon \in (0, 1/2]$. Then there exists a cut-off function $\widehat{\chi} \in C^1(\widehat{K})$ with

$$\widehat{\chi}|_{\partial\widehat{K}} = 0, \quad \nabla\widehat{\chi}|_{\partial\widehat{K}} = \mathbf{0},$$

and

$$\widehat{\chi}|_{\{\widehat{a}\}\times(0,1)} = \widehat{b}^{\beta}\widehat{v}|_{(0,1)}.$$

Furthermore, there hold the estimates

$$\|\widehat{\chi}\|_{0,\widehat{K}} \leqslant C_{\widehat{a},\epsilon}\left\|\widehat{b}^{\beta/2}\widehat{v}\right\|_{0,(0,1)}, \quad \|\Delta\widehat{\chi}\|_{0,\widehat{K}} \leqslant C_{\widehat{a},\epsilon}\, p^{4-\beta}\left\|\widehat{b}^{\beta/2}\widehat{v}\right\|_{0,(0,1)}, \tag{3.16}$$

and

$$\|\widehat{v}\|_{0,(0,1)} \leqslant C_{\widehat{a},\epsilon}\, p^{\beta}\left\|\widehat{b}^{\beta/2}\widehat{v}\right\|_{0,(0,1)}, \quad \|\nabla\widehat{\chi}\|_{0,(0,1)} \leqslant C_{\widehat{a},\epsilon}\, p^{2-\beta}\left\|\widehat{b}^{\beta/2}\widehat{v}\right\|_{0,(0,1)}.$$

Here $C_{\widehat{a},\epsilon} > 0$ is a constant independent of $\widehat{v}$ and p.

Proof. On $\widehat{K}$ we define

$$\widehat{\chi}(\widehat{x},\widehat{y}) = \widehat{b}^{\beta}(\widehat{a})^{-1}\widehat{b}^{\beta}(\widehat{x})\widehat{b}^{\beta}(\widehat{y})\widehat{v}(\widehat{y}).$$

The lemma is again proved by referring to suitable 1D results from Bernardi *et al.* (2001, Lemmas 4 and 5). □

Proof of Theorem 3.4. We show each of the bounds (a)–(c) in a separate step.

Proof of (*a*). For $K \in \mathscr{T}_h$ let us define the volume residual

$$R_K = \Delta u_{\mathrm{DG}} - c_K u_{\mathrm{DG}} + \Pi_h f \in \mathbb{Q}_{p_K}(K).$$

Then, for $\beta = 3/2 + \epsilon$, $\epsilon \in (0, 5/2]$, using Lemma 3.5, together with a scaling argument, there exists a cut-off function B_K^{β} satisfying $B_K^{\beta}|_{\partial K} = 0$ and $\nabla B_K^{\beta}|_{\partial K} = \mathbf{0}$, and

$$\left\|\Delta\left(B_K^{\beta} R_K\right)\right\|_{0,K} \leqslant C_{\epsilon} h_K^{-2} p_K^{4-\beta}\left\|B_K^{\beta/2} R_K\right\|_{0,K}, \tag{3.17}$$

and

$$\|R_K\|_{0,K} \leqslant C_{\epsilon}\, p_K^{2\beta}\left\|B_K^{\beta/2} R_K\right\|_{0,K}, \tag{3.18}$$

with a constant $C_{\epsilon} > 0$ independent of h_K and p_K. Moreover, there holds

$$\begin{aligned}\left\|B_K^{\beta/2} R_K\right\|_{0,K}^2 &= \int_K B_K^{\beta} R_K(\Delta u_{\mathrm{DG}} - c_K u_{\mathrm{DG}} + \Pi_h f)\,\mathrm{d}\boldsymbol{x}\\ &= -\int_K B_K^{\beta} R_K(\Delta e_{\mathrm{DG}} - c_K e_{\mathrm{DG}} + f - \Pi_h f)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Note that since $\Delta e_{\mathrm{DG}} = \Delta u - \Delta u_{\mathrm{DG}} = c_K u - f - \Delta u_{\mathrm{DG}} \in L^2(K)$ we may apply Lemma 2.7 to see that

$$-\int_K B_K^{\beta} R_K\, \Delta e_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x} = \int_K \nabla(B_K R_K)\cdot\nabla e_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x}.$$

Here we observe that

$$\nabla\left(B_K^{\beta} R_K\right)\big|_{\partial K} = B_K^{\beta}\big|_{\partial K}\nabla R_K\big|_{\partial K} + R_K|_{\partial K}\nabla B_K^{\beta}\big|_{\partial K} = \mathbf{0}.$$

Hence, integrating by parts once again and recalling (3.17), yields

$$\left|\int_K B_K^\beta R_K \Delta e_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x}\right| = \left|\int_K e_{\mathrm{DG}} \Delta\Big(B_K^\beta R_K\Big)\mathrm{d}\boldsymbol{x}\right| \leqslant \|e_{\mathrm{DG}}\|_{0,K}\left\|\Delta\Big(B_K^\beta R_K\Big)\right\|_{0,K}$$
$$\leqslant C_\epsilon h_K^{-2} p_K^{4-\beta}\|e_{\mathrm{DG}}\|_{0,K}\left\|B_K^{\beta/2}R_K\right\|_{0,K}.$$

Moreover,

$$\left|c_K\int_K B_K^\beta R_K e_{\mathrm{DG}}\,\mathrm{d}\boldsymbol{x}\right| \leqslant |c_K|\left\|B_K^\beta R_K\right\|_{0,K}\|e_{\mathrm{DG}}\|_{0,K} \leqslant C\left\|B_K^{\beta/2}R_K\right\|_{0,K}\|e_{\mathrm{DG}}\|_{0,K}.$$

Similarly,

$$\left|\int_K B_K^\beta R_K(\Pi_h f - f)\,\mathrm{d}\boldsymbol{x}\right| \leqslant \left\|B_K^\beta R_K\right\|_{0,K}\|\Pi_h f - f\|_{0,K} \leqslant C\left\|B_K^{\beta/2}R_K\right\|_{0,K}\|\Pi_h f - f\|_{0,K}.$$

Therefore,

$$\left\|B_K^{\beta/2}R_K\right\|_{0,K}^2 \leqslant C_\epsilon\left\|B_K^{\beta/2}R_K\right\|_{0,K}\Big(h_K^{-2}p_K^{4-\beta}\|e_{\mathrm{DG}}\|_{0,K} + \|\Pi_h f - f\|_{0,K}\Big).$$

Dividing the above inequality by $\|B_K^{\beta/2}R_K\|_{0,K}$ and employing (3.18) we deduce the inequality in (a).

Proof of (*b*). We define, for any $e \in \mathscr{E}_{\mathscr{I}}$, the edge residual

$$E_e = [\![\nabla_h u_{\mathrm{DG}}]\!] \in \mathbb{P}_{p_e}(e),$$

where $p_e := \max(p_K, p_{K'})$. For simplicity, we make the assumption that $\mathscr{T}_h$ is a regular mesh (otherwise, the mesh may be suitably regularized, see Houston *et al.*, 2008, Remark 3.9). In this case ω_e can be affinely mapped to the unit square $\widehat{K} = (0,1)^2$. By the same mapping the intersection $e = (\partial K \cap \partial K')^\circ$ is transformed to a unit edge $\{\widehat{a}\} \times (0,1) \subset \widehat{K}$, with $0 < \widehat{a} < 1$, i.e., K is mapped to $(0,\widehat{a}) \times (0,1)$ and K' is mapped to $(\widehat{a},1) \times (0,1)$. Hence, we may apply Lemma 3.6 (with $\epsilon \in (0, 1/2]$) to obtain a cut-off function $\chi_e \in C^1(\omega_e)$ and a bubble function b_e^β on e with

$$\chi_e|_{\partial\omega_e} = 0, \quad \nabla\chi_e|_{\partial\omega_e} = \mathbf{0},$$

and

$$\|\chi_e\|_{0,\omega_e} \leqslant C h_K^{1/2}\left\|b_e^{\beta/2}E_e\right\|_{0,e}, \quad \|\Delta\chi_e\|_{0,\omega_e} \leqslant C h_K^{-3/2}p_K^{4-\beta}\left\|b_e^{\beta/2}E_e\right\|_{0,e}. \tag{3.19}$$

Note that, due to (3.1) and (3.2), it holds that $h_K \sim h_{K'}$, and $p_e \sim p_K \sim p_{K'}$. Moreover,

$$\|E_e\|_{0,e} \leqslant C p_K^\beta\left\|b_e^{\beta/2}E_e\right\|_{0,e}, \quad \|\nabla\chi_e\|_{0,e} \leqslant C h_K^{-1}p_K^{2-\beta}\left\|b_e^{\beta/2}E_e\right\|_{0,e}. \tag{3.20}$$

Here $C > 0$ is a constant independent of E_e, p_K and $p_{K'}$. Also, we have that $\widehat{a} \sim h_K/h_K + h_{K'}$, which, with (3.1), is bounded away from 0 and 1 (independently of $\boldsymbol{h}$), and hence, C does not depend on the element sizes either. Furthermore, we have that

$$\left\|b_e^{\beta/2}E_e\right\|_{0,e}^2 = \int_e \chi_e E_e\,\mathrm{d}s = \int_{\partial K}\chi_e(\nabla_h u_{\mathrm{DG}}\cdot\boldsymbol{n}_K)\mathrm{d}s + \int_{\partial K'}\chi_e(\nabla_h u_{\mathrm{DG}}\cdot\boldsymbol{n}_{K'})\,\mathrm{d}s.$$

Elementwise integration by parts yields

$$\left\| b_e^{\beta/2} E_e \right\|_{0,e}^2 = \int_{\omega_e} \nabla_h \chi_e \cdot \nabla_h u_{\mathrm{DG}} \, \mathrm{d}\boldsymbol{x} + \int_{\omega_e} \chi_e \Delta_h u_{\mathrm{DG}} \, \mathrm{d}\boldsymbol{x}.$$

Using Lemma 2.7 and integrating by parts again we obtain that

$$\begin{aligned}
\left\| b_e^{\beta/2} E_e \right\|_{0,e}^2 &= -\int_{\omega_e} \nabla_h \chi_e \cdot \nabla_h e_{\mathrm{DG}} \, \mathrm{d}\boldsymbol{x} - \int_{\omega_e} \chi_e \Delta_h e_{\mathrm{DG}} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\omega_e} e_{\mathrm{DG}} \Delta_h \chi_e \, \mathrm{d}\boldsymbol{x} - \int_e \nabla_h \chi_h \cdot [\![e_{\mathrm{DG}}]\!] \, \mathrm{d}s - \int_{\omega_e} c \chi_e e_{\mathrm{DG}} \, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\omega_e} \chi_e (\Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}} + \Pi_h f) \, \mathrm{d}\boldsymbol{x} + \int_{\omega_e} \chi_e (f - \Pi_h f) \, \mathrm{d}\boldsymbol{x} \\
&\leqslant C \big(\| e_{\mathrm{DG}} \|_{0,\omega_e} \| \Delta_h \chi_e \|_{0,\omega_e} + \| \nabla_h \chi_h \|_{0,e} \| [\![e_{\mathrm{DG}}]\!] \|_{0,e} + \| e_{\mathrm{DG}} \|_{0,\omega_e} \| \chi_e \|_{0,\omega_e} \\
&\quad + \| \Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}} + \Pi_h f \|_{0,\omega_e} \| \chi_e \|_{0,\omega_e} + \| f - \Pi_h f \|_{0,\omega_e} \| \chi_e \|_{0,\omega_e} \big).
\end{aligned}$$

Applying (3.19), (3.20) and using the bound from (a) for the elementwise volume residual results in (b).

Proof of (*c*). This follows directly from (3.13) and (3.14). □

REMARK 3.7 We note that the dependence of the lower bounds in Theorem 3.4 is suboptimal with respect to the polynomial degrees. This effect has been observed earlier in the *a posteriori* error analysis of hp-methods; see, e.g., Melenk & Wohlmuth (2001) and Houston *et al.* (2007, 2008). We remark, however, that the p-suboptimality is less pronounced in *energy norm* lower bounds. We also mention the alternative approach presented in Braess *et al.* (2009) in the context of spectral methods.

3.4 *Numerical example*

On the rectangle $\Omega = (-1, 1) \times (0, 1)$ we consider the PDE problem: find u such that

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \Gamma. \end{aligned}$$

We choose the Dirichlet boundary data g in such a way that the analytical solution is given by

$$u(r, \theta) = \frac{1}{\pi} \theta,$$

where (r, θ) denote polar coordinates in $\mathbb{R}^2$. Note that g is smooth on Γ except at the point $(0, 0)$. Indeed, in Cartesian coordinates, we have that

$$g(x, y = 0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad (x, y) \in \Gamma.$$

In addition, we remark that $u \notin H^1(\Omega)$. However, there holds $u \in H_\alpha^{1,1}(\Omega)$ for any $\alpha \in (0, 1)$, where the weight function for this problem is given by $\Phi_\alpha(\boldsymbol{x}) = |\boldsymbol{x}|^\alpha$. Furthermore, u is analytic away from $(0, 0)$ and belongs to the Babuška–Guo space (see, e.g., Babuška & Guo, 1988)

$$B_{\boldsymbol{\alpha}}^1(\Omega) = \left\{ v \in L^2(\Omega) \colon |v|_{H_{\boldsymbol{\alpha}}^{k,1}(\Omega)} \leqslant C d^k k! \quad \forall k \geqslant 1, \text{ and constants } C, d > 0 \right\}.$$

With this in mind, we might therefore be able to achieve exponential convergence when hp-refinement is employed; cf. Schötzau & Schwab (2001).

Firstly, however, we investigate the practical performance of the *a posteriori* error estimate derived in Theorem 3.2 within an automatic h-version adaptive refinement procedure which is based on 1-irregular quadrilateral elements. The h-adaptive meshes are constructed by marking the elements for refinement/derefinement according to the size of the local error indicators η_K; this is done by employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively.

In Fig. 1(a) we show the initial mesh and computed DG solution based on employing $p = 2$, i.e., biquadratic polynomials. Furthermore, the computational mesh and DG solution are depicted in Fig. 1(b and c) after 4 and 9 adaptive refinements have been undertaken, respectively. Here we observe that the

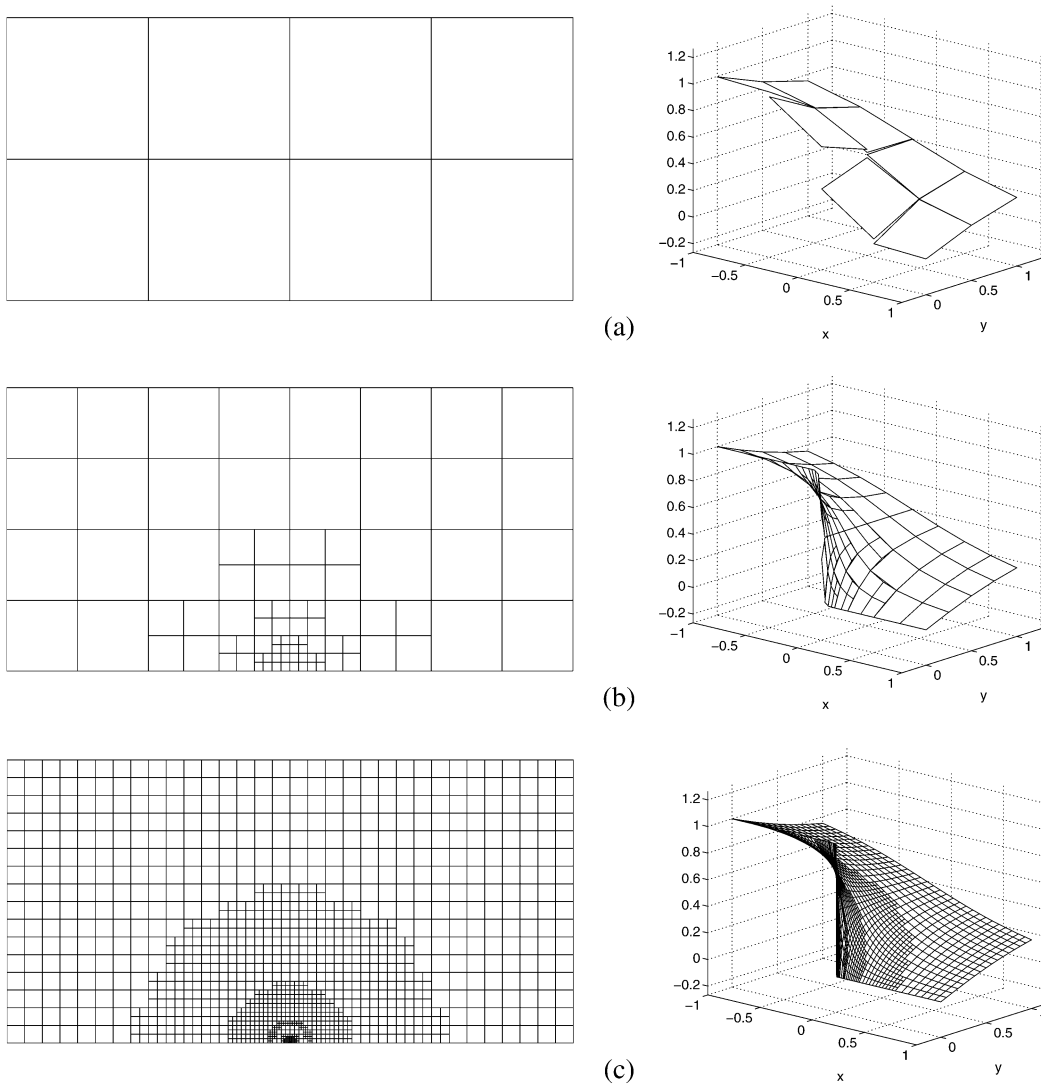

FIG. 1. h-Refinement. (a) Initial mesh and solution with 8 elements; mesh and solution after: (b) 4 adaptive refinements with 86 elements and (c) 9 adaptive refinements with 1286 elements.

mesh has been significantly refined in the vicinity of the discontinuity present in g as we would expect. Figure 2(a) shows the history of the actual and estimated $|||\cdot|||_{0,h,\Omega}$-norm of the error on each of the meshes generated based on employing h-adaptive refinement. Here we observe that the *a posteriori* bound over estimates the true error by a consistent factor. Indeed, the effectivity index tends to a value of around 1.5 as the mesh is adaptively refined, cf. Fig. 2(b).

We now turn our attention to hp-mesh adaptation. Here we again mark elements for refinement/derefinement according to the size of the local error indicators η_K based on employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively. Once an element $K \in \mathscr{T}_h$ has been flagged for refinement or derefinement, a decision must be made whether the local mesh size h_K or the local degree p_K of the approximating polynomial should be adjusted accordingly. The choice to perform either h-refinement/derefinement or p-refinement/derefinement is based on estimating the local smoothness of the (unknown) analytical solution. To this end, we employ the hp-adaptive strategy developed in Houston & Süli (2005), where the local regularity of the analytical solution is estimated from truncated local Legendre expansions of the computed numerical solution; see, also, Houston *et al.* (2003).

In Fig. 3(a) we present a comparison of the actual and estimated $|||\cdot|||_{0,h,\Omega}$-norm of the error versus the third root of the number of degrees of freedom in the finite element space $V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ on a linear-log scale for the sequence of meshes generated by our hp-adaptive algorithm. We remark that the third root of the number of degrees of freedom is chosen on the basis of the *a priori* error analysis carried out in Wihler *et al.* (2003); cf., also, Schötzau & Wihler (2003). Here we observe that the error bound over estimates the true error by a (reasonably) consistent factor; indeed, from Fig. 3(b), we see that the computed effectivity indices are in the range 1.5–2.5 as the mesh is refined. Moreover, from Fig. 3(a), we observe that the convergence lines using hp-refinement are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this problem.

In Fig. 4 we present a comparison between the actual error employing both h- and hp-refinement. In particular, we compute both the enhanced L^2-norm $|||u - u_{\mathrm{DG}}|||_{0,h,\Omega}$ for which the proposed *a posteriori* error indicators have been derived as well as $\|u - u_{\mathrm{DG}}\|_{H^{1,1}_{1/2}(\Omega,\mathscr{T}_h)}$. Here the norm $\|\cdot\|_{H^{1,1}_{1/2}(\Omega,\mathscr{T}_h)}$

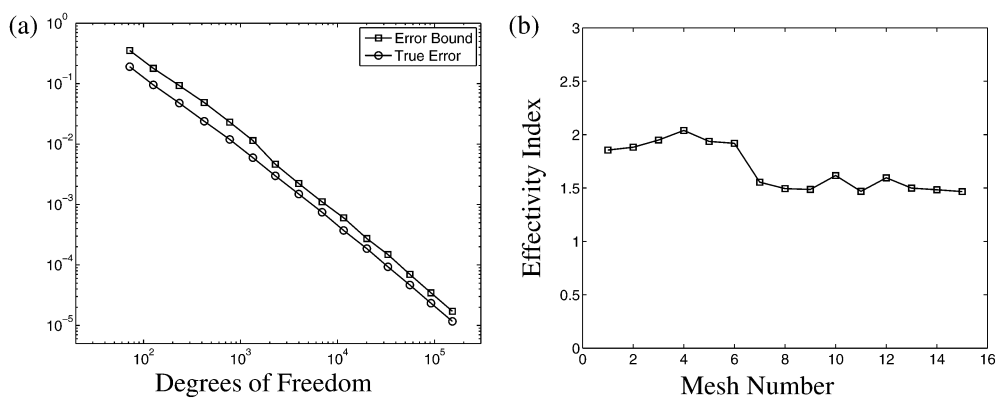

FIG. 2. h-Refinement. (a) Comparison of the actual and estimated $|||\cdot|||_{0,h,\Omega}$-norm of the error with respect to the number of degrees of freedom and (b) effectivity indices.

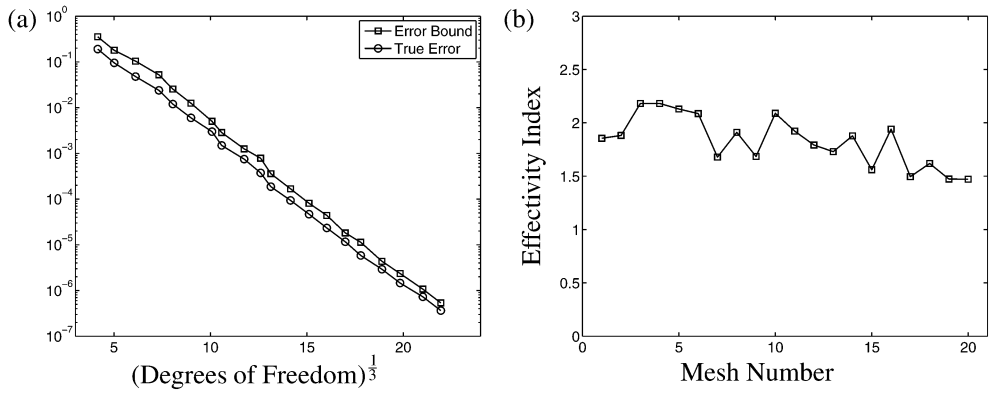

FIG. 3. hp-Refinement. (a) Comparison of the actual and estimated $|||\cdot|||_{0,h,\Omega}$-norm of the error with respect to the (third root of the) number of degrees of freedom and (b) effectivity indices.

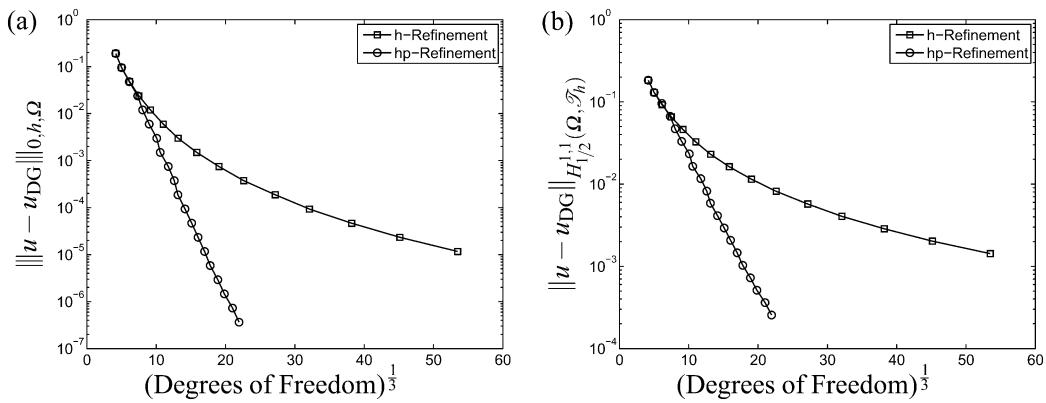

FIG. 4. Comparison between h- and hp-refinement: (a) $|||u - u_{\mathrm{DG}}|||_{0,h,\Omega}$ and (b) $\|u - u_{\mathrm{DG}}\|_{H^{1,1}_{1/2}(\Omega,\mathcal{T}_h)}$.

represents the broken variant of the $H^{1,1}_{1/2}(\Omega)$-norm, i.e.,

$$\|v\|^2_{H^{1,1}_{1/2}(\Omega,\mathcal{T}_h)} = \|v\|^2_{0,\Omega} + \|\Phi_{1/2}\nabla_h v\|^2_{0,\Omega}.$$

In both cases we clearly observe the superiority of employing a grid adaptation strategy based on exploiting hp-adaptive refinement: on the final mesh, $|||u - u_{\mathrm{DG}}|||_{0,h,\Omega}$ computed using hp-refinement is around three orders of magnitude smaller than the corresponding quantity computed when h-refinement is employed alone; an improvement of almost two orders of magnitude in $\|u - u_{\mathrm{DG}}\|_{H^{1,1}_{1/2}(\Omega,\mathcal{T}_h)}$ is observed when using hp-refinement in contrast to h-refinement. In addition, this demonstrates, for at least the problem at hand, that the proposed error indicator is capable of automatically guiding the adaptive process to ensure convergence of the error measured in the $H^{1,1}_{1/2}(\Omega, \mathcal{T}_h)$-norm. In general, however, a reliable upper bound on the the error computed in terms of the $H^{1,1}_{1/2}(\Omega, \mathcal{T}_h)$-norm would be necessary to ensure convergence of the adaptive algorithm, cf. below.

Finally, in Figs. 5 and 6, we show the mesh generated using the proposed hp-version *a posteriori* error indicator stated in Theorem 3.2 after 9 and 14 hp-adaptive refinement steps, respectively. For clarity, we also show the h-mesh alone as well as a zoom of the mesh in the vicinity of the origin. Here we observe that geometric h-refinement has been performed in the vicinity of the discontinuity present

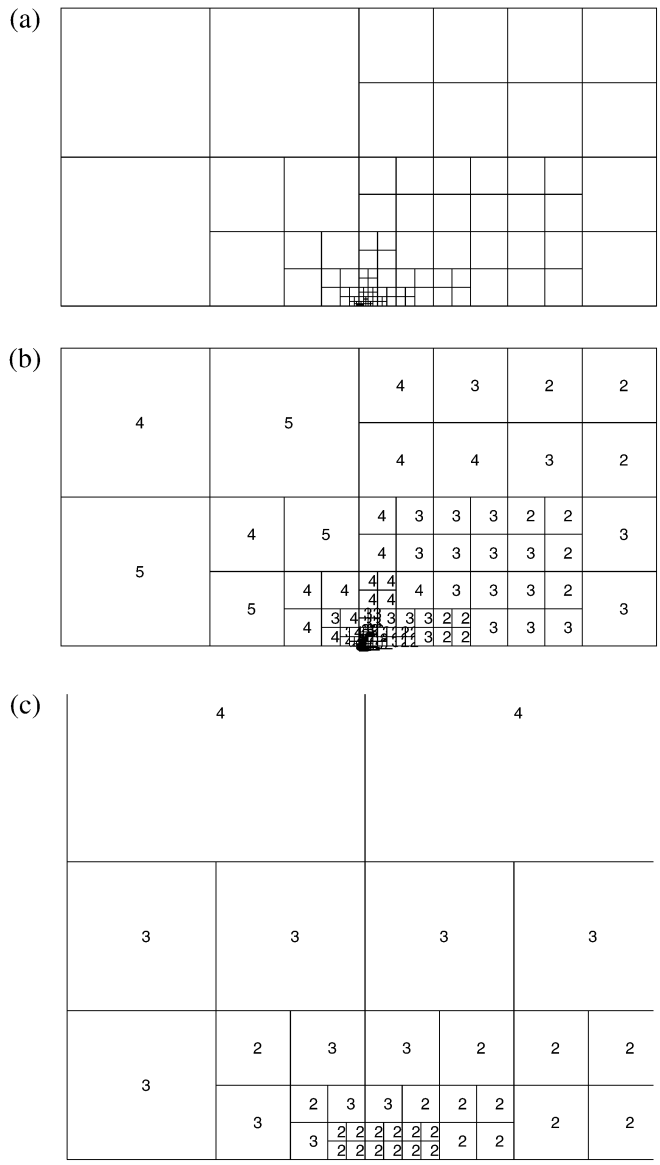

FIG. 5. hp-Mesh distribution after 9 adaptive refinements with 134 elements and 2002 degrees of freedom: (a) h-mesh alone, (b) hp-mesh and (c) zoom of (b).

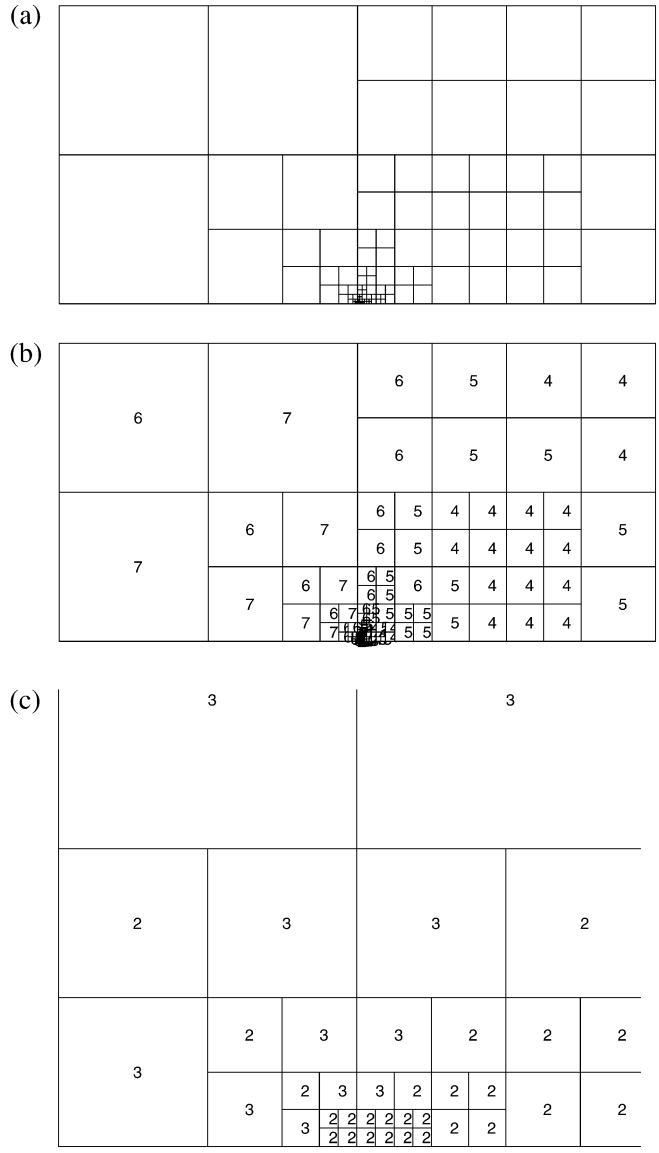

FIG. 6. *hp*-Mesh distribution after 14 adaptive refinements with 206 elements, 4904 degrees of freedom: (a) *h*-mesh alone, (b) *hp*-mesh and (c) zoom of (b).

in g, cf. above. Within this region, the polynomial degree has been kept at 2. Away from this region, the *hp*-adaptive algorithm increases the degree of the approximating piecewise polynomials where the analytical solution is smooth. This corresponds to the *a priori hp*-approximation strategies proposed in, e.g., Guo & Babuška (1986a,b), Schwab (1998); see further Schwab (1998), Schötzau & Schwab (2001), Wihler (2002), Schötzau & Wihler (2003), Wihler *et al.* (2003), Houston *et al.* (2004).

4. Conclusions

In this work we have introduced a new variational framework for linear second-order elliptic PDEs with discontinuous Dirichlet boundary conditions based on locally weighted Sobolev spaces. In particular, we have proved the well-posedness of the new setting by means of an inf-sup condition. In addition, we have proposed the use of symmetric hp-version interior penalty DG methods for the numerical approximation of such problems. For this discretization scheme we have derived an (enhanced) L^2-norm *a posteriori* error analysis featuring upper and lower estimates. The performance of the resulting error indicators within h- and hp-adaptive refinement procedures has been displayed with a model numerical experiment. Future work will involve an error analysis with respect to the $H^{1,1}_{\boldsymbol{\alpha}}$-norm, and some extensions of the present setting to systems such as, e.g., the Stokes equations for cavity flow problems.

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